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Coupled cells with internal symmetry: I. Wreath products

Benoit Dionne†, Martin Golubitsky‡ and Ian Stewart§

† Department of Mathematics, University of Ottawa, Ottawa, ON K1N 6N5, Cannda
‡ Department of Mathematics, University of Houston, Houston, TX 77204-3476, USA
§ Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

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Abstract. In this paper and its sequel we study arrays of coupled identical cells that possess a 'global' symmetry group $\mathcal{G}$, and in which the cells possess their own 'internal' symmetry group $\mathcal{L}$. We focus on general existence conditions for symmetry-breaking steady-state and Hopf bifurcations. The global and internal symmetries can combine in two quite different ways, depending on how the internal symmetries affect the coupling. Algebraically, the symmetries either combine to give the wreath product $\mathcal{L} \wr \mathcal{G}$ of the two groups or the direct product $\mathcal{L} \times \mathcal{G}$. Here we develop a theory for the wreath product: we analyse the direct product case in the accompanying paper (henceforth referred to as II).

The wreath product case occurs when the coupling is invariant under internal symmetries. The main objective of the paper is to relate the patterns of steady-state and Hopf bifurcation that occur in systems with the combined symmetry group $\mathcal{L} \wr \mathcal{G}$ to the corresponding bifurcations in systems with symmetry $\mathcal{L}$ or $\mathcal{G}$. This organizes the problem by reducing it to simpler questions whose answers can often be read off from known results.

The basic existence theorem for steady-state bifurcation is the equivariant branching lemma, which states that under appropriate conditions there will be a symmetry-breaking branch of steady states for any isotropy subgroup with a one-dimensional fixed-point subspace. We call such an isotropy subgroup *axial*. The analogous result for equivariant Hopf bifurcation involves isotropy subgroups with a two-dimensional fixed-point subspace, which we call **C**-*axial* because of an analogy involving a natural complex structure. Our main results are classification theorems for axial and **C**-axial subgroups in wreath products.

We study some typical examples, rings of cells in which the internal symmetry group is $\mathbf{O}(2)$ and the global symmetry group is dihedral. As these examples illustrate, one striking consequence of our general results is that systems with wreath product coupling often have states in which some cells are performing nontrivial dynamics, while others remain quiescent. We also discuss the common occurrence of heteroclinic cycles in wreath product systems.

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1. Internal and global symmetries

Arrays of coupled oscillators have been studied by many authors [1, 2, 11]. It has been noted that when the oscillators are identical, symmetries are induced into the associated system of differential equations [13] and these symmetries depend on the exact pattern of coupling. For example, one popular configuration is a system of N cells coupled in a ring [2, 11]; this system has dihedral $\mathbf{D}_N$ symmetry. Another popular pattern of coupling is *all to all* coupling where each cell is coupled to every other cell [16, 4]; this type of coupling of N cells induces $\mathbf{S}_N$ permutation symmetry. We call symmetries induced by the pattern of coupling *global symmetries*; the group of global symmetries is always a finite subgroup $\mathcal{G}$ of $\mathbf{S}_N$.

There is another set of symmetries of coupled cells that has been considered less frequently. They occur when the differential equations governing the dynamics in each cell have their own symmetries [3]. This may happen, for example, when each cell is viewed as a geometric object having certain symmetry—such as a circular disc—and the dynamics in each cell are governed by partial differential equations that are invariant under that symmetry. Another common example is an array of coupled van der Pol oscillators, each of which has a reflectional symmetry $(x, y) \mapsto (-x, y)$ where (x, y) are the state variables of one of the oscillators. We call these symmetries *internal symmetries* and denote the group of internal symmetries by $\mathcal{L}$.

In this paper and in [7] we develop a theory of how patterns formed through steady-state and Hopf bifurcations in such systems depend upon both the internal and global symmetries. A subtlety that appears in this discussion is that the full group Γ of symmetries of the coupled system depends on the precise nature of the coupling. Although, in any coupled system, Γ is derived from $\mathcal{G}$ and $\mathcal{L}$, the precise way in which the groups combine depends on the form of the coupling.

There are two natural types of coupling that lead to two quite different groups Γ—one type leads to direct products and the other leads to wreath products. We illustrate these two types of coupling by assuming that the dynamics of each cell is governed by a PDE. In the first type of coupling, the cells are coupled pointwise (at least on the boundary). For example, here we imagine two biological cells having a common membrane that allows different ions to permeate at different rates. This type of coupling leads to a total symmetry group $\Gamma = \mathcal{L} \times \mathcal{G}$. Bifurcations based on these direct product symmetries are studied in [7]. For the second type, we imagine a kind of 'mean-field' coupling where the effects on one cell are felt uniformly in space and depend only on averaged quantities from the other cell or averaged quantities on its boundary. This type of coupling leads to the wreath product symmetry group $\mathcal{L} \wr \mathcal{G}$ which is the subject of this paper; wreath products are defined in section 2. Examples where such systems arise in applications are described in [12]. Bifurcations with specific wreath product symmetries have been studied in [14, 9, 10]

1.1. Axial subgroups

We will not attempt to find all possible branching patterns—the groups are too complicated and the irreducible representations that drive the bifurcations are of too high a dimension. Rather, we take a more restricted approach that will, nevertheless, yield interesting results. In steady-state bifurcations, it is well known that when isotropy subgroups have one-dimensional fixed-point subspaces, then generically the equivariant branching lemma [13] guarantees the existence of solutions with that symmetry. In this paper, when we study steady-state bifurcations, we look only for solutions corresponding to symmetries having one-dimensional fixed-point subspaces. These isotropy subgroups are always maximal isotropy subgroups and the one-dimensional fixed-point subspaces are axes of symmetry. With this in mind we define:

Definition 1.1. *A subgroup $\Sigma \subset \Gamma$ is* axial *if it is an isotropy subgroup having a one-dimensional fixed-point subspace.*

Similarly, when studying Hopf bifurcations, the equivariant Hopf theorem [13] states that branches of periodic solutions having symmetry Σ occur generically whenever Σ has a two-dimensional fixed-point subpace.

Definition 1.2. *A subgroup* $\Sigma \subset \Gamma \times \mathbf{S}^1$ *is* **C**-*axial if it is an isotropy subgroup having a two-dimensional fixed-point subspace.*

We will expand on this definition in sections 4 and 5.

We divide the paper as follows. In section 2 we describe properties that the coupling must have when the local and global symmetries combine to form direct products and wreath products. Section 3 addresses the representation theory of wreath products which determines the abstract behaviour of bifurcations. Axial subgroups for steady-state bifurcations are found in section 4 and **C**-axial subgroups for Hopf bifurcations are found in section 5. In both contexts the crucial data are the possible 'blocks' for the global group $\mathcal{G}$, which determine the general structure of axial and **C**-axial subgroups, and hence the range of patterns that occurs. A description of some of the more complicated dynamics that occur in systems with wreath product coupling is discussed briefly in section 6.

The answers to the corresponding questions when the coupling yields direct product symmetry groups requires more detailed information about real irreducible representations. This issue along with the classification of certain axial and **C**-axial subgroups for direct products is discussed in [7].

2. Coupled cells and ODEs

We begin by discussing a general form that the assumption of *identical cells* with *identical coupling* forces on systems of ODEs; this form will allow us to illustrate how the type of coupling changes the possible symmetries. In order to focus on the link between modelling assumptions and symmetry we discuss a specific, fairly natural, form of coupling. However, the theory that we develop applies to any form of coupling that possesses appropriate symmetry properties.

Let X_j denote the state variables of the jth cell and let $X = (X_1, \dots, X_N)$ be the state variables for the entire N-cell system. The assumption that the cells are identical implies that $X_j \in \mathbf{R}^k$ for each j and $X \in (\mathbf{R}^k)^N$. A system of ODEs

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X)$$

is a *system of coupled cells* if

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = f_j(X_j) + h_j(X)$$

where f_j governs the internal dynamics of the jth cell and h_j governs the coupling between cells. Since the cells are assumed to be identical, we assume that $f_j = f$ for all j.

We formulate our assumptions about coupling as follows. Define the *connection matrix* C by setting

$$C(i, j) = \begin{cases} 1 & \text{if cell } i \text{ is coupled to cell } j \\ 0 & \text{otherwise}. \end{cases}$$

To keep the motivating ideas simple we assume that the coupling has the form

$$h_j(X) = \sum_{i=1}^{N} C(i, j) h_{ij}(X_i, X_j)$$

where h_{ij} models the coupling of cell i to cell j. That is, we assume that the effect of coupling on the jth cell is found by just summing the influences of all cells coupled to the jth cell. The additive nature of this form of coupling is not an essential feature of the subsequent theory, nor is its restriction to pairwise interactions. Its role is to exhibit the symmetries clearly. The assumption that the cells are identically coupled implies that $h_{ij} = h$ for all i and j.

We next discuss the global permutation symmetries that are present in the system of ODEs

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = f(X_j) + \sum_{i=1}^{N} C(i, j)h(X_i, X_j)\,. \tag{2.1}$$

Let $\sigma \in \mathbf{S}_N$ be a permutation. The action of σ on state space is:

$$\sigma \cdot X = (X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(N)})\,.$$

Observe that σ is a symmetry of (2.1) if

$$\sigma C \sigma^{-1} = C \tag{2.2}$$

where σ is viewed as an $N \times N$ permutation matrix in (2.2). The *global* symmetry group $\mathcal{G}$ consists precisely of these permutation symmetries. It follows that

$$F(\sigma \cdot X) = \sigma \cdot F(X)$$

for all $\sigma \in \mathcal{G}$. This equivariance condition encodes the information that these symmetries permute the cells so that the differential equations do not change.

Next we discuss the local internal symmetry group $\mathcal{L} \subset \mathbf{O}(k)$. To be an *internal* symmetry we require that $\ell \in \mathcal{L}$ satisfy

$$f(\ell X_j) = \ell f(X_j)\,.$$

Whether internal symmetries are symmetries of (2.1) depends on properties of the coupling term h. As a minimum we require that when ℓ acts simultaneously on each cell, then it is a symmetry of the coupled cell system. That is, we require that

$$h(\ell X_i, \ell X_j) = \ell h(X_i, X_j)\,. \tag{2.3}$$

If we define

$$\ell \cdot X = (\ell X_1, \ldots, \ell X_N)$$

then

$$F(\ell \cdot X) = \ell \cdot F(X)$$

and ℓ is a symmetry of (2.1). It follows that $\mathcal{L} \times \mathcal{G}$ are symmetries of (2.1) where $\mathcal{L}$ is viewed as the diagonal subgroup of $\mathcal{L}^N$. Note that if the coupling term h is diagonal linear, that is,

$$h(X_i, X_j) = X_i - X_j$$

then the direct product is a symmetry group of (2.1).

However, we also consider coupled systems where the action of ℓ on each cell individually is a symmetry of (2.1). That is, we suppose

$$h(X_i, \ell X_j) = \ell h(X_i, X_j) \tag{2.4}$$

$$h(\ell X_i, X_j) = h(X_i, X_j)\,. \tag{2.5}$$

Any two of equations (2.3)–(2.5) imply the third. In this case, the group $\mathcal{L}^N$ is a symmetry group of (2.1). The *wreath product* $\mathcal{L} \wr \mathcal{G}$ is the symmetry group generated by the groups $\mathcal{L}^N$ and $\mathcal{G}$; under these assumptions it is a symmetry group of (2.1). See [17] for a discussion of the algebraic structure of wreath products.

An example of wreath product coupling is given by

$$h(X_i, X_j) = |X_i|^2 X_j\ . \tag{2.6}$$

The exact form of such a system is

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = f(X_j) + \sum_{i=1}^{N} C(i, j)|X_i|^2 X_j\ .$$

We have shown that if $\mathcal{L}$ denotes the internal symmetries and $\mathcal{G}$ denotes the global symmetries, then there are (at least) two natural types of coupling leading to two different symmetry groups Γ. The first type of coupling leads to the *direct product* $\Gamma = \mathcal{L} \times \mathcal{G}$, whereas the second type of coupling leads to the *wreath product* $\Gamma = \mathcal{L} \wr \mathcal{G}$. We discuss the wreath product coupling in the remainder of this paper and direct product coupling in II [7].

In order to simplify the analysis we shall assume that the global symmetries act transitively on the cells, that is, we assume

(H_T) $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$.

If the action of $\mathcal{G}$ is intransitive, consideration of group orbits of cells under $\mathcal{G}$ reduces the analysis to a finite list of cases in each of which (H_T) holds.

3. Linear theory for the wreath product

3.1. *Group structure of the wreath product*

In this section we study a network of coupled cells with wreath product coupling as described in section 2. Let $V = \mathbf{R}^k$; then V^N is the state space of the coupled system (2.1).

We begin by discussing the group structure of the wreath product $\mathcal{L} \wr \mathcal{G}$. The action of $\mathcal{L} \wr \mathcal{G}$ on V^N is given by

$$(\ell, \sigma) \cdot (x_1, x_2, \ldots, x_N) = (\ell_1 x_{\sigma^{-1}(1)}, \ell_2 x_{\sigma^{-1}(2)}, \ldots, \ell_N x_{\sigma^{-1}(N)}) \tag{3.1}$$

where $\ell \in \mathcal{L}^N$, $\sigma \in \mathcal{G}$ and $(x_1, x_2, \ldots, x_N) \in V^N$. The permutations act naturally on $\ell \in \mathcal{L}^N$ by

$$\sigma(\ell) = (\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(N)})\ .$$

With this definition it is easy to check that the group multiplication in the wreath product is given by

$$(h, \tau)(\ell, \sigma) = (h\tau(\ell), \tau\sigma)\ .$$

3.2. *The linear theory*

When considering steady-state bifurcation from a group-invariant equilibrium, we may make the generic hypothesis that

(H_S) $\Gamma = \mathcal{L} \wr \mathcal{G}$ acts absolutely irreducibly on the kernel of the linearized equations.

See [13], proposition 3.2, chapter XIII. Similarly, when considering Hopf bifurcation we may make the generic hypothesis that

(H_H) Γ acts Γ-simply on the centre subspace.

See [13], proposition 1.4, chapter XVI. In either case, we must first understand how Γ decomposes the state space V^N into irreducible subspaces.

Let $W \subset V^N$ be an irreducible subspace of Γ. It follows that W is an invariant subspace for the subgroup $\mathcal{L}^N \subset \Gamma$. If $\mathcal{L}^N$ acts trivially on W, then the local symmetries will have no affect on a bifurcation supported by this representation. Indeed, the bifurcation will be of the type studied in coupled cell systems with only the global symmetry group $\mathcal{G}$—which we assume has been studied previously. In this paper, we are interested only in studying bifurcations with combined local and global symmetries; therefore, we assume

$(H_{\mathcal{L}})$ $\mathcal{L}^N$ acts nontrivially on W.

Let $U_j = \{v_j \in V : (0, \ldots, v_j, \ldots, 0) \in W\}$. Each $U_j \subset W$ is an $\mathcal{L}^N$ invariant subspace. We assert:

Lemma 3.1. *Assume hypotheses* $(H_T), (H_S), (H_{\mathcal{L}})$. *Then*
(a) *U_j is $\mathcal{L}$-irreducible.*
(b) *All U_j are $\mathcal{L}$-isomorphic to a single $\mathcal{L}$-irreducible space U.*
(c) $W = U^N$.

Proof. By construction $W \supset U_1 \oplus \cdots \oplus U_N$. We claim that $W = U_1 \oplus \cdots \oplus U_N$. Note that $U_1 \oplus \cdots \oplus U_N$ is $\mathcal{G}$-invariant since $\mathcal{G}$ just permutes the subspaces U_j. Also, by construction, $U_1 \oplus \cdots \oplus U_N$ is $\mathcal{L}^N$ invariant. Hence $U_1 \oplus \cdots \oplus U_N$ is Γ-invariant since Γ is generated by $\mathcal{G}$ and $\mathcal{L}^N$. To verify the claim, we need only show that $U_1 \oplus \cdots \oplus U_N \neq 0$.

By assumption $\mathcal{L}^N$ acts nontrivially on W. Suppose $(v_1, v_2, \ldots, v_N) \in W \subset V^N$ and $\ell \in \mathcal{L}$. Then invariance implies that $(\ell v_1, v_2, \ldots, v_N) \in W$. Hence $(\ell v_1 - v_1, 0, \ldots, 0) \in W$ for all $\ell \in \mathcal{L}$. Also, we have assumed in $(H_{\mathcal{L}})$ that $\mathcal{L}$ acts nontrivially on W; without loss of generality, we may assume that $\mathcal{L}$ acts nontrivially on the first component of vectors in W. It follows that $U_1 \neq 0$, which verifies the claim.

The global symmetries $\mathcal{G}$ permute the U_j. Assumption (H_T) states that $\mathcal{G}$ acts transitively on the U_j and hence all of the U_j are $\mathcal{L}$-isomorphic. Finally, if $U_0 \subset U$ were $\mathcal{L}$-irreducible, then U_0^N would be Γ-invariant. The irreducibility of Γ on $W = U^N$ implies that $U_0 = U$ and U is $\mathcal{L}$-irreducible. $\square$

Next we show that Γ acts absolutely irreducible on U^N if and only if $\mathcal{L}$ acts absolutely irreducibly on U. Let $\mathcal{D}_\Gamma(W)$ be the space of linear mappings on W that commute with the action of Γ.

Lemma 3.2. *Assume that* $\mathrm{Fix}_U(\mathcal{L}) = \{0\}$. *Then*

$$\mathcal{D}_\Gamma(U^N) \cong \mathcal{D}_{\mathcal{L}}(U)\,.$$

Proof. Suppose that $A : U \to U$ is linear and commutes with $\mathcal{L}$. Then $A^N : U^N \to U^N$ commutes with Γ, since $\mathcal{G}$ just permutes the factors of U. This construction induces an injection of $\mathcal{D}_{\mathcal{L}}(U)$ into $\mathcal{D}_\Gamma(U^N)$.

Conversely, suppose that $B : U^N \to U^N$ is linear and commutes with Γ. In coordinates, let $B = (C_1, \ldots, C_N)$ and note that C_j commutes with the action of $\mathcal{L}^N$. In particular,

$$C_1(\ell_1 v_1, \ldots, \ell_N v_N) = \ell_1 C_1(v_1, \ldots, v_N)\,.$$

Next, let C denote one of the C_j, say C_1, and use linearity to write

$$C(v_1, \ldots, v_N) = D_1(v_1) + \cdots + D_N(v_N)\,.$$

Equivariance of C implies that each D_j for $j = 2, \ldots, N$ is $\mathcal{L}$-invariant. However, since $\mathrm{Fix}_U(\mathcal{L}) = \{0\}$, proposition 2.2, chapter XIII of [13] implies that all linear invariants vanish and $C(v_1, \ldots, v_N) = D_1(v_1)$. Hence

$$B(v_1, \ldots, v_N) = (A_1(v_1), \ldots, A_N(v_N))$$

where each $A_j : U \to U$ commutes with $\mathcal{L}$. Finally, since $\mathcal{G}$ acts transitively by (H_T), all the A_j are equal. □

Lemma 3.2 has implications for the form of the critical eigenspaces at points of steady-state or Hopf bifurcation. In the case of steady-state bifurcations the kernel of the linearization is generically absolutely Γ-irreducible. By (H_T) and $(H_{\mathcal{L}})$ this kernel must have the form U^N where U is an absolutely irreducible representation of $\mathcal{L}$.

Generically, in Hopf bifurcations, the centre subspace is Γ-*simple*; that is, the centre subspace either has the form $W \oplus W$ where W is absolutely Γ-irreducible or the centre subspace is a nonabsolutely Γ-irreducible subspace. Because of (H_T) and $(H_{\mathcal{L}})$, lemmas 3.2 and 3.1 imply that the centre subspace is either $(U\oplus U)^N \cong (U\otimes\mathbf{C})^N$ where U is absolutely $\mathcal{L}$-irreducible or the centre subspace is U^N where U is nonabsolutely $\mathcal{L}$-irreducible.

4. Steady-state bifurcation for wreath products

We assume that W is the kernel of the linearization of (2.1) at a Γ-invariant equilibrium. We make the generic hypothesis (H_S) that Γ acts absolutely irreducibly on W. We make the additional assumption $(H_{\mathcal{L}})$ that $\mathcal{L}^N$ acts nontrivially on W, which focuses attention on new patterns of bifurcation associated with wreath product symmetry. In particular, we can write $W = U^N$ where $\mathcal{L}$ acts absolutely irreducibly on U.

We divide this section into two subsections. In the first we discuss the axial subgroups of wreath products acting on W and in the second we discuss all isotropy subgroups and maximal isotropy subgroups.

4.1. Axial subgroups

We begin with a definition. A subset of indices $J \subset \{1, \ldots, N\}$ is a *block* if there exists a subgroup $\mathcal{H}$ of $\mathcal{G}$ that acts transitively on J. Note that singletons are blocks (take $\mathcal{H} = \mathbf{1}$). To each block J we associate the permutation subgroup

$$Q_J = \{\sigma \in \mathcal{G} : \sigma(J) = J\}$$

which acts transitively on J since it contains $\mathcal{H}$.

Let $A \subset \mathcal{L}$ be any subgroup and define

$$\Sigma(A, J) = (B_1 \times \cdots \times B_N) \dot{+} Q_J$$

where

$$B_j = \begin{cases} A & \text{if} \quad j \in J \\ \mathcal{L} & \text{if} \quad j \notin J\,. \end{cases}$$

Lemma 4.1. *For each block J and each axial subgroup $A \subset \mathcal{L}$ acting on U, the subgroup $\Sigma(A, J) \subset \mathcal{L} \wr \mathcal{G}$ is an axial subgroup.*

Proof. Let $x \in U$ be a nonzero vector fixed by A and let $\boldsymbol{x} = (x_1, \ldots, x_N)$ where

$$x_j = \begin{cases} x & \text{if} \quad j \in J \\ 0 & \text{if} \quad j \notin J\,. \end{cases}$$

Note that $\Sigma(A, J)$ fixes $\boldsymbol{x}$. Conversely, let $\boldsymbol{y} \in U^N$ be fixed by $\Sigma(A, J)$. Since $\boldsymbol{y}$ is fixed by $B_1 \times \cdots \times B_N$ it follows that $y_j = 0$ for $j \notin J$ and y_j is a multiple of x when $j \in J$. Since Q_J acts transitively on J it follows that all the nonzero y_j are equal and $\mathrm{Fix}_U(\Sigma(A, J)) = \mathbf{R}\{\boldsymbol{x}\}$.

To complete the proof we must show that $\Sigma(A, J)$ is the isotropy subgroup $\Sigma_{\boldsymbol{x}}$ of $\boldsymbol{x}$. The previous paragraph shows that $\Sigma(A, J) \subset \Sigma_{\boldsymbol{x}}$. Now suppose that (ℓ, σ) fixes $\boldsymbol{x}$. It follows that σ must preserve J and hence that $\sigma \in Q_J$. Thus $(\mathbf{1}, \sigma) \in \Sigma(A, J)$ and $(\ell, 1)$ fixes $\boldsymbol{x}$—from which it follows that $\ell \in B_1 \times \cdots \times B_n$. Thus $(\ell, \sigma) \in \Sigma(A, J)$, as required. □

We will show that all axial subgroups of the wreath product are conjugate to subgroups of the form $\Sigma(A, J)$. Let $\Pi_{\mathcal{G}} : \mathcal{L} \wr \mathcal{G} \to \mathcal{G}$ be projection and let

$$U_J = \{(x_1, \ldots, x_N) \in U^N : x_j = 0 \text{ for} j \notin J\}.$$

Lemma 4.2. *Suppose that Σ is an axial subgroup of $\mathcal{L} \wr \mathcal{G}$. Then $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on some block J, and* $\mathrm{Fix}_{U^N}(\Sigma) \subset U_J$.

Proof. Let $\boldsymbol{x}$ be a nonzero element of $\mathrm{Fix}_{U^N}(\Sigma)$, and let J be the set of all $j \in \{1, \ldots, N\}$ such that $x_j \neq 0$. We show that $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Since $\ell_j x_{\sigma^{-1}(j)} = 0$ if and only if $x_{\sigma^{-1}(j)} = 0$, we have that $\Pi_{\mathcal{G}}(\Sigma) J \subset J$. Suppose that there exist two disjoint subsets J_1 and J_2 of J such that $\Pi_{\mathcal{G}}(\Sigma) J_i \subset J_i$ for $i = 1, 2$. Then

$$y_1 = \begin{cases} x_j & \text{if} \quad j \in J_1 \\ 0 & \text{if} \quad j \notin J_1 \end{cases} \qquad \text{and} \qquad y_2 = \begin{cases} x_j & \text{if} \quad j \in J_2 \\ 0 & \text{if} \quad j \notin J_2 \end{cases}$$

are two linearly independent elements of $\mathrm{Fix}_{U^N}(\Sigma)$. By assumption this subspace is one dimensional, which is a contradiction. Thus $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J and J is a block. □

To simplify notation, we assume that if Σ is an axial subgroup of $\mathcal{L} \wr \mathcal{G}$, then the block J whose existence is guaranteed by lemma 4.2 is $J = \{1, \ldots, s\}$ where $s \leqslant N$.

Proposition 4.3. *Let $\Sigma \subset \mathcal{L} \wr \mathcal{G}$ be axial and let $\boldsymbol{x} \in \mathrm{Fix}_{U^N}(\Sigma)$ be nonzero. Relabel the cells, if necessary, so that $\boldsymbol{x} = (x_1, \ldots, x_s, 0, \ldots, 0)$. Let A be the isotropy subgroup of x_1 in $\mathcal{L}$. Then*
(a) $A \subset \mathcal{L}$ is axial,
(b) Σ is conjugate to $\Sigma(A, J)$.

Proof. We begin by showing that we can conjugate $\boldsymbol{x}$ to $(x_1, \ldots, x_1, 0, \ldots, 0)$. Since $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J, we can find for each $j \in J$ an element $(\ell, \sigma) \in \Sigma$ such that $\sigma(1) = j$. Thus $x_j = \ell_j x_{\sigma^{-1}(j)} = \ell_j x_1$. Let $\boldsymbol{h} = (\ell_1^{-1}, \ldots, \ell_s^{-1}, 1, \ldots, 1)$. Then $\boldsymbol{h}\Sigma\boldsymbol{h}^{-1}$ is an isotropy subgroup conjugate to Σ with

$$\begin{aligned} \mathrm{Fix}_{U^N}(\boldsymbol{h}\Sigma\boldsymbol{h}^{-1}) &= \mathbf{R}\{\boldsymbol{h}(x_1, \ldots, x_s, 0, \ldots, 0)\} \\ &= \mathbf{R}\{(x_1, \ldots, x_1, 0, \ldots, 0)\}. \end{aligned}$$

We may therefore assume that $\mathrm{Fix}_{U^N}(\Sigma) = \mathbf{R}\{(x, \ldots, x, 0, \ldots, 0)\}$ where $x = x_1$. Since Σ is the isotropy subgroup of $(x, \ldots, x, 0, \ldots, 0)$, it follows that $\Sigma \supset \Sigma(A, J)$. Lemma 4.1 states that $\Sigma(A, J)$ is a maximal isotropy subgroup from which it follows that $\Sigma = \Sigma(A, J)$, which verifies (b).

Now we show that

$$\mathrm{Fix}_{U^N}(\Sigma(A, J)) = \{(y_1, \ldots, y_1, 0, \ldots, 0) : y_1 \in \mathrm{Fix}_U(A)\}.$$

Let $\boldsymbol{y} = (y_1, \ldots, y_N)$ be in $\mathrm{Fix}_{U^N}(\Sigma(A, J))$. The action of $A^s \times \mathcal{L}^{N-s}$ forces y_j to be 0 when $j > s$, and it forces y_j to be fixed by A when $j \leqslant s$. Since Q_J acts transitively on J, we see that $y_1 = \cdots = y_s$.

Since $\Sigma = \Sigma(A, J)$ it follows that $\dim\mathrm{Fix}_{U^N}(\Sigma(A, J)) = 1$ and y_1 is a multiple of x. Then $\dim\mathrm{Fix}_U(A) = 1$ and A is axial, which verifies (a). □

4.2. An example

In order to clarify the implications of proposition 4.3 we describe its application to a typical example. We take $\mathcal{G} = \mathbf{D}_{15}$ and $\mathcal{L} = \mathbf{O}(2)$, both acting in their standard representations on $\mathbf{C} \cong \mathbf{R}^2$. (The same patterns arise if we take $\mathcal{G} = \mathbf{Z}_2$, with axial subgroup $\mathbf{1}$, but the internal symmetry of each cell that does not have full $\mathbf{Z}_2$ symmetry is then trivial.) Let $X = \{0, 1, 2, \ldots, 14\}$, on which $\mathbf{D}_{15}$ acts by cyclic permutation and inversion. We first classify the blocks $J \subset X$. The possible subgroups $\mathcal{H} \subset \mathcal{G}$ are (up to conjugacy) $\mathbf{1}, \mathbf{Z}_3, \mathbf{Z}_5, \mathbf{Z}_{15}, \mathbf{D}_1, \mathbf{D}_3, \mathbf{D}_5$ and $\mathbf{D}_{15}$. The blocks are the $\mathcal{H}$-orbits on X, which we list in table 1. Note that for the cyclic groups all orbits consist of equally spaced cells and, up to conjugacy, we may assume that cell 0 is in the block. For the dihedral groups we only list those blocks not obtained using the cyclic subgroups. For example, the orbits of $\mathbf{D}_5$ acting on X are $\{0, 3, 6, 9, 12\}$ and $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}$. Only the second block is new. From J we immediately compute Q_J, also shown in the table.

The only axial subgroup $A \subset \mathbf{O}(2)$, up to conjugacy, is $\mathbf{Z}_2^\kappa$. By lemma 4.1 and proposition 4.3 the axial subgroups of $\mathcal{L} \wr \mathcal{G}$, up to conjugacy, are the groups $\Sigma(\mathbf{Z}_2^\kappa, J)$. Such a group is a direct product of a number of copies of $\mathbf{Z}_2^\kappa$, one in each cell $j \in J$, and copies of $\mathbf{O}(2)$ in each remaining cell; all extended by Q_J. Let J' be the complement of J in X. Suppose that $x = (x_0, \ldots, x_{14}) \in \mathrm{Fix}(\Sigma(\mathbf{Z}_2^\kappa, J))$. Then $x_j \in \mathrm{Fix}(\mathbf{O}(2))$ whenever $j \in J'$; that is, $x_j = 0$ whenever $j \in J'$. We call such a cell *quiescent* and all other cells *active*. We may expect active cells typically to take up nonzero states. Moreover, since Q_J acts transitively on J, all the active x_j are equal for $j \in J$. Thus any state with isotropy subgroup $\Sigma(\mathbf{Z}_2^\kappa, J)$ corresponds to quiescent cells for $j \in J'$ and identical active cells for $j \in J$.

Table 1. Axial subgroups of $\mathbf{O}(2) \wr \mathbf{D}_{15}$ up to conjugacy.

$\mathcal{H}$	$J =$ active cells	Q_J
$\mathbf{1}$	$\{0\}$	$\mathbf{D}_1$
$\mathbf{Z}_3$	$\{0, 5, 10\}$	$\mathbf{D}_3$
$\mathbf{Z}_5$	$\{0, 3, 6, 9, 12\}$	$\mathbf{D}_5$
$\mathbf{Z}_{15}$	$\{0, \ldots, 14\}$	$\mathbf{D}_{15}$
$\mathbf{D}_1$	$\{\pm k\}, k = 1, \ldots, 7$	$\mathbf{D}_1$
$\mathbf{D}_3$	$\{1, 4, 6, 9, 11, 14\}$	$\mathbf{D}_3$
$\mathbf{D}_3$	$\{2, 3, 7, 8, 12, 13\}$	$\mathbf{D}_3$
$\mathbf{D}_5$	$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}$	$\mathbf{D}_5$

Figure 1 (top) illustrates the 14 different patterns of active/quiescent cells, up to conjugacy, that result from this classification. This list is typical for a ring of n cells with $\mathbf{O}(2)$ internal symmetry when n is odd. When n is even the classification is similar, but there are two distinct conjugacy classes of dihedral subgroups of some orders. Rather than writing down a complicated list of conditions, figure 1 (bottom) illustrates another typical case, when $n = 12$. This time there are 15 patterns (up to conjugacy).

More complicated internal symmetries just impose lots of possible choices for A. The crucial thing is the list of *blocks*, which depends only upon $\mathcal{G}$.

Note the prevalence of solutions in which some cells are quiescent, some active. Such states arise because the ‘invariant’ coupling rules for wreath products, which in suitable circumstances can effectively decouple quiescent states from their neighbours. More generally, assume for simplicity that $\mathrm{Fix}(\mathcal{G}) = 0$, and pick *any* subset $K \subset \{1, \ldots, N\}$,

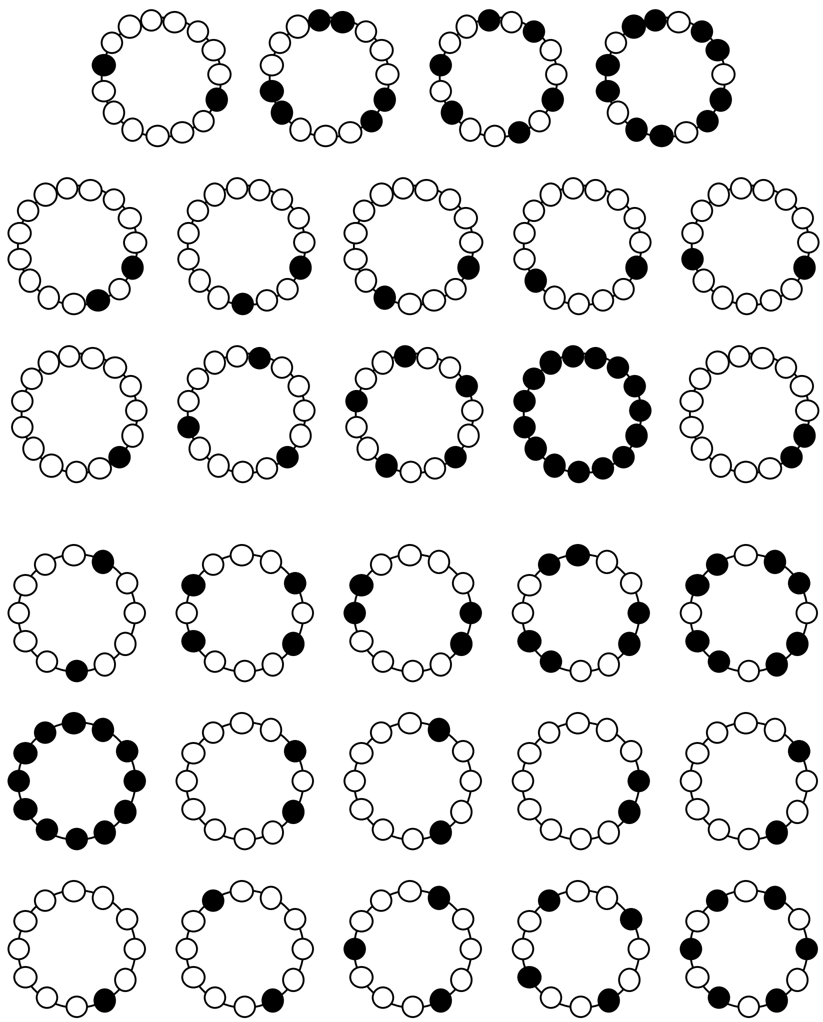

Figure 1. Top: the 14 patterns of active/quiescent cells in a ring of 15 identical cells with $\mathbf{O}(2)$ internal symmetry. Bottom: the 15 patterns of active/quiescent cells in a ring of 12 identical cells. Black cells are active, white cells are quiescent.

not necessarily a block. Consider a subgroup $\Upsilon \subset \mathcal{L} \wr \mathcal{G}$ of the form

$$\Upsilon = B_1 \times \cdots \times B_N$$

where

$$B_k = \begin{cases} \mathcal{G} & \text{for } k \in K \\ \mathbf{1} & \text{otherwise}. \end{cases}$$

Then

$$\mathrm{Fix}(\Upsilon) = V_1 \oplus \cdots \oplus V_N$$

where

$$V_k = \begin{cases} 0 & \text{for } k \in K \\ U & \text{otherwise}. \end{cases}$$

Because it is a fixed-point subspace, such a subspace is invariant under the dynamics. *Any* nonzero solution of the restriction of the original ODE to $\mathrm{Fix}(\Upsilon)$ is a dynamical state of the whole system in which the cells in K are all quiescent. (However, the active cells need no longer be in *identical* states.) It could therefore be possible, for example, to arrange for some cells to behave chaotically while neighbouring cells remain quiescent. It just requires arranging the appropriate dynamics for the restriction of the ODE to $\mathrm{Fix}(\Upsilon)$. Note that instead of choosing the B_k to be $\mathbf{1}$ for $k \notin K$, we can choose them to be arbitrary (not necessarily equal) subgroups of $\mathcal{L}$, and similar remarks apply. However, now the symmetry of each active cell is constrained. Of course, the possible states of this kind depend upon what is permitted by the restriction of the full ODE to the corresponding fixed-point subspace.

4.3. Isotropy subgroups and maximal isotropy subgroups

We begin as follows. Let

$$J = J_1 \cup \cdots \cup J_s$$

be a partition of $\{1, \ldots, N\}$. A subset J_i is called a *part* of the partition J. Let

$$Q_J = \{\sigma \in \mathcal{G} : \sigma J_i = J_i \text{ for } 1 \leqslant i \leqslant s\}.$$

To simplify the the indexing define

$$\chi : \{1, \ldots, N\} \to \{1, \ldots, s\}$$

by

$$\chi(i) = k \qquad \text{if} \quad i \in J_k.$$

So $\chi(i)$ denotes the part of J in which i sits.

Let $\Sigma_1, \ldots, \Sigma_s$ be isotropy subgroups of $\mathcal{L}$ acting on U and let

$$\Sigma_J = B_1 \times \cdots \times B_N$$

where $B_i = \Sigma_{\chi(i)}$. Finally, let

$$\Sigma = \Sigma_J \dot{+} Q_J.$$

Proposition 4.4. *Σ is an isotropy subgroup of $\Gamma = \mathcal{L} \wr \mathcal{G}$ acting on U^N and every isotropy subgroup of Γ is conjugate to such a Σ.*

Proof. Let $w_i \in U$ be a vector whose isotropy subgroup in $\mathcal{L}$ is Σ_i. Assume that the w_i all lie on distinct $\mathcal{L}$ group orbits. Let $v = (v_1, \ldots, v_N)$ where $v_i = w_{\chi(i)}$. By construction Σ fixes v. Since the w_i lie on distinct $\mathcal{L}$ group orbits, any element in $\mathcal{L} \wr \mathcal{G}$ that fixes v must preserve the partition J. It follows that no group element in addition to those in Σ fixes v and Σ is an isotropy subgroup.

Conversely, consider the isotropy subgroup of a vector $v = (v_1, \ldots, v_N) \in U^N$. Construct a partition J by putting two indices ℓ and m in the same part if v_ℓ and v_m lie on the same $\mathcal{L}$ orbit. Then conjugate v so that all v_i in the same part are equal. The isotropy subgroup of v is Σ. $\square$

This construction allows us to compute the fixed-point subspace of Σ. Refine the partition J to K where the permutation subgroup Q_J acts transitively on each part in K. Define $\rho(i) = j$ if i is in the jth part in the partition K. Then

$$\mathrm{Fix}_{U^N}(\Sigma) = \{(z_1, \ldots, z_N) \in U^N : z_i = z_j \text{ if } \rho(i) = \rho(j)\}\,.$$

We can also compute the dimension of $\mathrm{Fix}_{U^N}(\Sigma)$ as follows. Let J_i^K be the number of parts of the K partition that are contained in the J_i part in the J partition. Then

$$\dim\mathrm{Fix}_{U^N}(\Sigma) = \sum_{i=1}^{s} J_i^K \dim\mathrm{Fix}_U(\Sigma_i)\,. \tag{4.1}$$

We can now classify the maximal isotropy subgroups. Suppose that Σ is an isotropy subgroup corresponding to a partition J. We claim that if Σ is maximal, then it contains just two parts and one of the subgroups B_j must be $\mathcal{L}$. To verify the claim observe that Σ can always be enlarged by setting one of the B_is equal to $\mathcal{L}$. We may assume that $B_2 = \mathcal{L}$. Similarly, it follows that B_1 must be a maximal isotropy subgroup in the action of $\mathcal{L}$ on U.

Next we claim that if $\Sigma_J \dot{+} Q_J$ is a maximal isotropy subgroup (with $B_2 = \mathcal{L}$), then J_1 must be a block, that is, Q_J must act transitively on J_1. Indeed, we can refine the partition J so that Q_J acts transitively on the parts of the new partition in J_1. Then, again, we can enlarge the isotropy subgroup by setting $B_1 = \mathcal{L}$ on all parts in J_1 save one. We have proved:

Proposition 4.5. *Every maximal isotropy subgroup in Γ has the form $\Sigma_J \dot{+} Q_J$ where $J = \{J_1, J_2\}$ is a partition, J_1 is a block, $B_2 = \mathcal{L}$, and B_1 is a maximal isotropy subgroup of the action of $\mathcal{L}$ on U.*

5. Hopf bifurcation for wreath products

5.1. Complex structure

At points of Hopf bifurcation in Γ-equivariant systems, the centre subspace generically has a special form—it is Γ-simple. That is, the centre subspace either has the form $U \oplus U$ where Γ acts absolutely irreducibly on U, or it has the form U where Γ acts nonabsolutely irreducibly on U. In either case, there is a complex structure on these spaces and a natural action of the circle group $\mathbf{S}^1$. The complex structure is obtained as follows. Suppose the system of ODEs is written as

$$\dot{X} = F(X)$$

and that Hopf bifurcation is contemplated from the trivial solution $X = 0$. That is, we assume that $F(0) = 0$ and $J = DF(0)$ has purely imaginary eigenvalues which, after rescaling of time, are $\pm\mathrm{i}$. Then $a + \mathrm{i}b$ acts on the centre subspace by $aI + bJ$, and this defines the complex structure. Equivalently $(r\mathrm{e}^{\mathrm{i}\theta})X = r(\theta \cdot X)$ where the θ-action is via the groups $\mathbf{S}^1$. In the first case this complex structure can be written explicitly in coordinates using the identification $U \oplus U \cong U \otimes \mathbf{C}$. Then $\mathbf{S}^1$ acts on $\mathbf{C}$ as unit complex numbers. In both cases Γ has a complex irreducible representation on the centre subspace. Indeed, irreducibility of the complex representation of Γ is equivalent to Γ-simplicity of the real representation, by [13], proposition 3.5, chapter XVI.

By lemma 3.2 we can write the centre subspace as V^N, where either $\mathcal{L}$ acts nonabsolutely irreducibly on V, or $V = U \otimes \mathbf{C}$ and $\mathcal{L}$ acts absolutely irreducibly on U.

5.2. *Classification of* **C***-axial subgroups*

In this subsection we assume the generic hypothesis (H_H), so we consider Hopf bifurcation of $\mathcal{L}\wr\mathcal{G}$ acting Γ-simply on the centre subspace. First we show that for each block J and each **C**-axial subgroup $B^\Psi \subset \mathcal{L}\times\mathbf{S}^1$ there is a **C**-axial subgroup $\Sigma(B^\Psi, J) \subset (\mathcal{L}\wr\mathcal{G})\times\mathbf{S}^1$. The subgroup B^Ψ is the subgroup of $\mathcal{L}\times\mathbf{S}^1$ that is formed from the homomorphism $\psi : B \to \mathbf{S}^1$, that is

$$B^\Psi = \{(b, \psi(b)) : b \in B\}\,.$$

Such subgroups are said to be *twisted*, see [13], section 7, chapter XVI.

The group $\Sigma(B^\Psi, J)$ is the group generated by three subgroups indicated as follows:

$$\Sigma(B^\Psi, J) = (\mathbf{1}^N, Q_J, 0) + ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) + ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$$

where we assume without loss of generality that $J = \{1, \ldots, s\}$ and that Q_J consists of all permutations in $\mathcal{G}$ that preserves J. The subgroup $\hat{B}$ is defined by

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}\,.$$

Because ψ is a group homomorphism, $\hat{B}$ is a subgroup of B^s.

Proposition 5.1. *$\Sigma(B^\Psi, J)$ is* **C***-axial.*

Proof. Let $\Sigma = \Sigma(B^\Psi, J)$ and let $w = (w_1, \ldots, w_N) \in V^N$ be fixed by Σ. The fact that $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ fixes w implies that $w_{s+1} = \cdots = w_N = 0$. Thus

$$w = (w_1, \ldots, w_s, 0, \ldots, 0)\,.$$

Because w is fixed by $(\mathbf{1}^N, Q_J, 0)$ it follows that $w_1 = \cdots = w_s$. Finally, w is fixed by $((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$, so w_1 is fixed by B^Ψ. Since B^Ψ is **C**-axial, we see that $\mathrm{Fix}_{V^N}(\Sigma)$ is two dimensional.

To complete the proof we show that Σ is the isotropy subgroup of w, whence Σ is **C**-axial. Let Σ_w be the isotropy subgroup of $w = (w_1, \ldots, w_1, 0, \ldots, 0)$. The previous discussion shows that $\Sigma_w \supset \Sigma(B^\Psi, J)$. To verify the reverse inclusion we show that if $(\ell, \sigma, \theta) \in \Sigma_w$ then $(\ell, \sigma, \theta) \in \Sigma(B^\Psi, J)$. Now $(\mathbf{1}^N, Q_J, 0)$ and $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ each fix w, so $\sigma = ((\ell_1, \ldots, \ell_s, 1, \ldots, 1), 1, \theta)$ fixes w. But

$$\sigma w = ((\ell_1, \theta)w_1, \ldots, (\ell_s, \theta)w_1, 0, \ldots, 0)$$

and $(\ell_j, \theta) \in \mathcal{L}\times\mathbf{S}^1$ fixes w_1. Since B^Ψ is the isotropy subgroup of w_1, it follows that (ℓ_j, θ) is in B^Ψ and $\theta = \psi(\ell_j)$. Thus $\sigma \in ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$, so $\Sigma_w = \Sigma(B^\Psi, J)$ and $\Sigma(B^\Psi, J)$ is **C**-axial. □

Next we show that up to conjugacy we have found all of the **C**-axial subgroups of the wreath product.

Proposition 5.2. *Let $\Sigma \subset (\mathcal{L}\wr\mathcal{G})\times\mathbf{S}^1$ be* **C***-axial. Then Σ is conjugate to $\Sigma(B^\Psi, J)$ where B^Ψ is a* **C***-axial subgroup of $\mathcal{L}\times\mathbf{S}^1$ and J is a block.*

Proof. Let $w \neq 0$ be a vector fixed by Σ and let $Q = \Pi_{\mathcal{G}}(\Sigma)$. As in steady-state bifurcation Q decomposes $\{1, \ldots, N\}$ into a union of blocks. Since Σ is **C**-axial, w is supported on precisely one of these blocks J. Without loss of generality we take $J = \{1, \ldots, s\}$ and $w = (w_1, \ldots, w_s, 0, \ldots, 0)$. It follows directly that $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ is in Σ.

Since Q acts transitively on J there exists a permutation $q_j \in Q$ such that $(1, q_j, 0)w = (w_j, w_{q_j^{-1}(2)}, \ldots, w_{q_j^{-1}(s)}, 0)$. Moreover there exists $(\ell, q_j, \theta) \in \Sigma$, so $(\ell_1, \theta)w_j = w_1$. We can now conjugate w to have the form $w = (w_1, \ldots, w_1, 0, \ldots, 0)$. It follows directly that the new conjugated Σ (which we still call Σ) contains the subgroup $(\mathbf{1}^N, Q_J, 0)$.

Next suppose that $(\ell, \sigma, \theta) \in \Sigma$. The previous discussion shows that

$$((\ell_1, \ldots, \ell_s, 1, \ldots, 1), 1, \theta) \in \Sigma \,.$$

Hence $(\ell_j, \theta) \in B^\Psi$ for each j where B^Ψ is the isotropy subgroup of w_1. Indeed, $\psi(\ell_j) = \theta$ and $(\ell_1, \ldots, \ell_s) \in \hat{B}$. Hence $\Sigma = \Sigma(B^\Psi, J)$, as required.

It remains only to show that B^Ψ is $\mathbf{C}$-axial. If B^Ψ fixes an element w_2 that is not a multiple of w_1, then Σ fixes $(w_2, \ldots, w_2, 0, \ldots, 0)$. But this contradicts Σ being $\mathbf{C}$-axial. Hence $\mathrm{Fix}_V(B^\Psi)$ is two dimensional and B^Ψ is $\mathbf{C}$-axial. □

The interpretation of the corresponding patterns depends upon the blocks in the same manner as for the steady-state case, and will not be discussed further.

6. Heteroclinic cycles

There seems to be a tendency for heteroclinic cycles to occur in systems with wreath product symmetry. Perhaps the best known example of a structurally stable heteroclinic cycle in a symmetric system is the one abstracted by Guckenheimer and Holmes [14] from a model by Busse and Heikes [5] on rotating convection. In the experiment the dynamics of the convection system passes near three roll patterns—each rotated by 120° from the previous one. Guckenheimer and Holmes observed that the model in [5] can be abstracted using a certain 24 element symmetry group; this symmetry group is just $\mathbf{Z}_2 \wr \mathbf{Z}_3$. The system of ODEs has the form of a system of three coupled cells with one internal state variable ($k = 1$) and one nontrivial internal symmetry ($\mathbf{Z}_2$). Due to the rotation in the model, the coupling from cell i to cell j is not equal to the coupling from cell j to cell i. See figure 2. Thus the symmetry in this system is that of a directed ring.

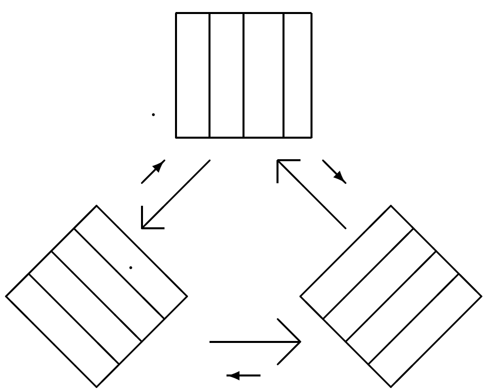

Figure 2. Rolls at 120° and 240° with two-way coupling.

The existence of heteroclinic cycles may be related to the coupling pattern. Examples of Field and Richardson [8] on symmetry groups $\mathbf{Z}_2 \wr \mathbf{Z}_N$ substantiate this point of view. The ‘instant chaos’ scenario of Guckenheimer and Worfolk [15] involves a subgroup of index two in $\mathbf{Z}_2 \wr \mathbf{Z}_4$. In another direction, the numerical experiments of [6] show that the cycling phenomenon in coupled cell systems which connects equilibria can also connect chaotic invariant sets leading to the notion of cycling chaos. We also note that the symmetry group of the cube is the wreath product $\mathbf{Z}_2 \wr \mathbf{D}_3$.

We now consider the Guckenheimer and Holmes construction in more detail. As we noted above the system of differential equations has three state variables (x_1, x_2, x_3), and the symmetries are generated by

$$(x_1, x_2, x_3) \to (\pm x_1, \pm x_2, \pm x_3)$$
$$(x_1, x_2, x_3) \to (x_2, x_3, x_1).$$

The steady-state bifurcation results of section 4 imply that we can expect equilibria where one cell is active and the other two are quiescent. Guckenheimer and Holmes [14] prove that for an open set of cubic order coefficients in these coupled cell systems, there is an asymptotically stable (structurally stable) heteroclinic cycle connecting these three equilibria.

To third order the differential equations with $\mathbf{Z}_2 \wr \mathbf{Z}_3$ symmetry are:

$$\dot{x_1} = (\lambda + \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2)x_1$$
$$\dot{x_2} = (\lambda + \gamma x_1^2 + \alpha x_2^2 + \beta x_3^2)x_2$$
$$\dot{x_3} = (\lambda + \beta x_1^2 + \gamma x_2^2 + \alpha x_3^2)x_3\,.$$

To obtain the pure form of a coupled cell system with identical coupling, such as appears in figure 3, we set $\gamma = 0$. We can write this pure form as a coupled cell system with wreath product coupling, as follows:

$$f(X_j) = (\lambda + \alpha x_j^2)x_j \qquad h(x_i, x_j) = \beta x_i^2 x_j\,.$$

Note that this h has the same form as the sample wreath product h in (2.6). Heteroclinic cycles exist when $\lambda < 0$ and $\beta < \alpha \ll 0$. See [14] for details.

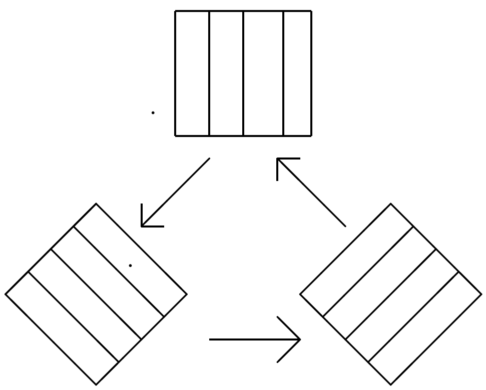

Figure 3. Rolls at 120° and 240° with one-way coupling.

The cycling form of heteroclinic connections between equilibria should persist even when the dynamics in individual cells is more complicated than equilibria. This observation has been substantiated in the numerical work of [6], where the internal cell dynamics is assumed to be a Lorentz attractor or a Chua circuit, and leads to the phenomenon of cycling chaos.

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