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Bifurcations with local gauge symmetries in the Ginzburg–Landau equations

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An interesting class of physical systems are those that exhibit *local gauge symmetries*: internal invariances that can be implemented independently at any space–time point. Systems in which these symmetries are spontaneously broken exhibit remarkable properties such as superconductivity, and if such systems also possess spatial symmetry, pattern formation can accompany the gauge symmetry-breaking. We conduct a careful analysis of a well-known example of this phenomenon: the formation of the *Abrikosov vortex lattice* in the Ginzburg–Landau model of Type-II superconductors. The study of this system has a long history and our principal contribution is to put the analysis rigorously into the context of steady-state equivariant bifurcation theory by the proper implementation of a gauge-fixing procedure. This example may be typical of the way that gauge and spatial symmetries intertwine to produce spatial patterns.

1. Introduction

In this article we wish to study the bifurcation behavior in the Ginzburg–Landau equations using the methods of bifurcation theory in the presence of both spatial and local gauge symmetries. The original Ginzburg–Landau equations [6] are a model for gauge symmetry breaking in superconductivity and are formulated in terms of two unknowns: the *electromagnetic vector potential* A which is a mapping of $\mathbb{R}^3 \to \mathbb{R}^3$, and the *Cooper pair wave function* ϕ which is a mapping of $\mathbb{R}^3 \to \mathbb{C}$. The main qualitative feature of these equations is that they are gauge, Euclidean and time-reversal invariant, and it is this feature that we use to study the existence of equilibria near a "trivial" invariant equilibrium.

It is well known that equations with Euclidean equivariance, such as the Boussinesq equations which model convection, support spatially periodic steady-state solutions and that the origin of these solutions can be traced to the Euclidean symmetry [9, 4]. It is also known that the Ginzburg–Landau equations support solutions with a type of spatial periodicity, called *Abrikosov's vortex lattice* [1, 10]; these states have also been observed in experiments [5]. A reasonable conjecture is that the spatial periodicity in Abrikosov's solution arises for the same reason that spatially periodic solutions to the Boussinesq equations arise. This conjecture is false. What we will show in section 6, however, is that bifurcation problems having both Abelian gauge symmetries and Euclidean invariance typically reduce to a bifurcation analysis either like that which leads to rolls and hexagons in the Boussinesq equations or like the one that we will describe that leads to Abrikosov's vortex lattice in the Ginzburg–Landau equations.

In this article we follow and build on a bifurcation analysis of Lasher [12] and Odeh [13] to show how the spatial periodicity in Abrikosov's solution arises. The story itself is based on an interesting utilization of both the gauge and spatial Euclidean symmetries. Making this connection precise in the mathematical sense will allow us to use standard techniques to study bifurcations found in the Ginzburg–Landau equations.

The setting in which Abrikosov's solution occurs is the following. One assumes that in the interior of a horizontal plane layer of (possibly) superconducting material, the observed fields are constant in the vertical direction. One also assumes that a magnetic field is applied to the plane layer in the vertical direction. What is observed is that when the strength of that magnetic field is large the material is not superconducting, but when that strength is decreased slowly to a temperature lower than some critical temperature, a bifurcation to superconductivity is observed. Moreover, depending on the value of a certain parameter χ, the *Ginzburg–Landau parameter*, when this transition to superconductivity occurs either the magnetic field is excluded (the Meissner effect or Type I superconductivity) or the magnetic field forms a hexagonal lattice of vortices (Type II superconductivity). In our discussion here we focus only on Type II superconductors.

The consequence of our assumption of independence of the fields in the vertical direction is to reduce the Ginzburg–Landau equations to equations whose unknowns are defined on the plane rather than in three-dimensional space. Of course this reduces the Euclidean symmetry to two dimensions as well.

The main analytic question concerns how we can reduce the question of bifurcation of equilibria in the Ginzburg–Landau equations to a finite-dimensional bifurcation problem. We now give an overview of this process.

Abstractly, the bifurcation problem we consider is finding new critical points of the Ginzburg–Landau functional (restricted to two dimensions, as noted previously) as the strength k_{ext} of the external magnetic field is varied. We look for these critical points near the "trivial" non-superconducting state where the wave function ϕ is 0 and the electromagnetic vector potential A is just the potential of the external magnetic field. As we discuss in section 7 the external magnetic potential for a constant external field may be written in complex coordinates (after a preliminary gauge transformation) as $-\mathrm{i}k_{\text{ext}}\bar{z}$.

The major assumption that we make is that we are going to look only for critical points that correspond to fields whose observables are spatially doubly periodic. Of course, when we make this assumption we also have to choose the lattice $\mathscr{L}$ on which this double periodicity is to be observed. We call fields whose observables are $\mathscr{L}$-periodic *gauge $\mathscr{L}$-periodic* fields; these fields are defined and discussed in section 3. The external magnetic potential is gauge $\mathscr{L}$-periodic for *any* lattice $\mathscr{L}$.

Since the Ginzburg–Landau functional has both gauge and Euclidean symmetries – so that any state that is gauge equivalent or Euclidean equivalent to a critical point is also a critical point – we may first perform preliminary gauge equivalences to specialize the form of A. This procedure is called *fixing the gauge*. The symmetries and their precise actions on the fields ϕ and A are described in section 2, and the gauge fixing procedure is described in section 4. The phrase "gauge fixing" is common in the literature and we wish to stress that our notion of gauge fixing is stronger in a way that we now describe. Most calculations in non-relativistic electrodynamics are done in the so-called *Coulomb* gauge, which is defined by $\operatorname{div} A = 0$. In fact this is only a partial fixing – since it leaves an infinite dimensional local gauge group generated by the harmonic functions. We regard the gauge as fixed only when the residual gauge symmetry is a finite dimensional Lie group. In order to perform a bifurcation analysis we need more – we need the total symmetry group consisting of both residual gauge and spatial symmetries to form a finite dimensional compact Lie group. See ref. [9].

The main result on gauge fixing (see theorem 4.1) is that after a preliminary gauge transformation any gauge $\mathscr{L}$-periodic field may be assumed to have the form where ϕ is an $\mathscr{L}$-theta function (see definition 4.2) and A is the sum of an $\mathscr{L}$-periodic, divergence free, mean zero field and $-\mathrm{i}k\bar{z}$ for some real constant k. That is, we can assume that the Ginzburg–Landau potential is defined on $\mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}} \times \mathbb{R}$ where $\mathscr{T}_{\mathscr{L}}$ is the space of $\mathscr{L}$-theta functions; $\mathscr{X}_{\mathscr{L}}$ is the space of $\mathscr{L}$-periodic, divergence free, mean zero fields; and k is a constant. There are, however, some intrigues associated with this constant k.

The proof that gauge fixing is possible depends on being able to solve Laplace's equation in a parallelogram using non-standard boundary conditions (that generalize periodic boundary conditions). Although the proof of existence and uniqueness of solutions uses methods that are known in PDE theory, the actual theorem seems not to have been stated in the literature. The proof of this result is included in an appendix co-authored by Roland Glowinski and Helen Lopes.

The total field A may have a part that is externally applied. By hypothesis we assume this part to be spatially constant. Thus the constant k is actually the sum of two constants k_{ext} – the strength of the external magnetic field – and k_{int} – the strength of an internally induced field. A remarkable feature of the constant k is that it is fixed by the lattice $\mathscr{L}$; this point is discussed fully in section 5.

Gauge fixing allows us to make a rigorous reduction to a bifurcation problem in finite dimensions; basically the domain of the function spaces $\mathscr{T}_{\mathscr{L}}$ and $\mathscr{X}_{\mathscr{L}}$ are now compact so that integration may be properly defined. Next we discuss what fields we allow to vary in our variational procedure. Basically we consider perturbations of $(0,0)$ in $\mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}}$ and perturbations of k_{int}. We allow this last perturbation by fixing the geometry of the lattice $\mathscr{L}$ (that is, by considering only hexagonal lattices) and then varying the size of the lattice. This allows k and hence k_{int} to vary. The variational formulation of the Ginzburg–Landau equations is given in sections 7 and 8.

The actual bifurcation problem that results depends on the linearization of the Ginzburg–Landau equations – which is presented in section 9. Much of the bifurcation analysis then depends on the kernel of the linearized Ginzburg–Landau equations at the point of bifurcation. When symmetries are present we may expect these symmetries to act irreducibly on the kernel. Indeed, since both the Euclidean group and the group of gauge transformations are non-compact we may expect infinite dimensional kernels. Because of the compactification given by the gauge fixing only a small number of these symmetries remain. Indeed the total remaining symmetries (essentially) form the group O(2); there are the global gauge transformations (S^1) and a reflectional symmetry (associated with the time-reversal symmetry). These residual symmetries are discussed in section 6.

The end result of this reduction procedure is that the expected bifurcation is a steady state bifurcation with O(2) symmetry. That is, the kernel should be one or two-dimensional, and since we are expecting new solutions that break the global gauge symmetry we expect a two-dimensional kernel. This is what happens and the actual finite dimensional bifurcation problem is actually quite easy to analyze. The resulting branch of solutions consists of the Abrikosov vortex lattice solutions. This analysis is presented in section 11.

2. The symmetries

The Ginzburg–Landau equations are usually derived via a formal variational procedure applied to an appropriately constructed free energy density. One requirement on this density is that it must be both Euclidean and gauge invariant. We begin by presenting these symmetries. Note that our assumption that the fields are independent of the vertical direction allows us to reduce the space variables to the plane. In this plane we shall use complex notation, that is, we identify

$\mathbb{R}^2 \cong \mathbb{C}$. Indeed, as we will see below, A also becomes a two-dimensional vector function; that is, a map of $\mathbb{C} \to \mathbb{C}$. We indicate this by changing A to $\mathscr{A}$ at this point. To realize the reduction we decompose $\mathbb{R}^3 \cong \mathbb{R}^2 \oplus \mathbb{R}$. If $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then the first two coordinates correspond to $\mathbb{R}^2$ and the third component to $\mathbb{R}$. We identify $\mathbb{R}^2 \cong \mathbb{C}$ by the construction $z = x_1 + \mathrm{i}x_2 \in \mathbb{C}$.

The Euclidean and gauge symmetries act on the fields ϕ and $\mathscr{A}$ introduced previously. If we let z be in $\mathbb{C}$, then the Euclidean symmetries are defined as follows. Elements $R \in \mathrm{O}(2)$ act by

$$R \cdot (\phi, \mathscr{A})(z) = (\phi(Rz), R^{-1}\mathscr{A}(Rz))$$

while translations $t \in \mathbb{R}^2$ act by

$$t \cdot (\phi, \mathscr{A})(z) = (\phi(z+t), \mathscr{A}(z+t)).$$

To conform with standard notation we write the field $\mathscr{A}(z) = \mathscr{A}^1 - \mathrm{i}\mathscr{A}^2$ where $\mathscr{A}^1$ and $\mathscr{A}^2$ are real-valued functions. In this notation the action of $R \in \mathrm{O}(2)$ is generated by the following:

$$\theta \cdot (\phi, \mathscr{A})(z) = (\phi(\mathrm{e}^{\mathrm{i}\theta} z), \mathrm{e}^{\mathrm{i}\theta} \mathscr{A}(\mathrm{e}^{\mathrm{i}\theta} z)), \quad \theta \in \mathrm{SO}(2),$$

$$\kappa \cdot (\phi, \mathscr{A})(z) = (\phi(\bar{z}), \overline{\mathscr{A}(\bar{z})}),$$

where $\kappa z = \bar{z}$ is complex conjugation. In these coordinates we also have the following identities from vector analysis which are useful in what follows:

$$\mathrm{div}(\mathscr{A}) = 2(\partial_z \bar{\mathscr{A}} + \partial_{\bar{z}} \mathscr{A}),$$

$$|\mathrm{curl}(\mathscr{A})| = 2|\partial_z \bar{\mathscr{A}} - \partial_{\bar{z}} \mathscr{A}| \tag{2.1}$$

The gauge symmetries of the Ginzburg–Landau equations are the so-called *Abelian U*(1) gauge symmetries, whose action we now describe. Define the action of a differentiable function g: $\mathbb{C} \to \mathbb{R}$ on (ϕ, A) by

$$g \cdot (\phi, \mathscr{A}) = (\mathrm{e}^{\chi \mathrm{i} g} \phi, \mathscr{A} + \mathrm{grad}(g)).$$

where the Ginzburg–Landau parameter χ is a constant. Note that in complex notation we may replace the gradient by the partial derivative ∂_z. Physically, gauge symmetries are deemed unobservable, and all measurable quantities must be gauge invariant constructions of ϕ and $\mathscr{A}$. Examples are curl($\mathscr{A}$) and $|\phi|^2$.

There is one final symmetry, the *time-reversal* symmetry, that acts by

$$* \cdot (\phi, \mathscr{A})(z) = (\bar{\phi}(z), -\mathscr{A}(z)).$$

This is the standard time-reversal transformation for quantum systems, and does correspond to a symmetry of the time-dependent Ginzburg–Landau equations when coupled to the transformation $t \to -t$.

3. Gauge $\mathscr{L}$-periodicity

Our goal is to look for spatially doubly periodic extrema to the Ginzburg–Landau free energy, particularly for those solutions whose double periodicity is supported on the hexagonal lattice. We do not care whether the solutions are actually doubly periodic – only that the *observables* are doubly periodic. With this in mind we make:

Definition 3.1. Let $\mathscr{L}$ denote a planar lattice. A state $(\phi, \mathscr{A})$ is *gauge $\mathscr{L}$-periodic* if for every $t \in \mathscr{L}$ the translated state $(\phi, \mathscr{A})(z+t)$ is gauge equivalent to $(\phi, \mathscr{A})(z)$.

In this section we discuss some of the generalities of gauge $\mathscr{L}$-periodicity. We begin by describing the lattice $\mathscr{L}$ more precisely. Two-dimensional lattices have bases consisting of two vectors; we shall only consider here those lattices whose bases consist of vectors of the same length. Moreover, after rotation, we may always assume that the lattice $\mathscr{L}$ has a basis vector that is real. Thus we assume that $\mathscr{L}$ is generated by $r \in \mathbb{R}$ and $s = r\,\mathrm{e}^{\mathrm{i}\theta}$.

Note that definition 3.1 is satisfied if the two translated states $(\phi, \mathscr{A})(z+r)$ and $(\phi, \mathscr{A})(z+s)$ are gauge equivalent to $(\phi, \mathscr{A})(z)$.

We now show that even in this generality there is a restriction on the form that the gauge transformations can take. This restriction is called the *Legendre relation* in the mathematics literature [11] and corresponds to the *fluxoid quantization condition* in the physics literature [15]. Suppose that $(\phi, \mathscr{A})(z)$ satisfies definition 3.1. Then there is a family of functions $g^t: \mathbb{C} \to \mathbb{R}$ for each $t \in \mathscr{L}$ such that

$$\phi(z+t) = e^{\chi i g^t(z)} \phi(z), \tag{3.1}$$

$$\mathscr{A}(z+t) = \mathscr{A}(z) + \partial_z g^t(z). \tag{3.2}$$

The constant χ is introduced so that the form of gauge $\mathscr{L}$-periodicity defined here will be consistent with the units we choose for the Ginzburg–Landau potential described later.

It follows from (3.1), (3.2) that the gauge transformation associated with translation by the lattice vector $s+t$ is the same as the one obtained by first translating by s and then by t. Hence if $\phi(z)$ is non-zero, then (3.1) implies

$$g^{s+t}(z) = g^t(z+s) + g^s(z) + C_{s,t} \tag{3.3}$$

for some constant $C_{s,t} = (2\pi/\chi) m_{s,t}$ where $m_{s,t}$ is an integer.

It follows that if we define

$$g_{s,t}(z) \equiv g^t(z+s) + g^s(z),$$

then (3.3) implies

$$\partial_z g_{s,t} = \partial_z g_{t,s}. \tag{3.4}$$

Hence (3.4) implies that for fixed s and t

$$K_{s,t} \equiv g_{s,t} - g_{t,s} = C_{t,s} - C_{s,t} \tag{3.5}$$

is a real constant. We now verify:

Lemma 3.2. For any $s, t \in \mathscr{L}$ we have the Legendre relation

$$K_{s,t} = \frac{2\pi}{\chi} n(s,t)$$

where n is an integer-valued, antisymmetric, bilinear form defined on $\mathscr{L}$.

Proof. By construction the constant $K_{s,t}$ is real-valued, antisymmetric and an integer multiple of $2\pi/\chi$. We claim that $K_{s,t}$ is also bilinear. Eq. (3.3) implies

$$K_{r+s,t}(0) = g^t(r+s) + g^s(r) + g^r(0) - g^s(t+r) - g^r(t) - g^t(0).$$

A direct calculation shows that

$$K_{r+s,t}(0) = K_{s,t}(r) + K_{r,t}(0).$$

Since $K_{s,t}(z)$ is actually constant in z, it follows that

$$K_{r+s,t} = K_{s,t} + K_{r,t},$$

which establishes the claim. □

Note that the space of real-valued, antisymmetric, bilinear forms on $\mathbb{R}^2$ is one-dimensional. Hence, we can also write

$$K_{s,t} = \alpha i (s\bar{t} - \bar{s}t)$$

for some real scalar α. We will see in section 5 that α is proportional to the strength of the external magnetic field and so $K_{s,t}$ is proportional to the magnetic flux through a parallelogram spanned by s and t. Thus the Legendre relation expresses the condition that the flux through the parallelogram is quantized.

4. Fixing the gauge

Since the Ginzburg–Landau functional is gauge invariant we may perform preliminary gauge transformations to specialize the form of $\mathscr{A}$ and

ϕ. Recall that $r \in \mathbb{R}$ and $s = r\,e^{i\theta}$ are assumed to be a basis for the lattice $\mathscr{L}$.

Theorem 4.1. Suppose $(\psi, \mathscr{A})$ is gauge $\mathscr{L}$-periodic. Then $(\psi, \mathscr{A})$ is gauge equivalent to $(\phi, \mathscr{P} + C)$, where

(a) $\mathscr{P}$ is $\mathscr{L}$-periodic with mean zero,

(b) $\operatorname{div}(\mathscr{P}) = 0$

(c) $C(z) = -\mathrm{i}k\bar{z}$ for some real number k,

(d) $\phi(z+t) = e^{\chi k(\bar{t}z - t\bar{z})}\phi(z)$ for $t = r$ and $t = s$.

Since $\mathscr{P}$ is $\mathscr{L}$-periodic we may average $\mathscr{P}$ over any fundamental cell of the lattice and get the same answer. This number is the *mean* of $\mathscr{P}$. We denote the space of $\mathscr{L}$-periodic, divergence free, mean zero $\mathscr{P}$ by $\mathscr{X}_{\mathscr{L}}$.

Definition 4.2. A function ϕ is an *$\mathscr{L}$-theta function* if ϕ satisfies theorem 4.1d. We denote the space of $\mathscr{L}$-theta functions by $\mathscr{T}_{\mathscr{L}}$.

Remark 4.3. Eq. (2.1) implies that $\operatorname{curl}(C) = 4k$; hence k is related to the magnitude of the external magnetic field. In this way we may think of k as the bifurcation parameter in the Ginzburg–Landau equations. It is noteworthy that this parameter appears naturally in the decomposition given by theorem 4.1. Indeed, $\operatorname{curl}(\mathscr{P})$ may be interpreted as representing the internal magnetic field.

Proof of theorem 4.1. **Step 1.** The first step in this proof is to find a gauge equivalence that makes $\mathscr{A}$ divergence free. To do this we need to find a globally defined function ρ such that $\Delta\rho = \operatorname{div}(\mathscr{A})$. This is possible; hence the gauge equivalence $\mathscr{A} \to \mathscr{A} - \rho_z$ transforms $\mathscr{A}$ to a divergence free field. We now assume that $\mathscr{A}$ is divergence free.

Step 2. Next we show that $(\psi, \mathscr{A})$ is gauge equivalent to $(\phi, \mathscr{P} - \mathrm{i}k\bar{z})$ where $\mathscr{P}$ is divergence free and $\mathscr{L}$-periodic. To do this we need to find a harmonic function η and a real number k such that

$$\mathscr{A}(z) = \mathscr{P}(z) - \mathrm{i}k\bar{z} - \eta_z(z), \tag{4.1}$$

where $\mathscr{P}$ is $\mathscr{L}$-periodic. Our strategy is to first find k and then η. The function $\mathscr{P}$ is then determined by solving (4.1) for $\mathscr{P}$. At that point a calculation will show that $\mathscr{P}$ is actually $\mathscr{L}$-periodic.

Recall that gauge $\mathscr{L}$-periodicity implies that

$$\mathscr{A}(z+t) = \mathscr{A}(z) + g_z^t(z)$$

for all $t \in \mathscr{L}$. Henceforth we assume that the functions g^t are given; since $\operatorname{div}(\mathscr{A}) = 0$ the functions g^t are harmonic.

Assume now that $\mathscr{A}$ has the form given in (4.1). Coupling this form with gauge $\mathscr{L}$-periodicity leads to

$$\eta(z) - \eta(z+r) - \mathrm{i}k(\bar{r}z - r\bar{z}) + K_r = g^r(z), \tag{4.2}$$

$$\eta(z) - \eta(z+s) - \mathrm{i}k(\bar{s}z - s\bar{z}) + K_s = g^s(z), \tag{4.3}$$

where K_r and K_s are arbitrary constants. We can now solve for k explicitly. Just evaluate the expression

$$(4.2)(z=0) - (4.2)(z=s) - (4.3)(z=0) + (4.3)(z=r)$$

to obtain

$$k = \frac{g^r(0) - g^r(s) - g^s(0) + g^s(r)}{4\operatorname{Im}(\bar{r}s)} = \frac{K_{r,s}}{4\operatorname{Im}(\bar{r}s)}, \tag{4.4}$$

where the second equality follows from (3.5).

We let $\mathscr{C}$ be the unit cell of the lattice $\mathscr{L}$; that is,

$$\mathscr{C} = \{xr + ys : 0 \le x,\ y \le 1\},$$

where r, s are basis vectors for $\mathscr{L}$.

Next observe that if the harmonic function η is defined on the unit cell $\mathscr{C}$, then we can use (4.2), (4.3) to define η on the entire plane. Identity (3.3) will guarantee that the extended η is uniquely determined. This is checked by verifying that the double extensions $\eta((z+r)+s)$ and $\eta((z+s)+r)$ are equal. To carry out this calculation begin by using (4.2), (4.3) to compute

$$\begin{aligned}\eta((z+r)+s) &= \eta(z) - g^s(z+r) - g^r(z)\\ &\quad - \mathrm{i}k[(\bar{r}+\bar{s})z - (r+s)\bar{z}]\\ &\quad - \mathrm{i}k(\bar{s}r - s\bar{r}) + K_r + K_s,\\ \eta((z+s)+r) &= \eta(z) - g^r(z+s) - g^s(z)\\ &\quad - \mathrm{i}k[(\bar{r}+\bar{s})z - (r+s)\bar{z}]\\ &\quad - \mathrm{i}k(\bar{r}s - r\bar{s}) + K_r + K_s.\end{aligned}$$

It follows that $\eta((z+r)+s) = \eta((z+s)+r)$ if

$$K_{r,s} = 4k\,\mathrm{Im}(\bar{r}s),$$

where $K_{r,s}$ is defined in (3.5). Here we use the fact that $K_{r,s}$ is independent of z. The validity of this equation follows directly from (4.4).

We have reduced the question of finding η to the problem of finding a harmonic function η on $\mathscr{C}$. We now show that η must satisfy certain non-standard boundary conditions on $\partial\mathscr{C}$. We make this point precise, as follows. Define the harmonic functions

$$h^r(z) = g^r(z) + \mathrm{i}k(\bar{r}z - r\bar{z}) - K_r,$$
$$h^s(z) = g^s(z) + \mathrm{i}k(\bar{s}z - s\bar{z}) - K_s.$$

Note that h^r and h^s are themselves harmonic functions. In terms of the functions h^r and h^s equations (4.2), (4.3) for η become

$$\eta(z) - \eta(z+r) = h^r(z), \tag{4.5}$$
$$\eta(z) - \eta(z+s) = h^s(z). \tag{4.6}$$

It follows that η must satisfy the boundary conditions

$$\eta(ys) - \eta(ys+r) = h^r(ys), \tag{4.7}$$
$$\eta(xr) - \eta(xr+s) = h^s(xr) \tag{4.8}$$

on $\partial\mathscr{C}$.

Observe that (4.5), (4.6) imply that

$$h^r(z+s) - h^r(z) = h^s(z+r) - h^s(z). \tag{4.9}$$

The existence of a harmonic function η on $\mathscr{C}$ that satisfies the boundary conditions (4.7), (4.8) is proved in the appendix. This proof relies on the identity (4.9). The proof of uniqueness and smoothness of η up to the boundary of $\mathscr{C}$ (which is needed to make the extension of η to the plane described above harmonic) relies on specifying boundary conditions on the normal derivatives of η. These conditions are obtained by taking the normal derivatives of (4.5), (4.6). This proof of uniqueness and smoothness, which is also given in the appendix, relies on the fact that h^r and h^s are globally defined harmonic functions.

Finally, as indicated previously, we define $\mathscr{P}$ by (4.1) and calculate that $\mathscr{P}$ is actually $\mathscr{L}$-periodic.

Step 3. Next we apply a *linear* gauge equivalence (that is, a gauge equivalence in which $g(z)$ is linear) to ensure that $\mathscr{P}$ has zero mean.

Step 4. The last step in the proof of theorem 4.1 is the verification of part (d).

We begin by using the fact that $(\phi, \mathscr{P}+C)$ is gauge $\mathscr{L}$-periodic to show that

$$(\mathscr{P}+C)(z+t) = (\mathscr{P}+C)(z) + \partial_z g^t(z),$$

where $g^t(z)$ is a linear gauge. Indeed, a calculation shows that

$$\partial_z g^t(z) = -\mathrm{i}k\bar{t}.$$

It follows that

$$g^t(z) = -\mathrm{i}k(\bar{t}z - t\bar{z}) + \beta_t,$$

where β_t is a real constant.

The gauge $\mathscr{L}$-periodicity now implies that

$$\phi(z+t) = e^{i\chi g^t(z)}\phi(z) = e^{\chi k(\bar{t}z - t\bar{z})} e^{i\chi\beta_t}\phi(z).$$

To verify part (d) of the theorem we perform a transformation that allows us to set β_r and β_s to zero. The transformation that we use is a composition of a linear gauge transformation and a translation.

More precisely, let

$$h^\alpha(z) = \bar{\alpha}z + \alpha\bar{z}$$

and let ρ be the composition obtained by first applying the gauge transformation h^α and then the translation u. A calculation shows that if ρ is to preserve the form $\mathscr{P} - ik\bar{z}$ where $\mathscr{P}$ has zero mean, then

$$\alpha = -iku.$$

For notational convenience we define

$$q^u(z) = h^\alpha(z) = ik(\bar{u}z - u\bar{z}).$$

We can now explicitly compute $\hat{\phi}$, the result of the action of ρ on ϕ, as

$$\hat{\phi}(z) \equiv e^{i\chi q^u(z)}\phi(z+u). \tag{4.10}$$

A computation shows that $\hat{\phi}$ transforms by

$$\hat{\phi}(z+t) = e^{i\chi q^u(z+t)}\phi(z+u+t) = e^{\chi k(\bar{t}z - t\bar{z})}\hat{\phi}(z)\, e^{\chi k[2(\bar{t}u - t\bar{u})] + i\chi\beta_t}. \tag{4.11}$$

Theorem 4.1d will follow if u can be chosen so that

$$2k(\bar{t}u - t\bar{u}) + i\beta_t = 0$$

for the two basis vectors r and s. Indeed, this equation may be rewritten as

$$\mathrm{Im}(\bar{t}u) = \frac{\beta_t}{4k},$$

which is easily solved for u. □

5. The lattice size

We begin with the following:

Remark 5.1. Theorem 4.1d has the consequence of determining k from the size of the lattice $\mathscr{L}$. In particular, this means that k is an invariant of a gauge $\mathscr{L}$-periodic field.

Proof. Suppose a basis for the lattice $\mathscr{L}$ is given by $\{r, r\,e^{i\theta}\}$ for some angle $0 < \theta < \pi$ and some length $r > 0$. In the notation of (3.1),

$$g^t(z) = -ki(\bar{t}z - \bar{z}t).$$

Hence by (3.5)

$$K_{s,t} = 2ki(\bar{s}t - s\bar{t}).$$

It now follows that

$$K_{r,re^{i\theta}} = -4kr^2\sin\theta.$$

Lemma 3.2 implies that

$$k = \frac{-n\pi}{2\chi r^2\sin\theta}, \tag{5.1}$$

which determines k from the length of the lattice r. □

We now assume that we will be looking for gauge $\mathscr{L}$-periodic extrema of the Ginzburg–Landau functional that satisfy (5.1) with $n = -1$. Indeed, minima of the Ginzburg–Landau functional satisfy this property at the bifurcation point.

Identity (5.1) has several implications for the structure of the bifurcation problem we solve later which are worth mentioning here. As noted previously, we will show that the strength of the *external* magnetic field determines k. Once k has been fixed the size of the lattice $\mathscr{L}$ that will support $\mathscr{L}$-periodic solutions to the Ginzburg–Landau equations is also determined. (Here we assume that the material constant χ and the type of the lattice θ have been fixed.)

We note here two mathematical observations that guarantee that k cannot vary once the strength of the external magnetic field is chosen.

Lemma 4.2. The number k is an invariant of gauge equivalence.

Proof. Gauge orbits of gauge $\mathscr{L}$-periodic pairs $(\psi, \mathscr{A})$ are connected. By continuity and (5.1) the number n must be constant along gauge orbits. Hence k cannot vary along a gauge orbit. □

Lemma 5.3. Let $(\psi_\tau, \mathscr{A}_\tau)$ be a one parameter family of gauge $\mathscr{L}$-periodic fields, with the lattice $\mathscr{L}$ fixed. Then the number k obtained by gauge fixing is constant in τ.

Proof. Since the lattice $\mathscr{L}$ is fixed in this deformation the number k is determined by (5.1) independently of τ. □

Next we discuss how the bifurcation analysis will proceed. We denote the Ginzburg–Landau functional abstractly by $V(\psi, \mathscr{A})$. Formally we think of using variational theory to find critical gauge $\mathscr{L}$-periodic fields $(\psi, \mathscr{A})$ for this functional. However, these fields are defined on the whole plane and it is difficult to carry out this plan rigorously since the variational process involves integration of quantities like curl($\mathscr{A}$) over the whole plane. To remedy this we fix the gauge and reduce $(\psi, \mathscr{A})$ to $(\phi, \mathscr{P} - \mathrm{i}k\bar{z})$. This yields a functional involving integration only over a fundamental cell of the lattice $\mathscr{L}$. Hence, this variational problem can be completed rigorously.

Later, in section 7, it will be mathematically convenient to look for solutions to the Ginzburg–Landau equations on a fixed lattice $\mathscr{L}$ where

$$r = \frac{1}{\chi}\sqrt{\frac{2\pi}{\sin\theta}}\,; \tag{5.2}$$

that is, where $k = \frac{1}{4}\chi$. This will be accomplished by a rescaling of the space variables in the Ginzburg–Landau equations that depends on k.

We end this section with a discussion of the lattice transformations on the space $\mathscr{T}_{\mathscr{L}}$.

Lemma 5.4. The complete lattice transformation law for functions in $\mathscr{T}_{\mathscr{L}}$ is

$$\phi(z+t) = (-1)^{kl}\,\mathrm{e}^{-2\chi k\,\mathrm{Im}(t\bar{z})\mathrm{i}}\phi(z),$$

where $t = kr + lr\,\mathrm{e}^{\mathrm{i}\theta}$ is in $\mathscr{L}$.

Remark 5.5. We denote kl by $\langle t\rangle$ and call $(-1)^{\langle t\rangle}$ the *parity* of t.

Proof. We assume inductively that ϕ transforms as

$$\phi(z+t) = \epsilon_t\,\mathrm{e}^{-2\chi k\,\mathrm{Im}(t\bar{z})\mathrm{i}}\,\phi(z),$$

where $\epsilon_t = \pm 1$. Indeed, we know that this formula is valid for the basis elements of $\mathscr{L}$ with $\epsilon = 1$. We now compute:

$$\begin{aligned}\phi(z+t_1+t_2) &= \epsilon_{t_2}\,\mathrm{e}^{-2\chi k\,\mathrm{Im}(t_2(\overline{z+t_1}))\mathrm{i}}\,\phi(z+t_1)\\ &= \epsilon_{t_2}\epsilon_{t_1}\,\mathrm{e}^{-2\chi k\,\mathrm{Im}(t_2(\overline{z+t_1}))\mathrm{i}}\,\phi(z)\,\mathrm{e}^{-2\chi k\,\mathrm{Im}(t_1\bar{z})\mathrm{i}}.\end{aligned}$$

If we set $t_1 = m_1 r + n_1 r\,\mathrm{e}^{\mathrm{i}\theta}$ and $t_2 = m_2 r + n_2 r\,\mathrm{e}^{\mathrm{i}\theta}$, then we find

$$\epsilon_{t_1+t_2} = \epsilon_{t_1}\epsilon_{t_2}(-1)^{m_1 n_2 - m_2 n_1}.$$

Here we use the restriction on r (5.1). The result now follows. □

6. Symmetries of the restricted spaces

In sections 7 and 8 we show that the Ginzburg–Landau functional may be thought of as

$$\mathscr{V}: \mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}} \times \mathbb{R} \to \mathbb{R},$$

where $\mathscr{T}_{\mathscr{L}}$ is the space of $\mathscr{L}$-theta functions defined in theorem 4.1d, $\mathscr{X}_{\mathscr{L}}$ is the space of divergence free, mean zero, $\mathscr{L}$-periodic mappings, and $\mathbb{R}$ is the space of C's parametrized by k. On these spaces we can perform a reduction to finite dimensions. Before doing that, we wish to discuss which of the symmetries introduced in section 2 still operate on $\mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}} \times \mathbb{R}$, the space on which the operator $\mathscr{V}$ is defined. We will also discuss how these symmetries can be expected to affect the bifurcation behavior of $\mathscr{V}$.

We begin by observing that the global gauge symmetries leave these spaces invariant for any lattice $\mathscr{L}$. Recall that the *global* gauge transformations are the gauge symmetries for which g is constant, and such transformations act by

$$\theta \cdot (\phi, \mathscr{P}, k) = (\mathrm{e}^{\chi i \theta} \phi, \mathscr{P}, k).$$

We let S^1 denote the circle group of global gauge symmetries.

The existence of these global gauge symmetries on the restricted functional $\mathscr{V}$ leads to a fundamental dichotomy in the expected bifurcation behavior of equations like the Ginzburg–Landau equations. Roughly speaking, these symmetries force the bifurcation behavior to be either like the one that produces Abrikosov vortex lattice solutions in the Ginzburg–Landau equations or like the one that produces rolls and hexagons in the Boussinesq equations. This dichotomy is seen by looking at the action of S^1 on the kernel of the linearized Ginzburg–Landau equations, as we now explain.

We denote the Ginzburg–Landau equations by $\mathscr{G}(\phi, \mathscr{P}, k) = 0$, where $\mathscr{G} = \nabla \mathscr{V}$. Since $\mathscr{V}$ is invariant under S^1, $\mathscr{G}$ must commute with this symmetry. It follows that at a bifurcation point $(0, 0, k)$, which is fixed by S^1, the kernel K of $(d\mathscr{G})$ must be S^1-invariant.

We claim that

$$\mathrm{K} = \mathrm{K}_{\phi} \oplus \mathrm{K}_{\mathscr{P}},$$

where K_{ϕ} consists only of kernel vectors of the form $(\psi, 0)$ in $\mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}}$, and $\mathrm{K}_{\mathscr{P}}$ consists of kernel vectors of the form $(0, \mathscr{Q})$. This decomposition follows since S^1 acts trivially on $\mathrm{K}_{\mathscr{P}}$ and nontrivially on each nonzero vector in K_{ϕ}.

Now observe that all of the other symmetries listed in section 2 leave both spaces K_{ϕ} and $\mathrm{K}_{\mathscr{P}}$ invariant. It follows that for codimension one bifurcations we expect that (generically) the kernel K will equal either K_{ϕ} or $\mathrm{K}_{\mathscr{P}}$. We will show that in the former case it is natural to expect bifurcations like those used to produce Abrikosov's vortex lattice, while in the latter we may expect bifurcations typical of those found in planar Bénard convection (with periodic boundary conditions).

To verify the latter observe that the set of vectors $\mathrm{V}_{\mathscr{P}} = \{(0, \mathscr{P}, k)\}$ is just the subspace of $\mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}} \times \mathbb{R}$ fixed by the global gauge symmetries S^1. The space $\mathrm{V}_{\mathscr{P}}$ is an invariant subspace for the nonlinear operator $\mathscr{G}$, since fixed point subspaces are always invariant subspaces. It follows that should $\mathrm{K} = \mathrm{K}_{\mathscr{P}}$ then the wave function ϕ is completely removed from the problem, and the bifurcation problem proceeds as if it were posed on the lattice $\mathscr{L}$ with only the Euclidean symmetries present. That is, just like the bifurcation behavior in the planar Bénard problem. See ref. [9].

Next we show that for the two planar lattices of real interest, the hexagonal and the square lattices, the group of symmetries leaving the space $\mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}}$ invariant is at least $\mathrm{O}(2) \times \mathbb{Z}_2$ for the hexagonal lattice and the semidirect product $\mathrm{O}(2) \dot{\times} \mathrm{D}_4$ for the square lattice. For each of these lattices, there is also a reflection symmetry (involving time reversibility) that does not commute with the global gauge group and that leaves the space $\mathcal{T}_{\mathcal{L}} \times \mathcal{X}_{\mathcal{L}} \times \mathbb{R}$ invariant. For the hexagonal lattice this symmetry is

$$\tau_{\text{hex}} \equiv * \cdot \kappa \cdot R_{\pi/3},$$

and for the square lattice it is

$$\tau_{\text{square}} \equiv * \cdot \kappa,$$

where κ is the reflection of the plane $z \to \bar{z}$ and $*$ is the time-reversal symmetry. These symmetries act on $(\phi, \mathscr{P}, k)$ by

$$\tau_{\text{hex}} \cdot (\phi, \mathscr{P}, k)(z) = \left(\overline{\phi(e^{-i\pi/3}\bar{z})}, -\overline{R_{\pi/3}\mathscr{P}(e^{-i\pi/3}\bar{z})}, k\right),$$

and

$$\tau_{\text{square}} \cdot (\phi, \mathscr{P}, k)(z) = (\overline{\phi(\bar{z})}, -\overline{\mathscr{P}(\bar{z})}, k).$$

Together the global gauge symmetries and these symmetries generate the group O(2).

In addition, there is a subgroup of the holohedry of the lattice that leaves the function spaces invariant. For the hexagonal lattice it is generated by the rotation through the angle π, that acts by

$$\sigma_{\text{hex}} = R_{\pi} \cdot (\phi, \mathscr{P}, k)(z) = (\phi(-z), -\mathscr{P}(-z), k),$$

while for the square lattice it is generated by the rotation by $\frac{1}{2}\pi$, that acts by

$$\sigma_{\text{square}} = R_{\pi/2} \cdot (\phi, \mathscr{P}, k)(z) = (\phi(iz), i\mathscr{P}(iz), k).$$

Observe that these symmetries commute with the global gauge symmetries. Also, σ_{hex} commutes with τ_{hex} and these symmetries generate the group $O(2) \times \mathbb{Z}_2$, as claimed. On the square lattice, however, σ_{hex} (which is of order four) and τ_{square} (which is of order two) do not commute. It is easy to check that these symmetries generate the group D_4. It follows that on the square lattice these symmetries generate the group $O(2) \dot{\times} D_4$, as desired.

We now state the following. Let G_{symm} be the group of symmetries that both leave the space $\mathcal{T}_{\mathcal{L}} \times \mathcal{X}_{\mathcal{L}} \times \mathbb{R}$ invariant and leave the trivial equilibrium $ik\bar{z}$ invariant.

Theorem 6.1. For hexagonal and square lattices the groups of symmetries G_{symm} are just the group of symmetries described above.

It is a straightforward exercise to check that these symmetries have the desired properties; it is more difficult to show that these are the only such symmetries. In our discussion of the Abrikosov vortex lattice we use only the existence of these symmetries. For this reason we only state this theorem here without proof.

Finally we discuss the implication of this theorem for the bifurcation behavior of the Ginzburg–Landau equations. We expect in one-parameter (codimension one) bifurcations that the kernel K will be an irreducible representation of the symmetry group G_{symm}. (This has not been proved – but is consistent with *expected* transversality properties of the Ginzburg–Landau equations.) Moreover, the global gauge group S^1 must act nontrivially on K if $K = K_{\phi}$ and hence dim K is even. Now the even dimensional irreducible representations of G_{symm} are all two-dimensional for the hexagonal lattice and either two- or four-dimensional for the square lattice. Moreover, when $\dim K = 2$ it follows that the expected codimension one bifurcation in the pitchfork of revolution which produces one nontrivial branch of (group orbits of) solutions.

The remainder of this paper is devoted to setting up the Ginzburg–Landau bifurcation problem more precisely and checking that the Ginzburg–Landau equation actually satisfies the conclusions of this discussion. The one nontrivial branch of solutions is the Abrikosov vortex lattice solutions.

We also note that when $\dim K = 2$ there are some symmetries that must act trivially on K, since the largest compact group that acts faithfully on a two-dimensional space is O(2). In particular, this implies that the σ symmetries will act trivially on the eigenfunctions of such a bifurcation. We will verify this point in section 9.

7. The variational formulation

The Ginzburg–Landau equations are derived formally using the so-called *Ginzburg–Landau free energy*. As we now show, this derivation is rigorous once one makes the ansatz of restricting to gauge $\mathscr{L}$-periodic fields. One can think of the Ginzburg–Landau free energy as a mapping from the space of fields $(\phi, \mathscr{A})$ to the space of mappings from $\mathbb{C} \to \mathbb{R}$.

To introduce the explicit form of the Ginzburg–Landau free energy, we return briefly to three dimensions. Since we wish to interpret our results as describing thermodynamic phase transitions in superconductors in the presence of an external magnetic field, we construct the Ginzburg–Landau free energy as the *Gibbs* free energy of nonrelativistic scalar electrodynamics. (Cf. ref. [15], section 8.3.) This can be written as

$$\mathscr{E}(\phi, A) = \left|\left(\frac{\mathrm{i}}{\chi}\nabla + A\right)\phi\right|^2 + \epsilon|\phi|^2 + \tfrac{1}{2}|\phi|^4 + |\nabla\times(A - A_0)|^2, \tag{7.1}$$

where $A = A_{\mathrm{int}} + A_0$. In electromagnetism, curl(A) is the (physically observable) *magnetic field*, and curl(A_0) is the part of the field that is externally applied. By definition curl(A_{int}) is the magnetization – or the internal part of the magnetic field. The constant χ is the same as that which occurs in the gauge transformation law and is interpreted here as a *material constant* (like density). Recalling remark 1.5, this means that we will identify the decomposition $A = A_{\mathrm{int}} + A_0$ with the decomposition $\mathscr{A} = \mathscr{P} + C$ derived in section 4, with A_{int} identified with $\mathscr{P}$ and A_0 identified with C. This will be seen explicitly below after the reduction to two dimensions. From the point of view of bifurcation theory, the field configuration $(\phi, A) = (0, A_0)$ is the trivial solution from which the bifurcation will occur. Prior to gauge fixing this is a relative equilibrium consisting of a gauge orbit of fields equivalent to A_0. By hypothesis, we assume that A_0 describes a spatially constant field pointing in the vertical direction which upon gauge fixing can be written in complex coordinates as $\mathscr{A}_0 = -\mathrm{i}k_0\bar{z}$. Note that the symmetries defined above actually act on A; their action on A_0 and A_{int} follow naturally.

Finally the parameter ϵ is a measure of the *temperature* and is proportional to $T - T_{\mathrm{c}}$ where T_{c} is the critical temperature of the superconductor. For $\epsilon > 0$ the only extremum of the free energy density is the trivial extremum $(\phi, A) = (0, A_0)$. When $\epsilon < 0$, the possibility of superconduction occurs, as can be seen from the existenced of the extremum $(\phi, A) = (\sqrt{-\epsilon}, 0)$ when $A_0 = 0$. We wish to study bifurcation to superconducting extrema as the strength of the external field is varied. For present purposes we assume that $\epsilon < 0$ and we choose the temperature scale so that $\epsilon = -1$. These are the units that were originally introduced by Abrikosov [2]. (This form of the Ginzburg–Landau free energy is obtained from the form usually found in physics texts by a temperature dependent scaling of the fields and coordinates. Cf. ref. [3])

In the notation of section 2 the free energy density becomes

$$\begin{aligned}\mathscr{E}(\phi, A) = {} & 2\left|\left(\frac{\mathrm{i}}{\chi}\partial_z + \mathscr{A}\right)\phi\right|^2 + 2\left|\left(\frac{\mathrm{i}}{\chi}\partial_{\bar{z}} + \bar{\mathscr{A}}\right)\phi\right|^2 \\ & - |\phi|^2 + \tfrac{1}{2}|\phi|^4 \\ & + 4\left|\partial_{\bar{z}}(\mathscr{A} - \mathscr{A}_0) - \partial_z\left(\bar{\mathscr{A}} - \bar{\mathscr{A}}_0\right)\right|^2 \\ & + \left|\left(\frac{\mathrm{i}}{\chi}\partial_3 + A_3\right)\phi\right|^2 \\ & + 4|\partial_3(\mathscr{A} - \mathscr{A}_0) - \partial_z(A_3 - A_{0,3})|^2.\end{aligned} \tag{7.2}$$

We now specialize to a two-dimensional system where

$$\partial_3\phi = \partial_3\mathscr{A} = \partial_3\mathscr{A}_0 = A_3 = 0,$$

so that the free energy density is

$$\mathscr{E}(\phi,\mathscr{A}) = 2\left|\left(\frac{\mathrm{i}}{\chi}\partial_z + \mathscr{A}\right)\phi\right|^2 + 2\left|\left(\frac{\mathrm{i}}{\chi}\partial_{\bar z} + \bar{\mathscr{A}}\right)\phi\right|^2 - |\phi|^2 + \tfrac{1}{2}|\phi|^4 + 4\left|\partial_{\bar z}(\mathscr{A}-\mathscr{A}_0) - \partial_z(\bar{\mathscr{A}}-\bar{\mathscr{A}}_0)\right|^2. \tag{7.3}$$

We now implement the gauge fixing procedure by making the ansatz that $\mathscr{E}$ is defined only on the space of gauge $\mathscr{L}$-periodic fields. Adopting the notation of section 4 and denoting the gauge-fixed free energy as $\mathscr{V}$, we obtain

$$\mathscr{V}(\phi,\mathscr{P},k) = 2\left|\left(\frac{\mathrm{i}}{\chi}\partial_z + \mathscr{P} - \mathrm{i}k\bar z\right)\phi\right|^2 + 2\left|\left(\frac{\mathrm{i}}{\chi}\partial_{\bar z} + \bar{\mathscr{P}} + \mathrm{i}kz\right)\phi\right|^2 - |\phi|^2 + \tfrac{1}{2}|\phi|^4 + 4\left|\partial_{\bar z}\mathscr{P} - \partial_z\bar{\mathscr{P}} - 2\mathrm{i}(k-k_0)\right|^2. \tag{7.4}$$

The free energy (7.4) is equivariant under the action of the two-dimensional Euclidean group and is invariant under the two-dimensional gauge transformations.

It follows from the Euclidean equivariance of $\mathscr{V}$ that translations in the lattice $\mathscr{L}$ leave $\mathscr{V}$ fixed. With this observation we can set up a variational problem. Recalling that the size of the lattice is determined by k, it is convenient to rescale the complex coordinates so that the size of the lattice is fixed. In light of (5.1), an acceptable rescaling is to define $w = \sqrt{k\chi}\, z$, whence the size of the lattice is now $\sqrt{\pi/(2\sin\theta)}$. We also define

$$L_+ = \partial_{\bar w} + w, \quad L_- = \partial_w - \bar w,$$

and the free energy becomes

$$\mathscr{V}(\phi,\mathscr{P},k) = 2\left|\left(\mathrm{i}\sqrt{\frac{k}{\chi}}\,L_- + \mathscr{P}\right)\phi\right|^2 + 2\left|\left(\mathrm{i}\sqrt{\frac{k}{\chi}}\,L_+ + \bar{\mathscr{P}}\right)\phi\right|^2 - |\phi|^2 + \tfrac{1}{2}|\phi|^4 + 4\left|\sqrt{k\chi}\,(\partial_{\bar w}\mathscr{P} - \partial_w\bar{\mathscr{P}}) - 2\mathrm{i}(k-k_0)\right|^2. \tag{7.5}$$

Let $\mathscr{C}$ be a fundamental cell for the lattice $\mathscr{L}$. The variational functional is then:

$$\mathscr{V}_\varepsilon(\phi,\mathscr{P},k) \equiv \iint_{\mathscr{C}} \mathscr{V}(\phi + \varepsilon\zeta, \mathscr{P} + \varepsilon\mathscr{S}, k) \tag{7.6}$$

where $(\zeta,\mathscr{S})$ are arbitrary elements of $\mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}}$. Note that this integral converges since $\mathscr{C}$ is compact. The previous remark shows that the value of the integral in (7.6) is independent of the choice of the fundamental cell $\mathscr{C}$. This now rigorously defines the variational problem which allows us to derive the Ginzburg–Landau equations, which we do in the next section.

8. The Ginzburg–Landau equations

In this section we derive the Ginzburg–Landau equations as the equation for the critical points of the Ginzburg–Landau functional. First, we consider the determination of the parameter k in the context of the bifurcation problem. We are interested in the loss of stability of the trivial state $(\phi,\mathscr{P}) = (0,0)$ as the strength of the external field is varied. It is convenient to begin by determining the value of k in terms of k_0 in the trivial state. To do so, we require that

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathscr{V}_0(0,0,k) = 0$$

at the appropriate value of k. A simple calculation shows that this occurs for $k = k_0$, so that k is purely external, and the lattice is in fact determined solely by the value of the external field at the bifurcation point. We now dispense with the distinction between k and k_0, and refer only to k.

To proceed, we follow Lasher [12] and simplify the Ginzburg–Landau free energy by defining

$$\psi = \sqrt{\frac{\chi}{k}}\,\phi, \quad \mathscr{A} = \sqrt{\frac{\chi}{k}}\,\mathscr{P}, \quad \lambda = 1 - \frac{\chi}{4k}.$$

In these coordinates the Ginzburg–Landau free energy is simply

$$\begin{aligned} V(\psi,\mathscr{A}) = {} & |(L_- - \mathrm{i}\mathscr{A})\psi|^2 + |(L_+ - \mathrm{i}\bar{\mathscr{A}})\psi|^2 \\ & - 2(1-\lambda)|\psi|^2 + \tfrac{1}{4}|\psi|^4 \\ & + 2\chi^2 |\partial_{\bar{w}}\mathscr{A} - \partial_w \bar{\mathscr{A}}|^2. \end{aligned} \tag{8.1}$$

We digress briefly to discuss properties of the linear operators $L_\pm$. Define the *inner product* on complex-valued functions by

$$\langle \phi, \psi \rangle \equiv \int\!\!\int_{\mathscr{C}} \phi \bar{\psi}\,\mathrm{d}x\,\mathrm{d}y, \tag{8.2}$$

where $\mathscr{C}$ is the fundamental cell of the lattice $\mathscr{L}$. A calculation verifies

Lemma 8.1.

$$\langle \psi, \partial_z \phi \rangle = -\langle \partial_{\bar{z}} \psi, \phi \rangle \tag{8.3}$$

when $\phi, \psi \in \mathscr{T}_{\mathscr{L}}$ or $\phi, \psi \in \mathscr{X}_{\mathscr{L}}$.

It follows from lemma 8.1 that the adjoints of L_+ and L_- are

$$(L_+)^* = -L_-, \quad (L_-)^* = -L_+. \tag{8.4}$$

In this notation the Ginzburg–Landau variational functional is then

$$\mathscr{V}_\varepsilon(\zeta, \mathscr{S}) \equiv \int\!\!\int_{\mathscr{C}} V(\psi + \varepsilon\zeta, \mathscr{A} + \varepsilon\mathscr{S}), \tag{8.5}$$

where $(\zeta, \mathscr{S}) \in \mathscr{T}_{\mathscr{L}} \times \mathscr{X}_{\mathscr{L}}$ and $\mathscr{C}$ is the fundamental cell in the lattice $\mathscr{L}$.

The Ginzburg–Landau equations are obtained by finding functions $(\psi, \mathscr{A})$ for which $\mathscr{V}_0' = 0$ for all $(\zeta, \mathscr{S})$. Recalling that V is actually a function of $(\psi, \partial_w\psi, \partial_{\bar{w}}\psi, \mathscr{A}, \partial_w\mathscr{A}, \partial_{\bar{w}}\mathscr{A})$ and their complex conjugates, the total derivative of V is

$$\begin{aligned} \mathscr{V}_0' = {} & \int\!\!\int_{\mathscr{C}} \left(\frac{\partial V}{\partial \psi}\zeta + \frac{\partial V}{\partial(\partial_w \psi)}(\partial_w \zeta) \right. \\ & \left. + \frac{\partial V}{\partial(\partial_{\bar{w}}\psi)}(\partial_{\bar{w}}\zeta) \right) + \text{c.c.} \\ & + \int\!\!\int_{\mathscr{C}} \left(\frac{\partial V}{\partial \mathscr{A}}\mathscr{S} \right. \\ & \left. + \frac{\partial V}{\partial(\partial_w \mathscr{A})}(\partial_w \mathscr{S}) + \frac{\partial V}{\partial(\partial_{\bar{w}}\mathscr{A})}(\partial_{\bar{w}}\mathscr{S}) \right) + \text{c.c.} \end{aligned}$$

Using the inner product defined in (8.2) we obtain

$$\begin{aligned} \mathscr{V}_0' = {} & \left(\left\langle \frac{\partial V}{\partial \psi}, \bar{\zeta} \right\rangle + \left\langle \frac{\partial V}{\partial(\partial_w\psi)}, \partial_{\bar{w}}\bar{\zeta} \right\rangle \right. \\ & \left. + \left\langle \frac{\partial V}{\partial(\partial_{\bar{w}}\psi)}, \partial_w \bar{\zeta} \right\rangle \right) + \text{c.c.} \\ & + \left(\left\langle \frac{\partial V}{\partial \mathscr{A}}, \bar{\mathscr{S}} \right\rangle + \left\langle \frac{\partial V}{\partial(\partial_w\mathscr{A})}, \partial_{\bar{w}}\bar{\mathscr{S}} \right\rangle \right. \\ & \left. + \left\langle \frac{\partial V}{\partial(\partial_{\bar{w}}\mathscr{A})}, \partial_w \bar{\mathscr{S}} \right\rangle \right) + \text{c.c.} \end{aligned}$$

Since ψ, ζ are $\mathscr{L}$-theta functions and $\mathscr{A}, \mathscr{S}$ are $\mathscr{L}$-periodic functions we may integrate by parts to obtain

$$\begin{aligned} \mathscr{V}_0' = {} & \left\langle \frac{\partial V}{\partial \psi} - \partial_w \frac{\partial V}{\partial(\partial_w\psi)} - \partial_{\bar{w}} \frac{\partial V}{\partial(\partial_{\bar{w}}\psi)}, \bar{\zeta} \right\rangle + \text{c.c.} \\ & + \left\langle \frac{\partial V}{\partial \mathscr{A}} - \partial_w \frac{\partial V}{\partial(\partial_w\mathscr{A})} - \partial_{\bar{w}} \frac{\partial V}{\partial(\partial_{\bar{w}}\mathscr{A})}, \bar{\mathscr{S}} \right\rangle + \text{c.c} \end{aligned}$$

Upon substituting the form of V in (8.1) we obtain the Ginzburg–Landau equations

$$(L_-L_+-\lambda)\psi$$

$$=\left[\tfrac{1}{4}|\psi|^2+|\mathscr{Q}|^2+\mathrm{i}\left(\mathscr{Q}L_++\bar{\mathscr{Q}}L_-\right)\right]\psi, \tag{8.6}$$

$$\partial^2_{w\bar{w}}\mathscr{Q}=\frac{1}{8\chi^2}\left[\bar{\psi}(\mathrm{i}\partial_w-\mathrm{i}\bar{w}+\mathscr{Q})\psi\right.$$

$$\left.+\psi(-\mathrm{i}\,\partial_w-\mathrm{i}\bar{w}+\mathscr{Q})\bar{\psi}\right] \tag{8.7}$$

9. The linearized equations

We recall (5.2) that in the scaled variable w we are working on a lattice of fixed size $r=\sqrt{\pi/(2\sin\theta)}$. To obtain the linearized Ginzburg–Landau equations one just sets the right hand sides of (8.6) and (8.7) to zero. As we shall see a steady-state bifurcation takes place in this equation when $\lambda=0$.

At $\lambda=0$ we may rewrite the Ginzburg–Landau equations linearized about the trivial solution as

$$L_-L_+\psi=0, \tag{9.1}$$

$$\partial^2_{w\bar{w}}\mathscr{Q}=0. \tag{9.2}$$

Solutions $\mathscr{Q}$ to (9.2) must be quadratic in w and $\bar{w}$. Since these solutions are $\mathscr{L}$-periodic, they must be constant. Since they have zero mean, they must be identically zero.

Next we observe that if ψ is a solution to (9.1), then

$$L_+\psi=0; \tag{9.3}$$

the converse is trivially true.

To verify (9.3) we observe that if $L_-L_+\psi=0$, then lemma 8.1 implies

$$0=\langle L_-L_+\psi,\psi\rangle=-\langle L_+\psi,L_+\psi\rangle.$$

To solve (9.3) we make a simple calculation. Indeed, let

$$\psi(w)=\mathrm{e}^{-w\bar{w}}\xi(w);$$

then

$$L_+\psi(w)=\mathrm{e}^{-w\bar{w}}\partial_{\bar{w}}\xi(w).$$

It now follows that ψ is a solution to (9.1) if ξ is a complex analytic function.

Of course, we are looking for solutions ψ that are $\mathscr{L}$-theta functions. It is easy to check that ψ satisfies theorem 4.1(d) if ξ satisfies

$$\xi(w+t)=\mathrm{e}^{(2w+t)\bar{t}}\xi(w) \tag{9.4}$$

for all basic vectors $t\in\mathscr{L}$. Identity (9.4) just states that ξ is related to the third Jacobi theta function θ_3. Lang [11] shows that there is, up to a complex scalar, one such ξ satisfying (9.4).

It now follows that the kernel of the linearized Ginzburg–Landau equations when $\lambda=0$ is two (real) dimensions and has the form $(cv_0(w),0)$ where $c\in\mathbb{C}$ and

$$v_0(w)=\mathrm{e}^{w(w-\bar{w})}\theta_3\left(\frac{w}{r};\mathrm{e}^{\mathrm{i}\theta}\right)$$

where

$$\theta_3\left(\frac{w}{r};\mathrm{e}^{\mathrm{i}\theta}\right)=\sum_{n=-\infty}^{n=\infty}\exp\left(\frac{\mathrm{i}\pi}{r}(2nw+n^2r\,\mathrm{e}^{\mathrm{i}\theta})\right).$$

See ref. [16].

Recall from section 6 that when the kernel is two-dimensional certain symmetries (σ_{hex} and σ_{square}) must act trivially. In figs. 1–3 we give contour plots of v_0 in the square and hexagonal cases which verify this point. (We also remark that this invariance can be obtained from the uniqueness property of the Jacobi theta function.) We also present contours of the absolute values of these eigenfunctions to illustrate that the observables have the desired double periodicity.

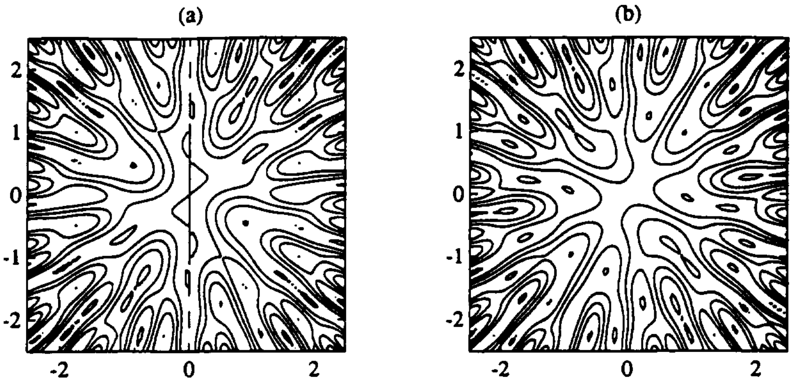

Fig. 1. Hexagonal case: (a) real part, (b) imaginary part.

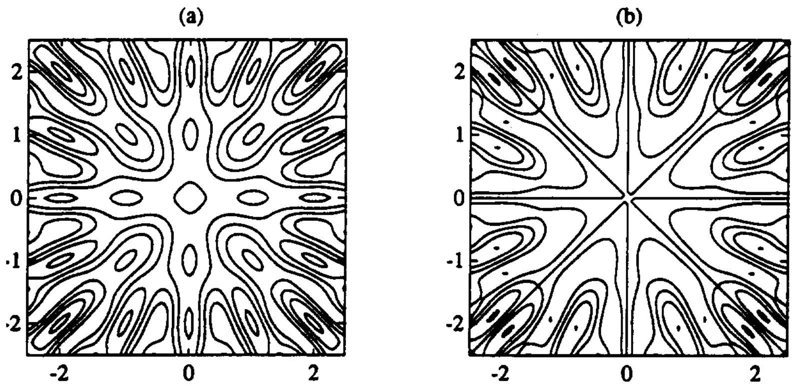

Fig. 2. Square case: (a) real part, (b) imaginary part.

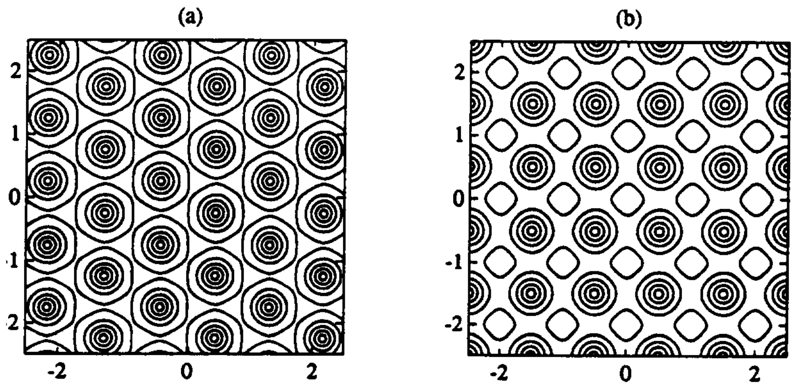

Fig. 3. Absolute values: (a) hexagonal case, (b) square case.

10. The reduced variational structure

One standard approach to determining the bifurcation behavior of the Ginzburg–Landau equations is to perform a splitting Lemma reduction on the Ginzburg–Landau functional and analyze the reduced potential defined on the kernel of the linearized equations. See refs. [7, 14]. Due to the SO(2) global gauge symmetry the reduced potential must have the form

$$v(z,\lambda) \equiv a(y,\lambda), \tag{10.1}$$

where $a\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is some smooth function and $y=|z|^2$. It is easy to see, however, that nontrivial critical points of (10.1) occur only when $a_y=0$. To see this note that $\nabla v(z,\lambda)=2a_y(y,\lambda)\,z$.

It then follows that the bifurcation equations for the Ginzburg–Landau equations must have the form

$$g(z,\lambda) \equiv a_y(|z|^2,\lambda)z=0, \tag{10.2}$$

and that the expected bifurcation is just a pitchfork bifurcation. Indeed, one only has to compute two numbers to check this; namely,

$$a_{yy}(0,0) \quad \text{and} \quad a_{y\lambda}(0,0). \tag{10.3}$$

Since the time-reversal symmetry implies that eq. (10.2) actually has O(2) symmetry, we may compute the numbers in (10.3) using a Lyapunov–Schmidt reduction – which is slightly easier to implement than the splitting lemma reduction. The numbers in (10.3) will be computed in the next section.

11. The branching of Abrikosov's solutions

We show now that the nontrivial branch of (Abrikosov's) solutions that bifurcate from the trivial uniform field solution appears for $\lambda<0$. To prove this we must show that both coefficients in (10.3) have the same sign. Abstractly the formulas for these coefficients are

$$a_{y\lambda}(0,0)=(I-E)\big[\mathrm{d}\hat{\mathscr{G}}(v_0) + (\mathrm{d}^2\hat{\mathscr{G}})(v_0,W_\lambda(0,0))\big]$$

and

$$a_{yy}(0,0)=(I-E)\big[3(\mathrm{d}^2\hat{\mathscr{G}})(v_0,W_{zz}(0,0)) + (\mathrm{d}^3\hat{\mathscr{G}})(v_0,v_0,v_0)\big],$$

where v_0 is the eigenfunction in the kernel of the linearized Ginzburg–Landau equations $(\mathrm{d}^m\hat{\mathscr{G}})$ is the mth derivative of $\hat{\mathscr{G}}$ evaluated at the bifurcation point, E is the projection onto the kernel, and W is the implicitly defined function in a Lyapunov–Schmidt reduction. See ref. [8], p. 32.

Since there is a trivial solution, $W_\lambda(0,0)=0$ (ref. [8], p. 33 (3.22b)); hence the quadratic terms in $\hat{\mathscr{G}}$ do not contribute to $a_\lambda(0,0)$. Since the quadratic terms in $\hat{\mathscr{G}}^{\mathscr{A}}$ involve only cross terms with both a $\mathscr{P}$ and a ϕ and the kernel vector v_0 has a zero $\mathscr{P}$ component we see that $(\mathrm{d}^2\hat{\mathscr{G}})(v_0,\cdot)\equiv 0$. Hence the quadratic terms do not contribute to the direction of branching of these equations.

One now calculates that

$$a_{yy}(0,0) = -\tfrac{3}{2}\int\int_{\mathscr{C}}|v_0|^4<0$$

and

$$a_{y\lambda}(0,0) = -\int\int_{\mathscr{C}}|v_0|^2<0.$$

So both coefficients have the same sign, as desired.

Acknowledgements

We are grateful to Roland Glowinski and Helena Lopes for their help in proving the PDE result that was needed to complete the proof of the gauge fixing theorem (theorem 4.1). Their work appears in an appendix to this paper. We also wish to thank Hansjörg Kielhöfer for helpful conversations.

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Appendix. The Laplace equation on a parallelogram with generalized periodic boundary data

R. Glowinski[1] *and H.J. Nussenzveig Lopes*[2]

Let r and s be linearly independent vectors and let h^r and h^s be two globally defined harmonic functions in the plane. Denote by $\mathscr{C}=\mathscr{C}(r,s)$ the open parallelogram generated by r and s. (See fig. A.1).

In this appendix we prove that there exists a harmonic function u in $\mathscr{C}$, with the difference of u on opposite sides of the parallelogram given by restrictions of h^r and h^s. By supplying the differences of normal derivatives and requiring that u has mean zero in $\mathscr{C}$ we obtain uniqueness. We then extend u to the whole plane by adding the given differences. The extension is harmonic and, therefore, smooth.

More precisely, we will prove the following proposition. In the statement we use the notation above.

Proposition 1. Let h^r and h^s satisfy the compatibility condition

$$h^r(z+s)-h^r(z)=h^s(z+r)-h^s(z),$$

for every $z\in\mathbb{R}^2$. Then there exists a unique

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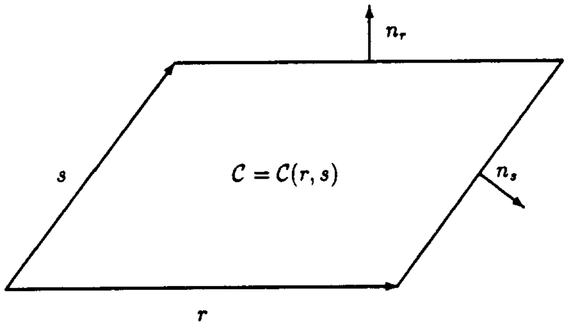

Fig. A.1. Parallelogram generated by r and s, with unit outward normals n_r and n_s.

smooth solution u to the following problem:

$$\begin{aligned}
&\Delta u = 0 && \text{in } \mathscr{C},\\
&u(s+tr) - u(tr) = h^s(tr) && \text{for } 0 \le t \le 1,\\
&u(r+ts) - u(ts) = h^r(ts) && \text{for } 0 \le t \le 1,\\
&\frac{\partial u}{\partial n_r}(s+tr) - \frac{\partial u}{\partial n_r}(tr) = \frac{\partial h^s}{\partial n_r}(tr) && \text{for } 0 \le t \le 1,\\
&\frac{\partial u}{\partial n_s}(r+ts) - \frac{\partial u}{\partial n_s}(ts) = \frac{\partial h^r}{\partial n_s}(ts) && \text{for } 0 \le t \le 1,\\
&\int_{\mathscr{C}} u = 0.
\end{aligned} \tag{1}$$

Here n_r denotes the outward unit normal to the side of the parallelogram parametrized by $s + tr$, and n_s is the outward unit normal to $r + ts$, $0 \le t \le 1$.

Moreover u has a harmonic extension to all of $\mathbb{R}^2$.

This problem has a variational formulation: the solution is a minimizer of an appropriate energy functional. The solution is originally obtained in the Sobolev space $\mathrm{H}^1(\mathscr{C})$ and the boundary data is interpreted in the trace sense. Uniqueness of the minimizer is also easily obtained. The difficulty is regularity at the boundary. We first extend the solution to adjacent parallelograms and show that the extension is smooth in the union of two such parallelograms.

Hereafter we will restrict our attention to the case $r = (1,0) \equiv e_1$ and $s = (0,1) \equiv e_2$; however, the proof of the general case is identical since the precise geometry of the parallelogram is not used. The formulation of this specific case is as follows.

Proposition 2. Let h^{e_1} and h^{e_2} satisfy the compatibility condition

$$\begin{aligned}
&h^{e_1}(x, y+1) - h^{e_1}(x,y)\\
&\quad = h^{e_2}(x+1, y) - h^{e_2}(x,y),
\end{aligned}$$

for every $(x, y) \in \mathbb{R}^2$. Then there exists a unique smooth solution u to the following problem.

$$\begin{aligned}
&\Delta u = 0 && \text{in } (0,1) \times (0,1),\\
&u(x,1) - u(x,0) = h^{e_2}(x,0) && \text{for } 0 \le x \le 1,\\
&u(1,y) - u(0,y) = h^{e_1}(0,y) && \text{for } 0 \le y \le 1,\\
&\frac{\partial u}{\partial y}(x,1) - \frac{\partial u}{\partial y}(x,0) = \frac{\partial h^{e_2}}{\partial y}(x,0) && \text{for } 0 \le x \le 1,\\
&\frac{\partial u}{\partial x}(1,y) - \frac{\partial u}{\partial x}(0,y) = \frac{\partial h^{e_1}}{\partial x}(0,y) && \text{for } 0 \le y \le 1,\\
&\int_{(0,1)\times(0,1)} u(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0.
\end{aligned} \tag{2}$$

Moreover u has a harmonic extension to all of $\mathbb{R}^2$.

We will extend the solution u above by defining it on an adjacent square to be the value of u on the "previous square" added to the value of the "difference function" on the previous square:

$$\begin{aligned}
u(x,y) &= u(x, y-1) + h^{e_2}(x, y-1)\\
&= u(x-1, y) + h^{e_1}(x-1, y).
\end{aligned}$$

Note that it is possible to compute $u(x+1, y+1)$ in terms of $u(x, y)$ using two different paths:

$$\begin{aligned}
&u(x+1, y+1)\\
&\quad = u(x,y) + h^{e_1}(x,y) + h^{e_2}(x+1,y)\\
&\quad = u(x,y) + h^{e_2}(x,y) + h^{e_1}(x,y+1).
\end{aligned}$$

Hence the need for the compatibility condition

$h^{e_1}(x, y+1) - h^{e_1}(x, y) = h^{e_2}(x+1, y) - h^{e_2}(x, y)$. It is possible to obtain an example of a pair of harmonic functions which satisfy this by adjusting coefficients of two linear functions.

The choice of conditions on the normal derivatives of u at the boundary of the square was dictated by the extension we made.

The first step in the proof of proposition 2 is a lemma establishing existence and uniqueness of a minimizer for the associated variational problem.

In what follows all restrictions of functions in H^1, and their normal derivatives, to the boundary of the unit square are to be interpreted in the trace sense. Given smooth functions f_1, f_2 we define the *admissible* set $\mathscr{A}$ to consist of those functions $v \in \mathrm{H}^1((0,1)\times(0,1))$ satisfying

(a) $\int_{(0,1)\times(0,1)} v(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$,

(b) $v(1, y) - v(0, y) = f_1(y)$,

(c) $v(x,1) - v(x,0) = f_2(x)$.

Remark 3. If $f_1(1) - f_1(0) = f_2(1) - f_2(0)$ then the admissible set $\mathscr{A}$ is nonempty.

Lemma 4. Let f_1, f_2 and g_1, g_2 be given smooth functions. If $\mathscr{A}$ is nonempty then there exists a unique minimizer $u \in \mathrm{H}^1((0,1)\times(0,1))$ of

$$I[v] = \tfrac{1}{2}\int_{(0,1)\times(0,1)} |\nabla v|^2\,\mathrm{d}x\,\mathrm{d}y - \int_0^1 v(x,0)\, g_1(x)\,\mathrm{d}x - \int_0^1 v(0, y)\, g_2(y)\,\mathrm{d}y \tag{3}$$

over $\mathscr{A}$.

Proof. The proof of existence is standard since it is easily verified that $I[\cdot]$ is weakly lower-semicontinuous over $\mathscr{A}$ and the admissible set is weakly closed. Uniqueness follows by contradiction: if v_1 and v_2 are distinct minimizers then $\frac{1}{2}(v_1 + v_2)$ is in the admissible set and $I[\frac{1}{2}(v_1 + v_2)] < \frac{1}{2}I[v_1] + \frac{1}{2}I[v_2]$ which cannot hold. □

The next lemma will show that, for specific f_1, f_2, g_1, g_2 the minimizer of the variational problem above solves (2). The Euler–Lagrange equation satisfied by the minimizer is the Laplace equation. The minimizer automatically satisfies the boundary condition by definition of the admissible set; the only remaining task is to recover the normal derivatives.

Let Ω be a domain in the plane with boundary Γ. Recall the following weak form of Green's formula:

$$\int_\Omega \nabla v\, \nabla u\,\mathrm{d}x = -\int_\Omega v\, \Delta u\,\mathrm{d}x + \int_\Gamma v \frac{\partial u}{\partial n}\,\mathrm{d}S, \tag{4}$$

valid for every u in $\mathrm{H}^1(\Omega)$ with Δu in $L^2(\Omega)$ and for every v in $\mathrm{H}^1(\Omega)$ (see pages 370–377 in ref. [1]). (The integral over Γ is, in effect, the pairing between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$: the restriction of $\partial u/\partial n$ to the boundary is, in the trace sense, in $\mathrm{H}^{-1/2}(\Gamma)$ whereas the restriction of v, in the trace sense, is in $\mathrm{H}^{1/2}(\Gamma)$).

Lemma 5. Choose $f_1(y) = h^{e_1}(0, y)$, $f_2(x) = h^{e_2}(x, 0)$, $g_1(x) = (\partial h^{e_2}/\partial y)(x, 0)$ and $g_2(y) = (\partial h^{e_1}/\partial x)(0, y)$. Then the minimizer of (3) solves (2) in $\mathrm{H}^1((0,1)\times(0,1))$.

Proof. First observe that the choice of f_1 and f_2 satisfy the condition in remark 3 and hence $\mathscr{A}$ is nonempty. Let u be the minimizer of (3). The Euler–Lagrange equation can be computed by introducing the variation $u + t\zeta$ for doubly periodic ζ (i.e. $\zeta(x,0) = \zeta(x,1)$ and $\zeta(0, y) = \zeta(1, y)$); this assures that the variation remains in the admissible set. Thus u satisfies the following Euler–Lagrange equation:

$$\int_{(0,1)\times(0,1)} \nabla u \cdot \nabla\zeta\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \zeta(0, y)\, g_2(y)\,\mathrm{d}y + \int_0^1 \zeta(x,0)\, g_1(x)\,\mathrm{d}x,$$

for every $\zeta \in C^1((0,1)\times(0,1))$, ζ doubly periodic. By taking the subclass of smooth test functions

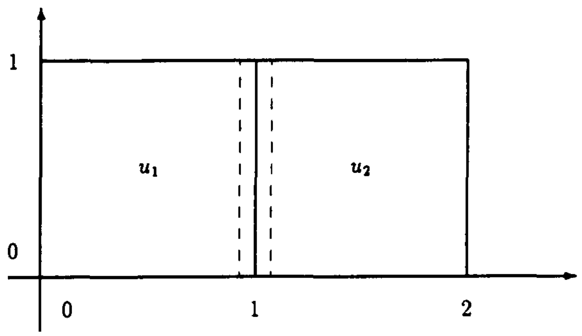

Fig. A.2. Regularity of two adjacent squares.

with compact support we observe that u is a weak solution of the Laplace equation (and hence a classical solution). Since $u \in \mathscr{A}$ it automatically satisfies

$$u(x,1) - u(x,0) = h^{e_2}(x,0),$$
$$u(1,y) - u(0,y) = h^{e_1}(0,y)$$

in the trace sense, in $H^{1/2}$ of the boundary. We then apply the formula in (4) to recover the conditions on the normal derivatives. □

We conclude with the proof of proposition 2.

Proof of proposition 2. The only remaining issue is regularity at the boundary. Classical theory shows that u is smooth in the interior of the square. We obtain regularity by considering two adjacent squares, say the unit square and $(1,2) \times (0,1)$ (see fig. A.2). First extend u to the second square as we described earlier. Call the restriction to the original square u_1 and the restriction to the second square u_2. By construction u_1 and u_2 are harmonic in each respective square and agree, in the trace sense in $H^{1/2}$, across the common boundary. Hence the function u whose restriction to the first square is u_1 and to the second square is u_2 belongs to $H^1((0,2) \times (0,1))$ (see lemma A.8 in ref. [2]). Also, by construction the normal derivatives of u_1 and u_2 agree, in the trace sense in $H^{-1/2}$ across the common boundary. Remove a small strip, with width ε, around this common boundary. Apply Green's formula (4) to u on the complement of the strip in the squares, against a smooth test function with support in the union of both original squares. By passing to the limit as ε goes to 0 and by using the two facts above we obtain that u satisfies the Laplace equation, in the weak sense, in $(0,2) \times (0,1)$. Thus u is smooth in the interior of the union of both squares. Of course this argument can be repeated in the two squares immediately above these two, so we now have a harmonic function in $H^1((0,1) \times (0,2))$ and another in $H^1((1,2) \times (0,2))$ whose restrictions to $(0,1) \times (0,1)$ and to $(1,2) \times (0,1)$ are u_1 and u_2, respectively. Due to the compatibility conditions on the difference functions h^{e_1} and h^{e_2} these new functions and their normal derivatives also agree, in the appropriate trace sense, across the extended common boundary $\{1\} \times (0,2)$. Hence, using the same argument above, we obtain a harmonic function in the interior of these four squares. Since this argument can be repeated inductively on the whole plane, u is a globally defined harmonic function. □

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