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# Classification and Unfoldings of Degenerate Hopf Bifurcations

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This paper initiates the classification, up to symmetry-covariant contact equivalence, of perturbations of local Hopf bifurcation problems which do not satisfy the classical non-degeneracy conditions. The only remaining hypothesis is that $\pm i$ should be simple eigenvalues of the linearized right-hand side at criticality. Then the Lyapunov–Schmidt method allows a reduction to a scalar equation $G(x, \lambda) = 0$, where $G(-x, \lambda) = -G(x, \lambda)$. A definition is given of the codimension of $G$, and a complete classification is obtained for all problems with codimension $\leqslant 3$, together with the corresponding universal unfoldings. The perturbed bifurcation diagrams are given for the cases with codimension $\leqslant 2$, and for one case with codimension 3; for this last case one of the unfolding parameters is a "modal" parameter, such that the topological codimension equals in fact 2. Formulas are given for the calculation of the Taylor coefficients needed for the application of the results, and finally the results are applied to two simple problems: a model of glycolytic oscillations and the Fitzhugh nerve equations.

## 1. INTRODUCTION

The classical theorem of Hopf [16, 25], describing the bifurcation of a periodic solution from an equilibrium point of a differential equation

$$\frac{dv}{dt} = f(v, \lambda), \qquad f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \tag{1.1}$$

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has found many applications. From a mathematical point of view, one reason for this fact is that the assumptions for Hopf bifurcation are generic (that is, true for most choices of $f$). One should observe, however, that if $f$ depends on extra parameters $\alpha$, (a rather typical situation in applications) then it is possible for the various hypotheses of Hopf to fail and to fail in a stable way. Recall that the theorem of Hopf is a local one (in $v$ and $\lambda$). For parameter values $\alpha$ close to a parameter value $\alpha_0$ where degeneracies occur, the results of Hopf are valid but valid only on an extremely small neighborhood, in fact a neighborhood of such small size that for practical purposes the conclusions of Hopf's theorem are invalidated. For this reason it has become clear that the study of various kinds of degenerate Hopf bifurcations is desirable for certain applications, and recently there have been a number of articles doing exactly this.

In this paper we shall study those degeneracies which still allow the Lyapunov–Schmidt method [35] or the alternative method [12] (these are equivalent for our purposes) to determine a real function $G(x, \lambda)$ whose nontrivial zeros correspond to periodic solutions to (1.1). Here $x$ corresponds—in a sense which will be made precise—to the amplitude of the associated periodic solution. Given the function $G$, one can use the singularity theory methods developed in [9, 10] to understand the structure of the periodic solutions as $\lambda$ is varied. One can do this not only for various degenerate Hopf problems but also for all small perturbations away from the degeneracy. This last observation allows one to obtain results which—although still local—are surely valid on a larger neighborhood than would be the case using the non-degenerate Hopf theorem. For example, we can simultaneously obtain two or more classical Hopf bifurcations in the same diagram. Although other authors have considered some of the degeneracies we consider, none have analysed the problem of perturbations away from the degeneracy in any but the simplest of cases. An advantage of the singularity theory approach is that it allows one to give—through the notion of codimension (properly interpreted)—the beginnings of a hierarchy of those degeneracies which are "likely" to occur, and to study with no extra effort the qualitative effects of arbitrary small perturbations.

In order to state our results more precisely, we quickly review the hypotheses of Hopf's theorem. We assume that $f(0, 0) = 0$.

(H1) The Jacobian $A \equiv (d_v f)(0, 0)$ has simple eigenvalues $\pm i$ (after rescaling $t$ if necessary) and no other eigenvalues of the form $ki$, where $k$ is an integer. In particular, 0 is not an eigenvalue of $A$.

From (H1) and the implicit function theorem, it follows that there exists a curve $v(\lambda)$ such that $f(v(\lambda), \lambda) \equiv 0$. Without loss of generality, we may assume $v(\lambda) \equiv 0$, that is,

$$f(0, \lambda) \equiv 0. \tag{1.2}$$

For $\lambda$ near 0, let $\sigma(\lambda) + i\omega(\lambda)$ be the unique smooth function satisfying $\sigma(0) = 0$ and $\omega(0) = 1$ such that $\sigma + i\omega$ is an eigenvalue for $(d_v f)(0, \lambda)$. The second Hopf hypothesis is

(H2) $\sigma'(0) \neq 0$ (transversality condition).

Assuming (H1) and (H2) Hopf proved the existence of a unique branch of non-constant periodic solutions of (1.1) with period near $2\pi$, bifurcating from $(v, \lambda) = (0, 0)$ and parameterized by the amplitude $x$. Furthermore he showed that along the periodic solution branch $\lambda$ is an even function of $x$ given by

$$\lambda = \mu_2 x^2 + \mu_4 x^4 + \cdots . \tag{1.3}$$

Assuming

(H3) $\mu_2 \neq 0$,

Hopf observed that this solution branch is either super- or subcritical depending on the sign of $\mu_2$, and he showed that the stability of the solution branch is determined via his exchange of stabilities formula.

In this paper we shall always assume that (H1) holds. This is sufficient information, as has been observed in [4, 21, 36, 40], to apply the Lyapunov–Schmidt-alternative method. In the next section we describe this approach, while fully exploiting the $SO(2)$ symmetry properties inherent in the problem.

The main theoretical results are given in Section 3, where we analyse the situation where hypotheses (H2) or (H3) or both fail. There are two principal theoretical results in this section.

(A) Theorem 3.19 gives a complete classification of bifurcation problems of codimension $\leqslant 3$ along with their universal unfoldings.

(B) Assuming that $\mu_2 = 0$ but that $\mu_4 \neq 0$ and no information about higher coefficients in (1.3) is needed, we give in Theorem 3.20 a complete classification of all bifurcation problems of finite codimension.

A modal parameter plays a crucial role for certain cases in both (A) and (B). Both of these results rely heavily on the description of bifurcation problems with symmetry given in [10]. Here the appropriate group is $\mathbb{Z}_2$. Our classifications appear to be substantially more complete than those given by other authors. From the point of view of applications, perhaps the most interesting example we find in our classification is the codimension 3 degeneracy in Theorem 3.19, Case (7). This degeneracy is the simplest case for which both hypotheses (H2) and (H3) fail, and it is also the simplest case involving a modal parameter. The complexity of the perturbed bifurcation diagrams near such a degeneracy is somewhat unexpected, see Figs. 4.6 and 4.7.

At the end of Section 3, in Theorem 3.47 we give explicit criteria for determining whether a given $G(x, \lambda)$ is equivalent to one of our normal forms, and in Theorem 3.48 we give tests for $F(x, \lambda, \alpha)$ to be a universal unfolding of such a $G(x, \lambda)$.

In Section 4 we present all the perturbed bifurcation diagrams for degeneracies of codimension $\leqslant 2$, as well as the one case of codimension 3 mentioned above. Moreover assuming that the stationary solution is stable before criticality and that the linearized operator $(d_u f)(0, 0)$ has no other pure imaginary eigenvalues except $\pm i$ we give the stability assignment of each periodic solution. Here we use Hopf's exchange of stability formula.

In Section 5 we show how to calculate the function $G(x, \lambda)$ for a given differential equation (1.1), and present explicit formulae for certain coefficients required for the application of the theoretical results of Section 3.

In the last section we show by examples how our results can be applied to real problems. For the sake of exposition we limit our applications to two simple models drawn from the recent literature: the Fitzhugh nerve impulse equations and a model of glycolytic oscillations.

In previous work, Andronov *et al.* [1] and Takens [33] have classified singularities of two-dimensional vector fields satisfying only (H1) though they have not treated in detail the role of a distinguished bifurcation parameter. Chafee [3, 4] has used the alternative method to determine the number of periodic orbits for any vector field in a neighborhood of one satisfying (H1). Göbber and Willamowski [8] studied (H3) degeneracies using Lyapunov functions. Flockerzi [7] considered both (H2) and (H3) degeneracies using an averaging procedure and Newton diagrams. Kielhöfer [21] extended Flockerzi's results to infinite dimensional spaces using a Lyapunov–Schmidt reduction, and laid bare how the degeneracy of (H2) reappears in the bifurcation equation. Hassard and Wan [14] first calculated the coefficient $\mu_4$ in (1.3). Vanderbauwhede's work [36, 37] uses the Lyapunov–Schmidt reduction and symmetry to obtain partial results for the (H3) degeneracy, and also anticipates the applicability of singularity theory to this problem. We would like to thank G. Iooss for bringing Vanderbauwhede's work to our attention. Over the past two decades, J. Hale has made contributions to bifurcation theory too numerous to list here, see [12, 13, 40, 41] and further references therein. Other authors who have contributed to this problem include Iooss [18], Joseph and Sattinger [20] and Schmidt [30].

We end this introduction with two remarks. One obtains entirely different degeneracies when (H1) is violated. Various problems with a simple zero eigenvalue in addition to $\pm i$ in (H1) have been studied in [11, 15, 23, 24, 32, 38, 39]. As shown in [23], the Lyapunov–Schmidt reduction applies to this case as well, so one can apply the singularity theory approach. The results in [10, 28, 29] therefore apply equally to the problems in [23, 24], and in fact these papers are identical in spirit though the techniques are different. The resonant situation with eigenvalues $\pm i$ and $\pm ki$ ($k \neq 0, 1$) is more complex and still under investigation.

Finally, we have presented our results in the setting of ordinary differential equations in $\mathbb{R}^n$. It is well known that the Lyapunov–Schmidt reduction of Section 2 also applies to differential equations in much more general settings, and leads to the same reduced bifurcation equations. Therefore our conclusions apply equally to these situations. We cite, for example, [5, 13, 21, 25, 27, 36].

## 2. The Lyapunov–Schmidt Reduction

In this section we summarize the well-known Lyapunov–Schmidt reduction as it applies to the Hopf bifurcation problem. We assume hypothesis (H1) but not (H2) or (H3), and place greater emphasis on symmetry than is usually the case. Most of the calculations have been suppressed for this presentation. However, explicit calculations which are needed for applications of the theory are given in Section 5.

Begin with the differential equation

$$\frac{dv}{dt} = f(v, \lambda), \qquad f(0, \lambda) \equiv 0, \tag{2.1}$$

where $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is $C^\infty$ and defined on a neighborhood of the origin $(0, 0)$. Hypothesis (H1) implies that the linearization of (2.1) at $(0, 0)$ has $2\pi$-periodic solutions. This motivates us to seek periodic solutions of (2.1) near $(0, 0)$ with period $2\pi/(1+\tau)$, for small $\tau$. Rescale time by setting

$$s = (1+\tau)t, \qquad u(s) = v(t) \tag{2.2}$$

so that such solutions $u$ will have period $2\pi$ in $s$. (The choice (2.2) rather than $t = (1+\tau)s$ gives slightly simpler formulae in Section 5.) Now rewrite (2.1) as the nonlinear operator equation

$$N(u, \lambda, \tau) \equiv (1+\tau)\frac{du}{ds} - f(u, \lambda) = 0. \tag{2.3}$$

Let $C_{2\pi}$, resp. $C^1_{2\pi}$, denote the Banach spaces of $2\pi$-periodic functions from $\mathbb{R}$ into $\mathbb{R}^n$ which are continuous, resp. continuously differentiable, with norm

$$\|u\| = \max_s |u(s)|, \qquad \text{resp. } \|u\|_1 = \|u\| + \left\|\frac{du}{ds}\right\|, \tag{2.4}$$

where $|\cdot|$ is a norm on $\mathbb{R}^n$. Note that

$$N: C^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to C_{2\pi} \tag{2.5}$$

is defined on a neighborhood of the origin (0, 0, 0).

The purpose of the Lyapunov–Schmidt reduction is to replace the problem of solving (2.3) with the problem of solving an equation of the form

$$g(x, y, \lambda, \tau) = 0, \tag{2.6}$$

where $g: \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ is $C^\infty$ and defined on a neighborhood of the origin (0, 0, 0). Moreover, as we shall stress in our exposition, the form of $g$ is far from arbitrary, as $g$ must commute with the symmetry group $SO(2)$ acting on $\mathbb{R}^2$. The structure of $g$ allows us to further reduce (2.6) to an equation of the form $G(x, \lambda) = 0$, where $G: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is odd in $x$. First we describe the construction of $g$.

Define the linear operator $L: C^1_{2\pi} \to C_{2\pi}$ by

$$L = (d_u N)(0, 0, 0) = \frac{d}{ds} - A, \qquad A \equiv (d_u f)(0,0). \tag{2.7}$$

Observe that $L$ is bounded with two dimensional nullspace $\mathcal{N}(L)$. Specifically, if $c$ is a complex eigenvector of $A$ satisfying $Ac = ic$, then $\mathcal{N}(L) = \operatorname{span}\{\phi_1, \phi_2\}$, where

$$\phi_1(s) = \operatorname{Re}(e^{is}c), \qquad \phi_2(s) = \operatorname{Im}(e^{is}c). \tag{2.8}$$

We define the inner product on $C_{2\pi}$

$$(u, v) = \frac{1}{2\pi} \int_0^{2\pi} v^*(s) u(s)\, ds \tag{2.9}$$

and define the formal adjoint of $L$, $L^*: C^1_{2\pi} \to C_{2\pi}$, by

$$L^* = -\frac{d}{ds} - A^*, \tag{2.10}$$

where $v^*$ and $A^*$ denote (conjugate) transpose of $v$ and $A$. The nullspace of $L^*$ is also two dimensional and is spanned by

$$\psi_1(s) = \operatorname{Re}(e^{is}d), \qquad \psi_2(s) = \operatorname{Im}(e^{is}d), \tag{2.11}$$

where $d$ is an eigenvector of $A^*$ corresponding to the eigenvalue $-i$. Furthermore, the Fredholm alternative holds:

$$\mathcal{R}(L) = \mathcal{N}(L^*)^\perp \equiv \{v \in C_{2\pi} \mid (v, \psi_i) = 0, i = 1, 2\}. \tag{2.12}$$

Note that the simplicity of the eigenvalue $i$ implies that $d^*c \neq 0$. Therefore we can normalize

$$d^*c = 2, \tag{2.13}$$

and it follows by direct calculation that

$$(\phi_i, \psi_j) = \delta_{ij}, \qquad i,j = 1, 2. \tag{2.14}$$

Therefore $C_{2\pi}$ has the direct sum decomposition

$$C_{2\pi} = \mathscr{N}(L) \oplus \mathscr{R}(L). \tag{2.15}$$

Define the (oblique) projectors on $C_{2\pi}$

$$\begin{aligned} Pv &\Rightarrow (v, \psi_1)\phi_1 + (v, \psi_2)\phi_2, \\ Q &= I - P, \end{aligned} \tag{2.16}$$

and note that $P$ has kernel $\mathscr{R}(L)$ and image $\mathscr{N}(L)$ and vice versa for $Q$. (This choice of oblique rather than orthogonal projections will help simplify the calculations in Section 5.) With the natural imbedding of $C^1_{2\pi} \subseteq C_{2\pi}$, let (2.15) induce the decomposition $C^1_{2\pi} = \mathscr{N}(L) \oplus W$, where

$$W = \{w \in C^1_{2\pi} \mid (w, \psi_i) = 0, i = 1, 2\}, \tag{2.17}$$

and clearly $W \subseteq \mathscr{R}(L)$.

The Lyapunov–Schmidt reduction begins with the observation that solving (2.3) is equivalent to solving the two equations

$$PN(u, \lambda, \tau) = 0, \tag{2.18}$$

$$QN(u, \lambda, \tau) = 0. \tag{2.19}$$

Now decompose $u \in C^1_{2\pi}$ by

$$u = x\phi_1 + y\phi_2 + w, \qquad w \in W, \tag{2.20}$$

and (2.9) has the form

$$M(w; x, y, \lambda, \tau) \equiv QN(x\phi_1 + y\phi_2 + w, \lambda, \tau) = 0, \tag{2.21}$$

where $M$: $W \times \mathbb{R}^4 \to \mathscr{R}(L)$. The Fréchet derivative $(d_wM)(0, 0)$: $W \to \mathscr{R}(L)$ is easily calculated from (2.3) to be $L$ restricted to $W$. Since this map has a bounded inverse, the implicit function theorem applied to (2.21) yields a unique solution $w = w(x, y; \lambda, \tau)$ near $(0, 0; 0, 0)$ satisfying $w(0, 0; 0, 0) = 0$. In fact this uniqueness, along with $M(0; 0, 0, \lambda, \tau) \equiv 0$ shows that

$$w(0, 0; \lambda, \tau) \equiv 0, \tag{2.22}$$

for small $\lambda, \tau$. One now substitutes this $w(x, y; \lambda, \tau)$ into Eq. (2.18) and defines $g: \mathcal{N}(L) \times \mathbb{R} \times \mathbb{R} \to \mathcal{N}(L)$ by

$$g = PN(x\phi_1 + y\phi_2 + w(x, y; \lambda, \tau), \lambda, \tau). \tag{2.23}$$

More explicitly, writing $g = g_1\phi_1 + g_2\phi_2$, in terms of coordinates on $\mathcal{N}(L)$, we have

$$g_i(x, y; \lambda, \tau) = \left((1+\tau)\frac{du}{ds} - f(u, \lambda), \psi_i\right) = 0, \qquad i = 1, 2, \tag{2.24}$$

where $u = = x\phi_1 + y\phi_2 + w(x, y; \lambda, \tau)$. This completes the Lyapunov–Schmidt reduction. Note that (2.12), (2.22) and $f(0, \lambda) \equiv 0$ imply that the linear terms in the Taylor expansion of $g$ at the origin all vanish.

We now consider the symmetry group $SO(2)$ and how it affects $g$. Define the shift operator $S_\theta$ on $C_{2\pi}$ by

$$S_\theta u(s) = u(s - \theta), \qquad \theta, s \in \mathbb{R}. \tag{2.25}$$

It is clear that $S_0 = I$, $S_{\theta+2\pi} = S_\theta$, and that $S_\theta$ induces a representation of the group of plane rotations $SO(2)$ on $C_{2\pi}$. Observe that since (2.3) is autonomous, whenever $u$ is a solution so is $S_\theta u$. In fact, for a $2\pi$-periodic solution $u$ of (2.3), all the solutions $S_\theta u$ define the same orbit in phase-space, and differ only by the choice of initial point $s = 0$. Thus it is natural to identify these solutions as one.

One can easily show that $S_\theta$ commutes with $d/ds$, $L$, $L^*$, $P$, and $Q$, and

$$(S_\theta u, S_\theta v) = (u, v). \tag{2.26}$$

Thus, $\mathcal{N}(L)$, $\mathcal{N}(L^*)$, $\mathcal{R}(L)$ and $W$ are all invariant subspaces for $S_\theta$. We denote by $S_\theta(x, y)$ the action of $S_\theta$ on $x\phi_1 + y\phi_2 \in \mathcal{N}(L)$ in terms of the coordinates $(x, y)$. One checks that this representation of $S_\theta$ is given in matrix form by the standard representation of $SO(2)$ on $\mathbb{R}^2$, i.e.,

$$S_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.27}$$

Now by the uniqueness of solutions to (2.21) given by the implicit function theorem, one has

$$w(S_\theta(x, y); \lambda, \tau) = S_\theta w(x, y; \lambda, \tau). \tag{2.28}$$

It follows that the reduced function $g$ satisfies

$$g(S_\theta(x, y); \lambda, \tau) = S_\theta g(x, y; \lambda, \tau), \tag{2.29}$$

that is, $g$ is $SO(2)$-equivariant. The consequences for $g$ of this equivariance are given in [10, p. 227] (using a result of Schwarz [31]). There it is shown that $g$ must have the coordinate form

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = p(x^2 + y^2, \lambda, \tau) \begin{pmatrix} x \\ y \end{pmatrix} + q(x^2 + y^2, \lambda, \tau) \begin{pmatrix} -y \\ x \end{pmatrix}, \tag{2.30}$$

where $p$ and $q$ are $C^\infty$ and defined near $(0, 0, 0)$. Moreover, since the linear terms of $g$ at the origin vanish, one has that $p(0) = q(0) = 0$. In this form, it is easy to see that solutions to $g = 0$ are given by either $x = y = 0$ (the trivial stationary solutions) or $p = q = 0$ (the periodic bifurcating solutions). Note that for a given solution $(x, y)$, the rotated solutions $S_\theta(x, y)$ correspond exactly to the previously discussed phase shift of periodic solutions to the original problem. It suffices to choose just one representative for each such orbit of solutions; we take $y = 0$, $x \geqslant 0$. The equation $g = 0$ from (2.30) is then

$$\begin{aligned} g_1 &= p(x^2, \lambda, \tau)x = 0, \\ g_2 &= q(x^2, \lambda, \tau)x = 0. \end{aligned} \tag{2.31}$$

A further reduction is now possible. One calculates directly from the definitions that

$$\frac{d\phi_1}{ds} = -\phi_2, \qquad \frac{d\phi_2}{ds} = \phi_1, \qquad \frac{d\psi_1}{ds} = -\psi_2, \qquad \frac{d\psi_2}{ds} = \psi_1. \tag{2.32}$$

Then differentiation of (2.23) using (2.22) and (2.32) gives

$$g_x(0, 0; 0, \tau) = P\tau \frac{d\phi_1}{ds} = -\tau\phi_2. \tag{2.33}$$

Therefore

$$p_\tau(0, 0, 0) = 0, \qquad q_\tau(0, 0, 0) = -1, \tag{2.34}$$

and the implicit function theorem guarantees that the equation $q(x^2, \lambda, \tau) = 0$ has a unique solution $\tau = \tau(x^2, \lambda)$ near $(0, 0, 0)$. Substituting this information into (2.31) yields a single scalar equation

$$G(x, \lambda) = a(x^2, \lambda)x = 0, \qquad a(0, 0)x = 0. \tag{2.35}$$

where $a(x^2, \lambda) = p(x^2, \lambda, \tau(x^2, \lambda))$. Note that (2.35) has a $\mathbb{Z}_2$ symmetry—if we allow negative as well as positive values of $x$—that is $G$ is odd in $x$. This is what remains of the $SO(2)$ symmetry in the original problem. The study of small nontrivial solutions of (2.35) is the object of the next section. It will become clear that Hopf's hypotheses (H2) and (H3) stated in the introduction are equivalent respectively to

$$a_\lambda(0, 0) \neq 0 \quad \text{and} \quad a_z(0, 0) \neq 0, \tag{2.36}$$

where $z \equiv x^2$. If $a_\lambda(0, 0) \neq 0$, then the implicit function theorem applied to $a(x^2, \lambda) = 0$ gives a unique small solution $\lambda = \lambda(x^2)$. If also $a_z(0, 0) \neq 0$, then the local behaviour of this solution is determined as in (1.3), i.e.,

$$\lambda = \mu_2 x^2 + \cdots, \qquad \mu_2 = -\frac{a_z(0, 0)}{a_\lambda(0, 0)} \neq 0. \tag{2.37}$$

## 3. Classification of Degenerate Bifurcations of Hopf Type

In the last section we showed that finding periodic solutions reduced to solving an equation of the form

$$G(x, \lambda) = a(x^2, \lambda)x = 0. \tag{3.1}$$

In particular $x = 0$ corresponds to a given steady-state solution and non-zero solutions $x$ to $a(x^2, \lambda) = 0$ correspond to periodic solutions. Our approach to (3.1) is to change coordinates so that $a(x^2, \lambda)$ has a simple polynomial form from which the solutions may easily be determined. This is the key to the use of singularity theory described in [9, 10]. The changes of coordinates we allow are:

Definition 3.2. $G$ and $H$ are $\mathbb{Z}_2$-*equivalent* if

$$H(x, \lambda) = T(x^2, \lambda)\, G(X(x^2, \lambda)x, \Lambda(\lambda)\, \lambda), \tag{3.3}$$

where

$$T(0, 0) \neq 0, X(0, 0) > 0, \text{ and } \Lambda(0) > 0.$$

The main result of this section is a classification by codimension of bifurcation problems (3.1). (Roughly speaking, the codimension of $G$ gives a measure of how difficult it is to find a given singularity.) In order to define codimension we introduce some algebraic notation.

Let $\mathscr{E}_2 = \mathscr{E}_{z,\lambda}$ be the ring of germs of $C^\infty$ functions mapping $(\mathbb{R}^2, 0) \to \mathbb{R}$ with coordinates $z$ and $\lambda$ on $\mathbb{R}^2$. Let $\mathscr{E}_\lambda$ be the space of germs depending on one variable $\lambda$. Let $\mathcal{M}$ denote the maximal ideal in $\mathscr{E}_{z,\lambda}$ consisting of germs which vanish at (0, 0). If $p_1,\dots,p_k$ are in $\mathscr{E}_{z,\lambda}$ we denote by $\langle p_1,\dots,p_k\rangle$ the ideal in $\mathscr{E}_{z,\lambda}$ generated by $p_1,\dots,p_k$. The product of two ideals is the ideal generated by products of germs, one from each of the two ideals. $\mathcal{M}^k$ is then the ideal in $\mathscr{E}_2$ of germs whose Taylor expansion begins with terms of degree $k$ or higher. In fact, Taylor's Theorem shows that $\mathcal{M}^k$ is generated by a basis for the homogeneous polynomials of degree $k$. An extremely useful tool in calculation is:

LEMMA 3.4 (Nakayama). *Let $\mathscr{U}=\langle p_1,\dots,p_k\rangle$ be an ideal in $\mathscr{E}_2$. Let $q_1,\dots,q_k$ be in $\mathcal{M}\cdot\mathscr{U}$. Then $\mathscr{U}=\langle p_1+q_1,\dots,p_k+q_k\rangle$.*

Let $z=x^2$. Associate to $G$ in (3.1) the ideal in $\mathscr{E}_2$

$$\tilde{\Gamma}G=\langle a(z,\lambda), za_z(z,\lambda)\rangle. \tag{3.5}$$

Also let

$$\Gamma G=\tilde{\Gamma}G+\mathscr{E}_\lambda\{a_\lambda(z,\lambda)\}, \tag{3.6}$$

where " + " indicates vector space sum and $\mathscr{E}_\lambda\{a_\lambda\}$ is the vector space of germs of the form $h(\lambda)a_\lambda(z,\lambda)$.

DEFINITION 3.7. The *codimension* of $G$ is dim $\mathscr{E}_2/\Gamma G$.

In [10], it is shown that codim $G$ is an invariant of $\mathbb{Z}_2$-equivalence. Another way of thinking of codim $G$ is: Suppose $V$ is a vector subspace of $\mathscr{E}_2$ such that

$$\mathscr{E}_2=\Gamma G\oplus V. \tag{3.8}$$

Then codim $G=\dim V$. We shall use $V$ to denote the choice of complementary subspace to $\Gamma G$ in $\mathscr{E}_2$ in future calculations.

A basic result from singularity theory [10] which enables one to compute $\mathbb{Z}_2$-equivalences in a somewhat mechanical fashion is:

PROPOSITION 3.9. *Suppose $G_t(x,\lambda)=H(x,\lambda)+tP(x,\lambda)$ and that either*

$$\tilde{\Gamma}G_t=\tilde{\Gamma}H \qquad \textit{for all } t \tag{3.9a}$$

or

$$\Gamma G_t=\Gamma H \qquad \textit{for all } t. \tag{3.9b}$$

*Then $G_t$ is $\mathbb{Z}_2$-equivalent to $H$ for all $t$.*

The situation in classical Hopf bifurcation is that

$$G(x,\lambda)=Bx^3+Cx\lambda+\cdots=0, \tag{3.10}$$

where $B \cdot C \neq 0$. We will show incidentally that under these assumptions $G$ is $\mathbb{Z}_2$-equivalent to

$$H(x, \lambda) = x^3 \pm \lambda x = 0. \tag{3.11}$$

Our main concern here is to find out what happens when either $B$ or $C$ or both is zero.

A second theoretical issue is to classify what bifurcation problems are possible when a given $G$ is subjected to small perturbations. The answer—given by singularity theory—is, in principle, quite straightforward. Suppose codim $G < \infty$ and let $P_1, \ldots, P_k$ be a basis for the complementary subspace $V$. Then

$$F(x, \lambda, \alpha) = G(x, \lambda) + \alpha_1 P_1(x, \lambda) + \cdots + \alpha_k P_k(x, \lambda) \tag{3.12}$$

is a universal unfolding of $G$. That is, every small perturbation of $G$ is $\mathbb{Z}_2$-equivalent to $F(\cdot,\cdot, \alpha)$ for some fixed $\alpha$ with $\alpha$ depending smoothly on the given perturbation.

As the above discussion suggests, one must assume that codim $G < \infty$ in order to make a complete singularity theory analysis of the problem. This is a very mild assumption which we henceforth make.

The assumption of finite codimension has two immediate consequences. To see this write $a(z, \lambda)$ as a power series in $z$ with coefficients in $\lambda$; that is,

$$a(z, \lambda) = a_0(\lambda) + a_1(\lambda) z + \cdots + a_n(\lambda) z^n + \cdots. \tag{3.13}$$

If $a_0(\lambda)$ vanishes to infinite order at $\lambda = 0$, then to any finite order $z$ is a factor of $a$, $za_z$, and $a_\lambda$. As a result $\lambda, \lambda^2, \lambda^3, \ldots$ are all independent vectors in $\mathscr{E}_2/\Gamma G$ and $G$ has infinite codimension. Next suppose that $a_i(0) = 0$ for all $i$. Then again $G$ has infinite codimension as now $\lambda$ is a factor of $a$ to any finite order. One can check $z, z^2, z^3, \ldots$ are independent in $\mathscr{E}_2/\Gamma G$ except for (perhaps) one relation (given by $a_\lambda$).

Let $m$ be the first integer for which $a_m(0) \neq 0$ and let $k$ be the integer such that $a_0(\lambda) = \lambda^k \tilde{a}_0(\lambda)$ with $\tilde{a}_0(0) \neq 0$. It is easy to show that $G$ is $\mathbb{Z}_2$-equivalent to $G$ with the form (divide by $a_m$ and rescale $\lambda$)

$$a(z, \lambda) = \pm \lambda^k + a_1(\lambda) z + \cdots + a_{m-1}(\lambda) z^{m-1} + z^m + a_{m+1}(z, \lambda) z^{m+1}. \tag{3.14}$$

We now present our results. As indicated above the case $m = k = 1$ is the situation of Hopf bifurcation. The following two propositions determine what happens when one or the other of these two conditions fails.

PROPOSITION 3.15. *Suppose $k = 1$. Then $G$ is $\mathbb{Z}_2$-equivalent to*

$$H(x, \lambda) = (z^m \pm \lambda)x, \tag{3.16}$$

*where* codim $H = m - 1$ *and a basis for $V$ is* $\{z, z^2, \ldots, z^{m-1}\}$.

PROPOSITION 3.17. *Suppose $m = 1$. Then $G$ is $\mathbb{Z}_2$-equivalent to*

$$H(x, \lambda) = (z \pm \lambda^k)x, \tag{3.18}$$

*where* codim $H = k - 1$ *and a basis for $V$ is* $\{1, \lambda, \ldots, \lambda^{k-2}\}$.

The beginning of a classification by codimension promised above is:

THEOREM 3.19. *Suppose that $G$ has codimension $\leqslant 3$. Then $G$* is $\mathbb{Z}_2$-*equivalent to one of the following $H$'s.*

| ($\nu$) | $H$ | codim $H$ | Basis for $V$ |
|---|---|---|---|
| (1) | $x^3 \pm \lambda x$ | 0 | |
| (2) | $x^3 \pm \lambda^2 x$ | 1 | $x$ |
| (3) | $x^5 \pm \lambda x$ | 1 | $x^3$ |
| (4) | $x^3 \pm \lambda^3 x$ | 2 | $x, \lambda x$ |
| (5) | $x^7 \pm \lambda x$ | 2 | $x^3, x^5$ |
| (6) | $x^3 \pm \lambda^4 x$ | 3 | $x, \lambda x, \lambda^2 x$ |
| (7) | $x^5 + 2b\lambda x^3 + \varepsilon\lambda^2 x$; $\varepsilon = \pm 1, b^2 \neq \varepsilon$, | 3* | $x, x^3, \lambda x^3$ |
| (8) | $x^5 \pm 2(\lambda \pm \lambda^2)x^3 + \lambda^2 x$ | 3 | $x, x^3, \lambda x^3$ |
| (9) | $x^5 \pm 2\lambda x^3 \pm \lambda^3 x$ | 3 | $x, \lambda x, \lambda^2 x$ |
| (10) | $x^7 \pm \lambda x^3 \pm \lambda^2 x$ | 3 | $x, x^3, x^5$ |
| (11) | $x^9 \pm \lambda x$ | 3 | $x^3, x^5, x^7$. |

In the next section, we shall describe the bifurcation diagrams associated with problems of codimension $\leqslant 2$ along with their universal unfoldings. We note here that in a very concrete sense normal form (7) has codimension two when $b \neq 0$. That is, the topological $\mathbb{Z}_2$-codimension of (7) is two even though the $C^\infty\mathbb{Z}_2$-equivalences we use here gives codimension 3. We shall discuss this point more fully in the next section. We also note that normal form (7) is the most likely case to occur if both conditions on Hopf bifurcation fail, that is, $m > 1$ and $k > 1$.

Several authors [8, 33, 37] have studied the case $m = 2$ and obtained the very beginnings of a classification of such bifurcation problems. Here, we extend this classification substantially giving a complete description of those problems where no higher order information in $z$ than $m = 2$ is needed.

THEOREM 3.20. *Assume that $m = 2$ and that $G$ has finite codimension. Then $G$ is $\mathbb{Z}_2$-equivalent to one of the following:*

| | $H$ | Restrictions | codim $H$ |
|---|---|---|---|
| (1) | $x^5 \pm \lambda^k x$ | | $2k-1$ |
| (2) | $x^5 \pm \lambda^l x^3 \pm \lambda^k x$ | $1 \leqslant l \leqslant k-1, k \neq 2l$ | $k+l-1$ |
| (3) | $x^5 + 2b\lambda^l x^3 + \varepsilon\lambda^{2l} x$ | $\varepsilon = \pm 1, b \neq 0, b^2\varepsilon \neq 1$ | $4l-1$ |
| (4) | $x^5 \pm 2(b\lambda^l \pm \lambda^N)x^3 + \varepsilon\lambda^{2l} x$ | $l \geqslant 2, l+1 \leqslant N \leqslant 2l-1$ $\varepsilon = \pm 1, b \neq 0, b^2\varepsilon \neq 1$ | $2l+N-1$ |
| (5) | $x^5 \pm 2(\lambda \pm \lambda^2)x^3 + \lambda^2 x$ | | 3 |
| (6) | $d(x^2, \lambda)x^9 + x^5 \pm 2(\lambda^l + c(\lambda)\lambda^{l+1})x^3 + \lambda^{2l} x$ | $l \geqslant 2$ | $\geqslant 5$ |
| (7) | $d(x^2, \lambda)x^9 + x^5 \pm 2(\lambda + c(\lambda)\lambda^3) + \lambda^2 x$ | | $\geqslant 7$ |

*Note* (A). To determine whether a given $G$ is $\mathbb{Z}_2$-equivalent to normal forms (1)–(4) one does *not* need any information about the Taylor expansion of $G$ involving terms of the form $z^p$ for $p \geqslant 3$. One does need such information to obtain normal form (5) or to proceed with the classification of problems of type (6) or (7). (See Proposition 3.47.)

*Note* (B). The type of higher order information needed to show that $G$ is $\mathbb{Z}_2$-equivalent to (5) is contained in a certain non-degeneracy condition (see Proposition 3.47(8).) That is to say, if $G$ satisfies certain lower order conditions, then generically (or almost always) $G$ will be $\mathbb{Z}_2$-equivalent to (5). This is in contrast to the type of information needed to pursue the classification of (6) and (7) where higher order information is needed in an essential manner.

*Proof of Proposition 3.15.* From (3.16) we see that

$$\tilde{\Gamma} H = \langle z^m, \lambda \rangle. \tag{3.21}$$

Thus $\lambda z$ and $z^{m+1}$ are in $\mathcal{M}.\tilde{\Gamma} H$. By Nakayama's Lemma $\tilde{\Gamma} G = \tilde{\Gamma} H$ regardless of the choice of $a_1, \ldots, a_{m-1}, a_{m+1}$ in (3.14). By Proposition 3.9(a), we see that $G$ is $\mathbb{Z}_2$-equivalent to $H$. Moreover $\{z, z^2, \ldots, z^{m-1}\}$ is a basis for a complementary space $V$ to $\Gamma H$.

*Proof of Proposition* 3.17. From (3.18) we see that

$$\tilde{\Gamma} H = \langle z, \lambda^k \rangle. \tag{3.22}$$

As $z^2 \in M\tilde{\Gamma} H$, it follows that $\tilde{\Gamma} G = \tilde{\Gamma} H$. So $G$ is $\mathbb{Z}_2$-equivalent to $H$. A simple calculation shows that $\{1, \lambda, \ldots, \lambda^{k-2}\}$ is a basis for $V$.

*Proof of Theorem* 3.19 *assuming Theorem* 3.20. Assume that $G$ has codimension $\leqslant 3$. If $m = 1$, then Proposition 3.17 shows that $G$ is $\mathbb{Z}_2$-equivalent to (1), (2), (4) or (6). If $m = 2$, then Theorem 3.20 states that $G$ is $\mathbb{Z}_2$-equivalent to (3), (7), (8) or (9). The calculations to obtain the basis for $V$ for these $H$'s may either be done directly or recovered from the proof of Theorem 3.20. So we may assume that $m \geqslant 3$. If $k = 1$, then Proposition 3.15 states that $G$ is $\mathbb{Z}_2$-equivalent to either (5) or (11).

It follows that we may assume that $G$ has the form

$$a(z, \lambda) = \pm \lambda^k + a_1(\lambda)z + a_2(\lambda)z^2 + a_3(z, \lambda)z^3, \tag{3.23}$$

where $k \geqslant 2$. First we show that if $k \geqslant 3$, then codim $G \geqslant 4$. To see this compute

$$\tilde{\Gamma}G + \mathcal{M}^3 = \langle \lambda^k, a_1(0)\lambda z \rangle + \mathcal{M}^3 = \langle a_1(0)\lambda z \rangle + \mathcal{M}^3. \tag{3.24}$$

Thus 1, $\lambda$, $\lambda^2$, $z$, $z^2$ are independent in $\mathscr{E}_2/(\tilde{\Gamma}G + \mathcal{M}^3)$. Now

$$a_\lambda(z, \lambda) \equiv \pm h\lambda^{k-1} + a_1'(\lambda)z + a_2'(0)z^2 \bmod \mathcal{M}^3 \tag{3.25}$$

while $\lambda a_\lambda$ is in $\tilde{\Gamma}G + \mathcal{M}^3$. Thus at least four of the five vectors are independent in $\mathscr{E}_2/(\Gamma G + \mathcal{M}^3)$ and thus in $\mathscr{E}_2/\Gamma G$.

Next we assume that $k = 2$ and $m > 3$. So $a$ has the form:

$$a(z, \lambda) = \pm \lambda^2 + a_1(\lambda)z + a_2(\lambda)z^2 + a_3(\lambda)z^3 + a_4(\lambda, z)z^4, \tag{3.26}$$

where $a_1(0) = a_2(0) = a_3(0) = 0$. Now compute:

$$\tilde{\Gamma}G + \mathcal{M}^4 = \langle \pm \lambda^2 - a_2'(0)\lambda z^2, a_1(\lambda)z + 2a_2'(0)\lambda z^2 \rangle + \mathcal{M}^4. \tag{3.27}$$

It follows that 1, $\lambda$, $z$, $z^2$, $z^3$ are independent in $\mathscr{E}/(\tilde{\Gamma}G + \mathcal{M}^4)$. Moreover,

$$\lambda a_\lambda(z, \lambda) \equiv \pm 2\lambda^2 + \lambda a_1'(\lambda)z + \lambda a_2'(0)z^2 \bmod \mathcal{M}^4 \tag{3.28}$$

is independent of the five terms above. Again at least four of these five terms are independent in $\mathscr{E}_2/(\Gamma G + \mathcal{M}^4)$. So codim $G \geqslant 4$.

We have reduced the Theorem to the case $k = 2$ and $m = 3$. We may assume $a(z, \lambda)$ has the form

$$a(z, \lambda) = \pm\lambda^2 + a_1(\lambda)z + a_2(\lambda)z^2 + z^3 + a_4(z, \lambda)z^4, \tag{3.29}$$

where $a_1(0) = a_2(0) = 0$. Now with an argument similar to the ones above, one shows that if $a_1'(0) = 0$, then codim $G \geqslant 4$. Here, one computes $\Gamma G + \mathcal{M}^3$. So we can assume $a_1'(0) \neq 0$. Now by scaling we have

$$a(z, \lambda) = \pm\lambda^2 \pm \lambda z + z^3 + p(\lambda)\lambda^2 z + q(\lambda)\lambda z^2 + r(z, \lambda)z^4. \tag{3.30}$$

Now compute $\tilde{\Gamma}H$ for $H$ the normal form (10) to obtain

$$\tilde{\Gamma}H = \langle -2z^3 \pm \lambda^2, 3z^3 \pm \lambda z \rangle. \tag{3.31}$$

Computing modulo $\mathscr{M}$. $\tilde{\Gamma}H$ one has

$$z^4 \equiv \pm \frac{\lambda^2 z}{2} \equiv \pm \frac{3\lambda z^3}{2} \equiv \pm \frac{qz^5}{2}. \tag{3.32}$$

Hence $z^4(1-(q/2)z) \in \mathscr{M}^2$. $\tilde{\Gamma}H$ which implies that $z^4$, $\lambda^2 z$ and $\lambda z^3$ are in $\mathscr{M}$. $\tilde{\Gamma}H$. Nakayama's lemma implies $\tilde{\Gamma}G$ is independent of $p$, $q$, and $r$ while Proposition 3.9(a) implies that $G$ is $\mathbb{Z}_2$-equivalent to $H$. Finally one computes the codimension of $H$ to be 3 and a basis for $V$ to be the one stated in the Theorem.

*Proof of Theorem* 3.20. Assuming $m = 2$ one has

$$a(z,\lambda) = \varepsilon\lambda^k + 2a_1(\lambda)z + z^2 + a_3(\lambda)z^3 + a_4(z,\lambda)z^4, \tag{3.33}$$

where $\varepsilon = \pm 1$. Choose $l$ so that $a_1 = \lambda^l \tilde{a}_1$, where $l \geqslant 1$. Consider the change of coordinates $z = \bar{z} - a_3(\lambda)\bar{z}^2/2$ yielding after a division by $1 - a_1 a_3$

$$a(\bar{z},\lambda) = \frac{\varepsilon\lambda^k}{1 - a_1 a_3} + \frac{2a_1}{1 - a_1 a_3}\bar{z} + \bar{z}^2 + a_4(\bar{z},\lambda)\bar{z}^4. \tag{3.34}$$

Now let $\bar{\lambda} = \lambda/(\sqrt[k]{1 - a_1 a_3})$, thus obtaining after dropping the bars

$$a(z,\lambda) = \varepsilon\lambda^k + 2\lambda^l\; b(\lambda)z + z^2 + d(z,\lambda)z^4 \equiv A(z,\lambda) + d(z,\lambda)z^4. \tag{3.35}$$

Observe that the computations of $k$, $l$, $\varepsilon$, and $b(0)$ are independent of $a_3$ and $a_4$ as $b(0) = \tilde{a}_1(0)$. In fact $\lambda$ as a function of $\bar{\lambda}$ has the form $\lambda = \bar{\lambda} + q(\bar{\lambda})\bar{\lambda}^{l+1}$, where $q(0)$ *does* depend on $a_3(0)$. From this one can see that the coefficients of $\lambda^j$ in $b$ for $j < l$ are independent of $q(0)$ and $a_3(0)$. However, the coefficient of $\lambda^l$ in $b$ does depend on $a_3(0)$.

We consider four cases:

$$\begin{array}{lll} \text{(i)} & l \geqslant k, & \\ \text{(ii)} & 1 \leqslant l \leqslant k-1, & 2l \neq k,\ b(0) \neq 0, \\ \text{(iii)} & 2l = k, & b^2(0)\varepsilon \neq 1,\ b(0) \neq 0, \\ \text{(iv)} & 2l = k, & b^2(0)\varepsilon = 1. \end{array} \tag{3.36}$$

Let $H$ denote the bifurcation problem associated to $A(z,\lambda)$ defined in (3.35). In all cases one has:

$$\tilde{\Gamma}H = \langle \lambda^l b(\lambda)z + z^2, z^2 - \varepsilon\lambda^k \rangle. \tag{3.37}$$

Thus

$$\lambda^{2k} \equiv z^4 \equiv \lambda^{2l} b^2(\lambda) z^2 \equiv \lambda^{2l+k} b^2(\lambda)\varepsilon \bmod \mathcal{M}\tilde{\Gamma}H. \tag{3.38}$$

Thus for cases (3.36) (i), (ii), and (iii) one has $\lambda^{2k} \in \mathcal{M}\tilde{\Gamma}H$ and $z^4 \in \mathcal{M}\tilde{\Gamma}H$. Proposition 3.9(a) shows that $G$ is $\mathbb{Z}_2$-equivalent to $H$ as $\tilde{\Gamma}G$ is independent of $d(z, \lambda)$ by Nakayama's lemma.

In case (i), one has

$$\varepsilon\lambda^k \equiv z^2 \equiv -\lambda^l b(\lambda) z \bmod \tilde{\Gamma}H. \tag{3.39}$$

As $l \geqslant k$, one has $\lambda^k$ and $z^2$ in $\tilde{\Gamma}H$. In fact

$$\tilde{\Gamma}H = \langle \lambda^k, z^2 \rangle. \tag{3.40}$$

As $\tilde{\Gamma}H$ is independent of $b(\lambda)$ we use Proposition 3.9(a) to show that $H$ is $\mathbb{Z}_2$-equivalent to (1) in the Theorem. Moreover, 1, $\lambda, \ldots, \lambda^{k-2}$, $z$, $z\lambda, \ldots, z\lambda^{k-1}\}$ is a basis for $V$.

In case (3.36)(ii) we prove below that

$$\Gamma H = \langle z^2, \lambda^k, z\lambda^l \rangle + \mathbb{R}\{2l\, b(0)\lambda^{l-1} z + \varepsilon k \lambda^{k-1}\}. \tag{3.41}$$

In this computation one uses only that $b(0) \neq 0$. Now one scales the variables so that $b(0) = \pm 1$ and applies Proposition 3.9(b) to see that it is $\mathbb{Z}_2$-equivalent to normal form (2) of the Theorem. Also the space $V$ is spanned by the $k + l - 1$ vectors 1, $\lambda, \ldots, \lambda^{k-1}$, $z$, $z\lambda \cdots z\lambda^{l-2}$. To prove (3.41) observe that

$$\lambda a_\lambda = \lambda^l \tilde{b}(\lambda) z + k\lambda^k, \tag{3.42}$$

where $l\tilde{b}(0) = b(0)$. As $\lambda^l\, b(\lambda) z + \varepsilon\lambda^k$ is in $\tilde{\Gamma}H$ one sees that $\lambda^k$ and thus $z^2$, $z^3$ and $z\lambda^l$ are in $\Gamma H$. As all multiples of vectors in $\Gamma G$ by $\lambda$ remain in $\Gamma G$, we have proved (3.41). Recall (3.38).

We now consider case (ii). Let the two generators of $\tilde{\Gamma}H$ in (3.37) be $p$ and $q$. It is straightforward to show that $zp$, $\lambda^l p$, $zq$, and $\lambda^l q$ are independent and thus that $z^3$, $z^2\lambda^l$, $z\lambda^{2l}$, $\lambda^{3l}$ are in $\mathcal{M}\tilde{\Gamma}H$. Thus $H$ is $\mathbb{Z}_2$-equivalent to

$$a(z, \lambda) = \varepsilon\lambda^k + 2\lambda^l(b_l + b_{l+1}\lambda + \cdots + b_{2l+1}\lambda^{l-1})z + z^2.$$

If $b_{l+1} \cdots = b_{2l-1} = 0$, then $H$ is $\mathbb{Z}_2$-equivalent to the normal form (3) of the Theorem. If not, let $N$ be the integer such that $b_{l+1} = \cdots = b_{N-1} = 0$ and $b_N \neq 0$. Note that $l + 1 \leqslant N \leqslant 2l - 1$. Also one can scale $\lambda$ and $z$ so that $b_N = \pm 1$. Next observe that $z\lambda^N$ is in $\Gamma H$ from which it follows that $\lambda^{N+l}$ and $z^2\lambda^{N-l}$ are in $\Gamma H$. To see this compute

$$a - \frac{z a_z}{2} - \frac{\lambda a_\lambda}{2l} = \lambda^N z\, q(\lambda),$$

where $q(0) \neq 0$. One can now show that the subspace $\Gamma H$ does not depend on $b_{N+1},..., b_{2l-1}$. Apply Proposition 3.9(b) to show that $H$ is $\mathbb{Z}_2$-equivalent to the normal form (4). From the above discussion one sees that the $3N$ vectors $1, \lambda,..., \lambda^{N+l-1}, z, z\lambda,..., z\lambda^{N-1}, z^2, z^2\lambda,..., z^2\lambda^{N-l-1}$ span a subspace $W$ of $\mathscr{E}_2$ such that $\Gamma H + W = \mathscr{E}_2$. However, if $p$ and $q$ are the generators of $\tilde{\Gamma} H$ in (3.37) then the $2N - 2l + 1$ vectors $p, \lambda p,..., \lambda^{N-l-1}p, q, \lambda q,..., \lambda^{N-l-1}q, a_\lambda$ span a complement to $V$ in $W$. Thus $\dim V = 2l + N - 1$ as stated in the Theorem. Note that $1, \lambda,..., \lambda^{2l-2}, z, z\lambda,..., z\lambda^{N-1}$ span $V$. Taking $N = 2l$ gives the enumeration for normal form (3).

Finally we consider the case (3.36)(iv) where $b(0) = \pm 1$, $\varepsilon = 1$, and $k = 2l$. First we observe that if $l > 1$, then the computation of codimension is included in case (6). Taking $l = 1$, we have

$$a(z, \lambda) = z^2 + 2(b\lambda + c\lambda^2 + f(\lambda)\lambda^3)z + \lambda^2 + d(z, \lambda)z^4,$$

where $b = \pm 1$. If $c = 0$, then case (7) will show that codim $G \geqslant 4$. If $c \neq 0$, then $c$ may be scaled to $\pm 1$. Note that the computation of $c$ depends on the $z^3$ term in the original $G$; recall the discussion after (3.35). Consider the normal form $H$ given by

$$A(z, \lambda) = z^2 + 2(b\lambda + c\lambda^2)z + \lambda^2 = (z + b\lambda)^2 + 2c\lambda z^2. \tag{3.43}$$

We claim that $\mathscr{M}^4 \subset \mathscr{M}.\tilde{\Gamma} H$. Recall that

$$\tilde{\Gamma} H = \langle (z + b\lambda)^2 + 2c\lambda^2 z, z(z + b\lambda) + c\lambda^2 z \rangle \equiv \langle p, q \rangle. \tag{3.44}$$

Observe that $r = zp - (z + b\lambda)q = c\lambda^2 z^2 - bc\lambda^3 z$. By Nakayama's Lemma one sees that $\mathscr{M}^4 = \langle r, (z + b\lambda)^2 p, (z + b\lambda)^2 q, (z + b\lambda)zq, z^2 q \rangle$ thus proving the claim. Proposition 3.9(a) implies that $G$ is $\mathbb{Z}_2$-equivalent to $H$. It is an exercise to show that codim $H = 3$.

To show that bifurcation problems of the form

$$a(z, \lambda) = z^2 \pm 2(\lambda^l + c(\lambda)\lambda^{l+1})z \pm \lambda^{2l} + d(z, \lambda)z^4 \tag{3.45}$$

have codimension $\geqslant 5$, compute $\Gamma H + \mathscr{M}^4$ and see that $1, z, \lambda, z\lambda$, and $\lambda^2$ are contained in a complement $V$ to $\Gamma H + \mathscr{M}^4$. Finally, to show that bifurcation problems

$$a(z, \lambda) = z^2 \pm 2(\lambda + c(\lambda)\lambda^3)z + \lambda^2 + d(z. \lambda)z^4 \tag{3.46}$$

has codimension $\geqslant 7$, compute $\Gamma H + \mathscr{M}^5$ and see that $1, z, z^2, \lambda^2, z^3, z^3\lambda$, and $z^2\lambda^2$ are contained in a complement to $\Gamma H + \mathscr{M}^5$.

In order to use the results described in this section in a given application, one must answer the following two questions. Given $G(x, \lambda) = a(z, \lambda)x$, when is $G$ $\mathbb{Z}_2$-equivalent to one of the eleven normal forms listed in Theorem 3.19?

Suppose that $F(x, \lambda, \alpha)$ is a $k$ parameter unfolding of $G(x, \lambda)$, where $k = \text{codim } G$; that is, $\alpha \in \mathbb{R}^k$ and $F(x, \lambda, 0) = G(x, \lambda)$. Under what conditions is $F$ a universal $\mathbb{Z}_2$-unfolding of $G$? The answers to these questions were given—in principle—in the Proof of Theorem 3.19. For future reference we summarize the results in the following two propositions.

PROPOSITION 3.47. *$G$ is $\mathbb{Z}_2$-equivalent to normal form $(\nu)$ of Theorem* 3.19 *if and only if the following defining conditions and non-degeneracy conditions on $a(z, \lambda)$ are satisfied at $(z, \lambda) = (0, 0)$. Note $a(0, 0) = 0$ always.*

| $(\nu)$ | Defining conditions | Non-degeneracy conditions |
|---|---|---|
| (1) | None | $a_z \neq 0, a_\lambda \neq 0$ |
| (2) | $a_\lambda = 0$ | $a_z \neq 0, a_{\lambda\lambda} \neq 0$ |
| (3) | $a_z = 0$ | $a_{zz} \neq 0, a_\lambda \neq 0$ |
| (4) | $a_\lambda = a_{\lambda\lambda} = 0$ | $a_z \neq 0, a_{\lambda\lambda\lambda} \neq 0$ |
| (5) | $a_z = a_{zz} = 0$ | $a_{zzz} \neq 0, a_\lambda \neq 0$ |
| (6) | $a_\lambda = a_{\lambda\lambda} = a_{\lambda\lambda\lambda} = 0$ | $a_z \neq 0, a_{\lambda\lambda\lambda\lambda} \neq 0$ |
| (7) | $a_z = a_\lambda = 0$ | $a_{zz} \neq 0, a_{\lambda\lambda} \neq 0, b^2\varepsilon \neq 1$ |

*where* $b = a_{z\lambda}/|a_{zz} \cdot a_{zz}|^{1/2}$ *and* $\varepsilon = \text{sgn}(a_{zz} \cdot a_{\lambda\lambda})$.

(8) $\quad a_z = a_\lambda = 0; \varepsilon = +1, b^2 = 1 \qquad a_{zz} \neq 0, a_{\lambda\lambda} \neq 0$

*and choose* $v \neq 0$ *so that* $D^2a(v, v) = 0$, *then* $D^3a(v, v, v) \neq 0$.

(The first $\pm$ sign is given by $b$; the second $\pm$ sign depends on the sign of $D^3a(v, v, v)$.)

(9) $\quad a_z = a_\lambda = a_{\lambda\lambda} = 0 \qquad a_{zz} \neq 0, a_{z\lambda} \neq 0, a_{\lambda\lambda\lambda} \neq 0.$

(The first $\pm$ sign is $\text{sgn}(a_{z\lambda} \cdot a_{zz})$; the second $\pm$ sign is $\text{sgn}(a_{\lambda\lambda\lambda} \cdot a_{zz})$.)

(10) $\quad a_z = a_{zz} = a_\lambda = 0 \qquad a_{zzz} \neq 0, a_{\lambda\lambda} \neq 0, a_{z\lambda} \neq 0$

(The first $\pm$ sign is $\text{sgn}(a_{z\lambda} \cdot a_{zzz})$; the second $\pm$ sign is $\text{sgn}(a_{\lambda\lambda} \cdot a_{zzz})$.)

(11) $\quad a_z = a_{zz} = a_{zzz} = 0 \qquad a_{zzzz} \neq 0; a_\lambda \neq 0.$

The answer for the second question—when is $F$ a universal $\mathbb{Z}_2$-unfolding of $G$—is somewhat obvious for the first five normal forms; namely, when the perturbation parameters fill in the missing lower order terms in $G$. We list the results, for these cases and normal form (7) as these are the cases most likely to occur in applications.

PROPOSITION 3.48. *$F(x, \lambda, \alpha, \beta, \ldots) = e(z, \lambda, \alpha, \beta, \ldots)x$ is a universal unfolding of $G(x, \lambda)$—assumed $\mathbb{Z}_2$-equivalent to normal form ($\nu$)—if and only if the following conditions hold.*

| ($\nu$) | Non-degeneracy condition |
|---|---|
| (1) | None |
| (2) | $e_\alpha \neq 0$ |
| (3) | $e_{\alpha z} \neq 0$ |
| (4) | $\det\begin{pmatrix} e_\alpha & e_{\alpha\lambda} \\ e_\beta & e_{\beta\lambda} \end{pmatrix} \neq 0$ |
| (5) | $\det\begin{pmatrix} e_{\alpha z} & e_{\alpha zz} \\ e_{\beta z} & e_{\beta zz} \end{pmatrix} \neq 0$ |
| (7) | $\det\begin{pmatrix} 0 & 0 & 0 & a_{zz} & a_{z\lambda} & a_{\lambda\lambda} \\ 0 & 0 & 0 & 2a_{zz} & a_{z\lambda} & 0 \\ 0 & a_{z\lambda} & a_{\lambda\lambda} & a_{zz\lambda} & a_{z\lambda\lambda} & a_{\lambda\lambda\lambda} \\ e_\alpha & e_{\alpha z} & e_{\alpha\lambda} & e_{\alpha zz} & e_{\alpha z\lambda} & e_{\alpha\lambda\lambda} \\ e_\beta & e_{\beta z} & e_{\beta\lambda} & e_{\beta zz} & e_{\beta z\lambda} & e_{\beta\lambda\lambda} \\ e_\gamma & e_{\gamma z} & e_{\gamma\lambda} & e_{\gamma zz} & e_{\gamma z\lambda} & e_{\gamma\lambda\lambda} \end{pmatrix} \neq 0$ |

As we have indicated above, normal form (7) has topological codimension 2 (as long as $b \neq 0$, that is, $a_{z\lambda} \neq 0$). As we shall show in the next section, only the range of $b$ matters when classifying the perturbed bifurcation diagrams. In this case ($b \neq 0$) only two perturbation parameters are needed to obtain (up to topological $\mathbb{Z}_2$-equivalence) all the perturbed bifurcation diagrams. Thus when $b \neq 0$ a universal $\mathbb{Z}_2$ (topological) unfolding is given by $F = ex$ as long as $e$ satisfies

$$\det\begin{pmatrix} 0 & a_\lambda & a_{\lambda\lambda} \\ e_\alpha & e_{\alpha z} & e_{\alpha\lambda} \\ e_\beta & e_{\beta z} & e_{\beta\lambda} \end{pmatrix} \neq 0. \tag{3.49}$$

As a final remark we note, that if $b = 0$ ($= a_{z\lambda}$) then the matrix condition (7) of Proposition 3.48 simplifies to

$$\det\begin{pmatrix} 0 & 0 & a_{\lambda\lambda} & a_{z\lambda\lambda} \\ e_\alpha & e_{\alpha z} & e_{\alpha\lambda} & e_{\alpha z\lambda} \\ e_\beta & e_{\beta z} & e_{\beta\lambda} & e_{\beta z\lambda} \\ e_\gamma & e_{\gamma z} & e_{\gamma\lambda} & e_{\gamma z\lambda} \end{pmatrix} \neq 0 \tag{3.50}$$

So, in practice, the computation of the determinant of the full $6 \times 6$ matrix for unfoldings of this normal form is never necessary.

## 4. The Bifurcation Diagrams

In this section we present the bifurcation diagrams associated to the universal unfoldings of problems with codimension less than or equal to two. We also include the one problem of codimension three whose topological codimension is two. (We are reasonably sure that we have found all bifurcation problems of topological codimension less than three, though we have not proved this.)

Specifically, we consider the following universal unfolding normal forms from Proposition 3.19. Here $\varepsilon = \pm 1$, $\alpha$ and $\beta$ are unfolding parameters, and $b$ is a modal parameter.

$$x^3 + \varepsilon\lambda x = 0, \tag{4.1}$$

$$x^3 + \varepsilon(\lambda^2 + \alpha)x = 0, \tag{4.2}$$

$$x^5 + 2\alpha x^3 + \varepsilon\lambda x = 0, \tag{4.3}$$

$$x^3 + \varepsilon(\lambda^3 + \beta\lambda + \alpha)x = 0, \tag{4.4}$$

$$x^7 + \beta x^5 + \alpha x^3 + \varepsilon\lambda x = 0, \tag{4.5}$$

$$x^5 + 2b\lambda x^3 + (\lambda^2 + \operatorname{sgn}(b)\beta\lambda + \alpha)x = 0, \qquad b \neq 0, \pm 1, \tag{4.6}$$

$$x^5 + 2b\lambda x^3 - (\lambda^2 + \operatorname{sgn}(b)\beta\lambda + \alpha)x = 0, \qquad b \neq 0. \tag{4.7}$$

The various bifurcation diagrams associated with these normal forms are given in Figs. 4.1 to 4.7, respectively. We choose $x$ as the vertical axis and $\lambda$ as the horizontal axis. We also assume that $x \geqslant 0$.

The degenerate Hopf bifurcation diagrams obtained by [7, 21] correspond to the cases $\alpha = \beta = 0$ in our figures, though even here they did not consider the effect of the modal parameter $b$ in (4.6) and (4.7).

There are two pieces of information contained in each of these figures: the qualitative structure of the zero sets and the stability of each solution branch. We will explain below in more detail how the zero sets are found. We have

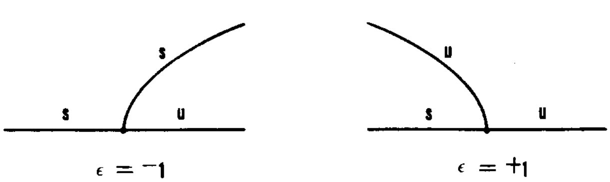


Fig. 4.1. Bifurcation diagrams for normal form (4.1).

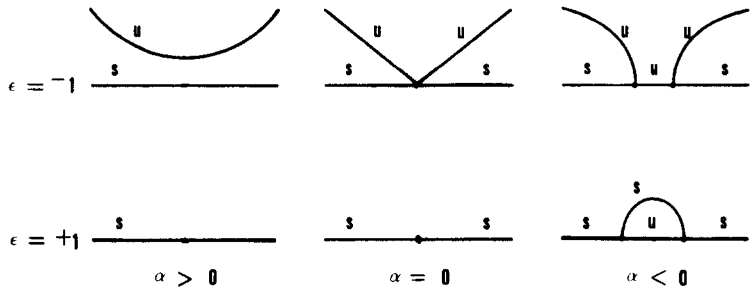


FIG. 4.2. Bifurcation diagrams for normal form (4.2).

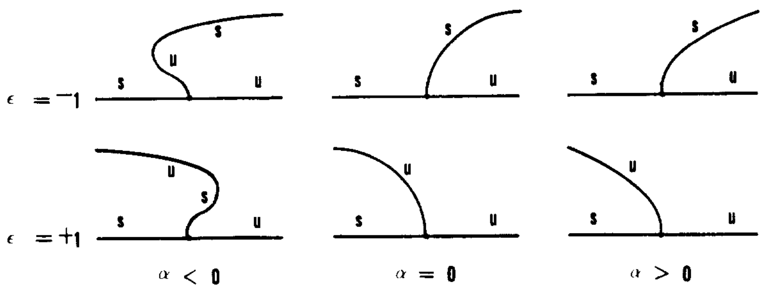


FIG. 4.3. Bifurcation diagrams for normal form (4.3).

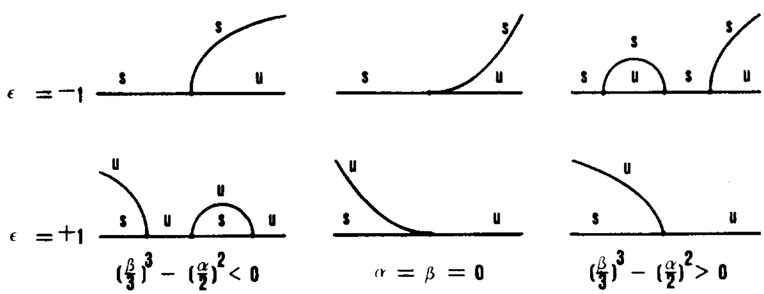


FIG. 4.4. Bifurcation diagrams for normal form (4.4).

used the notation that "$s$" means stable and "$u$" means unstable; this notation has the obvious meanings for both the stationary and periodic solutions.

For the stability assignments of "$s$" and "$u$", we assume that $(d_u f)(0, 0)$ has all eigenvalues except $\pm i$ with negative real parts and we have made the convention that the steady state solutions $x = 0$ for $\lambda < 0$ are stable. If those

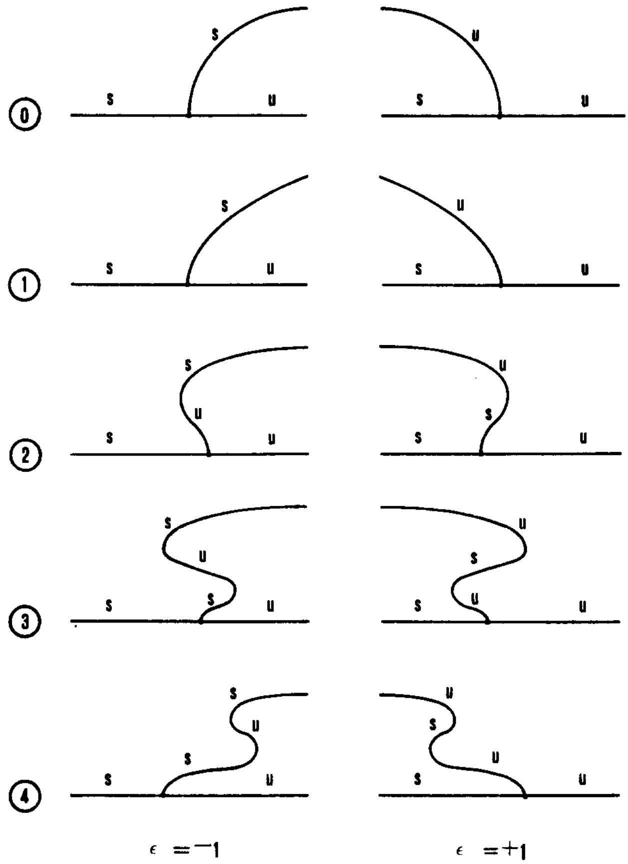


FIG. 4.5. Bifurcation diagrams for normal form (4.5). Case 0: $\alpha = \beta = 0$.

solutions are in fact unstable then the stability assignments for any given diagram are obtained by interchanging "$s$" and "$u$" throughout that diagram.

One should also observe that we have not drawn all possible perturbations; that is, diagrams for all choices of $\alpha, \beta$ in (4.4)–(4.7). We have however drawn the qualitative pictures for an open dense set of $(\alpha, \beta)$'s; namely, those diagrams which, except for $(\alpha, \beta) = (0, 0)$, contain no singularities other than limit points and non-degenerate Hopf bifurcations from the steady state. It is well known through Hopf's exchange of stability formula how the stability of periodic solutions is determined for such diagrams, at least for the connected component of the diagram containing the $x = 0$ solutions. The local nature of the universal unfolding allows one—by continuity arguments in $\alpha, \beta$—to obtain the stability assignments for other components as well.

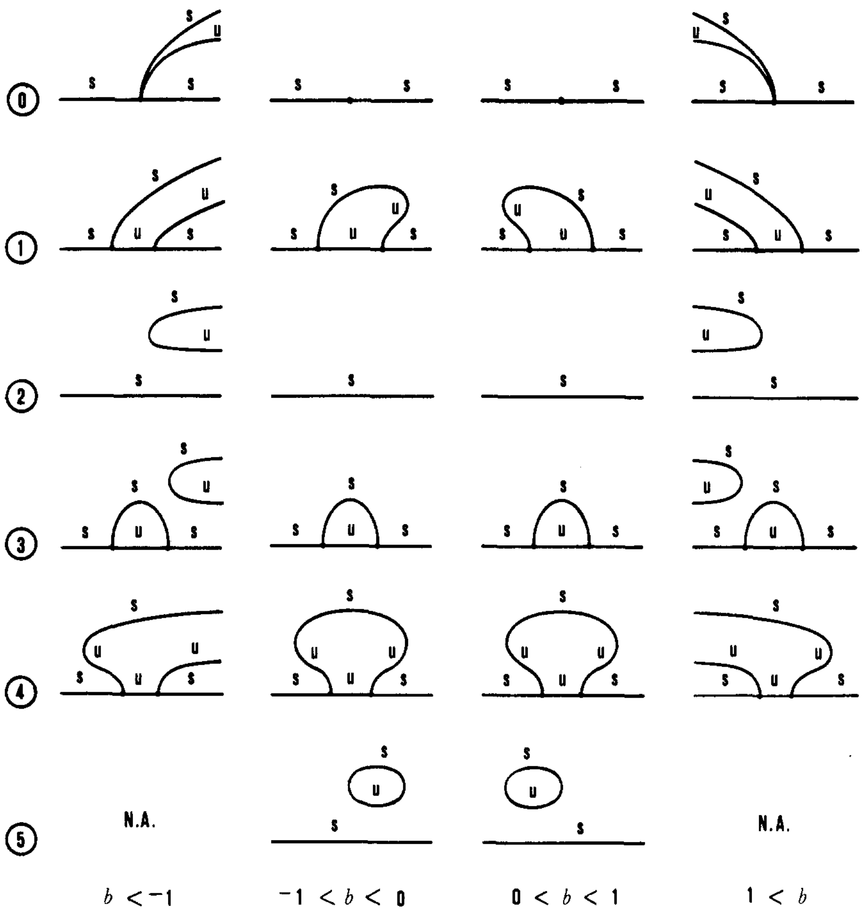


FIG. 4.6. Bifurcation diagrams for normal form (4.6). Case 0: $\alpha = \beta = 0$: See Fig. 4.10(a). Cases 1 to 5: See Figs. 4.11(a) and 4.12(a).

The diagrams (4.1), (4.2), (4.4) are obtained in a straightforward manner. Note that $x^3 + Ax = 0$ has three solutions when $A < 0$ and only the trivial solution $x = 0$ when $A > 0$.

A number of people [8, 33, 37] have made a similar observation when $x^5$ is the initial non-zero term. Takens, in particular, has observed that this discussion can be generalized to higher order initial terms. The universal perturbation is described by

$$x^5 + 2Bx^3 + Cx = 0. \tag{4.8}$$

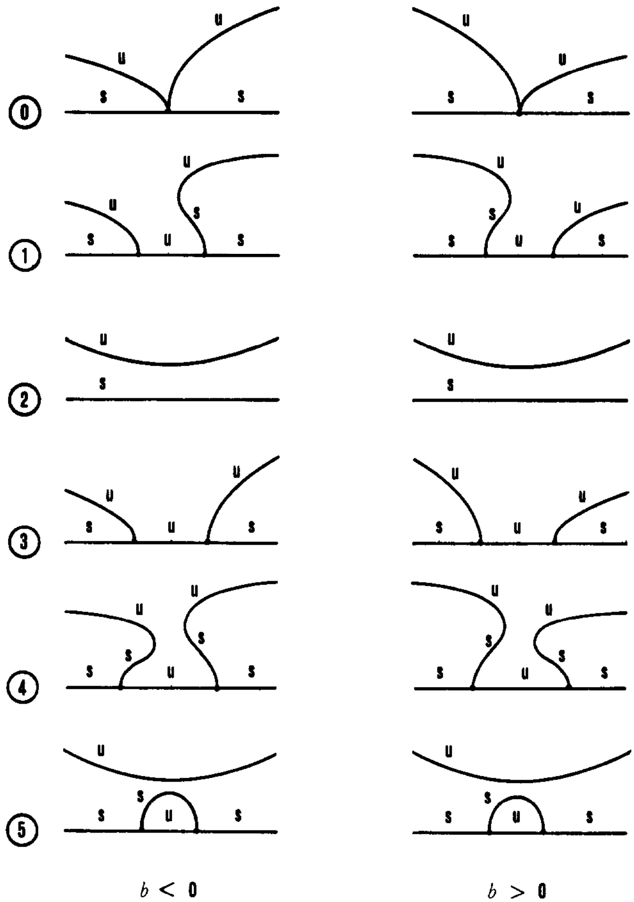


FIG. 4.7. Bifurcation diagrams for normal form (4.7). Case 0: $\alpha=\beta=0$: See Fig. 4.10(b). Cases 1 to 5: See Figs. 4.11(b) and 4.12(b).

One obtains the diagram pictured in Fig. 4.8 showing the regions in the $BC$-plane corresponding to 1, 3 or 5 real roots $x$ in (4.8). Note that on boundary (i) 0 is a triple root and $\pm\sqrt{-2B}$ are simple roots. On boundary (iii) 0 is a triple root and there are no other real roots. On boundary (ii) there are two double roots at $\pm\sqrt{-B}$.

The bifurcation diagrams for (4.3), (4.6) and (4.7) may now be obtained by analysing paths through Fig. 4.8. More precisely, these normal forms give $B$ and $C$ as functions of $\lambda$ for fixed choices of $\alpha$, $\beta$, and $b$. The effect on a bifurcation diagram corresponding to a given path when that path crosses one of the boundaries (i)–(iii) transversely is given in Fig. 4.9. In this figure

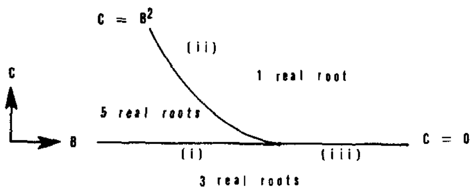


FIG. 4.8. Roots of Eq. (4.8).

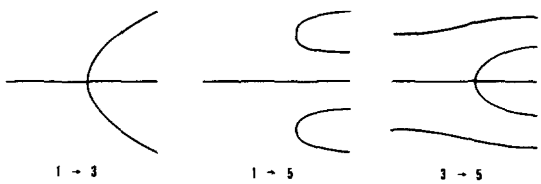


FIG. 4.9. Crossings of boundaries in Fig. 4.8.

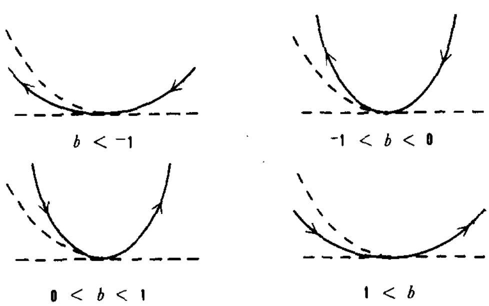


FIG. 4.10(a). Paths through Fig. 4.8 for normal form (4.6) with $\alpha = \beta = 0$.

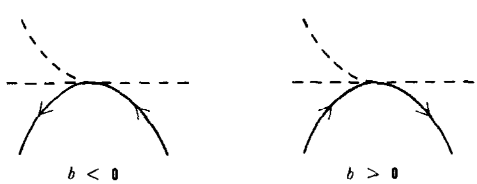


FIG. 4.10(b). Paths through Fig. 4.8 for normal form (4.7) with $\alpha = \beta = 0$.

we use the notation $p \to q$ to indicate a path going from the region with $p$ roots to the region with $q$ roots.

It should be clear that the qualitative type of a bifurcation diagram corresponding to a path in the $BC$-plane which intersects the boundaries (i)–(iii) transversely and never intersects the origin is determined by which regions the path starts and ends in, along with the order of intersections (in $\lambda$) of the path with the boundaries. Such paths we call *stable*. The set of $(\alpha, \beta)$ which correspond to stable paths is open and dense in $(\alpha, \beta)$ space.

It is now relatively easy to draw the bifurcation diagrams corresponding to stable paths in (4.3), (4.6), (4.7). In (4.3) the path corresponding to $\alpha = 0$ is the vertical line in the $BC$-plane containing the origin. The perturbation parameter $\alpha$ moves that line to the left ($\alpha < 0$) or to the right ($\alpha > 0$). The sign of $\varepsilon$ determines the direction of the path along this line.

Problems (4.6) and (4.7) are slightly more complicated (though the effect of the modal parameter $b$ can now be readily ascertained). Setting $\alpha = \beta = 0$ one obtains parabolas as paths. The relative position of these parabolas vis-a-vis the boundaries (i)–(iii) is determined by $b$ and summarized in Fig. 4.10.

One now observes that when $\alpha$ and $\beta$ are non-zero one also obtains a parabola as a path in the $BC$-plane. The net effect of $\alpha$ and $\beta$ is to (arbitrarily) move the vertex of the parabola from the origin. In Fig. 4.11 we list the kinds of stable paths which can occur. Note that we consider $b$ fixed—though no change would occur if $b$ varied as long as one did not change the interval in which $b$ was originally found.

For completeness we list in Fig. 4.12 the regions of the $\alpha\beta$-plane which correspond to the stable paths given in Fig. 4.11. Note that in each case

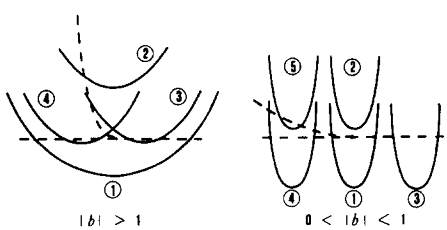


FIG. 4.11(a). Stable paths for normal form (4.6).

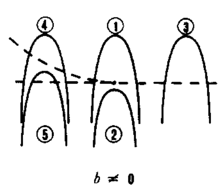


FIG. 4.11(b). Stable paths for normal form (4.7).

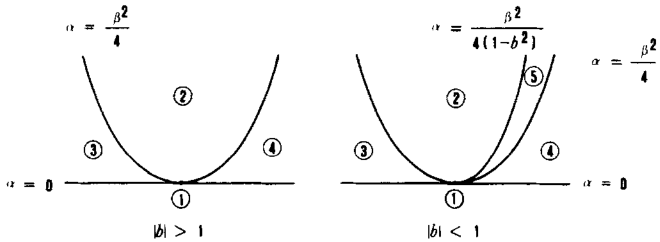


FIG. 4.12(a). Regions corresponding to paths in Fig. 4.11(a).

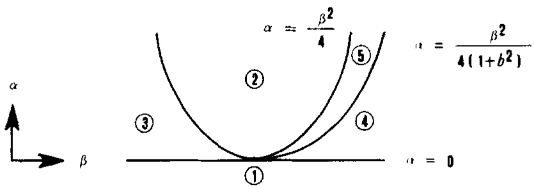


FIG. 4.12(b). Regions corresponding to paths in Fig. 4.11(b).

regions 1 and 2 correspond to these diagrams which are most likely to occur as a small perturbation from $\alpha = \beta = 0$.

To draw the bifurcation diagrams associated with normal form (4.5) observe that (4.5) has the form

$$x(z^3 + \beta z^2 + \alpha z + \varepsilon\lambda) = 0, \tag{4.9}$$

where $z = x^2$. The solutions of (4.9) are

$$-\varepsilon\lambda = z^3 + \beta z^2 + \alpha z, \qquad \text{and} \qquad x = 0. \tag{4.10}$$

The non-trivial branch in (4.10) is just an arbitrary cubic through 0 whose asymptotic behavior at $\infty$ is determined by $\varepsilon$. The diagrams in Fig. 4.5 are now easily obtained.

## 5. Calculation of the Coefficients

In order to facilitate applications of our theoretical results, we describe in this section a simple procedure for calculating the defining and non-degeneracy coefficients of Proposition 3.47, and we present explicit formulae for all but one of the coefficients necessary for the classification of bifurcation diagrams given in Section 4. Formulae for the lowest order coefficients, sufficient only for the non-degenerate normal form (4.1), are given in

[3, 5, 13, 16, 17, 19, 25, 26, 27]. A numerical implementation of the Lyapunov–Schmidt method assuming (H2) but not (H3) is given in [22]. Calculations of various higher order coefficients can be found in [4, 7, 8, 14, 19, 21, 30, 40]. Our approach is based on that used in [19, 26] and by Howard and Kopell in [25]. Similar formulae are given by Kielhöfer [21].

Recall from Section 2 the bifurcation equations have component form

$$g_1 = p(x^2, \lambda, \tau)x = 0,$$
$$g_2 = q(x^2, \lambda, \tau)x = 0. \tag{5.1}$$

The Taylor series for $p$ and $q$, with $z \equiv x^2$, are

$$p(z, \lambda, \tau) = \Sigma p_{jkl} z^j \lambda^k \tau^l,$$
$$q(z, \lambda, \tau) = \Sigma q_{jkl} z^j \lambda^k \tau^l, \tag{5.2}$$

where the coefficients are given by

$$p_{jkl} = \frac{1}{(2j+1)!\, k!\, l!} \frac{\partial^{2j+1+k+l} g_1(0,0,0)}{\partial x^{2j+1}\, \partial\lambda^k\, \partial\tau^l}, \tag{5.3}$$

and similarly for $q_{jkl}$ and $g_2$. As in Section 2, we can solve $q = 0$ for $\tau = \tau(x^2, \lambda)$ and substitute into $p$ to obtain the single equation

$$G(x, \lambda) = a(x^2, \lambda)x = 0, \tag{5.4}$$

where we write

$$a(z, \lambda) = \Sigma a_{jk} z^j \lambda^k. \tag{5.5}$$

It is these coefficients $a_{jk}$ which we need to calculate.

For convenience of notation, define the symmetric $k$-linear form $f^k\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ by $f^k(v^1,..., v^k) = (d_u^k f)_{(0,0)}(v^1,..., v^k)$, with $i$th component

$$f_i^k(v^1,..., v^k) = \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_k=1}^{n} \frac{\partial^k f_i(0,0)}{\partial u_{\alpha_1} \cdots \partial u_{\alpha_k}} v^1_{\alpha_1} \cdots v^k_{\alpha_k}, \tag{5.6}$$

and similarly define

$$f^{k,l}(v^1,..., v^k) = \left(d_u^k \frac{\partial^l f}{\partial\lambda^l}\right)_{(0,0)} (v^1,..., v^k). \tag{5.7}$$

Recall that $Av = f^1(v)$.

Now from Eq. (2.33) we immediately obtain the coefficients

$$p_{00l}=0,\ q_{00l}=-\delta_{1l}, \qquad l=0,1,2,\ldots. \tag{5.8}$$

In similar fashion we show that

$$p_{01l}=0=q_{01l}, \qquad l=1,2,\ldots. \tag{5.9}$$

Recall that the Lyapunov–Schmidt procedure has replaced the single equation

$$\begin{aligned} 0 &= (1+\tau)\frac{d}{ds}(x\phi_1+w)-f(x\phi_1+w,\lambda) \\ &\equiv N(x\phi_1+w,\lambda,\tau) \end{aligned} \tag{5.10}$$

by the two equations: $QN=0$ which defines $w(x,\lambda,\tau)$ via the implicit function theorem, and $PN(x\phi_1+w(x,\lambda,\tau),\lambda,\tau)\equiv g(x,\lambda,\tau)=0$ which is the bifurcation equation. Note that

$$\begin{aligned} 0 &= QN_x(0,0,\tau) \\ &= Q\left[(1+\tau)\frac{d}{ds}(\phi_1+w_x)-A(\phi_1+w_x)\right] \\ &= Lw_x+\tau Q\frac{dw_x}{ds}. \end{aligned} \tag{5.11}$$

This has the unique solution for small $\tau$

$$w_x(0,0,\tau)\equiv 0. \tag{5.12}$$

Now

$$\begin{aligned} g_{x\lambda}(0,0,\tau) &= PN_{x\lambda}(0,0,\tau) \\ &= P\left[(1+\tau)\frac{d}{ds}w_{x\lambda}-Aw_{x\lambda}-f^{1,1}(\phi_1+w_x)\right] \\ &= -Pf^{1,1}\phi_1. \end{aligned} \tag{5.13}$$

Since this is independent of $\tau$, (5.9) follows. Equations (5.8) and (5.9) simplify the calculation of the coefficients $a_{jk}$. Those required for the cases analysed in Section 4 are

$$\begin{aligned} a_{00} &= 0, \\ a_{10} &= p_{100}, \\ a_{01} &= p_{010}, \end{aligned}$$

$$\begin{aligned} a_{20} &= p_{200} + p_{101} q_{100}, \\ a_{11} &= p_{110} + p_{101} q_{010}, \\ a_{02} &= p_{020}, \\ a_{30} &= p_{300} + p_{101} [q_{200} + q_{101} q_{100}] \\ &\qquad + p_{201} q_{100} + p_{102} q_{100}^2, \\ a_{03} &= p_{030} + p_{021} q_{010}. \end{aligned} \tag{5.14}$$

The coefficients $a_{10} \equiv a_2(0, 0)$ and $a_{01} \equiv a_\lambda(0, 0)$ are equivalent to those calculated by Hopf, corresponding to (H3) and (H2) respectively; see (2.36), (2.37). The first calculation of $a_{20}$ was in [14]; we give an alternate derivation. The labor in calculating $a_{j0}$ increases rapidly with $j$, and we have not explicitly calculated $a_{30}$ for normal form (4.5). On the other hand (as has been shown by Kielhöfer [21]) the first nonvanishing $a_{0k}$ is relatively easy to calculate, this facilitates applications of Proposition 3.17 and of Proposition 3.20 with $l = 1$.

The modal parameter $b$ in (4.6), (4.7) is

$$b = a_{11} \operatorname{sgn} a_{20}/(2|a_{20} a_{02}|^{1/2}). \tag{5.15}$$

The remaining calculations proceed more efficiently in complex notation. Define

$$\begin{aligned} \Phi(s) &= e^{is} c = \phi_1 + i\phi_2, \\ \Psi(s) &= e^{is} d = \psi_1 + i\psi_2. \end{aligned} \tag{5.16}$$

Then the projector $P$ in (2.16) can be written

$$Pv = \operatorname{Re}(v, \Psi)\Phi. \tag{5.17}$$

We illustrate the technique in deriving the well-known formulae for $p_{010}$ and $q_{010}$. From (5.13),

$$\begin{aligned} g_{x\lambda}(0, 0, 0) &= -Pf^{1,1}\phi_1 \\ &= -\operatorname{Re}(f^{1,1}\phi_1, \Psi)\,\Phi \\ &= -\operatorname{Re} \frac{1}{2\pi} \int_0^{2\pi} e^{-is} d^* f^{1,1} \left( \frac{e^{is}c + e^{-is}\bar{c}}{2} \right) ds\, \Phi \\ &= -\tfrac{1}{2} \operatorname{Re} d^* f^{1,1}(c) \Phi. \end{aligned} \tag{5.18}$$

But differentiation with respect to $\lambda$ of the identity

$$(d_u f)_{(0,\lambda)}\, c(\lambda) = [\sigma(\lambda) + i\omega(\lambda)]\, c(\lambda) \tag{5.19}$$

leads to

$$\begin{aligned} d^*f^{1,1}c &= [\sigma'(0)+i\omega'(0)]\,d^*c - d^*[A-iI]c'(0) \\ &= 2[\sigma'(0)+i\omega'(0)]. \end{aligned} \tag{5.20}$$

Substituting into (5.18) and taking components gives

$$\begin{aligned} a_{01}=p_{010} &= -(f^{1,1}\phi_1,\psi_1) = -\tfrac{1}{2}\,\mathrm{Re}\,d^*f^{1,1}c = -\sigma'(0), \\ q_{010} &= -(f^{1,1}\phi_1,\psi_2) = \tfrac{1}{2}\,\mathrm{Im}\,d^*f^{1,1}c \ \ = \omega'(0). \end{aligned} \tag{5.21}$$

Next we calculate $p_{100}$ and $q_{100}$. From (5.10) we compute

$$N_{xx}(0) = \left[\frac{d}{ds} - A\right] w_{xx} - f^2(\phi_1,\phi_1) = 0, \tag{5.22}$$

$$\begin{aligned} N_{xxx}(0) &= \left[\frac{d}{ds} - A\right] w_{xxx} - 3f^2(\phi_1, w_{xx}) - f^3(\phi_1,\phi_1,\phi_1). \\ &= 0. \end{aligned} \tag{5.23}$$

Note $Pf^2(\phi_1,\phi_1)=0$ from the symmetry, so from (5.22), $w_{xx}$ satisfies the differential equation

$$\left[\frac{d}{ds} - A\right] w_{xx} = \frac{1}{4}\,[e^{2is}f^2(c,c) + 2f^2(c,\bar{c}) + e^{-2is}f^2(\bar{c},\bar{c})]. \tag{5.24}$$

This has unique solution in $W$ given by

$$w_{xx} = a_0 + e^{2is}\,a_2 + e^{-2is}\,\bar{a}_2, \tag{5.25}$$

where $a_0 \in \mathbb{R}^n$, $a_2 \in \mathbb{C}^n$ satisfy

$$\begin{aligned} Aa_0 &= -\tfrac{1}{2}f^2(c,\bar{c}), \\ [A-2iI]a_2 &= -\tfrac{1}{4}f^2(c,c). \end{aligned} \tag{5.26}$$

On substituting (5.25) into (5.23), projecting $PN_{xxx} = g_{xxx}$, integrating, taking components and applying (5.3), we find

$$\begin{aligned} a_{10}=p_{100} &= -\tfrac{1}{4}\,\mathrm{Re}\,d^*[f^2(c,a_0) + f^2(\bar{c},a_2) + \tfrac{1}{4}f^3(c,c,\bar{c})], \\ q_{100} &= +\tfrac{1}{4}\,\mathrm{Im}\,d^*[f^2(c,a_0) + f^2(\bar{c},a_2) + \tfrac{1}{4}f^3(c,c,\bar{c})]. \end{aligned} \tag{5.27}$$

Higher order coefficients are obtained by the same procedure, we omit the details. For the second order $a_{jk}(j+k=2)$ in (5.14) one first computes vectors analogous to (5.26):

$$
\begin{aligned}
[A-iI]a_1 &= -(3/2)[f^2(c,a_0)+f^2(\bar{c},a_2)+\tfrac{1}{4}f^3(c,c,\bar{c})] \\
&\quad -3[p_{100}-iq_{100}]c, \qquad d^*a_1=0, \\
[A-3iI]a_3 &= -\tfrac{1}{2}[3f^2(c,a_2)+\tfrac{1}{4}f^3(c,c,c)], \\
A\,b_0 &= -2[f^2(c,\bar{a}_1)+f^2(\bar{c},a_1)] \\
&\quad -3[f^2(a_0,a_0)+2f^2(a_2,\bar{a}_2)+f^3(c,\bar{c},a_0)] \\
&\quad -(3/2)[f^3(c,c,\bar{a}_2)+f^3(\bar{c},\bar{c},a_2)] \\
&\quad -(3/8)\,f^4(c,c,\bar{c},\bar{c}), \\
[A-2iI]b_2 &= -2[f^2(c,a_1)+f(\bar{c},a_3)+3f^2(a_0,a_2)] \\
&\quad -3[f^3(c,\bar{c},a_2)+\tfrac{1}{2}f^3(c,c,a_0)] \\
&\quad -\tfrac{1}{4}f^4(c,c,c,\bar{c}), \\
[A-iI]c_1 &= -f^{1,1}c+[\sigma'(0)+i\omega'(0)]c, \qquad d^*c_1=0, \\
[A-2iI]c_2 &= 2ia_2, \\
Ad_0 &= -\tfrac{1}{2}[f^2(c,\bar{c}_1)+f^2(\bar{c},c_1)+f^{2,1}(c,\bar{c})] \\
&\quad -f^{1,1}a_0, \\
[A-2iI]d_2 &= -[\tfrac{1}{2}f^2(c,c_1)+\tfrac{1}{4}f^{2,1}(c,c)+f^{1,1}a_2].
\end{aligned} \tag{5.28}
$$

Equations (5.28) may be easier to solve if one makes a linear change of variables $u=T\tilde{u}$ throughout the original problem, to bring $A$ to a block-diagonal form $\tilde{A}=T^{-1}AT$. Our formulae apply whether or not this has been done, unlike [14, 17]. Note also that if one has not rescaled $t$ as in (H1) to make the basic frequency $\omega(0)=1$, then the only change in our formulae is to replace $kiI$ by $k\,\omega(0)iI$, $k=1$, 2, 3 on the left side of all of Eqs. (5.26) (5.28) (5.31), and also on the right side of the equations for $c_2$ and $d_1$ only. Now we have

$$
\begin{aligned}
p_{200} = (-1/4!)\,\mathrm{Re}\,d^*[&\tfrac{1}{2}f^2(c,b_0)+\tfrac{1}{2}f^2(\bar{c},b_2) \\
&+2(f^2(a_0,a_1)+f^2(a_2,\bar{a}_1)+f^2(\bar{a}_2,a_3)) \\
&+\tfrac{1}{2}f^3(c,c,\bar{a}_1)+f^3(c,\bar{c},a_1)+\tfrac{1}{2}f^3(c,\bar{c},a_3) \\
&+3(f^3(c,a_2,\bar{a}_2)+f^3(\bar{c},a_0,a_2)+\tfrac{1}{2}f^3(c,a_0,a_0)) \\
&+\tfrac{1}{4}(f^4(c,c,c,\bar{a}_2)+3f^4(c,c,\bar{c},a_0)+3f^4(c,\bar{c},\bar{c},a_2)) \\
&+(1/16)\,f^5(c,c,c,\bar{c},\bar{c})],
\end{aligned}
$$

$$p_{110} = -\tfrac{1}{4}\operatorname{Re} d^*[f^2(c_1, a_0) + f^2(\bar{c}_1, a_2) + f^2(c, d_0) + f^2(\bar{c}, d_2)$$
$$+ f^{2,1}(c, a_0) + f^{2,1}(\bar{c}, a_2) + (2/3) f^{1,1}a_1$$
$$+ \tfrac{1}{4}f^3(c, c, \bar{c}_1) + \tfrac{1}{2}f^3(c, \bar{c}, c_1) + \tfrac{1}{4}f^{3,1}(c, c, \bar{c})],$$
$$p_{101} = -\tfrac{1}{4}\operatorname{Re} d^* f^2(\bar{c}, c_2),$$
$$p_{020} = -\tfrac{1}{2}\sigma''(0) = -\tfrac{1}{2}\operatorname{Re} d^*[f^{1,1}c_1 + \tfrac{1}{2}f^{1,2}c],$$
$$q_{020} = +\tfrac{1}{2}\omega''(0) = +\tfrac{1}{2}\operatorname{Im} d^*[f^{1,1}c_1 + \tfrac{1}{2}f^{1,2}c],$$
$$p_{011} = 0,$$
$$p_{002} = 0. \qquad (5.29)$$

Each $q_{jkl}$ is obtained from the corresponding $p_{jkl}$ by reversing the sign and taking the imaginary instead of the real part.

The only third order coefficient which we calculate is

$$a_{03} = -1/3!\ \sigma'''(0) + \tfrac{1}{2}\sigma'(0) \operatorname{Im} d^* f^{1,1} d_1, \qquad (5.30)$$

where

$$[A - iI]d_1 = ic_1, \qquad d^* d_1 = 0. \qquad (5.31)$$

This simplifies if $\sigma'(0) = 0$, the only case in which we would need $a_{03}$. In fact as shown by Kielhöfer [21], if

$$\sigma(0) = \sigma'(0) = \cdots = \sigma^{(m-1)}(0) = 0 \qquad (5.32)$$

then $a_{01}$ through $a_{0,m-1}$ are also zero, and

$$a_{0m} = -\frac{1}{m!}\sigma^{(m)}(0). \qquad (5.33)$$

## 6. Examples

We have chosen two simple biochemical models from the recent literature which illustrate the two types of codimension one and one case of codimension two degenerate Hopf bifurcations. They also show how our techniques are used to provide useful information. Other more complex models are under investigation.

### *Glycolytic Oscillations*

A number of differential equation models have been proposed to explain the oscillations which have been observed in experimental studies of glycolysis. One of the simplest [2, 34] is

$$\frac{dX}{dt} = \lambda - \kappa X - XY^2,$$
$$\frac{dY}{dt} = \kappa X + XY^2 - Y. \tag{6.1}$$

These equations model a product activated reaction in scaled concentration variables $X$, $Y$, with feed rate $\lambda > 0$ and low activity reaction rate $\kappa \in (0, 1)$.

In the experiments, if $\lambda$ is gradually increased, it is possible for a spontaneous oscillation to arise and grow to a maximum amplitude, then shrink and eventually disappear for larger values of $\lambda$.

Our analysis begins as in [2]. The steady states are given by

$$X_0 = \frac{\lambda}{\lambda^2 + \kappa}, \qquad Y_0 = \lambda. \tag{6.2}$$

Defining new variables $u_1 = X - X_0$, $u_2 = Y - Y_0$, we obtain

$$\frac{du}{dt} = Au + \begin{pmatrix} -1 \\ 1 \end{pmatrix} h(u), \tag{6.3}$$

where

$$A = \begin{pmatrix} -(\lambda^2 + \kappa) & \dfrac{-2\lambda^2}{\lambda^2 + \kappa} \\ \lambda^2 + \kappa & \dfrac{\lambda^2 - \kappa}{\lambda^2 + \kappa} \end{pmatrix},$$
$$h(u) = 2Y_0 u_1 u_2 + X_0 u_2^2 + u_1 u_2^2. \tag{6.4}$$

One calculates

$$\det A = \lambda^2 + \kappa > 0, \tag{6.5}$$

$$\operatorname{tr} A = -(\lambda^2 + \kappa) + (\lambda^2 - \kappa)/(\lambda^2 + \kappa). \tag{6.6}$$

Therefore $A$ has eigenvalues $\sigma \pm i\omega$ satisfying $\sigma = 0$, $\omega > 0$ if and only if $\operatorname{tr} A = 0$, i.e.,

$$\lambda^4 + (2\kappa - 1)\lambda^2 + \kappa(\kappa + 1) = 0,$$
$$\lambda_\pm^2 = \tfrac{1}{2}(1 - 2\kappa \pm \sqrt{1 - 8\kappa}), \qquad \lambda_\pm > 0. \tag{6.7}$$

The degeneracy $\sigma'(\lambda_\pm) = 0$ occurs when the discriminant of (6.7) vanishes, giving

$$\kappa_c = \frac{1}{8}, \qquad \lambda_c^2 = \frac{3}{8}. \tag{6.8}$$

In the notation of Section 5 we now have

$$a_{01} = 0. \tag{6.9}$$

Since there are no more parameters in the problem, we do not expect to find any higher degeneracy. Indeed, at $(\kappa_c, \lambda_c)$ the formulae of Section 5 give

$$\begin{aligned} a_{10} &= \tfrac{3}{8}, \\ a_{02} &= -\tfrac{1}{2}\sigma''(\lambda_c) = \tfrac{3}{2}. \end{aligned} \tag{6.10}$$

Therefore this example has normal form (4.2) with $\varepsilon = +1$ and bifurcation diagrams in Fig. 4.2. From (6.6) we calculate

$$\frac{\partial \sigma}{\partial \kappa}(\kappa_c, \lambda_c) = -2 < 0. \tag{6.11}$$

By Proposition 3.48, $\kappa - \kappa_c$ is an unfolding parameter, and it has the same sign as $\alpha$ in Fig. 4.2. One easily verifies the stability assignments using (6.6). We conclude that for $\kappa > \frac{1}{8}$ there is no Hopf bifurcation, but in a neighborhood $\kappa < \frac{1}{8}$ we have the existence of two Hopf bifurcations which are directed toward each other and connected by a branch of stable periodic solutions. Hence the model duplicates the experimental phenomena described above. This qualitative behaviour has been found in the model by other authors [2, 34]. In addition to being more direct, our method explains what happens to the periodic solutions at $\kappa = \frac{1}{8}$, where the classical Hopf theorem fails.

*Fitzhugh Equations*

A simplified model of the nerve impulse has been proposed by Fitzhugh [6], and has been analysed by a number of authors, see [8, 17] and further references therein. The equations are

$$\begin{aligned} \frac{dx}{dt} &= \gamma + y + x - \frac{x^3}{3}, \\ \frac{dy}{dt} &= \rho(\delta - x - \beta y). \end{aligned} \tag{6.12}$$

The parameters satisfy $\gamma, \delta \in \mathbb{R}$ and

$$0 < \beta, \rho < 1. \tag{6.13}$$

We begin with the observation that $\delta$ can be eliminated by setting $y = Y + \delta/\beta$, $\gamma = \lambda - \delta/\beta$, $x = X$ to obtain

$$\frac{dX}{dt} = \lambda + Y + X - \tfrac{1}{3}X^3,$$
$$\frac{dY}{dt} = -\rho(X + \beta Y). \tag{6.14}$$

We take $\lambda$ as the bifurcation parameter. Note that (6.14) has the $\mathbb{Z}_2$ symmetry

$$(X, Y, \lambda) \to (-X, -Y, -\lambda). \tag{6.15}$$

Given one solution of (6.14) we obtain another by applying (6.15), so we need only consider $\lambda \geqslant 0$.

The steady states $(X_0, Y_0)$ are given by

$$Y_0 = -X_0/\beta, \tag{6.16}$$

$$\lambda = \frac{1}{3} X_0^3 + \frac{(1-\beta)}{\beta} X_0. \tag{6.17}$$

Note that

$$\frac{d\lambda}{dX_0} = X_0^2 + \frac{1-\beta}{\beta} > 0, \tag{6.18}$$

so (6.16) (6.17) define a unique steady state for every $\lambda$, assuming (6.13). Translate variables $X = X_0 + u_1$, $Y = Y_0 + u_2$ and get the equation

$$\frac{du}{dt} = Au + \begin{pmatrix} -1 \\ 0 \end{pmatrix} h(u), \tag{6.19}$$

where

$$A = \begin{pmatrix} 1 - X_0^2 & 1 \\ -\rho & -\rho\beta \end{pmatrix},$$
$$h(u) = X_0 u_1^2 + \tfrac{1}{3} u_1^3. \tag{6.20}$$

We calculate

$$\det A = \rho\beta \left[ X_0^2 + \frac{(1-\beta)}{\beta} \right] > 0, \tag{6.21}$$

$$\operatorname{tr} A = 1 - \rho\beta - X_0^2. \tag{6.22}$$

Therefore $A$ has eigenvalues $\sigma \pm i\omega$ with $\sigma = 0$ if and only if $\operatorname{tr} A = 0$, i.e.,

$$X_0^2 = 1 - \rho\beta, \tag{6.23}$$

which is positive by (6.13). We select the positive root $X_0$ of (6.23), in accord with our convention $\lambda \geqslant 0$ and the oddness of (6.17).

Note that hypothesis (H2) always holds for this model, since at $\sigma = 0$

$$\sigma' = \frac{1}{2}\frac{\partial \operatorname{tr} A}{\partial \lambda} = -X_0 \frac{\partial X_0}{\partial \lambda} < 0. \tag{6.24}$$

This gives $a_{01} > 0$, from (5.21).

Now consider hypothesis (H3). The calculation described in Section 5 gives

$$a_{10} = \frac{\rho}{8\omega^2}[\rho\beta^2 - 2\beta + 1]. \tag{6.25}$$

(Eigenvector $c$ was normalized to have first component 1, because only the first component enters in the calculation of $a_{10}$.) Expressions equivalent to (6.25) are given in [8, 17]. The condition $a_{10} = 0$ defines a unique $\rho \in (0, 1)$ for each $\beta \in (\frac{1}{2}, 1)$, but $a_{10}$ is positive for all $(\beta, \rho) \in (0, \frac{1}{2}) \times (0, 1)$. We solve simultaneously the conditions for the steady-state (6.16), (6.17), for $\sigma = 0$ (6.23) and for $a_{10} = 0$ (6.25), and obtain the critical values of our variables in parametric form for $\beta \in (\frac{1}{2}, 1)$:

$$\begin{aligned}
\rho_c &= \frac{2\beta - 1}{\beta^2}, \\
\lambda_c &= \frac{4}{3}\left(\frac{1-\beta}{\beta}\right)^{3/2}, \\
X_c &= \left(\frac{1-\beta}{\beta}\right)^{1/2}, \\
Y_c &= -\frac{1}{\beta}\left(\frac{1-\beta}{\beta}\right)^{1/2}, \\
\omega_c &= \frac{1}{\beta}[2(2\beta - 1)(1-\beta)]^{1/2}.
\end{aligned} \tag{6.26}$$

With these values, one can proceed to calculate $a_{20}(\beta)$ using the formulae of Section 5. Preliminary calculations indicate that $a_{20}(\beta)$ has a unique simple zero $\beta = \beta_0 \in (\frac{1}{2}, 1)$, and that $a_{20}(\beta) < 0$ on $(\frac{1}{2}, \beta_0)$ and $a_{20}(\beta) > 0$ on $(\beta_0, 1)$. Then for $\beta \neq \beta_0$, we obtain the bifurcation diagrams of Fig. 4.3 with

$\varepsilon = \operatorname{sign} a_{20}$. On differentiating (6.25) and applying Proposition 3.48 we see that we have a universal unfolding with unfolding parameter $\rho$ related to $\alpha$ in (4.3) to first order by

$$\alpha = \frac{(2\beta - 1)(\rho - \rho_c)}{16\omega_c^2 a_{20}(\beta)}, \qquad \beta \neq \beta_0. \tag{6.27}$$

For the unique choice $\beta = \beta_0$ such that $a_{20} = 0$, we obtain the codimension two normal form (4.5). Again according to preliminary calculations, the parameters $\beta$ and $\rho$ provide a universal unfolding of the singularity at $(\beta_0, \rho_c(\beta_0))$. The resulting bifurcation diagrams are shown in Fig. 4.5. This implies for example that the model could exhibit three concentric limit cycles, two of which are stable giving hysteresis, a possibility not suggested in [8, 17].

We remark that this model contains additional degeneracies if one weakens conditions (6.13). The symmetry property (6.15) implies that there are two Hopf bifurcations, occurring at $\pm\lambda_c$. These bifurcations can coalesce as in the previous example, but only at $\lambda = 0$ for which, from (6.17), (6.23) and (6.24),

$$X_0 = 0 = Y_0, \qquad \rho\beta = 1, \qquad \sigma' = 0. \tag{6.28}$$

But then (6.21) implies $\beta < 1 < \rho$ and so $a_{10}$ cannot vanish. In this situation the bifurcation diagrams are like those of the glycolysis model. Only by taking $\rho = \beta = 1$ can we obtain degeneracy of both (H2) and (H3), but then the matrix $A$ is singular so our hypothesis (H1) is violated also. What is happening is that the steady-state solutions have a hysteresis-point bifurcation for these same parameter values. This type of multiple degeneracy is under investigation. For this model, one observes that there are not enough parameters for a universal unfolding of the degeneracy at $(\beta, \rho) = (1, 1)$, so the model is not structurally stable there and the point should be excluded for this reason alone.

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