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on Semialgebraic Sets
for Nonnegative Polynomials
Finding Representations
Introducing Polynomial Optimization

We would like to know a lower bound for $f$. For example, if $f$ is such a bound

\[ f > t + f \geq 0 \]

If we have a method of certifying nonnegativity, then we want to find the least $t$ that we can with our method.

\[ t \]

Differentiating, but this is impractical.

So minimizing is closely related to certifying nonnegativity.
Ahomogeneous form in \( n \) variables is called positivesemidefinite (or \( \text{psd} \)) if it takes nonnegative valuesforall \( \mathbf{x} \in \mathbb{R}^n \).

If a polynomial is a sum of products of the \( p_i \)'s with nonnegative coefficients, it is nonnegative on \( S \) ●

If a polynomial is a sum of products of other polynomials, it is nonnegative on \( S \) ●

\[ \{0 \leq (\mathbf{x}, \cdots, \mathbf{x})^T \mathbf{d}, \cdots , 0 \leq (\mathbf{x}, \cdots, \mathbf{x})^T \mathbf{d} | (\mathbf{x}, \cdots, \mathbf{x}) \} = S \] ●

inequalities:

We may restrict it to a semialgebraic set, defined by polynomial inequalities.

So far we haven't constrained the region over which to bound \( f \)●

If a polynomial is a sum of squares (or sos) of other polynomials, it is nonnegative on \( S \) ●

\[ \mathbf{x} \in \mathbb{R}^n \] (or \( \text{psd} \)) if it takes nonnegative values for all \( \mathbf{x} \in \mathbb{R}^n \).

\[ \mathbf{x}, \cdots, \mathbf{x} \] ∈ \( \mathbb{R}^n \) (or \( \text{psd} \)) if it takes nonnegative values for all \( \mathbf{x} \in \mathbb{R}^n \).

A homogeneous form in \( n \) variables is called positive semidefinite ●

Certifying Nonnegativity
History:

In 1888, Hilbert showed that every psd ternary quartic form is a sum of squares of quadratic forms.

At the same time, he proved that some psd forms are not the sum of three squares of quadratic forms.

At the same time, he proved that some psd forms are not the sum of squares (or sos) of any other forms.

Suppose \( f \) is nonnegative. Does there exist a representation

\[
\sum_{i=1}^{r} \frac{b_i}{c_i} x_i^2 = f
\]

of \( f \) as the sum of squares of rational functions with real coefficients?

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- Suppose \( f \) is nonnegative. Does there exist a representation of \( f \) as the sum of squares of rational functions with real coefficients?

- In 1888, Hilbert showed that every psd ternary quartic form is a sum of squares of quadratic forms.
In 1926, Artin answered affirmatively to the question of whether the representation of a semialgebraic set $S$ can be represented in the form

$$\{x \in S \mid \exists \alpha \in \mathbb{R}^n, \exists \beta \in \mathbb{R}^m : x \alpha + \beta = 0\} = \{x \in \mathbb{R}^n \mid \exists \alpha \in \mathbb{R}^n, \exists \beta \in \mathbb{R}^m : x \alpha + \beta = 0\}$$

where the set $S$ is semialgebraic.
Compact Sets Lead To Nicer Representations

In 1990, Schmüdgen showed that in particular, if $S$ is compact, $f$ can be represented in the form:

$$\sum_{m \in \mathbb{N}^+} d \cdot \left( \sum_{i \in \mathbb{Z}} a_i f \right)$$

In 1988, Handelman showed that in particular, if $S$ is a polyhedron, then $f$ can be represented in the form:

$$\sum_{m \in \mathbb{N}^+} d \cdot \left( \sum_{i \in \mathbb{Z}} a_i f \right)$$
Another Nice Representation

Return to the situation of Schmüdgen’s theorem. In 1993, Putinar showed that if all the $d_i$’s have even degree, and their highest degree homogeneous parts do not have common zeros, then $f$ can be represented in the form $\sum w \sum \frac{f}{f} \sum$. 

Note that strict positivity is required, so if we are trying to find a lower bound for $f$ by representing $f$ in one of the above ways, there may not be a representation which gives us the exact minimum.

All these proofs are nonconstructive.
In 1983, Becker and Schwarz gave a short algebraic proof of the Kadison-Dubois Representation Theorem (actually due to Kadison and Dubois)
Kadison-Dubois Representation Theorem

Let $R$ be a ring, $P \subseteq R$ an archimedean preprime and $X(P) := \{ \phi \in \text{Hom}(R, R) | \phi(P) \subseteq R_+ \}$ the space of representations. If $f \in R$ satisfies $\phi(f) \geq 0$ for all $\phi \in X(P)$, then for any $n \in \mathbb{N}$, we have $1 + nf \in P$. 

• Theorem
Algebraic Proofs of Representation Theorem

Becker and Schwarz actually proved more than this, but this was all that was needed for:

Polya’s theorem

In aid of this, he gave a short algebraic proof of Schmudgen’s theorem in 1998

Vermann to give a short algebraic proof of

but this was all that was needed for:

Becker and Schwarz actually proved more than this,
Pólya’s Theorem

Let $f \in \mathbb{R}[x]$, be homogeneous and strictly positive on the closed positive orthant minus the origin. Then for some $N \in \mathbb{N}$, the polynomial $P_n(x)$ is a positive linear combination of monomials in the $x_i$'s.

There are bounds on the number $N$, called the Pólya exponent, but these vary inversely with the minimum value of $f$ on $S$.

Pólya’s Theorem
Schweighofer's Constructive Proof

Since $S$ is compact, there is a number $R$ such that $\forall x \in R - R\), it is strictly positive on $S$. So, it is so.

Schweighofer made W"ormann's proof constructive.
Now Polya’s theorem applies.

Let $\phi$ be a nonnegative polynomial on the variety in $\mathbb{R}^l$ corresponding to $\ker \phi$. Then any element of $\mathbb{R}^l$ is strictly positive on the variety in $\mathbb{R}^l$.

Define $R[\lambda_1, \lambda_2, \ldots, \lambda_l] \leftarrow \phi$. By $[\lambda_1, \lambda_2, \ldots, \lambda_l] \leftarrow \phi$. By $[\lambda_1, \lambda_2, \ldots, \lambda_l] \leftarrow \phi$.
Schweighofer’s Algorithm Implemented

The LP minimizes $t$ – nonnegativity to minimization

$s$ is a constant added to $f$ to convert certification of

$\phi$ elements of the Gröbner basis for $\ker f$

$t$ is a nonnegative multiplier of a sum of squares of

I have implemented this, solving an LP in $s$ and $t$

- Schweighofer’s Algorithm gives a Handelman representation on the original $p$'s
- Schweighofer’s Algorithm gives a Handelman representation on the $q$'s, or a and a Schm¨udgen

I have implemented this, solving an LP in $s$ and $t$
I have found some problems with it in practice. The outcome depends heavily on which element of $f_{-1} \phi$ was chosen to start with. A Handelman-type representation is found, restricting the form of the representation means the size of the representation will be bigger. In fact, it is even worse: requiring the representation to be homogeneous makes the representation invariably blow up.

I have found some problems with it in practice. Schweighofer’s Algorithm Critiqued.
How To Find A Representation?

Once we know a representation exists, we can search for the best one.

The Gram matrix method, which can be solved by SDP, allows us to compute the sum of squares representations. We can search for the best one.

I have implemented search for Handelman representations.

Schmüdgen representations can be found by the Gram matrix method, which can be solved by SDP in the unconstrained case. I have implemented this in the unconstrained case.

Schmüdgen and am working on the extension to Putinar and

I have implemented search for Handelman representations.
Some of the interest of these methods lies in their ability to give exact, trustworthy answers. However, all current SDP implementations are based on interior-point methods and use floating-point arithmetic. Alternatively, a floating-point implementation could be used as an "oracle", as in some exact LP solvers. In 1994 Gabor Pataki proposed a simplex-type method for solving SDPs for theoretical purposes. Exact vs. Floating-Point Computations