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Polynomial Graphs With Applications in Game Theory

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Polynomial Graph

To a system of n polynomial equations $f_1 = 0, \ldots, f_n = 0$ in n unknowns $\sigma_1, \ldots, \sigma_n$, we can associate a (non-unique) graph, the polynomial graph on n vertices, as follows:

- To each vertex i assign one of the unknowns, σ_i, and one of the equations, f_i.
- Draw an edge from vertex j to vertex k if and only if σ_k occurs in f_j.

“Fat” Edges: A Not So Special Case

Let $\{1,\ldots,d\} = \coprod_{i=1}^{N} T_i$ *be a partition of* $\{1,\ldots,d\}$, *denoted by* $\mathbf{P}$.
Write $d_i = |T_i|$.
The graph $\mathbf{G}$ *on* d *vertices has edges fattened by* $\mathbf{P}$ *iff:*
for any v_j *and* T_i,
if there is some $k \in T_i$ *such that*
there is an edge from v_j *to* v_k *in* $\mathbf{G}$,
then for every $k \in T_i$ *there is an edge from* v_j *to* v_k *in* $\mathbf{G}$.
Every graph has edges fattened by the trivial partition into singletons.

Example

$$\bullet c + \bullet d = 0; \tag{1a}$$

$$\bullet c + \bullet d = 0; \tag{1b}$$

$$\bullet e + \bullet f = 0; \tag{2c}$$

$$\bullet e + \bullet f = 0; \tag{2d}$$

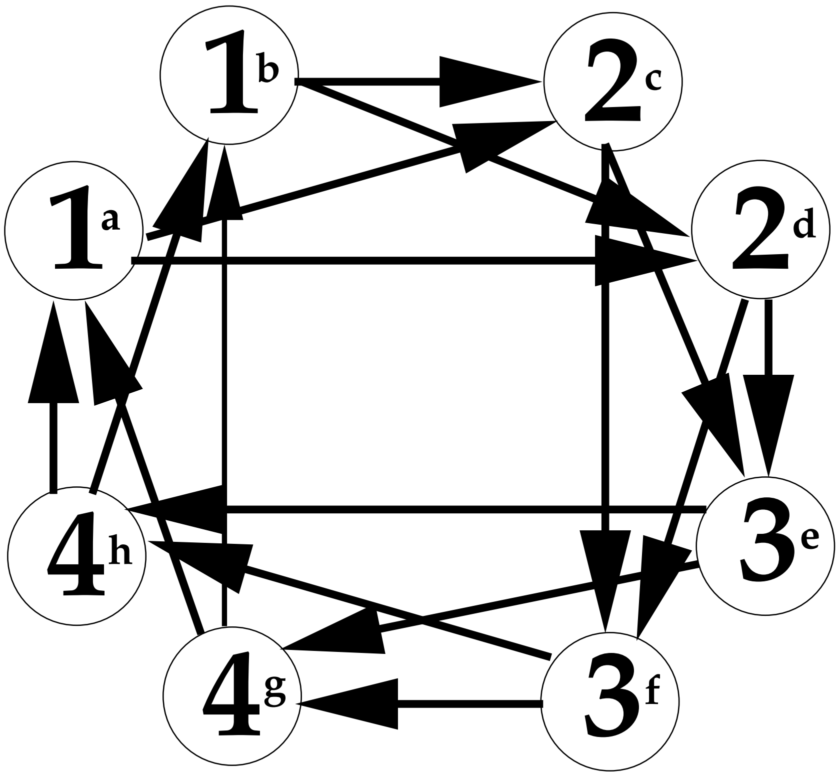
$$\bullet g + \bullet h = 0; \tag{3e}$$

$$\bullet g + \bullet h = 0. \tag{3f}$$

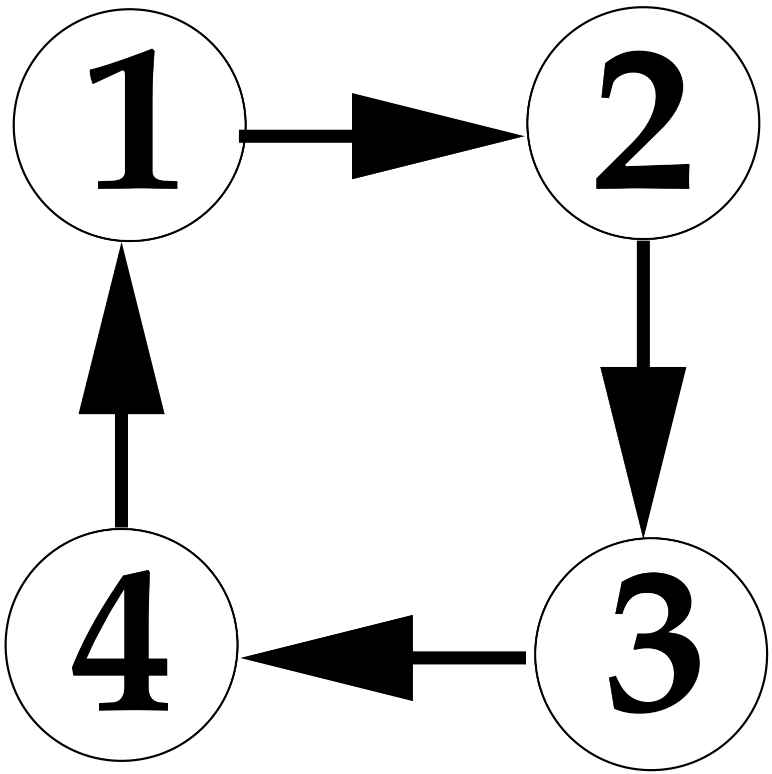
$$\bullet a + \bullet b = 0; \tag{4g}$$

$$\bullet a + \bullet b = 0; \tag{4h}$$

A Fat Polynomial Graph



A Thinner Graph



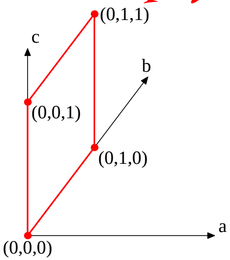
Newton Polytope

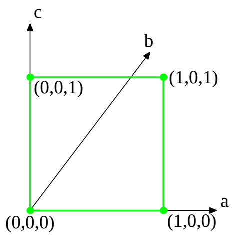
Polytope: convex hull of a finite set of points in affine space.

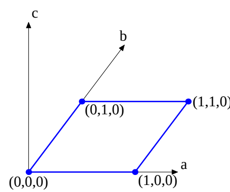
Each monomial $a^{\alpha}b^{\beta}c^{\gamma}\cdots$ in n variables is associated with a lattice point $(\alpha, \beta, \gamma, \ldots) \in \mathbb{N}^n$.

Support of polynomial: monomials occurring with nonzero coefficient.

Newton polytope of polynomial: convex hull of lattice points in its support.





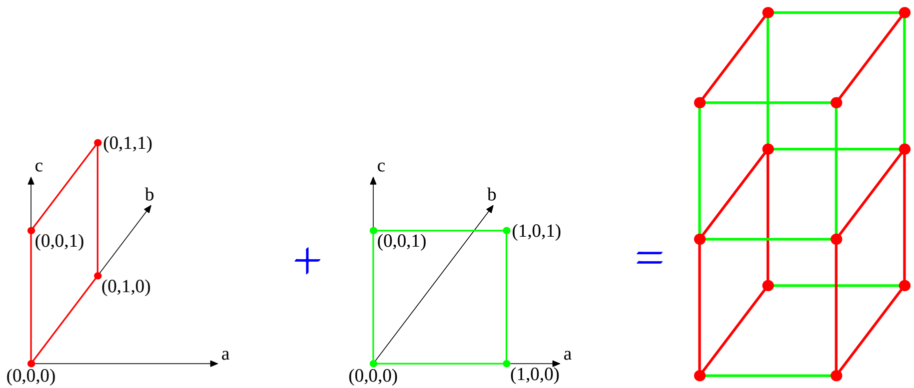


$\bullet bc - \bullet b - \bullet c + \bullet$ $\qquad$ $\bullet ac - \bullet a - \bullet c + \bullet$ $\qquad$ $\bullet ab - \bullet a - \bullet b + \bullet$

Minkowski Sum and Mixed Subdivision

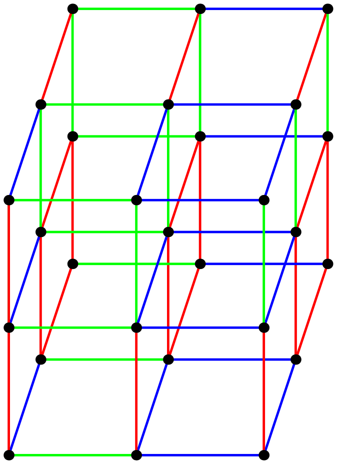
Minkowski sum of polytopes $P_1, \ldots, P_n$ is convex hull of $v_1 + \cdots + v_n$ where v_i is a vertex of P_i.

Translate faces of P_i along edges of P_j to get decomposition of Minkowski sum into *mixed subdivision* (not unique).



Bernstein-Kouchnirenko Theorem

The number of roots of a generic sparse system of polynomials is given by the mixed volume of their Newton polytopes.



Computing the mixed volume is not easy!

Permanent of a Matrix

The determinant of a matrix is
a sum of signed products of entries from the matrix.

The permanent of a matrix is
the sum of the same products,
without the signs.

Setup for Theorem

Let

$$f_1(\sigma_1, \ldots, \sigma_d) = 0,$$

$$f_2(\sigma_1, \ldots, \sigma_d) = 0,$$

$$\vdots$$

$$f_d(\sigma_1, \ldots, \sigma_d) = 0$$

be a system of d polynomial equations in d variables $\sigma_1, \ldots, \sigma_d$ *which is generic up to the following constraints.*

Theorem (D, 2003)

Suppose the variables can be partitioned into sets $T_1, \ldots, T_N$ *of cardinalities* $d_1, \ldots, d_N$ *such that*

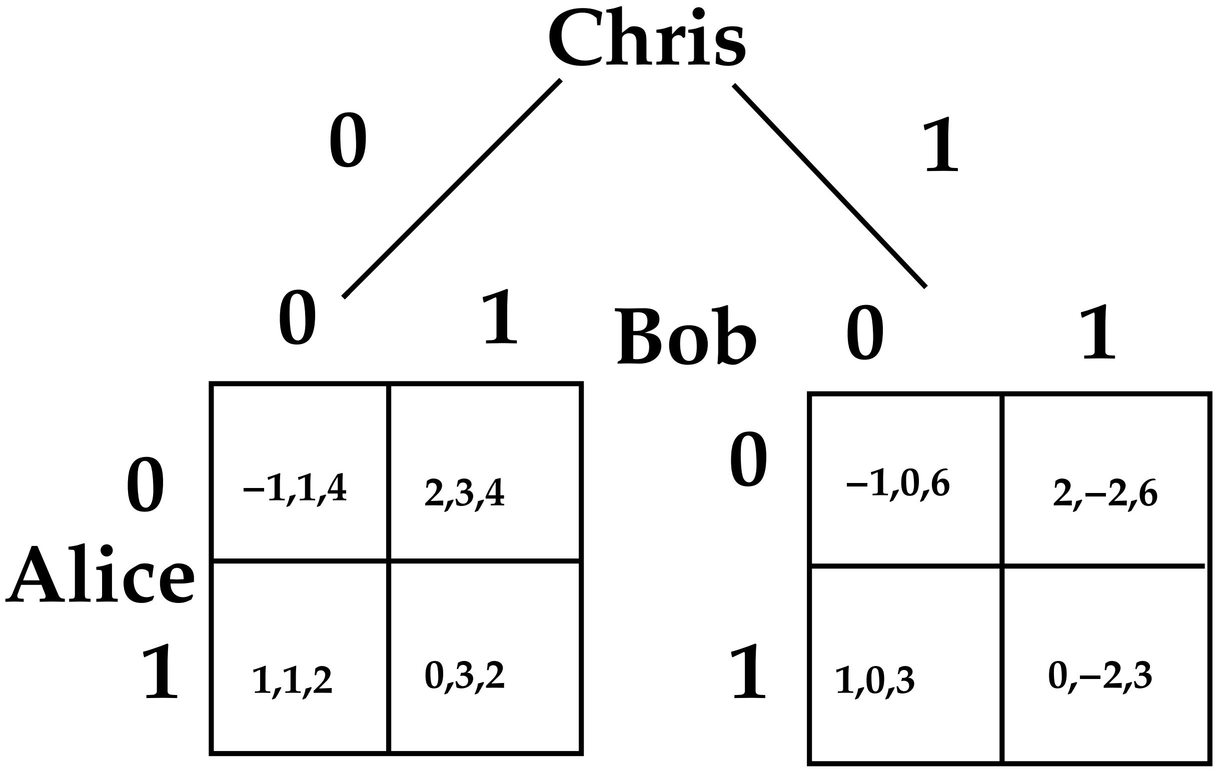
1) All monomials occurring in the f_i*'s are squarefree*

2) If $\sigma_j, \sigma_k \in T$ *with* $j \neq k$ *then* σ_j *and* σ_k *do not both occur in any monomial of any of the* f_i*'s*

3) If there is some $k \in T_i$ *such that there is an edge from* v_j *to* v_k *in* G*, then for every* $k \in T_i$ *there is an edge from* v_j *to* v_k *in* G*.*

Then if the polynomial system is 0*-dimensional, the number of its solutions in* $(\mathbb{C}^*)^d$ *is the permanent of the adjacency matrix of* G*, divided by* $\prod_{i=1}^{N} ((d_i)!)^{1/d_i}$*.*

Running Example: 3 X 2 game



Geometric Picture

Players: Alice, Bob, Chris

Two pure strategies each:

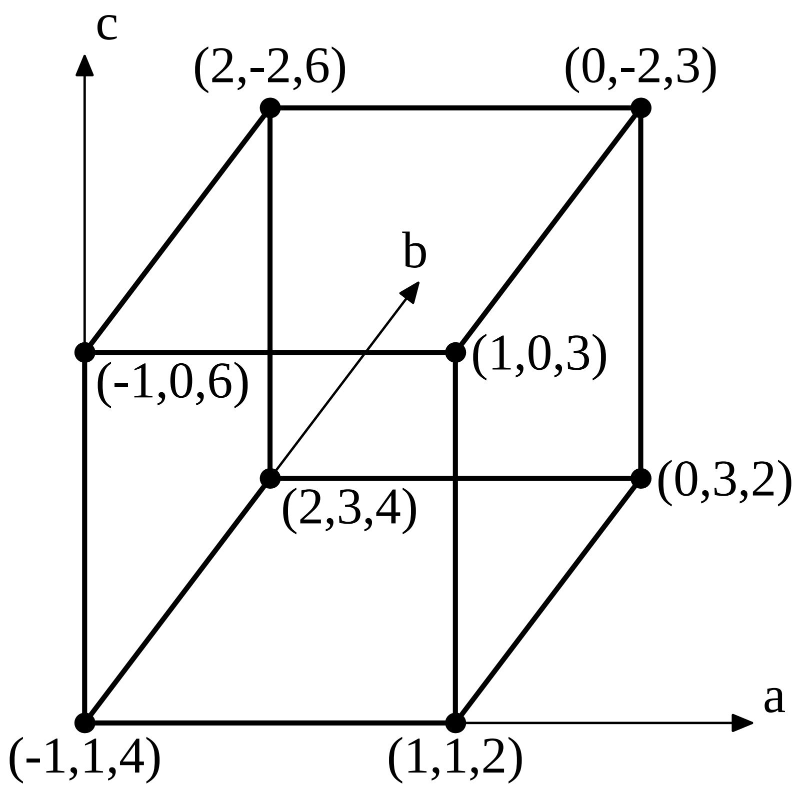
0 *and* 1

$a = \mathbf{Pr}[$*Alice chooses* 1$]$

$b = \mathbf{Pr}[$*Bob chooses* 1$]$

$c = \mathbf{Pr}[$*Chris chooses* 1$]$

If Alice chooses 1*,*
Bob chooses 0*,*
and Chris chooses 1*,*
then Alice's payoff is 1*,*
Bob's payoff is 0*,*
and Chris's payoff is 3.



Expected Payoff Functions

Payoff to Alice of picking pure strategy 0, is expected payoff conditioned on the event Alice chooses 0:

$$2bc + 2b(1-c) + -1(1-b)c + (-1)(1-b)(1-c) = 2b - (1-b) = 3b - 1$$

Similarly, payoff to Alice from picking pure strategy 1:

$$0bc + 0b(1-c) + 1(1-b)c + 1(1-b)(1-c) = 1 - b$$

Payoff to Bob of picking pure strategy 0:

$$0ca + 0c(1-a) + 1(1-c)a + 1(1-c)(1-a) = 1 - c$$

Payoff to Bob of picking pure strategy 1:

$$-2ca - 2c(1-a) + 3(1-c)a + 3(1-c)(1-a) = -2c + 3(1-c) = 3 - 5c$$

More Expected Payoff Functions

Payoff to Critter of picking pure strategy 0*:*

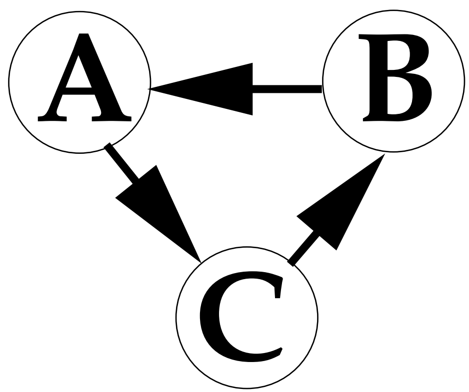
$$2ab + 2a(1-b) + 4(1-a)b + 4(1-a)(1-b) = 2a + 4(1-a) = 4 - 2a$$

Payoff to Critter of picking pure strategy 1*:*

$$ab + a(1-b) + 5(1-a)b + 5(1-a)(1-b) = a + 5(1-a) = 5 - 4a$$

Graphical Game

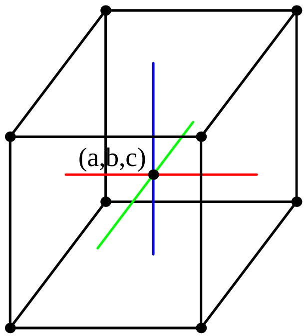
In a graphical game, the payoff to each player depends only on the actions of certain other players. We can draw the dependencies as a directed graph. Our example obeys the following graph:



Totally Mixed Nash Equilibria

Nash equilibrium: No player could unilaterally improve own payoff.

Totally mixed Nash equilibria are those in interior of cube.



Payoffs to each player of own pure strategies are equal.

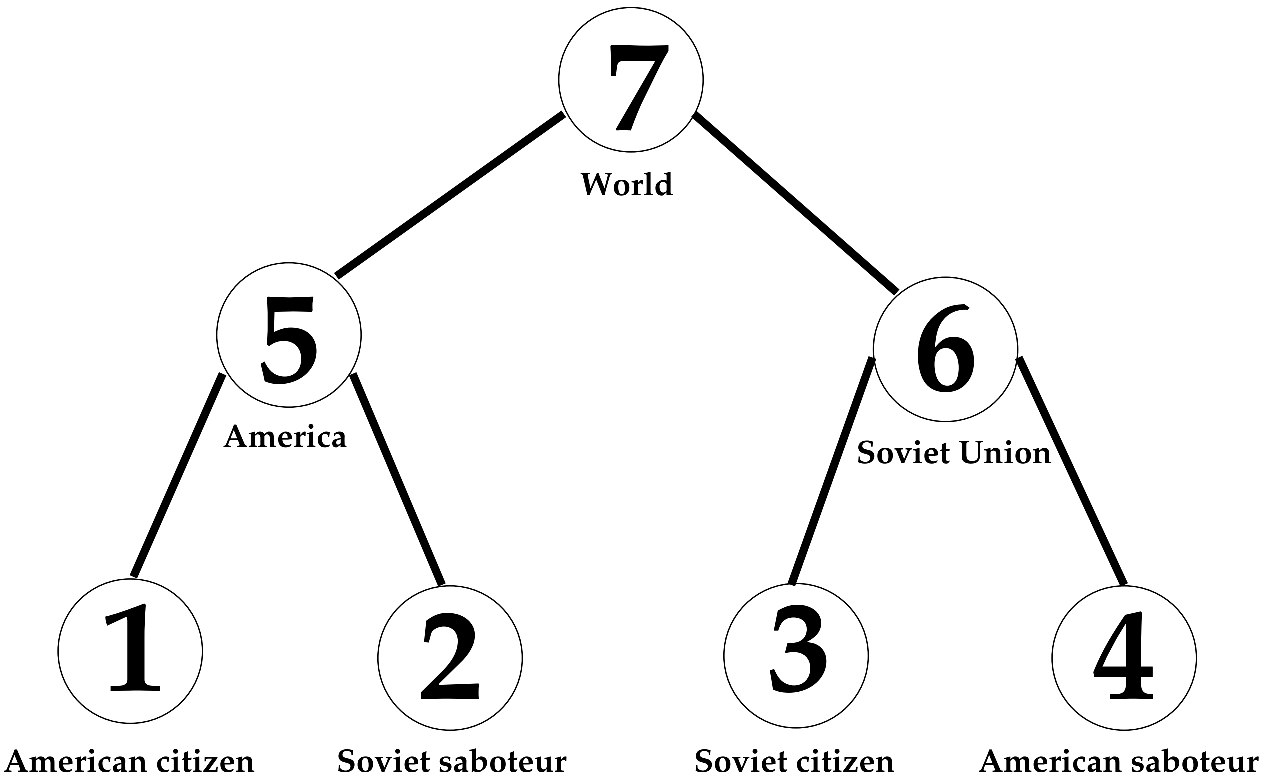
(1) $3b - 1 = 1 - b$, *i.e.*, $b = 1/2$

(2) $1 - c = 3 - 5c$, *i.e.*, $c = 1/2$

(3) $4 - 2a = 5 - 4a$, *i.e.*, $a = 1/2$

Single totally mixed Nash equilibrium: $(1/2, 1/2, 1/2)$.

A Game With Emergent "Players"



Emergent Node Tree Structure

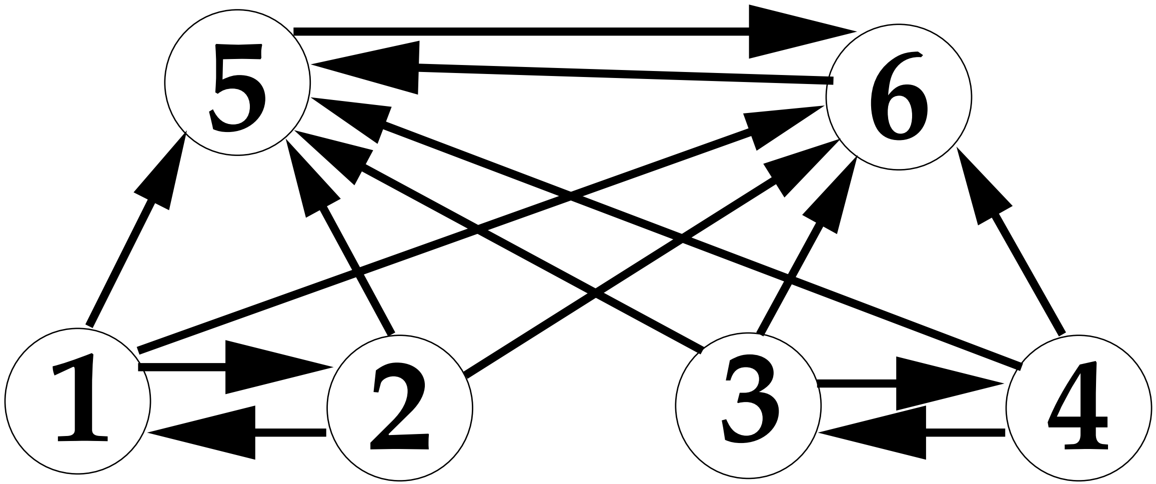
The saboteur game has an emergent node tree structure, or ENT, corresponding to the graph.

The actual players are the leaves of the tree.

The strategies of the emergent players
(neither the root nor the leaves)
are determined (possibly stochastically)
by the strategies of their children.

The payoffs to each player depend only
on the strategies of its siblings
and the payoffs to its non-root ancestors.

Polynomial Graph of Emergent Game



Hierarchically Perfect Nash Equilibria

A Nash equilibrium of a game with an ENT structure is
hierarchically perfect,
if it would be a Nash equilibrium
if the emergent players were actual players.

Generically, there may be none.
But a nearby game might have some.

This gives a new model of bounded rationality:
computing a Nash equilibrium of the actual game
may be too hard for the players to do.
So they might aim for a hierarchically perfect one of a nearby game.