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# Using Computer Algebra To Compute Nash Equilibria

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# Running Example

Players: Alice, Bob, Chris

Two pure strategies each:

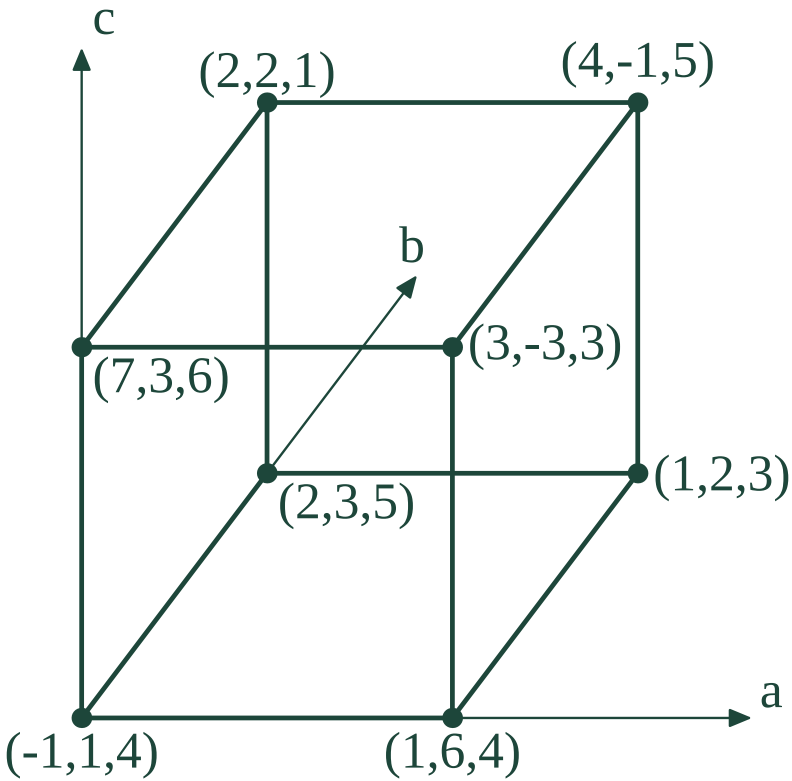
0 and 1

$a = \mathbf{Pr}[\text{Alice chooses } 1]$

$b = \mathbf{Pr}[\text{Bob chooses } 1]$

$c = \mathbf{Pr}[\text{Chris chooses } 1]$

If Alice chooses 1,
Bob chooses 0,
and Chris chooses 1,
then Alice's payoff is 3,
Bob's payoff is $-3$,
and Chris's payoff is 3.



# *Expected Payoff Functions*

*Payoff to Alice of picking pure strategy* 0*, is expected payoff conditioned on event Alice chooses* 0*:*

$$2bc + 2b(1-c) + 7(1-b)c + (-1)(1-b)(1-c) = -1 + 3b + 8c - 8bc$$

*Similarly, payoff to Alice from picking pure strategy* 1*:*

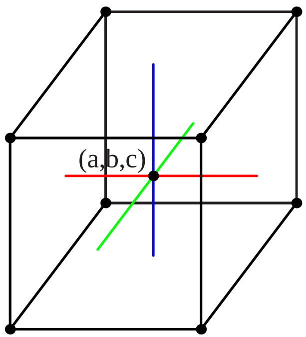
$$4bc + b(1-c) + 3(1-b)c + (1-b)(1-c) = 1 + 2c + bc$$

# Totally Mixed Nash Equilibria

*Nash equilibrium: No player could unilaterally improve own payoff.*

*Need procedure to find totally mixed Nash equilibria: those in interior of cube.*

*Using this, can find all Nash equilibria by stratification.*




*Payoffs to each player of own pure strategies are equal.*

(1) $9bc - 3b - 6c + 2 = 0$, *i.e.*, $(3b - 2)(3c - 1) = 0$

(2) $9ac - 6a - 3c + 2 = 0$, *i.e.*, $(3a - 1)(3c - 2) = 0$

(3) $9ab - 3a - 6b + 2 = 0$, *i.e.*, $(3a - 2)(3b - 1) = 0$

*Two totally mixed Nash equilibria:* $(1/3, 1/3, 1/3)$ *and* $(2/3, 2/3, 2/3)$.

# The Status Quo: Gambit

Start with cube containing product of simplices.

Look for single root in cube numerically.

Check that root is in product of simplices.

If intersection of linear hypersurfaces is $0$-dimensional, it's a single point. So check linear Taylor approximation to system: good enough within cube? Then this is only root in this cube, so done.

Otherwise, subdivide cube into smaller cubes, start again.

Problem: neither fast nor robust.

# Grobner Bases

*Eliminate terms through computations like these:*

$$
\begin{array}{rrrrrrrrr}
 & & & 9bc & & -3b & -6c+2 & =0 & (1)\\
 & & 9ac & & -6a & & -3c+2 & =0 & (2)\\
 & 9ab & & & -3a & -6b & +2 & =0 & (3)\\
 & & & & & & & & \\
9abc- & 3ab- & 6ac & & +2a & & & =0 & a*(1)\\
-9abc+ & 6ab & & +3bc & & -2b & & =0 & -b*(2)\\
\hline
 & 3ab- & 6ac & +3bc & +2a & -2b & & =0 & (4)\\
 & & & & & & & & \\
 & 9ab & -18ac & +9bc & +6a & -6b & & =0 & 3*(4)\\
 & -9ab & & & +3a & +6b & -2 & =0 & -(3)\\
\hline
 & & -18ac & +9bc & +9a & & -2 & =0 & (5)
\end{array}
$$

# Grobner Bases (continued)

$$\begin{aligned}
-18ac + 9bc + 9a \qquad - 2 &= 0 && (5)\\
18ac \qquad - 12a \qquad - 6c + 4 &= 0 && 2*(2)
\end{aligned}$$

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$$9bc - 3a - 6c + 2 = 0 \qquad (6)$$

$$\begin{aligned}
9bc - 3a \qquad - 6c + 2 &= 0 && (6)\\
-9bc \qquad + 3b + 6c - 2 &= 0 && -(1)
\end{aligned}$$

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$$-3a + 3b = 0 \qquad (7)$$

# Elimination Theory

End up with triangular sets:

equation in one variable

$$9c^2 - 9c + 2 = 0$$

equation in two variables

$$3b - 3c = 0$$

equation in three (here two) variables

$$3a - 3c = 0$$

Find roots of univariate equation, substitute into next to get another univariate equation, repeat.

# Homotopy Continuation

Choose a start system similar to the desired system, but such that the roots are easy to find.

Then deform it gradually towards the desired system, computing the new roots at each step.

Starting from each old root, predictor step aims at initial guess for new root, corrector step actually finds it iteratively.

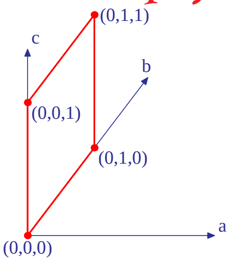
# Newton Polytope

*Polytope: convex hull of a finite set of points in affine space.*

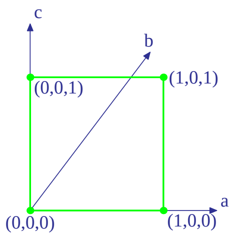
*Each monomial* $a^\alpha b^\beta c^\gamma \cdots$ *in* $n$ *variables is associated with a lattice point* $(\alpha, \beta, \gamma, \ldots) \in \mathbb{N}^n$.

*Support of polynomial: monomials occurring with nonzero coefficient.*

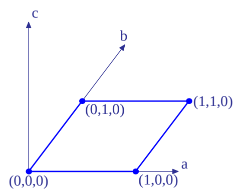
*Newton polytope of polynomial: convex hull of lattice points in its support.*




$9bc - 3b - 6c + 2$




$9ac - 6a - 3c + 2$

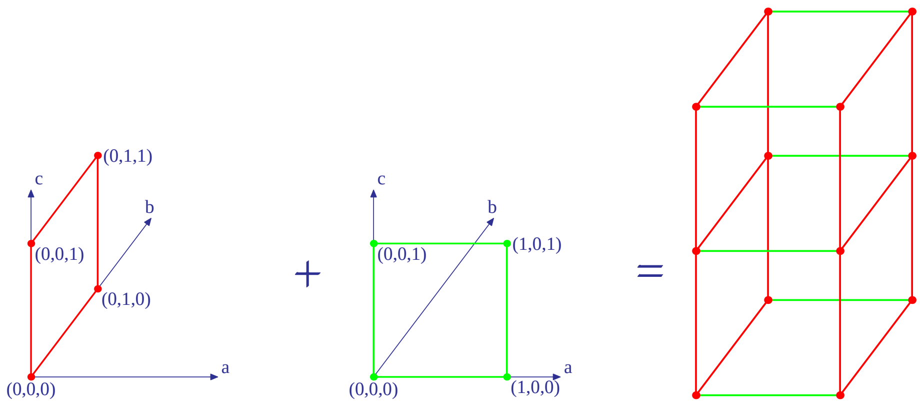



$9ab - 3a - 6b + 2$

# Minkowski Sum and Mixed Subdivision

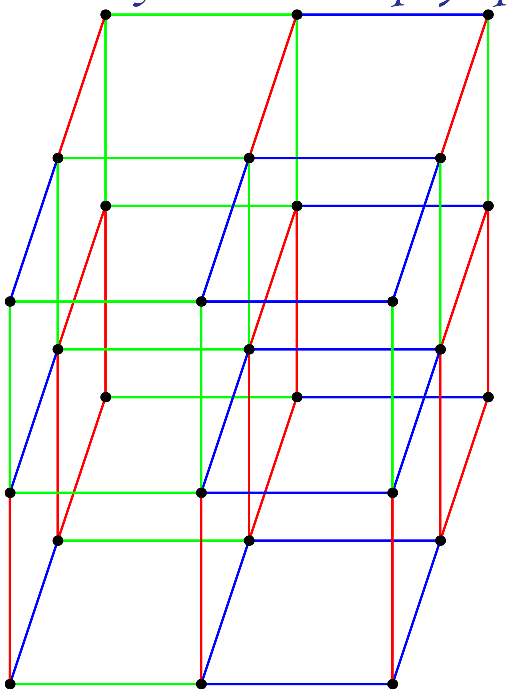
*Minkowski sum* of polytopes $P_1, \ldots, P_n$ is convex hull of $v_1 + \cdots + v_n$ where $v_i$ is a vertex of $P_i$.

Translate faces of $P_i$ along edges of $P_j$ to get decomposition of Minkowski sum into *mixed subdivision* (not unique).



# Polyhedral Homotopy Continuation

The number of roots of a generic sparse system of polynomials is given by the mixed volume of their Newton polytopes.



Choose a start system
with the same support
as the desired system,
so that the roots are easy to find.

# Performance

Used Singular for Grobner bases. Used PHC for polyhedral homotopy continuation (a new alternate implementation, PHoM, is parallelizable).

| # players | # pure strategies each | # roots | Gambit | Singular | PHC |
|---|---|---|---|---|---|
| 3 | 2 | 2 | 60-160ms | 10ms | 20ms |
| 4 | 2 | 9 | * | 70ms | 260ms |
| 3 | 3 | 10 | * | 1150ms | 350ms |
| 5 | 2 | 44 | * | ** | 7s200ms |
| 3 | 4 | 56 | * | ** | 13s280ms |
| 6 | 2 | 265 | * | ** | 7m10s790ms |
| 4 | 3 | 297 | * | ** | 4m3s220ms |
| 3 | 5 | 346 | * | ** | 3m19s540ms |
| 3 | 6 | 2252 | * | ** | 48m41s870ms |
| 4 | 4 | 13833 | * | ** | 7h2m20s780ms |