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Homogeneous three-cell networks

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Abstract

A cell is a system of differential equations. Coupled cell systems are networks of cells. The architecture of a coupled cell network is a graph indicating which cells are identical and which cells are coupled to which. In this paper we continue the work of Stewart, Golubitsky, Pivato and Török by classifying all homogeneous three-cell networks (where each cell has at most two inputs) and classifying all generic codimension one steady-state and Hopf bifurcations from a synchronous equilibrium. We use combinatorial arguments to show that there are 34 distinct homogeneous three-cell networks as opposed to only three such two-cell networks.

We show that codimension one bifurcations in homogeneous three-cell networks can exhibit interesting features that are due to network architecture. Indeed, network architecture determines, even at linear level, the kind of generic transitions from a synchronous equilibrium that can occur as we vary one parameter and plays a crucial role in establishing how the solutions on the bifurcating branches manifest themselves in each cell.

Mathematics Subject Classification: 34A34, 34C14, 34C15, 37G15, 34K18

1. Introduction

Networks of nonlinear differential equations are currently a topic of considerable interest, mainly, because they can naturally model applications in a wide variety of fields [15, 17]. Because of this, Stewart *et al* [12, 13, 16] have attempted to develop a theory of *coupled cell systems*, where a *cell* is just a system of differential equations. This theory is based on the *architecture* of a network: a graph that indicates which cells are coupled to which, which cells have the same state variables and which couplings are identical [13, 16].

In coupled systems the cells provide a canonical set of coordinates, which can be compared. For example, in a given solution, two cells are *synchronous* if the dynamics in each cell are the same for all time, and time periodic states are *phase-related* if the time series from two cells differ by a phase shift. Stewart *et al* have derived necessary and sufficient conditions for synchronous dynamics to appear robustly in a coupled cell system.

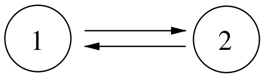

Figure 1. The graph associated with a symmetric two identical cell network.

In this context it is natural to ask a more general question.

> Which properties of the dynamics observed in a coupled cell system are inherent to the network architecture and which are related to the specific dynamics of the cells and form of couplings?

In this paper we attempt to answer part of this question by classifying the codimension one bifurcations from a synchronous equilibrium in homogeneous, identically coupled, three-cell networks. To help define these terms and to help discuss our results in more detail, we recall two standard examples: the symmetric two-cell network and the three-cell bidirectional ring.

Two-cell network. The simplest example of a coupled cell network is the two identical cell, identically coupled system

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2), \\ \dot{x}_2 &= f(x_2, x_1), \end{aligned} \tag{1.1}$$

where $x_1, x_2 \in \boldsymbol{R}^k$ for some k and $f : \boldsymbol{R}^k \times \boldsymbol{R}^k \to \boldsymbol{R}^k$. The network architecture for this system is given by the graph in figure 1. The class of systems of differential equations in (1.1) are the coupled cell systems *associated with* the two-cell network pictured in figure 1.

It follows from (1.1) that the diagonal subspace $x_1 = x_2$ is flow-invariant for any f. Hence, *synchronous* solutions, where $x_1(t) = x_2(t)$ for all t, are expected in this cell system. Indeed, such systems can be expected to have a synchronous equilibrium, and at such an equilibrium, the Jacobian matrix has the form

$$J = \begin{bmatrix} Q & R \\ R & Q \end{bmatrix},$$

where Q is the $k \times k$ matrix of linearized internal dynamics and R is the $k \times k$ linearized coupling matrix.

Suppose that f depends on a parameter λ and that the equilibrium is a point of Hopf bifurcation; that is, J has purely imaginary eigenvalues. Since the $2k$ eigenvalues of J are just the eigenvalues of the matrices $Q + R$ (with eigenvectors of the form $(x, x)^t$ for some $x \in \boldsymbol{C}^k$) and $Q - R$ (with eigenvectors of the form $(x, -x)^t$), there are two types of Hopf bifurcation: those where the critical eigenvalues are eigenvalues of $Q + R$ and those where the critical eigenvalues are eigenvalues of $Q - R$. As is well known, the first case leads to synchronous periodic solutions, whereas the second case leads to half-period out of phase T-periodic solutions in which $x_2(t) = x_1(t + T/2)$. Observe that the existence of synchronous equilibria and the fact that Hopf bifurcation from those equilibria divides into two types depends only on network architecture and not on the particular form of f.

Earlier theoretical work on the dynamics of coupled cell systems emphasized symmetric networks, where the symmetry of the network is the only mathematical structure used in the analysis. Indeed, the analysis of this two-cell system can proceed using symmetry arguments alone. The general theory is described in Golubitsky and Stewart [11]. The work of Stewart, Golubitsky, Pivato, and Török [8, 13, 16] shows that the class of vector fields associated with

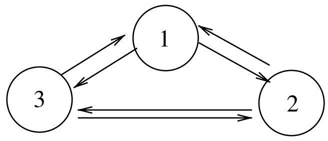

Figure 2. Three identical cell bidirectional ring.

a given coupled cell network is completely described by its symmetry groupoid [4], which can be thought of as a set of local symmetries. This less stringent form of symmetry shows that global group-theoretic symmetry is not the only mechanism that can lead to patterns of synchronized cells in coupled cell systems. Their work also shows that the bifurcation theory for coupled cell systems is different from that which occurs either in general systems or in symmetric systems [7, 8].

The bidirectional ring. A *homogeneous* network is one in which the associated differential equations have the form $\dot{x}_i = f(x_i, \ldots)$ where f is independent of the cell index i. In this paper we focus on homogeneous identically coupled three-cell networks, where we now explain what we mean by the term 'identically coupled.' Consider the homogeneous bidirectional ring pictured in figure 2. The associated systems of differential equations have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3), \\ \dot{x}_2 &= f(x_2, x_3, x_1), \\ \dot{x}_3 &= f(x_3, x_1, x_2), \end{aligned} \tag{1.2}$$

where $x_1, x_2, x_3 \in \boldsymbol{R}^k$ and $f : \boldsymbol{R}^k \times \boldsymbol{R}^k \times \boldsymbol{R}^k \to \boldsymbol{R}^k$. In the bidirectional ring we assume that the coupling from cells 2 and 3 to cell 1 are identical; that is

$$f(u, v, w) = f(u, w, v).$$

In a homogeneous identical coupled system, we assume that the vector field f is invariant under permutations of the coupling cell variables.

The *valency* of the network is the number of signals received by each cell; the two-cell system has valency 1 and the bidirectional ring has valency 2. In this paper we will classify all homogeneous identically coupled three-cell networks with valency 1 or 2. Before presenting this classification we must discuss 'self-coupling' and 'multiple arrows.' It is straightforward to check that $x_2 = x_3$ is a flow-invariant subspace for the system (1.2). It follows that synchronous solutions of the type $(x_1(t), x_2(t), x_2(t))$ are to be expected in the bidirectional ring. Moreover, the system of differential equations restricted to this flow-invariant subspace has the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_2), \\ \dot{x}_2 &= f(x_2, x_1, x_2). \end{aligned} \tag{1.3}$$

Note that the restricted system that describes synchronous dynamics is a system associated with the two-cell system in figure 3 and this network has both self-coupling (the arrow from cell 2 to itself follows since x_2 appears as a coupling variable in the $\dot{x}_2$ equation) and multiple arrows (the two arrows from cell 2 to cell 1 follows since the coupling variable x_2 appears twice in the $\dot{x}_2$ equation). The theory of such network architectures is developed in [13]. Networks with multiple arrows and self-coupling always describe synchronous dynamics on larger networks without these features.

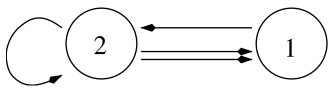

Figure 3. Two-cell system obtained from synchronous dynamics in the bidirectional ring.

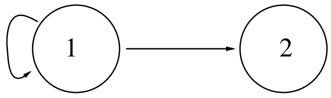

Figure 4. Homogeneous two-cell feed forward network.

In this paper we address three problems.

(a) Classification of all homogeneous three-cell networks with identical coupling (including multiple arrows and self-coupling) and valency 1 or 2.
(b) Classification of all codimension one steady-state and Hopf bifurcations from a synchronous equilibria in these three-cell networks.
(c) Determination of how the different solutions manifest themselves in the three cells.

The work of Stewart, Golubitsky and co-workers answer some of this questions, but there are many others that are still open.

We focus on homogeneous three-cell networks for three reasons. First, the classification and codimension one bifurcations of homogeneous two-cell systems is relatively easy to understand and the three-cell systems are the next ones to explore. Second, we focus on homogeneous networks because the differential equations associated with such networks naturally support synchronous equilibria. Third, it is now known that certain small subnetworks appear with higher frequency than random in large networks, and these subnets are called *network motifs* [14, 3]. This discovery leads to a general problem of understanding the role of motifs in the dynamics of large networks.

Enumeration and linearized systems. We enumerate homogeneous three-cell networks with valency 1 or 2 in section 2. We use combinatorial arguments and a result of Dias and Stewart [5] to show that there are 34 distinct connected networks. These networks are listed in figure 5. This result should be contrasted with the fact that a similar classification of two-cell networks leads to just three networks: the symmetric network in figure 1, the asymmetric network in figure 3, and the feed forward network in figure 4.

Propositions 3.3 and 3.1 show that there are some networks where the Jacobian J at a synchronous equilibrium is forced to have complex eigenvalues, some networks where J is forced to have multiple eigenvalues and some networks where J is forced to be nilpotent. (None of these features are present in the corresponding two-cell networks.) This fact shows that there are features of network architecture (unrelated to symmetry) that constrain the coupled cell system, even at linear level. Since the eigenvalues control bifurcations from a synchronous equilibrium, it follows that network architecture effectively constrains the generic synchrony-breaking bifurcations. Elmhirst and Golubitsky [6] show that additional constraints can also occur at nonlinear level.

Codimension one synchrony-breaking bifurcations. In this paper we will classify both codimension one steady-state (see section 4) and codimension one Hopf bifurcations (see

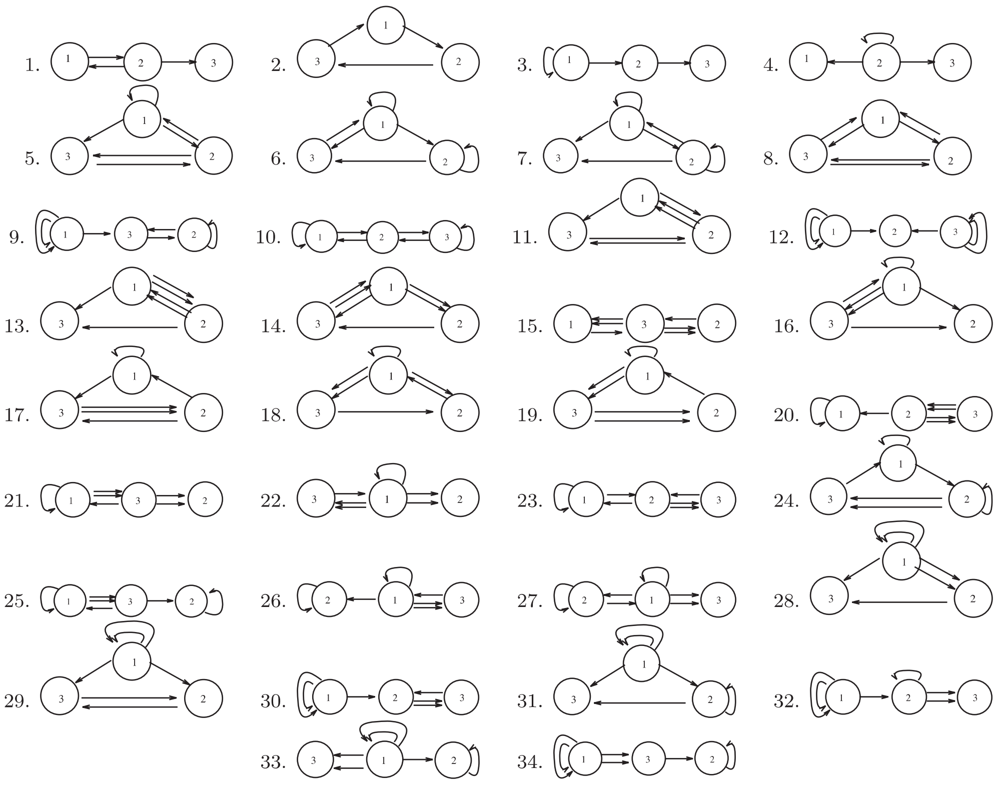

Figure 5. The distinct homogeneous three-cell networks with valency $n = 1, 2$.

section 5) from a synchronous equilibrium in the 34 networks. In this introduction we illustrate several interesting features of these bifurcations (mostly using Hopf bifurcation).

Codimension one synchrony preserving Hopf bifurcation occurs in all homogeneous networks through simple critical eigenvalues and the application of standard Hopf theory. These bifurcations lead to a unique branch of synchronous periodic solutions.

There are six phenomena associated with synchrony-breaking bifurcations that we discuss in this introduction.

- Patterns of oscillation associated with simple eigenvalue bifurcations.
- Tori of periodic solutions associated with certain feed-forward networks.
- Periodic solutions that are constant in certain cells.
- Bifurcating solutions whose amplitudes grow at different rates in different cells.
- Multiple eigenvalues and multiple bifurcating branches.
- Some cells are related by symmetry in an asymmetric network.

Patterns of oscillations. In most networks codimension one synchrony-breaking Hopf bifurcations occur through simple critical eigenvalues and standard theory leads to a unique branch of asynchronous periodic solutions. However, there is an approximate pattern to these asynchronous oscillations that depends only on network architecture, as we now explain.

For example, the unidirectional ring, figure 5(2), has $\mathbf{Z_3}$-symmetry and it is well known that symmetry-breaking Hopf bifurcations in such systems lead to periodic solutions that are

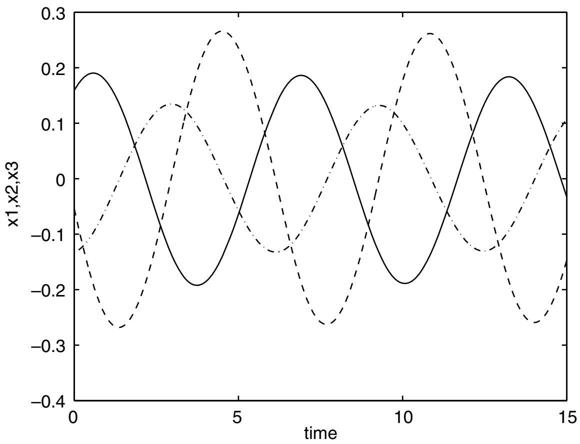

Figure 6. Time series from network 14; cell 1 is solid, cell 2 is dashed and cell 3 is dashed-dotted.

discrete rotating waves [11]. More precisely, $x_2(t)$ is the same as $x_1(t)$ with a third-period phase shift and $x_3(t)$ is the same as $x_2(t)$ with a third-period phase shift. It is perhaps surprising that when Hopf bifurcation occurs with simple critical eigenvalues in each of the other networks, the periodic solutions (at lowest order in the bifurcation parameter λ) have well-defined amplitude and phase relations between cells.

We illustrate this feature by considering network 14 (see theorem 5.1 for details). Standard Hopf theory shows that we can write periodic solutions on the bifurcating branch at lowest order in λ as

$$X(t) \approx (A_1 \cos(2\pi(t+\phi_1)/T), A_2 \cos(2\pi(t+\phi_2)/T), A_3 \cos(2\pi(t+\phi_3)/T))\lambda^{1/2},$$

where T is the period; A_1, A_2, A_3 are the amplitudes of oscillation in each cell and ϕ_1, ϕ_2, ϕ_3 are the phases of oscillation in each cell. The actual values of the amplitudes and the phase shifts depend on the specific differential equation f, but it is less obvious that the ratios of the amplitudes and the differences in the phases depend only on network architecture. In the unidirectional ring, the amplitude ratios are all equal to unity and the phase difference are all either $1/3$ or $2/3$. In section 5.1 we show that for network 14 with the critical eigenvalue being $+i$ the amplitude ratios and the phase shift between cells are

$$\frac{A_2}{A_1} = \sqrt{2} \qquad \frac{A_3}{A_1} = \frac{\sqrt{2}}{2} \qquad \phi_2 - \phi_1 = \frac{5}{8} \qquad \phi_3 - \phi_1 = \frac{3}{8}.$$

The results of numerical simulation using

$$f(x_1, x_2, x_3) = -x_1 - 1.02(x_2 + x_3) - 0.8x_1^2 - x_1^3 \tag{1.4}$$

for network 14 is shown in figure 6. The amplitude ratios and phase differences obtained from the time series are

$$\frac{A_2}{A_1} \approx 1.42 \qquad \frac{A_3}{A_1} \approx 0.71 \qquad \phi_2 - \phi_1 \approx 0.63 \qquad \phi_3 - \phi_1 \approx 0.39,$$

which is a good approximation to the analytic result.

Tori of periodic solutions. The bidirectional ring, network 8, has $\boldsymbol{D_3}$-symmetry and is an example of a network where (because of symmetry) Hopf bifurcations occur with double critical eigenvalues. It is well known that generically these symmetry-breaking bifurcations lead to three types of periodic solutions and that the patterns of oscillations associated with such periodic solutions are determined by symmetry [11].

Hopf synchrony-breaking bifurcations in networks 4 and 11 also occur with double critical eigenvalues and lead generically to a unique branch of tori foliated by periodic solutions. More precisely, these families of periodic solutions have two cells with the same wave-form, but the phase shift between the cells is arbitrary. The phase shift that is seen depends on initial conditions. See sections 5.3 and 5.5.

Constant versus periodic. The existence of periodic solutions that are constant in certain cells was observed for the feed-forward network 3 in [7]. The existence of such periodic solutions is due to the existence of nontrivial flow-invariant subspaces forced by the skew-product form of feed forward coupled cell systems. For example, networks 31 (see section 5.2) and 4 (see section 5.3) also yield such solutions.

Distinct growth rates. In [6,7] it was also shown that generically periodic solutions emanating from the nilpotent Hopf bifurcation in the feed-forward network 3 have amplitudes that grow at different rates in different cells. In particular, in that network the amplitude in cell 2 grows at the standard rate of $\lambda^{1/2}$, whereas the amplitude in cell 3 grows with the unexpected rate of $\lambda^{1/6}$, where λ is the bifurcation parameter and $\lambda = 0$ is the point of Hopf bifurcation. This phenomenon also occurs in networks 27 and 28. A similar phenomenon is seen in synchrony-breaking steady-state bifurcations that occur in these networks.

Multiple bifurcating branches. Network 28 in figure 5 is an example where the steady-state synchrony-breaking bifurcation occurs generically with double critical eigenvalues and a deficiency of eigenvectors (section 4.4). These codimension one bifurcations lead to a transcritical branch and a pitchfork branch of equilibria (see theorem 4.6), where cells 1 and 2 are identically zero in the transcritical branch and cell 1 is identically zero in the pitchfork branch. As mentioned previously for synchrony-breaking Hopf bifurcations, this phenomenon is due to the skew-product form of the coupled cell system, which is forced by the feed-forward architecture.

Symmetry. It is well known that symmetry can affect the types of bifurcation that occur in codimension one bifurcations. Symmetry-breaking Hopf bifurcations in the bidirectional ring 8 is such an example. The other networks in figure 5 that have nontrivial symmetries are 2, 4, 7, 8, 10, 12, 13 and 15. It was also shown in [8] that network interior symmetries can affect codimension one bifurcations. The networks in figure 5 that have nontrivial interior symmetries are 1, 5, 11, 20 and 27.

Many of these symmetric networks have a feed-forward structure, where two symmetric cells force a third one. Consequently, there are asymmetric networks which have a two-cell symmetric subnet that forces a third cell. Network 27 is an example of such a network. It follows that there are equilibria in bifurcating branches with cells related by symmetry. Similarly, in Hopf bifurcation, there is a branch of periodic solutions where cells 1 and 2 are a half-period out of phase each with amplitude growth rate $\lambda^{1/2}$, whereas cell 3 grows at rate $\lambda^{1/6}$ [6].

To reiterate, the enumeration of networks is given in section 2, the eigenvalue structure of the Jacobian at a synchronous equilibrium is computed in section 3, codimension one steady-state bifurcations are discussed in section 4 and codimension one Hopf bifurcations are discussed in section 5. For the most part standard bifurcation theory techniques are combined with network architecture to determine the nonstandard results in the last two sections.

2. Enumeration of three-cell homogeneous networks

In this section we classify the connected homogeneous three-cell networks with one kind of coupling and valency n equal to either 1 or 2. The enumeration, up to permutation of cells, is made in section 2.1 for $n = 1$ and in section 2.2 for $n = 2$. We use combinatorial arguments to show that there are 42 such networks. Aldosray and Stewart [2] obtain the same result by different methods.

In section 2.4 we show that some of these networks are redundant in the sense that they define the same systems of differential equations. The 34 nonredundant networks with valency 1 or 2 are listed in figure 5.

In section 2.1 we define a homogeneous three-cell network by its adjacency matrix A (see definition 2.1). The matrices A associated with the networks in figure 5 and the eigenvalues and eigenvectors of A are given in table 1. The eigenvalues of A will be used in section 3 when we discuss the types of codimension one bifurcations that can occur from synchronous equilibria. These eigenvalues and eigenvectors are obtained by direct calculation.

2.1. Networks with one input arrow

In this section we enumerate the homogeneous three-cell networks with valency 1. This identification uses the adjacency matrix, defined as follows.

Definition 2.1. Given a homogeneous three-cell network. Let a_{ij} be the number of inputs that cell i receives from cell j. The 3×3 matrix of nonnegative integers $A = (a_{ij})$ is the *adjacency* matrix. The *valency* n of the network satisfies

$$a_{i1} + a_{i2} + a_{i3} = n \tag{2.1}$$

for $i = 1, 2, 3$.

The associated graph is given in figure 7.

Definition 2.2. Two networks are *isomorphic* if their adjacency matrices are conjugate by a permutation matrix.

Theorem 2.3. *Up to isomorphism there are four connected homogeneous three-cell networks with valency 1. These networks are shown in figures 5(1–4).*

Proof. Since the valency is 1, each row of A must have one entry equal to 1 and the other entries equal to 0. We divide the proof into two cases: networks without self-coupling and networks with self-coupling.

Networks without self-coupling. The diagonal entries in A are zero in such a network and up to conjugacy we may assume that $a_{21} = 1$, that is cell 2 connects to cell 1. Hence

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 1 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}.$$

Table 1. Adjacency matrices A for networks in figure 5 and their eigenvalues and eigenvectors. The first eigenvalue corresponds to μ_1. The superscript 2 and subscript $\star$ indicate an algebraic multiplicity of two and a geometric multiplicity of one.

$A_\#$	E'vals	Eigenvectors	$A_\#$	E'vals	Eigenvectors
$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 0 -1	$(1, 1, 1)$ $(0, 0, 1)$ $(1, -1, 1)$	$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 $\dfrac{-1-\sqrt{3}i}{2}$ $\dfrac{-1+\sqrt{3}i}{2}$	$(1, 1, 1)$ $\left(1, -\dfrac{1+\sqrt{3}i}{2}, -\dfrac{1-\sqrt{3}i}{2}\right)$ $\left(1, -\dfrac{1-\sqrt{3}i}{2}, -\dfrac{1+\sqrt{3}i}{2}\right)$
$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 $0^2_\star$	$(1, 1, 1)$ $(0, 0, 1)$	$A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 0 0	$(1, 1, 1)$ $(1, 0, 0)$ $(0, 0, 1)$
$A_5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 0 -1	$(1, 1, 1)$ $(-1, 1, 1)$ $(1, -2, 1)$	$A_6 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 $0^2_\star$	$(1, 1, 1)$ $(1, -1, -1)$
$A_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 0 0	$(1, 1, 1)$ $(1, -1, 0)$ $(0, 0, 1)$	$A_8 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 -1 -1	$(1, 1, 1)$ $(1, -1, 0)$ $(0, 1, -1)$
$A_9 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 $\dfrac{1}{2} + \dfrac{\sqrt{5}}{2}$ $\dfrac{1}{2} - \dfrac{\sqrt{5}}{2}$	$(1, 1, 1)$ $(0, 2, -1+\sqrt{5})$ $(0, 2, -1-\sqrt{5})$	$A_{10} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	2 1 -1	$(1, 1, 1)$ $(1, -2, 1)$ $(1, 0, -1)$
$A_{11} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 $-1^2_\star$	$(1, 1, 1)$ $(2, -1, -1)$	$A_{12} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$	2 2 0	$(1, 1, 1)$ $(2, 1, 0)$ $(0, 1, 0)$

Table 1. (Continued)

$A_\#$	E'vals	Eigenvectors	$A_\#$	E'vals	Eigenvectors
$A_{13} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 0 -2	$(1, 1, 1)$ $(0, 0, 1)$ $(1, -1, 0)$	$A_{14} = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 $-1+i$ $-1-i$	$(1, 1, 1)$ $\left(1, -(1+i), -\frac{1-i}{2}\right)$ $\left(1, -(1-i), -\frac{1+i}{2}\right)$
$A_{15} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$	2 0 -2	$(1, 1, 1)$ $(1, -1, 0)$ $(1, 1, -1)$	$A_{16} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$	2 0 -1	$(1, 1, 1)$ $(0, 1, 0)$ $(1, 1, -2)$
$A_{17} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$	2 0 -1	$(1, 1, 1)$ $(-1, 1, 0)$ $(1, -2, 1)$	$A_{18} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$	2 $\frac{-1-\sqrt{3}i}{2}$ $\frac{-1+\sqrt{3}i}{2}$	$(1, 1, 1)$ $\left(1, \frac{-3+\sqrt{3}i}{2}, -1-\sqrt{3}i\right)$ $\left(1, \frac{-3-\sqrt{3}i}{2}, -1+\sqrt{3}i\right)$
$A_{19} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$	2 $\frac{-1-\sqrt{7}i}{2}$ $\frac{-1+\sqrt{7}i}{2}$	$(1, 1, 1)$ $\left(1, \frac{-3+\sqrt{7}i}{2}, \frac{-1-\sqrt{7}i}{2}\right)$ $\left(1, \frac{-3-\sqrt{7}i}{2}, \frac{-1+\sqrt{7}i}{2}\right)$	$A_{20} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$	2 1 -2	$(1, 1, 1)$ $(1, 0, 0)$ $(1, -3, 3)$
$A_{21} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$	2 0 -1	$(1, 1, 1)$ $(0, 1, 0)$ $(1, -3, 3)$	$A_{22} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	2 0 -1	$(1, 1, 1)$ $(0, 1, 0)$ $(1, 4, -2)$

Table 1. (Continued)

$A_\#$	E'vals	Eigenvectors	$A_\#$	E'vals	Eigenvectors
$A_{23} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$	2 $\frac{-1+\sqrt{5}}{2}$ $\frac{-1-\sqrt{5}}{2}$	$(1, 1, 1)$ $\left(1, \frac{\sqrt{5}-3}{2}, 2\frac{3-\sqrt{5}}{1-\sqrt{5}}\right)$ $\left(1, -\frac{\sqrt{5}+3}{2}, 2\frac{3+\sqrt{5}}{1+\sqrt{5}}\right)$	$A_{24} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	2 i $-i$	$(1, 1, 1)$ $\left(1, -\frac{1+i}{2}, -(1-i)\right)$ $\left(1, -\frac{1-i}{2}, -(1+i)\right)$
$A_{25} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$	2 1 -1	$(1, 1, 1)$ $(0, 1, 0)$ $(1, 1, -2)$	$A_{26} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	2 1 -1	$(1, 1, 1)$ $(0, 1, 0)$ $(-2, 1, 4)$
$A_{27} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	2 $0^2_\star$	$(1, 1, 1)$ $(0, 0, 1)$	$A_{28} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 $0^2_\star$	$(1, 1, 1)$ $(0, 0, 1)$
$A_{29} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 1 -1	$(1, 1, 1)$ $(0, 1, 1)$ $(0, 1, -1)$	$A_{30} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$	2 $-\sqrt{2}$ $\sqrt{2}$	$(1, 1, 1)$ $(0, 1, \sqrt{2})$ $(0, 1, -\sqrt{2})$
$A_{31} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 1 0	$(1, 1, 1)$ $(0, 1, 1)$ $(0, 0, 1)$	$A_{32} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	2 0 1	$(1, 1, 1)$ $(0, 1, 2)$ $(0, 0, 1)$
$A_{33} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	2 1 0	$(1, 1, 1)$ $(0, 1, 0)$ $(0, 0, 1)$	$A_{34} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$	2 1 0	$(1, 1, 1)$ $(0, 1, 0)$ $(0, -1, 1)$

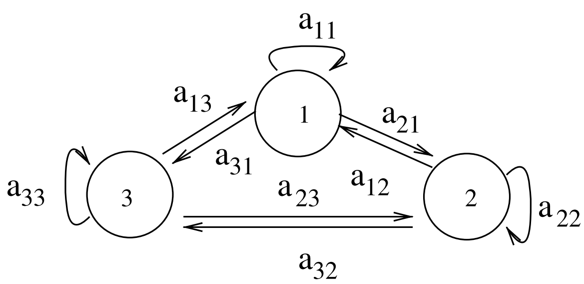

Figure 7. The graph associated with a homogeneous three-cell network.

Thus, the possible choices for A are A_1, A_2,

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

However, both B_1 and B_2 are conjugate to A_1.

Networks with self-coupling. Without loss of generality, we assume cell 1 is self-coupled. Then

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

If $a_{21} = a_{31} = 0$, then cell 1 is disconnected from cells 2 and 3. After a conjugacy, if needed, we can assume $a_{21} = 1$. There are three possible adjacency matrices: A_3 and

$$B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B_4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Matrix B_3 is conjugate to A_3 and matrix B_4 is conjugate to A_4. □

2.2. *Networks with two input arrows*

Theorem 2.4. *Up to isomorphism there are 38 connected homogeneous three-cell networks where the number of input arrows in each cell is 2. Those networks are shown in figures 5(5–34) and 8.*

Remark 2.5. In section 2.4 we will show that networks listed in figure 8 are redundant. ♢

Proof. First, we suppose that the networks have all arrows double. The proof is the same as in theorem 2.3. The resulting networks are shown in figures 8(39–42).

Next, we assume that at least one arrow is single. As before we divide the proof into two cases: networks without self-coupling and networks with self-coupling.

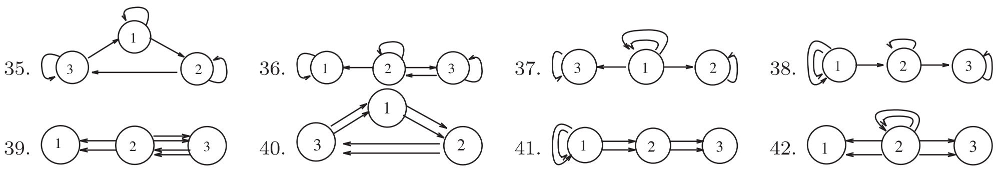

Figure 8. Eight of 38 networks classified in theorem 2.4.

Networks without self-coupling. In this case the diagonal entries of A are equal to zero. At least one cell receives two single arrows and we may assume that cell is cell 3. So, A has the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ 1 & 1 & 0 \end{bmatrix}.$$

If all arrows are single the adjacency matrix is A_8. So one cell receives a double arrow, which we may assume is cell 1. There are six possible choices for A: A_{11}, A_{13}, A_{14}, A_{15} and

$$B_5 = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B_6 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Matrix B_5 is conjugate to A_{11} and matrix B_6 is conjugate to A_{14}.

Networks with self-coupling. When there exist self-coupling in the network we study four cases: one cell with two self-coupling inputs and no other self-coupled cell, one cell with single self-coupling input and no other self-coupled cell, two self-coupled cells and all cells with self-coupling input.

(a) *One cell with two self-coupling inputs and no other self-coupled cell.* We can assume that the double self-coupling is in cell 1. Since there is a single arrow, we may assume that cell 2 receives a single arrow. Thus

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ a_{31} & a_{32} & 0 \end{bmatrix}.$$

There are three possible choices for A: A_{29}, A_{30} and

$$B_7 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Matrix B_7 is conjugate to A_{28}.

(b) *One cell with single self-coupling input and no other self-coupled cell.* We assume that cell is cell 1. Either $a_{12} = 1$ or $a_{13} = 1$. Without loss of generality we may assume $a_{12} = 1$. Hence

$$A = \begin{bmatrix} 1 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}.$$

There are six possible valency 2 choices for A: A_5, A_{17}, A_{18}, A_{19}, A_{20} and A_{23}.

(c) *Two self-coupled cells.* Without loss of generality, we assume those cells are cells 1 and 2. Then, the possible choices for A are

$$C = \begin{bmatrix} 2 & 0 & 0 \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \qquad B_8 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}.$$

The matrix B_8 is conjugate to A_{12}. The possible choices of C are A_9, A_{31}, A_{32}, A_{33}, A_{34} and the disconnected graph

$$B_9 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

There are 12 possible choices for D: A_6, A_7, A_{24}, A_{25}, A_{26}, A_{27} and

$$B_{10} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \qquad B_{11} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B_{12} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$B_{13} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \qquad B_{14} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \qquad B_{15} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Matrix B_{12} is conjugate to B_{11}, which is conjugate to A_{10}. Matrices B_{10}, B_{14} are conjugate to A_{25}, A_{26}. Matrices B_{13}, B_{15} are conjugate to A_{24}, A_6.

(d) *All cells are self-coupled.* First assume that there are no double self-coupling arrows. Two different cells must be coupled and we can assume that cell 2 is coupled to cell 1.
There are four possible choices of A:

$$B_{16} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad B_{17} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B_{18} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad B_{19} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The network associated with B_{16} is isomorphic to the one given in figure 8(35). Networks associated with B_{17}, B_{18}, B_{19} are isomorphic to the network shown in figure 8(36).
Next assume that precisely one cell, namely cell 1, is double self-coupled. There are again four possible choices for A:

$$B_{20} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad B_{21} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad B_{23} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The network corresponding to B_{20} is given by figure 8(37). Matrices B_{21} and B_{22} represent, up to permutation of cells 2 and 3, the network shown in figure 8(38). The network corresponding to B_{23} is disconnected.

Finally, we may assume that at least two cells, cells 1 and 2, are double self-coupled. Then A has the form

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where a_{33} is nonzero. So either $a_{31} = 0$ or $a_{32} = 0$ and the network is disconnected. $\square$

2.3. *Admissible vector fields*

Stewart *et al* [13, 16] show that a unique class of coupled cell systems can be associated with every network architecture (directed graph) and the differential equations in this class are called *admissible*. Abstractly coupled cell systems consist of all vector fields that commute with the symmetry groupoid of the graph. However, for homogeneous networks with one kind of coupling (the kinds of networks we consider) this identification is straightforward.

The phase space for each network in figure 5 is $P = (\boldsymbol{R}^k)^3$, where k can be any positive integer. We call k the *dimension of the internal dynamics* of a cell. The coupled cell systems associated with such networks have the form $\dot{X} = F(X)$, where $X = (x_1, x_2, x_3) \in P$ and the three coordinate functions of F have a special structure. For valency 1 networks these vector fields have the form

$$\dot{x}_i = f(x_i, x_j), \tag{2.2}$$

where $f : (\boldsymbol{R}^k)^2 \to \boldsymbol{R}^k$ and j is the unique cell coupled to cell i. For valency 2 networks

$$\dot{x}_i = f(x_i, \overline{x_j, x_l}), \tag{2.3}$$

where $f : (\boldsymbol{R}^k)^3 \to \boldsymbol{R}^k$ and j, l are the cells coupled to cell i. The bar over the second and third coordinates in (2.3) indicates that $f(u, v, w) = f(u, w, v)$ and reflects the fact that there is just one type of coupling. In both cases f is smooth. It is now straightforward to give the form of the admissible vector fields for each network in figure 5 and these vector fields are listed in table 2.

2.4. *Enumeration up to ODE-equivalence*

The space of admissible vector fields for two different networks can be identical [13], and hence the dynamics and bifurcations in these networks are the same. More generally, two networks are called *ODE-equivalent* if after a permutation of cells the spaces of admissible vector fields are identical. In proposition 2.8 we show that each of the three-cell networks in figure 8 is ODE-equivalent to a network in figure 5 and is hence redundant.

Dias and Stewart [5] call two networks *linearly-equivalent* if after a permutation of cells the corresponding vector spaces of linear admissible vector fields are identical. Moreover, theorem 7.1 in [5] states the following.

Theorem 2.6. *Two networks are ODE-equivalent if and only if they are linearly equivalent.*

Remark 2.7. Corollary 7.9 in [5] states that linear equivalence needs to be verified just for the case when $k = 1$. $\lozenge$

Table 2. Admissible systems for networks in figure 5.

#	Equations	#	Equations	#	Equations
1	$\dot{x}_1 = f(x_1, x_2)$ $\dot{x}_2 = f(x_2, x_1)$ $\dot{x}_3 = f(x_3, x_2)$	2	$\dot{x}_1 = f(x_1, x_3)$ $\dot{x}_2 = f(x_2, x_1)$ $\dot{x}_3 = f(x_3, x_2)$	3	$\dot{x}_1 = f(x_1, x_1)$ $\dot{x}_2 = f(x_2, x_1)$ $\dot{x}_3 = f(x_3, x_2)$
4	$\dot{x}_1 = f(x_1, x_2)$ $\dot{x}_2 = f(x_2, x_2)$ $\dot{x}_3 = f(x_3, x_2)$	5	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	6	$\dot{x}_1 = f(x_1, \overline{x_1, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$
7	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	8	$\dot{x}_1 = f(x_1, \overline{x_2, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	9	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_2, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$
10	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_2, x_3})$	11	$\dot{x}_1 = f(x_1, \overline{x_2, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	12	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_3, x_3})$
13	$\dot{x}_1 = f(x_1, \overline{x_2, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_1})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	14	$\dot{x}_1 = f(x_1, \overline{x_3, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_1})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	15	$\dot{x}_1 = f(x_1, \overline{x_3, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_3, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$
16	$\dot{x}_1 = f(x_1, \overline{x_3, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_3, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	17	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_3, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	18	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$
19	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_3, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$	20	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_3, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_2, x_2})$	21	$\dot{x}_1 = f(x_1, \overline{x_1, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_3, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$
22	$\dot{x}_1 = f(x_1, \overline{x_1, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_1})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$	23	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_2, x_2})$	24	$\dot{x}_1 = f(x_1, \overline{x_1, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_2, x_2})$
25	$\dot{x}_1 = f(x_1, \overline{x_1, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_2, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$	26	$\dot{x}_1 = f(x_1, \overline{x_1, x_3})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$	27	$\dot{x}_1 = f(x_1, \overline{x_1, x_2})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$
28	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_1})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	29	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	30	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_2, x_2})$
31	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_2})$	32	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_2, x_2})$	33	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_2})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$
34	$\dot{x}_1 = f(x_1, \overline{x_1, x_1})$ $\dot{x}_2 = f(x_2, \overline{x_2, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_1})$	—	—	—	—

Using the results of Dias and Stewart we prove the following proposition.

Proposition 2.8. *The following triples of networks in figures 5 and 8 are ODE-equivalent:* $(1, 36, 39)$, $(2, 35, 40)$, $(3, 38, 41)$, $(4, 37, 42)$.

Proof. Using remark 2.7, we set $k = 1$. Observe that, after permuting cells 1 and 3 in network 1, the linear admissible vector fields for the four triples $(1, 36, 39)$, $(2, 35, 40)$, (3, 38, 41), (4, 37, 42) have, respectively, the form

$$(\alpha x_1 + \beta x_2, \alpha x_2 + \beta x_3, \alpha x_3 + \beta x_2),$$
$$(\alpha x_1 + \beta x_3, \alpha x_2 + \beta x_1, \alpha x_3 + \beta x_2),$$
$$(\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_1, \alpha x_3 + \beta x_2),$$
$$(\alpha x_1 + \beta x_2, \alpha x_2 + \beta x_2, \alpha x_3 + \beta x_2)$$

where $\alpha, \beta \in \boldsymbol{R}$. It follows from theorem 2.6 that each triple consists of ODE-equivalent networks. □

3. Codimension one bifurcations

For every homogeneous coupled cell system, the diagonal subspace

$$\Delta = \{(x, x, x) : x \in \boldsymbol{R}^k\} \subset (\boldsymbol{R}^k)^3 \tag{3.1}$$

is flow-invariant. Moreover, for any homogeneous network the class of admissible vector fields restricted to $\triangle$ is the set of all vector fields on $\triangle$. Thus, it is reasonable to assume that there exists a synchronous equilibrium in $\triangle$, which we may assume, after a change of coordinates, is at the origin.

Let $F : (\boldsymbol{R}^k)^3 \times \boldsymbol{R} \to (\boldsymbol{R}^k)^3$ be an admissible vector field depending on a bifurcation parameter λ. Let $J = (dF)_{(0,0)}$ and $J^c = J|E^c$, where E^c denotes the centre subspace. In this section, using the Jordan normal form of J^c, we classify the types of local codimension one bifurcations that can occur from a synchronous equilibrium in homogeneous three-cell networks.

Codimension one bifurcations divide into *steady-state* (J^c has a zero eigenvalue) and *Hopf* bifurcation (J^c has purely imaginary eigenvalues). Each of these bifurcation types divide into *synchrony-preserving* ($E^c \subset \triangle$) and *synchrony-breaking* ($E^c \not\subset \triangle$). Given one of the homogeneous three-cell networks in figure 5, we address the following: classify the generic codimension one bifurcations from a synchronous equilibrium within the class of admissible vector fields for that network.

Note that codimension one synchrony-preserving bifurcations are easily classified. Since the restriction of the general F to Δ is the general vector field on Δ, the only codimension one synchrony-preserving steady-state bifurcation is a saddle node bifurcation and the only codimension one synchrony-preserving Hopf bifurcation is a standard simple eigenvalue Hopf bifurcation. The new steady states and periodic solutions that emanate from these bifurcations are themselves synchronous solutions. For the remainder of this paper we focus on synchrony-breaking bifurcations from a synchronous equilibrium.

In this section we group networks by eigenvalue type of the linearized coupled cell system J. Propositions 3.3 and 3.1 show that when $k = 1$ there are asymmetric networks for which J is forced to have complex eigenvalues and multiple eigenvalues (in some cases J is nilpotent). Thus, there are features of network architecture not related to symmetry that constrain the coupled cell system at linear level and hence the associated synchrony-breaking bifurcations.

We classify bifurcation types by the eigenvalue structure of J^c. This is accomplished in three steps. First, in proposition 3.1, we relate the eigenvalues of J and their associated eigenvectors to those of the adjacency matrix A. Second, we discuss the eigenvalue structure of A. Finally, we show that there are four types of synchrony-breaking steady-state bifurcation (S1–S4) and five types of synchrony-breaking Hopf bifurcation (H1–H5). See proposition 3.3 and the classification of bifurcations that follows it.

The computation of the eigenvalues of J is best done using tensor products. Observe that the state space of a three-cell homogeneous network is $\boldsymbol{R}^{3k} = \boldsymbol{R}^k \otimes \boldsymbol{R}^3$, where $\boldsymbol{R}^k$ is the phase space of internal dynamics for each cell and 3 is the number of cells.

Let f be defined as in (2.2) and (2.3). Let $Q = (d_{x_i} f)_0$ be the linearized internal dynamics and let $R = (d_{x_j} f)_0 = (d_{x_l} f)_0$ be the linearized coupling. Note that Q and R are $k \times k$ matrices. Using tensor product notation

$$J = Q \otimes I + R \otimes A, \tag{3.2}$$

where I is the 3×3 identity matrix. Denote the three eigenvalues of A by μ_1, μ_2, μ_3, where μ_1 corresponds to the synchrony eigenvector $(1, 1, 1) \in \triangle$ and is equal to the valency of the network.

Proposition 3.1. *The eigenvalues of J are the union of the eigenvalues of the three $k \times k$ matrices $Q + \mu_j R$, $j = 1, 2, 3$, including algebraic multiplicity. The eigenvectors of J are the vectors $u \otimes w$, where $u \in \boldsymbol{C}^k$ is an eigenvector of Q and $w \in \boldsymbol{C}^3$ is an eigenvector of A.*

Proof. Suppose $\mu \in \boldsymbol{C}$ is an eigenvalue of A with eigenvector $w \in \boldsymbol{C}^3$. Let

$$Y_w = \{u \otimes w : u \in \boldsymbol{C}^k\}.$$

We claim that the subspace $Y_w \subset \boldsymbol{C}^k \otimes \boldsymbol{C}^3$ is J-invariant and that $J|Y_w = Q + \mu_j R$. It then follows that the k eigenvalues of $Q + \mu_j R$ are eigenvalues of J. To verify the claim, calculate

$$J(u \otimes w) = Qu \otimes Iw + Ru \otimes Aw = Qu \otimes w + \mu Ru \otimes w = (Q + \mu R)u \otimes w.$$

Therefore, $J|Y_w$ is the matrix $Q + \mu R$.

Next observe that if $w_1, w_2, w_3 \in \boldsymbol{C}^3$ are linearly independent, then

$$\boldsymbol{C}^k \otimes \boldsymbol{C}^3 = Y_{w_1} \oplus Y_{w_2} \oplus Y_{w_3}. \tag{3.3}$$

It follows from (3.3) that if the eigenvalues μ_j have a complete set of eigenvectors, then the algebraic multiplicity associated with each eigenvalue μ_j contributes to the algebraic multiplicity of the eigenvalues of J.

This last comment proves the theorem for all networks except those for which $\mu_2 = \mu_3$ and the geometric multiplicity of μ_2 is one. In such a case, let w_2 be an eigenvector and w_3 be a generalized eigenvector such that $Aw_3 = w_2 + \mu_2 w_3$. A calculation shows that $Y_{w_2} \oplus Y_{w_3}$ is J-invariant and that $J|Y_{w_2} \oplus Y_{w_3}$ has the matrix form

$$\begin{bmatrix} Q + \mu_2 R & R \\ 0 & Q + \mu_2 R. \end{bmatrix}. \tag{3.4}$$

Hence, each eigenvalue of $Q + \mu_2 R$ appears with multiplicity 2 in J. □

Remark 3.2. Suppose that μ_2 is an eigenvalue of A with algebraic multiplicity 2 and geometric multiplicity 1. Then (3.4) shows that the Jordan normal form associated with one of the eigenvalues of $Q + \mu_2 R$ need not be nilpotent, though it will be nilpotent for a generic choice of the coupling matrix R.

Proposition 3.3 groups matrices A by eigenvalue type; its proof follows from table 1.

Proposition 3.3. *Networks in figure 5 can be grouped as follows:*

(a) $\mu_1 \in \boldsymbol{R}$ is double with a complete set of eigenvectors: network 12.
(b) $\mu_3 = \mu_2 \in \boldsymbol{R}$ with a complete set of eigenvectors outside Δ: 4, 7, 8.
(c) $\mu_3 = \mu_2 \in \boldsymbol{R}$ with an incomplete set of eigenvectors: 3, 6, 11, 27, 28.
(d) $\mu_3 = \overline{\mu}_2 \in \boldsymbol{C}$ with eigenvectors outside Δ: 2, 14, 18, 19, 24.
(e) Three real unequal eigenvalues: all remaining networks.

Next we classify synchrony-breaking steady-state and Hopf codimension one bifurcations for the admissible systems of differential equations shown in table 2.

3.1. Steady-state bifurcation

The eigenvalues of each of the three $k \times k$ matrices $B_j = Q + \mu_j R$ are generically simple. So the possible steady-state bifurcation types do not depend on k, and we may assume $k = 1$. In this case the 3×3 matrix J has three eigenvalues $\gamma_j = Q + \mu_j R$, where Q and R are 1×1 matrices, and γ_1 corresponds to the synchrony eigenvector $(1, 1, 1) \in \Delta$. Thus, for the synchrony-breaking steady-state bifurcations, either γ_2 or γ_3 equals zero, and the possible generic steady-state bifurcation types are divided into

- **S1** simple eigenvalues: networks 1, 5, 9, 10, 11, 13, 15–17, 20–23, 25, 26, 29–34,
- **S2** double eigenvalues and two eigenvectors: networks 4, 7, 8,
- **S3** double eigenvalues and one eigenvector: networks 3, 6, 11, 27, 28,
- **S4** double eigenvalue and one eigenvector in Δ: network 12.

It follows from proposition 3.3 (d) that there are no codimension one steady-state synchrony-breaking bifurcations in networks, 2, 14, 18, 19, 24, since the synchrony-breaking eigenvalues of the Jacobian are generically complex for these networks.

3.2. Hopf bifurcations

We begin by considering the minimum value of k for which Hopf bifurcations occur. When $k = 1$ it follows from proposition 3.3 that Hopf bifurcation can occur only when $\gamma_3 = \overline{\gamma}_2$ is purely imaginary. Next we consider $k = 2$. Purely imaginary eigenvalues can occur in cases (a, b, c, d) of proposition 3.3. In summary the possible generic Hopf bifurcation types are

- **H1** simple complex eigenvalues for A: networks 2, 14, 18, 19, 24,
- **H2** simple real eigenvalues for A: networks 1, 5, 9, 10, 12, 13, 15–17, 20–23, 25, 26, 29–34,
- **H3** double eigenvalues and two eigenvectors: networks 4, 7, 8,
- **H4** double eigenvalues and one eigenvector: networks 3, 6, 11, 27, 28,
- **H5** double eigenvalue and one eigenvector in Δ: network 12,

An argument similar to the one used for steady-state bifurcations shows that when $k > 2$, generically, no new types are found.

4. Codimension one steady-state bifurcations

In this section we classify the local steady-state synchrony-breaking codimension one bifurcations from a synchronous equilibrium that occur in the networks listed in figure 5. We do this by considering each of the bifurcation types S1–S4 identified in section 3.1.

In this section we also assume that the dimension of the internal dynamics of each cell is $k = 1$. As noted in section 3.1, this assumption will not change the classification of codimension one bifurcations, though it can affect (in standard ways) the discussion of the stability of bifurcating solutions. We return to this point below. For each bifurcation type we address the following questions.

1. How many new branches of solutions arise?
2. What is the stability of solutions on each new branch?
3. How do the new states manifest themselves in each individual cell?

4.1. General background

In this section we provide some general background on the steady-state bifurcations we consider. We discuss trivial equilibria, generalities concerning the stability of solutions and why branches of equilibria can have different properties when viewed in different cells.

Recall from table 2 that the valency 1 networks are defined by a single function f that depends on the internal cell variables, which we denote by u, and the coupling variables from one cell, which we denote by v. For the bifurcation problems considered in the remainder of this paper, we write

$$f = f(u, v, \lambda),$$

where λ is the bifurcation parameter. For valency 2 networks, f also depends on the coupling variables from a second cell, which we denote by w. Thus

$$f = f(u, v, w, \lambda),$$

where the fact that the two couplings are assumed identical leads to the identity

$$f(u, v, w, \lambda) = f(u, w, v, \lambda). \tag{4.1}$$

Identity (4.1) forces some partial derivatives of f to be equal at the origin. For example, $f_v(0) = f_w(0)$.

The existence of a trivial branch. There is a trivial branch of equilibria at every synchrony-breaking codimension one bifurcation except for network 11. To see this write the three-cell system in the general form

$$\dot{X} = F(X, \lambda), \tag{4.2}$$

where $\lambda \in \boldsymbol{R}$ is a bifurcation parameter and $X = (x_1, x_2, x_3) \in (\boldsymbol{R}^k)^3$. We assume that (4.2) has a codimension one steady-state bifurcation from a synchronous equilibrium at $\lambda = 0$ which, after an affine linear change of coordinates, we can assume is at the origin. So $J = (dF)_{0,0}$ has a zero eigenvalue. It follows from proposition 3.3 that, except for network 12 considered in section 4.5, the intersection of the centre subspace E^c of J with the synchrony subspace Δ is the origin. Hence, $J|\Delta$ is nonsingular, and the implicit function theorem implies that there is a unique branch of synchronous equilibria parametrized by λ, which, after a λ-dependent affine linear change in coordinates, we can assume is at $X = 0$. Thus, we assume

$$F(0, \lambda) \equiv 0. \tag{4.3}$$

Stability. There are three issues that need to be discussed in order to see that the analysis of linearized stability of equilibria for $k > 1$ is determined by the analysis for $k = 1$. First, we showed in proposition 3.1 that the eigenvalues of the $(3k) \times (3k)$ Jacobian matrix J are the eigenvalues of three $k \times k$ block matrices. In some networks the eigenvalues of two of the three blocks are forced to be equal, thus leading to the various multiple eigenvalue bifurcation problems. Nevertheless, the eigenvalues within each block are themselves arbitrary (depending only on the linearized internal dynamics and the linearized coupling). It follows that in codimension one bifurcations degeneracies come from multiple eigenvalues forced by equality of eigenvalues in different blocks, and these degeneracies are present when $k = 1$.

Second, as with all bifurcation problems, the bifurcation analysis only keeps track of the movement of critical eigenvalues, and this analysis is the same for all k. Thus, the stability of bifurcating solutions is determined by the signs of the real parts of the noncritical eigenvalues of the Jacobian matrix at the bifurcation point (which is arbitrary) and the detailed way in which the critical eigenvalues of the Jacobian matrix along the bifurcating solutions change (which is a form of exchange of stability).

Third, as we will discuss in section 4.2, when $k = 1$, the coupled cell structure forces an ordering of the eigenvalues of J and that ordering forces unstable equilibria for certain bifurcations. This ordering is not present when $k > 1$.

This discussion shows that when determining the stability characteristics of bifurcating equilibria, we can still assume that $k = 1$.

Manifestation of bifurcating equilibria in different cells. There are two issues that we need to discuss. First, network architecture can force certain flow-invariant polydiagonals [13, 16]. When bifurcating equilibria are forced to lie within these subspaces, cell coordinates in two different cells will be forced to be equal; that is, partial synchrony remains in the bifurcating equilibria.

Second, the rate in λ at which the bifurcating equilibria in synchrony-breaking codimension one bifurcations deviate from the synchronous equilibria is different in different cells, and these rates depend only on network architecture and not on the specific system. This phenomenon was already noted in synchrony-breaking Hopf bifurcation in network 3 in [6, 7]. The questions of partial synchrony and growth rate of bifurcating solutions need to be resolved separately for each network.

For homogeneous three-cell systems the question of partial synchrony is easy to resolve using the notion of balanced colourings presented in [13, 16]. We now discuss the question of how we measure the deviation of the bifurcating state from the synchronous state. For S1–S3 bifurcations we can assume that the branch of trivial synchronous equilibria is at the origin. Thus, in these systems, deviation of the bifurcating synchrony-broken state from the synchronous state can be measured by deviation of the bifurcating state from zero, and this deviation can be measured in each cell.

There are three network specific observations that impact how the deviation rates depend on the cells. First, the type of bifurcation (transcritical, pitchfork, etc) plays a role. For example, suppose we denote a transcritical branch of bifurcating equilibria by $X(\lambda)$, where

$$X(\lambda) = \lambda C p + O(\lambda^2), \tag{4.4}$$

$p \in \boldsymbol{R}^3$ is a critical eigenvector of J and $C \in \boldsymbol{R}$ is a constant that depends on the particular choice of differential equation. As we have seen, the eigenvector p is dictated by the network and the type of bifurcation (in this case simple eigenvalue); *p does not* depend on the specific choice of the associated differential equations. So we expect that the solution branch will deviate from synchrony at rate λ, and that we should see this deviation in each cell. In particular, if the coordinate p_j of p, which corresponds to cell j, is nonzero, then the rate of deviation of the two solutions will be of order λ when viewed in cell j.

However, if $p_j = 0$ (a network issue), then we would expect that the rate of deviation in cell j will be of order λ^2. This happens in some networks, but in other networks yet another phenomenon is possible. In feed-forward type networks the subpace $p_j = 0$ can be flow-invariant and when that happens the deviation between the synchronous and the nonsynchronous solutions will be identically zero in cell j.

For pitchfork bifurcations the branch of solutions has the form

$$X(\lambda) = \lambda^{1/2} C p + O(\lambda). \tag{4.5}$$

So again, if a cell coordinate of p is nonzero, the deviation is at rate $\lambda^{1/2}$, whereas if a cell coordinate of p is zero, then the deviation rate is of order λ. As in the case of transcritical bifurcations, the deviation can in fact be identically zero. Bifurcation of type S3 in network 27 can lead to a exotic rate of $\lambda^{1/4}$ in cell 3.

We present the results by grouping networks according to proposition 3.3. The answers are given in tables 3 and 4. The information in these tables includes the type of bifurcation that can occur, the critical eigenvalue at the bifurcation and associated eigenvector, the type of branches of equilibria and the type of partial synchrony, if any. Synchrony-breaking bifurcations are classified in sections 4.2 (S1), 4.3 (S2), 4.4 (S3), and 4.5 (S4). We note that each S1 bifurcation leads to a unique branch of solutions, which can be either transcritical or pitchfork. Each S2 bifurcation leads to three possible nontrivial branches of solution, and each S3 bifurcation leads to two nontrivial branches.

4.2. S1: simple eigenvalue

In this section we classify the S1 bifurcations, those bifurcations for which the critical eigenvalue is simple. These bifurcations lead to unique bifurcating branches that are either transcritical if no symmetry is present or of pitchfork type if symmetry is present. For each network that can have an S1 bifurcation (see section 3), we list in table 3 the networks, the eigenvalues and the type of bifurcation.

Networks with $\mathbf{Z}_2$ symmetry have pitchfork bifurcations precisely when that symmetry acts as -1 on the kernel of J. When the action is trivial transcritical bifurcations are found. The simple eigenvalues of S1 bifurcations imply that the reduced equation obtained by applying Liapunov–Schmidt reduction to the coupled cell system has the general form

$$K(y,\lambda) = 0, \tag{4.6}$$

where $y \in \mathbf{R}$, $K : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is smooth, $K_y(0) = 0$, and $K(0,\lambda) = 0$ (since the trivial solution is at $X = 0$). We must solve $K(y,\lambda) = 0$ near the origin.

Existence of transcritical bifurcation. Consider any bifurcation in table 3 that is listed as having a transcritical bifurcation. We claim that generically these systems satisfy

$$K_{yy}(0) \neq 0 \neq K_{y\lambda}(0) \tag{4.7}$$

from which it follows that transcritical bifurcation occurs.

To compute $K_{y\lambda}(0)$ and $K_{yy}(0)$ we use the formulaes for the derivatives of the Liapunov–Schmidt reduced equation given, for example, in [9, Chapter 1 p 33]. Since $K(0,\lambda) = 0$, it follows that

$$K_{yy}(0) = \langle v_0^*, d^2F(v_0, v_0)\rangle \text{ and } K_{y\lambda}(0) = \langle v_0^*, dF_\lambda(v_0)\rangle, \tag{4.8}$$

where $v_0 \in \ker J$ and $v_0^* \in (\operatorname{range} J)^\perp$.

We illustrate the calculations by considering network 1 when $f_u(0)$ is the critical eigenvalue. The form of F in network 1 is given by

$$F(X,\lambda) \equiv (f(x_1, x_2, \lambda), f(x_2, x_1, \lambda), f(x_3, x_2, \lambda)). \tag{4.9}$$

Since $f_u(0) = 0$ it follows that

$$J = f_v(0)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Observe that $v_0 = (0, 0, 1)$ and $v_0^* = (-1, 0, 1)$. A straightforward calculation using (4.9) and the formula for $K_{y\lambda}(0)$ in (4.8) shows that

$$K_{y\lambda}(0) = f_{u\lambda}(0),$$

which is generically nonzero.

Table 3. Coupled cell systems for networks in figure 5 that can have S1 bifurcations. Generically the solutions on the pitchfork and transcritical grow, respectively, with standard rate of $\lambda^{1/2}$ and λ. If the eigenvector has a 0 component and the corresponding cell is identically 0, then that component is denoted by $0^\star$. Otherwise the component grows with rate λ and λ^2 when the branch is pitchfork and transcritical, respectively.

Net	Eigenvalues	Eigenvectors	Bifurcation type	Synchrony
1	$f_u(0)$	$(0^\star, 0^\star, 1)$	transcritical	$x_1 = x_2$
	$f_u(0) - f_v(0)$	$(1, -1, 1)$	pitchfork	$x_1 = x_3$
5	$f_u(0) - f_v(0)$	$(1, -2, 1)$	transcritical	$x_1 = x_3$
	$f_u(0)$	$(-1, 1, 1)$	pitchfork	$x_2 = x_3$
9	$f_u(0) + \dfrac{1+\sqrt{5}}{2} f_v(0)$	$(0^\star, 2, -1+\sqrt{5})$	transcritical	—
	$f_u(0) + \dfrac{1-\sqrt{5}}{2} f_v(0)$	$(0^\star, 2, -1-\sqrt{5})$	transcritical	—
10	$f_u(0) - f_v(0)$	$(1, -2, 1)$	transcritical	$x_1 = x_3$
	$f_u(0) + f_v(0)$	$(1, 0, -1)$	pitchfork	
12	$f_u(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
13	$f_u(0)$	$(0^\star, 0^\star, 1)$	transcritical	$x_1 = x_2$
	$f_u(0) - 2f_v(0)$	$(1, -1, 0)$	pitchfork	
15	$f_u(0) - 2f_v(0)$	$(1, 1, -1)$	transcritical	$x_1 = x_2$
	$f_u(0)$	$(1, -1, 0)$	pitchfork	
16	$f_u(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
	$f_u(0) - f_v(0)$	$(1, 1, -2)$	transcritical	$x_1 = x_2$
17	$f_u(0)$	$(-1, 1, 0)$	transcritical	
	$f_u(0) - f_v(0)$	$(1, -2, 1)$	transcritical	$x_1 = x_3$
20	$f_u(0) + f_v(0)$	$(1, 0^\star, 0^\star)$	transcritical	$x_2 = x_3$
	$f_u(0) - 2f_v(0)$	$(1, -3, 3)$	pitchfork	
21	$f_u(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
	$f_u(0) - f_v(0)$	$(1, 4, -2)$	transcritical	
22	$f_u(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
	$f_u(0) - f_v(0)$	$(-1, 2, 2)$	transcritical	$x_2 = x_3$
23	$f_u(0) - \dfrac{1-\sqrt{5}}{2} f_v(0)$	$\left(1, \dfrac{\sqrt{5}-3}{2}, 2\dfrac{3-\sqrt{5}}{1-\sqrt{5}}\right)$	transcritical	—
	$f_u(0) - \dfrac{1+\sqrt{5}}{2} f_v(0)$	$\left(1, -\dfrac{\sqrt{5}+3}{2}, 2\dfrac{3+\sqrt{5}}{1+\sqrt{5}}\right)$	transcritical	—
25	$f_u(0) + f_v(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
	$f_u(0) - f_v(0)$	$(1, 1, -2)$	transcritical	$x_1 = x_2$
26	$f_u(0) + f_v(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
	$f_u(0) - f_v(0)$	$(-2, 1, 4)$	transcritical	
29	$f_u(0) + f_v(0)$	$(0^\star, 1, 1)$	transcritical	$x_2 = x_3$
	$f_u(0) - f_v(0)$	$(0^\star, 1, -1)$	pitchfork	
30	$f_u(0) + \sqrt{2} f_v(0)$	$(0^\star, 1, \sqrt{2})$	transcritical	—
	$f_u(0) - \sqrt{2} f_v(0)$	$(0^\star, 1, -\sqrt{2})$	transcritical	—
31	$f_u(0) + f_v(0)$	$(0^\star, 1, 1)$	transcritical	$x_2 = x_3$
	$f_u(0)$	$(0^\star, 0^\star, 1)$	transcritical	$x_1 = x_2$
32	$f_u(0) + f_v(0)$	$(0^\star, 1, 2)$	transcritical	
	$f_u(0)$	$(0^\star, 0^\star, 1)$	transcritical	$x_1 = x_2$
33	$f_u(0) + f_v(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
	$f_u(0)$	$(0^\star, 0^\star, 1)$	transcritical	$x_1 = x_2$
34	$f_u(0) + f_v(0)$	$(0^\star, 1, 0^\star)$	transcritical	$x_1 = x_3$
	$f_u(0)$	$(0^\star, -1, 1)$	transcritical	

Table 4. Form of asynchronous branches of equilibria for S2–S4 bifurcation type. The nonzero terms in x_1, x_2, x_3 indicated in last column are the approximation at lowest order in λ.

Bifurcation type	Net	Zero Ev	Eigenvectors	Synchrony	Growth rates		
S2	4	$f_u(0)$	$(0, 0, 1)$	$x_1 = x_2$	0	0	λ
			$(1, 0, 0)$	$x_2 = x_3$	λ	0	0
				$x_1 = x_3$	λ	0	λ
	7	$f_u(0)$	$(0, 0, 1)$	$x_1 = x_2$	0	0	λ
			$(1, -1, 0)$		$\lambda^{1/2}$	$\lambda^{1/2}$	$\lambda^{1/2}$
					$\lambda^{1/2}$	$\lambda^{1/2}$	$\lambda^{1/2}$
	8	$f_u(0) - f_v(0)$	$(1, -1, 0)$	$x_1 = x_2$	λ	λ	λ
			$(0, 1, -1)$	$x_1 = x_3$	λ	λ	λ
				$x_2 = x_3$	λ	λ	λ
S3	3, 28	$f_u(0)$	$(0, 0, 1)$	$x_1 = x_2$	0	0	λ
					0	λ	$\lambda^{1/2}$
	6	$f_u(0)$	$(1, -1, -1)$	$x_2 = x_3$	$\lambda^{1/2}$	$\lambda^{1/2}$	$\lambda^{1/2}$
					λ	λ	λ
	11	$f_u(0) - f_v(0)$	$(2, -1, -1)$	$x_2 = x_3$	λ	λ	λ
					λ	λ	λ
	27	$f_u(0)$	$(0, 0, 1)$	$x_1 = x_2$	0	0	λ
					$\lambda^{1/2}$	$\lambda^{1/2}$	$\lambda^{1/4}$
S4	12	$f_u(0) + 2f_v(0)$	$(1, 1, 1)$	$x_1 = x_2 = x_3$	$\lambda^{1/2}$	$\lambda^{1/2}$	$\lambda^{1/2}$
			$(1, 0, -1)$		$\lambda^{1/2}$	λ	$\lambda^{1/2}$

Next we calculate $K_{yy}(0)$. Since $v_0^* = (-1, 0, 1)$, it follows that

$$K_{yy}(0) = -d^2F_1(v_0, v_0) + d^2F_3(v_0, v_0). \tag{4.10}$$

Let $\rho = (\rho_1, \rho_2, \rho_3)$ and $\nu = (\nu_1, \nu_2, \nu_3)$ be vectors in $\boldsymbol{R}^3$. By definition

$$d^2F_l(\rho, \nu) = \sum_{i,j=1}^{3} \frac{\partial^2 F_l}{\partial x_i \partial x_j}(0)\rho_i \nu_j \tag{4.11}$$

with an analogous formula holding for $d^3F_l(\rho, \nu, \eta)$. Using (4.11) we calculate $d^2F_l(v_0, v_0)$ and substitute the result into (4.10). Thus

$$K_{yy}(0) = f_{uu}(0),$$

which is also generically nonzero.

Existence of pitchfork bifurcation. Consider the coupled cell systems in table 3 that are forced to have an S1 pitchfork bifurcation. Networks 10, 13, 15, 29 have a $\mathbf{Z}_2$-symmetry that permits the Liapunov reduction to be an odd function. Similarly, networks 1, 5, 20 have a $\mathbf{Z}_2$-symmetry on an invariant subspace ($x_1 = x_3$, $x_2 = x_3$, $x_2 = x_3$, respectively) that forces the Liaponov reduction to be odd. In each of these cases symmetry forces $K_{yy} = 0$. We claim that generically

$$K_{yyy}(0) \neq 0 \neq K_{y\lambda}(0). \tag{4.12}$$

Under assumption (4.12), it is known ([19, chapter 20, section 1D]) that a unique pitchfork branch bifurcates from the trivial branch.

We illustrate the calculations involved by considering network 13 when $f_u(0) - 2f_v(0) = 0$. Since $K(0, \lambda) = 0$, the formulaes to compute $K_{y\lambda}(0)$ and $K_{yyy}(0)$ are, respectively, (4.8) and

$$K_{yyy}(0) = \langle v_0^*, d^3F(v_0, v_0, v_0) - 3d^2F(v_0, J^{-1}Ed^2F(v_0, v_0))\rangle,$$

where $E : \boldsymbol{R}^3 \to \text{range } J$ with $\ker E = N$ and $\boldsymbol{R}^3 = N \oplus \text{range } J$.

The admissible vector field for network 13 is

$$F(X, \lambda) \equiv (f(x_1, x_2, x_2, \lambda), f(x_2, x_1, x_1, \lambda), f(x_3, \overline{x_1, x_2}, \lambda)). \tag{4.13}$$

Since $f_u(0) - 2f_v(0) = 0$ it follows that

$$J = f_v(0)\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Without loss of generality we let $f_v(0) = 1$. Observe that v_0, v_0^* solve, respectively, $Jv_0 = 0$ and $J^Tv_0^* = 0$. Hence, $v_0 = v_0^* = (1, -1, 0)$. A straightforward calculation using (4.13) and the formula for $K_{y\lambda}(0)$ in (4.8) shows that

$$K_{y\lambda}(0) = 2(f_{u\lambda}(0) - f_{v\lambda}(0) - f_{w\lambda}(0)).$$

The invariance of f under the permutation $(v, w) \mapsto (w, v)$ implies that

$$K_{y\lambda}(0) = 2f_{u\lambda}(0) - 4f_{v\lambda}(0),$$

which is generically nonzero.

Since the $\mathbf{Z}_2$-symmetry acts as $-I$ on $\ker J$ it follows that $K_{yy}(0) = 0$. So we calculate $K_{yyy}(0)$ by first calculating

$$A \equiv \langle v_0^*, d^3F(v_0, v_0, v_0)\rangle = 2(f_{uuu} + 6f_{uvv} - 6f_{uuv} - 6f_{vvw} + 6f_{uvw} - f_{vvv})$$

$$B \equiv \langle v_0^*, d^2F(v_0, J^{-1}Ed^2F(v_0, v_0))\rangle = \tfrac{1}{2}(f_{uu} + 2f_{vv} + 2f_{vw} - 4f_{uv})(f_{uu} - 2f_{vv} - 2f_{vw}), \tag{4.14}$$

where all derivatives of f are evaluated at the origin. Hence, generically $A - 3B \neq 0$, which implies that $K_{uuu}(0)$ is nonzero.

We verify (4.14). Since $v_0^* = (1, -1, 0)$, it follows that

$$A = d^3F_1(v_0, v_0, v_0) - d^3F_2(v_0, v_0, v_0)$$

and

$$B = d^2F_1(v_0, J^{-1}Ed^2F(v_0, v_0)) - d^2F_2(v_0, J^{-1}Ed^2F(v_0, v_0)). \tag{4.15}$$

A straightforward calculation using the formula for d^3F and the invariance of f under the permutation $(v, w) \mapsto (w, v)$ shows that

$$A = 2(f_{uuu} - f_{vvv} - 6f_{vvw} - 6f_{uuv} + 6f_{uvv} + 6f_{uvw}),$$

which verifies (4.14)

To find B we need to know $J^{-1} : \text{range } J \to M$, where M satisfies $\boldsymbol{R}^3 = \ker J \oplus M$. A straightforward calculation shows that M is spanned by $(1, 1, 1)$ and $(1, 1, 0)$. Hence, in this basis

$$J^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Thus, given $v = (v_1, v_1, v_2) \in \text{range } J$, $\overline{v} \in M$ has the form

$$\overline{v} = \tfrac{1}{4}(v_1, v_1, 2v_2 - v_1). \tag{4.16}$$

We use (4.11) and the invariance of f under the $(v, w) \mapsto (w, v)$ to compute

$$d^2F(v_0, v_0) = (d^2F_1(v_0, v_0), d^2F_2(v_0, v_0), d^2F_3(v_0, v_0)).$$

We obtain that $d^2F(v_0, v_0) = (a, a, b)$, where

$$a = f_{uu}(0) + 2f_{vv}(0) + 2f_{vw}(0) - 4f_{uv}(0),$$

$$b = 2(f_{vv}(0) - 2f_{vw}(0)).$$

Hence, $d^2F(v_0, v_0) \in$ range J. Therefore, use (4.16) and note that $E|_{\text{range } J} = I$ to obtain

$$u_0 = J^{-1}Ed^2F(v_0, v_0) = \tfrac{1}{4}(a, a, 2b - a). \tag{4.17}$$

We calculate $d^2F_1(v_0, u_0)$ and $d^2F_2(v_0, u_0)$ using (4.11). Substitution of the resulting expressions into (4.15) leads to

$$B = \tfrac{1}{2}a(f_{uu}(0) - 2f_{vv}(0) - 2f_{vw}(0)),$$

which verifies (4.14).

Ordering of eigenvalues when $k = 1$. When $k = 1$ and the eigenvalues of J are simple, the ordering of the eigenvalues of J is restricted. To see this we begin by stating lemma 4.1, which concerns eigenvalues of the adjacency matrix A of a homogeneous N-cell network G.

Lemma 4.1. *Consider the homogeneous N-cell network G with valency n. Let $k = 1$ and denote an eigenvalue of A by μ. Then, $|\mu| \leqslant n$. Moreover, if γ is an eigenvalue of J then $|\gamma - f_u(0)| \leqslant n|f_v(0)|$.*

Proof. Recall that a network is *l-regular* if every node lies at the head of exactly l edges, for fixed l. Since G is homogeneous, the network with adjacency matrix A is n-regular. Thus, the results follow from Andrásfai [1, theorem 3.45]. □

Remark 4.2. Assume $k = 1$. In the networks in which S1 bifurcations can occur in two ways, the stability of equilibria is determined by three eigenvalues that are close to $\gamma_1 = f_u(0) + nf_v(0)$, γ_2, γ_3. Observe that $\gamma_1 = 0$ for a synchrony preserving bifurcation and $\gamma_1 \neq 0$ for an S1 bifurcation. Let

$$\tau_2 = \min\{\gamma_2, \gamma_3\} \text{ and } \tau_3 = \max\{\gamma_2, \gamma_3\}.$$

In these networks the eigenvectors for τ_2 and τ_3 are fixed for all admissible vector fields, and, as we have seen, it is these eigenvectors that characterize the type of bifurcation that occurs. We claim that stable bifurcating equilibria are possible only when $f_v(0) < 0$ and $\tau_3 = 0$. Then the stability of bifurcating equilibria are determined by standard exchange of stability type arguments.

We begin by noting that at S1 bifurcations γ_1 and one of τ_2 and τ_3 are nonzero. The nonzero eigenvalues must be negative in order for any branch of equilibria to be stable. In particular, to find bifurcations where stable solutions appear, we must assume $\gamma_1 < 0$.

Suppose that γ is an eigenvalue of J. Then

$$\begin{aligned} &\mathrm{Re}(\gamma) \leqslant \gamma_1 \qquad \text{if } f_v(0) > 0, \\ &\mathrm{Re}(\gamma) \geqslant \gamma_1 \qquad \text{if } f_v(0) < 0. \end{aligned} \tag{4.18}$$

To prove (4.18) note that lemma 4.1 implies that $|\gamma - f_u(0)| \leqslant nf_u(0)$ when $f_v(0) > 0$. Hence

$$\mathrm{Re}(\gamma) - f_u(0) \leqslant \sqrt{(\mathrm{Re}(\gamma) - f_u(0))^2 + \mathrm{Im}(\gamma)^2} \leqslant |\gamma - f_u(0)| \leqslant nf_v(0).$$

Hence, $\mathrm{Re}(\gamma) \leqslant f_u(0) + nf_v(0) = \gamma_1$, as desired. A similar argument works in the case $f_v(0) < 0$.

It follows from (4.18) that

$$\tau_2 < \tau_3 < \gamma_1$$

when $f_v(0) > 0$, and no S1 bifurcation leading to stable solutions is possible. So we assume $f_v(0) < 0$. Then the eigenvalues satisfy

$$\gamma_1 < \tau_2 < \tau_3.$$

If τ_2 is the critical eigenvalue, then all equilibria near this bifurcation must be unstable since $\tau_3 > 0$. So stable solutions can occur only when the critical eigenvalue is τ_3. Then $\tau_2 < 0$ and in this case it follows that stable equilibria can occur near the bifurcation and stability is determined by standard exchange of stability for transcritical bifurcations.

4.3. S2: double eigenvalue, two eigenvectors; networks 4, 7, 8

We show that S2 bifurcations lead to multiple nontrivial bifurcating branches as opposed to S1 bifurcations which lead to a unique nontrivial branch. The nontrivial branches are either transcritical or pitchfork. Recall that the form of the coupled systems associated with each network are given in table 2.

Network 8: critical eigenvalue $f_u(0)-f_v(0)$; symmetry group $\boldsymbol{S}_3$. Network 8 has $\boldsymbol{S}_3$ symmetry and these bifurcations have been studied in [11, Chapter 1]. Codimension one bifurcations lead to three nontrivial transcritical symmetry related branches whose form is given in table 4. The existence of these branches is guaranteed by the equivariant branching lemma. These nontrivial equilibria are unstable near the origin [11, Chapter 1].

Network 4: critical eigenvalue $f_u(0)$; symmetry $\sigma(x_1, x_3) = (x_3, x_1)$.

Theorem 4.3. *Assume that the coupled cell system defined by $f(u, v, \lambda)$ associated with network 4 satisfies*

$$f_u(0) = 0 \qquad f_v(0) \neq 0 \qquad f_{u\lambda}(0) \neq 0 \qquad f_{uu}(0) \neq 0. \tag{4.19}$$

Then there are three transcritical branches bifurcating from the trivial solution of the form

$$(0, 0, x(\lambda), \lambda) \qquad (x(\lambda), 0, 0, \lambda) \qquad (x(\lambda), 0, x(\lambda), \lambda), \tag{4.20}$$

where $x(0) = 0$ and $x'(0) \neq 0$. Only the third solution in (4.20) can be stable near bifurcation.

Proof. Since $f_u(0) + f_v(0) \neq 0$, the implicit function theorem implies that the cell 2 equation $f(x_2, x_2, \lambda) = 0$ has only the trivial solution $x_2 = 0$ near bifurcation. So we may assume that $x_2 = 0$. The equations for cells 1 and 3 are identical to the form $f(u, 0, \lambda) = 0$. Conditions (4.19) imply that this equation has a unique nontrivial solution $f(x(\lambda), 0, \lambda) = 0$ satisfying $x(0) = 0$ and $x'(0) \neq 0$. Thus, cells 1 and 3 can equal either $u = 0$ or $u = x(\lambda)$, which leads to four solutions. The three nontrivial solutions are shown in (4.20).

In the generic bifurcation problem, only the equilibria on the trivial and the transcritical branch $(x(\lambda), 0, x(\lambda), \lambda)$ can be stable near the origin. To prove this, verify the expansion of eigenvalues of the Jacobian matrix (at lowest nonzero order) along each solution branch, as given in table 5. □

Table 5. Eigenvalues along solution branches at lowest nonzero order for network 4.

Branches of equilibria	Signs of eigenvalues		
$(0, 0, 0, \lambda)$	f_v	$f_{u\lambda}\lambda$	$f_{u\lambda}\lambda$
$(0, 0, x(\lambda), \lambda)$		$-f_{u\lambda}\lambda$	$f_{u\lambda}\lambda$
$(x(\lambda), 0, 0, \lambda)$		$-f_{u\lambda}\lambda$	$f_{u\lambda}(0)\lambda$
$(x(\lambda), 0, x(\lambda), \lambda)$		$-f_{u\lambda}\lambda$	$-f_{u\lambda}(0)\lambda$

Network 7: critical eigenvalue $f_u(0)$; symmetry $\sigma(x_1, x_2) = (x_2, x_1)$. This generic codimension one bifurcation in network 7 leads to three nontrivial branches in a way that we now describe. Observe that the equations for cells 1 and 2 decouple from cell 3 and have a permutation symmetry. It follows that when viewed in cells 1 and 2, a codimension one synchrony-breaking bifurcation is just a symmetry-breaking pitchfork bifurcation. Thus, in the equations for the first two cells there are two branches: a trivial branch and a parabolic pitchfork branch.

Each of these branches can be entered into the equation for cell 3. The trivial branch leads to a trancritical bifurcation in the cell 3 equation and hence to two branches: a trivial branch and a transcritical branch. The parabolic branch in the equations for cells 1 and 2 also leads to two branches in the cell 3 equation both of which are parabolic. Next we discuss this bifurcation in detail.

The equations for equilibria in cells 1 and 2, which do not depend on x_3 (see table 2), have the form

$$f(x_1, \overline{x_1, x_2}, \lambda) = 0,$$
$$f(x_2, \overline{x_2, x_1}, \lambda) = 0. \tag{4.21}$$

The Jacobian for this system at the origin is

$$J_2 = \begin{bmatrix} f_u(0) + f_v(0) & f_v(0) \\ f_v(0) & f_u(0) + f_v(0) \end{bmatrix}$$

with eigenvalues $f_u(0)$ and $f_u(0) + 2f_v(0)$. The corresponding eigenvectors are, respectively, $(1, -1)$ and $(1, 1)$. A symmetry-breaking bifurcation occurs when $f_u(0) = 0$. Assume the following nondegeneracy conditions:

$$f_v \neq 0 \qquad f_{u\lambda} \neq 0 \qquad f_{uu} \neq 0 \qquad A \neq 0 \qquad B \neq 0, \tag{4.22}$$

where

$$A = 2f_{uuu} + 12f_{uvv} - 12f_{uvw} - 3(f_{uu} + 2f_{uv})(f_{uu} + 2f_{uv} - 2f_{vw}),$$
$$B = 2f_{vw} - 2f_{uv} - 2f_{vv} - f_{uu} \tag{4.23}$$

and all derivatives of f are evaluated at the origin.

Theorem 4.4. *Consider the coupled cell system associated with network 7 satisfying the nondegeneracy conditions (4.22). Then, a transcritical branch and two pitchfork branches bifurcate from the trivial solution. See table 6 for the form of the solutions branches and their stabilities.*

Proof. On substituting $x_1 = x_2 = 0$ into the equation for cell 3, we obtain

$$f(x_3, 0, 0, \lambda) = 0.$$

Table 6. List of branches of equilibria for network 7 and eigenvalues along these branches (to lowest nonzero order). All derivatives of f are evaluated at 0; B is defined in (4.23).

Branches of equilibria	Type	sgn(eigenvalues)		
$(0, 0, 0, \lambda)$	trivial	f_v	$f_{u\lambda}\lambda$	$f_{u\lambda}\lambda$
$(0, 0, X_3(\lambda), \lambda)$	transcritical	f_v	$f_{u\lambda}\lambda$	$-f_{u\lambda}\lambda$
$(x_1, X_2(x_1), C_+(x_1), \Lambda(x_1))$	pitchfork	f_v	$f_v f_{uu} B$	$f_{uu}x_1$
$(x_1, X_2(x_1), C_-(x_1), \Lambda(x_1))$	pitchfork	f_v	$f_v f_{uu} B$	$-f_{uu}x_1$

This equation leads to a transcritical bifurcation if $f_{uu}(0)$ and $f_{u\lambda}(0)$ are nonzero, which is assumed valid in (4.22). Denote the transcritical branch by $x_3 = X_3(\lambda)$, where

$$X_3(0) = 0 \text{ and } X_3'(0) = -2\frac{f_{u\lambda}(0)}{f_{uu}(0)}.$$

We next consider the pitchfork branch in (4.21). This system has $\mathbf{Z}_2$-symmetry given by $\sigma_1(x_1, x_2) = (x_2, x_1)$, where σ acts as $-I$ on $\boldsymbol{R}\{(1, -1)\}$, which is the kernel of J_2. Liapunov–Schmidt reduction and symmetry leads to the equation

$$K(y, \lambda) = 0,$$

where $K(-y, \lambda) = -K(y, \lambda)$ and $K(0) = K_y(0) = 0$. This equation has a pitchfork bifurcation if $K_{yyy}(0) = A \neq 0$ and $K_{y\lambda}(0) = f_{u\lambda}(0) \neq 0$. Both of these conditions are assumed valid in (4.22). It follows that the pitchfork branch in (4.21) is parametrized by

$$x_1 \mapsto (x_1, X_2(x_1), \Lambda(x_1)),$$

where $X_2(0) = 0$, $X_2'(0) = -1$, $\Lambda(0) = \Lambda'(0) = 0$ and

$$\Lambda''(0) = -\frac{1}{3}\frac{A}{f_{u\lambda}(0)}. \tag{4.24}$$

We substitute the pitchfork solution in cells 1 and 2 into the cell 3 equation, obtaining

$$h(x_1, x_3) \equiv f(x_3, \overline{x_1, X_2(x_1)}, \Lambda(x_1)) = 0.$$

It follows easily that $h(0) = h_{x_1}(0) = h_{x_3}(0) = h_{x_1x_3}(0) = 0$ and $h_{x_3x_3}(0) = f_{uu}(0)$. Hence

$$f(x_3, \overline{x_1, X_2(x_1)}, \Lambda(x_1)) = \tfrac{1}{2}(h_{x_1x_1}(0)x_1^2 + f_{uu}(0)x_3^2) + \cdots. \tag{4.25}$$

We claim that $h_{x_1x_1}(0) = -f_{uu}(0)$. Since we assume that $f_{uu}(0) \neq 0$, it follows from (n7) that there are two curves of solutions $x_3 = C_\pm(x_1)$, where $C_\pm(0) = 0$ and $\boldsymbol{C}_\pm'(0) = \pm 1$. The branches are parametrized by

$$(x_1, X_2(x_1), C_\pm(x_1), \Lambda(x_1)),$$

and these branches of equilibria are of pitchfork type.

In order to verify the claim we must first compute the constant $X_2''(0)$, which can be found by implicit differentiation of the first equation in (4.21). Indeed,

$$X_2''(0) = \frac{2f_{vw}(0) - f_{uu}(0) - 2f_{vv}(0)}{f_v(0)}.$$

Observe that

$$h_{x_1x_1}(0) = 2f_{vv}(0) - 2f_{vw}(0) + f_w(0)X_2''(0),$$

which, on substitution of $X_2''(0)$, yields

$$h_{x_1x_1}(0) = -f_{uu}(0).$$

as claimed. □

Remark 4.5. In the generic bifurcation problem governed by theorem 4.4 with $k = 1$ the transcritical solutions are never stable, whereas one-half branch of one of the two pitchfork branches can be stable. This assertion is proved using the expansion of eigenvalues of J (at lowest nonzero order) along each solution branch given in table 6. From this table we see that $f_v(0) < 0$ is needed in order to have stable equilibria on any of the bifurcating branches.

The eigenvalue calculations in table 6 are performed as follows. The Jacobian of the system at an arbitrary equilibrium is

$$\begin{bmatrix} (f_u + f_v)(x_1, \overline{x_1, x_2}, \lambda) & f_v(x_1, \overline{x_1, x_2}, \lambda) & 0 \\ f_v(x_2, \overline{x_2, x_1}, \lambda) & (f_u + f_v)(x_2, \overline{x_2, x_1}, \lambda) & 0 \\ - & - & f_u(x_3, \overline{x_1, x_2}, \lambda) \end{bmatrix}.$$

The third eigenvalue is obtained by finding the sign of $f_u(x_3, \overline{x_1, x_2}, \lambda)$ to lowest order along each branch. The results are tabulated in the last column of table 6.

At constant order the 2×2 submatrix in the upper left is

$$\begin{bmatrix} f_v(0) & f_v(0) \\ f_v(0) & f_v(0) \end{bmatrix}.$$

Since the eigenvalues of this matrix are $2f_v(0)$ and 0, one of the eigenvalues of all solutions has sign determined by $\operatorname{sgn}(f_v(0))$, as shown in the table. Note that the other eigenvalue is zero to constant order. Moreover, this 2×2 matrix does not depend on x_3 so that the calculation along the branch $(x_1, X_2(x_1), C_\pm(x_1), \Lambda(x_1))$ is independent of the function $C_\pm(x_1)$. Similarly, since $\Lambda'(0) = 0$ the first order expansion does not depend on Λ. To linear order in x_1 the 2×2 matrix has the form

$$\begin{bmatrix} f_v + \alpha x_1 & f_v + \beta x_1 \\ f_v - \beta x_1 & f_v - \alpha x 1 \end{bmatrix},$$

whose determinant is $(\beta^2 - \alpha^2)x_1^2$ where

$$\alpha = f_{uu} + f_{uv} + f_{vv} - f_{vw} \text{ and } \beta = -f_{uv} - f_{vv} + f_{vw}.$$

Thus,

$$\beta^2 - \alpha^2 = (\beta + \alpha)(\beta - \alpha) = f_{uu}(2f_{vw} - 2f_{uv} - 2f_{vv} - f_{uu}) = f_{uu}B,$$

and the sign of the remaining eigenvalue is $\operatorname{sgn}(f_v f_{uu} B)$, which is assumed to be nonzero. $\Diamond$

4.4. S3: nilpotent double eigenvalue; networks 3, 6, 11, 27, 28

We show that the generic synchrony-breaking bifurcations of type S3 lead to two nontrivial bifurcating branches. We also show that in networks 3, 27 and 28 at least one nontrivial branch has solutions whose rates of growth are different in distinct cells, while in networks 6 and 11 the growth rate of solutions in a given branch is equal for all cells. However, in network 6 one of the bifurcating branches is transcritical and the other is pitchfork, whereas in network 11 the two nontrivial branches are transcritical.

Network 3: critical eigenvalue $f_u(0)$; eigenvector $(0, 0, 1)$. Synchrony-breaking bifurcation occurs when $f_u(0) = 0$.

Theorem 4.6. *We assume that network 3 satisfies the nondegeneracy conditions*

$$f_v(0) \neq 0 \qquad f_{u\lambda}(0) \neq 0 \qquad f_{uu}(0) \neq 0 \tag{4.26}$$

Then, there are two branches of asynchronous solutions bifurcating from the trivial solution: when viewed in cell 3 *one is transcritical and the other is pitchfork.*

Proof. Observe that equilibria of the coupled cell systems associated with network 3 have the skew-product form:

$$f(x_1, x_1, \lambda) = 0,$$

$$f(x_2, x_1, \lambda) = 0,$$

$$f(x_3, x_2, \lambda) = 0.$$

This form allows us to find equilibria branches by solving the equations in turn. Since $f_v(0) \neq 0$ the first equation has a unique solution $x_1 = 0$. The second equation $f(x_2, 0, \lambda) = 0$ has a trivial solution $x_2 = 0$. Since $f_u(0) = 0$ and $f_{uu}(0) \neq 0$, the second equation also has a nontrivial branch of solutions $x_2 = X_2(\lambda)$, where $X_2(0) = 0$ and

$$X_2'(0) = -2\frac{f_{u\lambda}(0)}{f_{uu}(0)} \neq 0. \tag{4.27}$$

Next we consider the cell 3 equation. There are two possibilities for equilibria, namely,

$$f(x_3, 0, \lambda) = 0 \text{ and } f(x_3, X_2(\lambda), \lambda) = 0.$$

In the first case, we can use the same argument as for cell 2 and see that the equilibria for the cell 3 equation are $x_3 = 0$ or $x_3 = X_2(\lambda)$. Thus, the nontrivial branch has the form $(0, 0, X_2(\lambda), \lambda)$ and is transcritical.

In the second case we need to solve

$$g(x_3, \lambda) \equiv f(x_3, X_2(\lambda), \lambda) = 0.$$

Straightforward calculations show that

$$g(0) = 0 \qquad g_{x_3}(0) = 0 \qquad g_\lambda(0) = -2f_v(0)\frac{f_{u\lambda}(0)}{f_{uu}(0)} \neq 0 \qquad g_{x_3x_3}(0) = f_{uu}(0) \neq 0.$$

The implicit function theorem guarantees the existence of a unique solution $\lambda = \Lambda(x_3)$ satisfying

$$f(x_3, X_2(\Lambda(x_3)), \Lambda(x_3)) \equiv 0$$

with $\Lambda(0) = 0$, $\Lambda'(0) = 0$ and

$$\Lambda''(0) = \frac{1}{2f_v(0)}\frac{f_{uu}^2(0)}{f_{u\lambda}(0)} \neq 0.$$

Therefore, the third branch of solutions is of pitchfork type. □

Remark 4.7. In the generic bifurcation problem governed by theorem 4.6 with $k = 1$, equilibria on all three branches of equilibria are unstable near the origin. To prove this assertion, note that the Jacobian matrix of these systems is always lower triangular with eigenvalues

$$(f_u + f_v)(x_1, x_1, \lambda) \qquad f_u(x_2, x_1, \lambda) \qquad f_u(x_3, x_2, \lambda).$$

Expansion of these three expressions to lowest order along the three branches of solutions yields the entries in table 7. ◊

Table 7. List of signs of the eigenvalues along solution branches at lowest nonzero order for networks 3 and 28. All derivatives of f are evaluated at the origin.

Solution type	sgn(eigenvalues)		
Trivial	f_v	$f_{u\lambda}\lambda$	$f_{u\lambda}\lambda$
Transcritical	f_v	$f_{u\lambda}\lambda$	$-f_{u\lambda}\lambda$
Pitchfork	f_v	$f_{uu}x_3$	$-f_v$

Network 28: critical eigenvalue $f_u(0)$; eigenvector $(0, 0, 1)$. Synchrony-breaking bifurcations occur when $f_u(0) = 0$. The analysis is virtually the same as in network 3.

Theorem 4.8. *We assume that network 28 satisfies the nondegeneracy conditions in theorem 4.6. Then, there are two branches of asynchronous solutions bifurcating from the trivial solution: when viewed in cell* 3 *one is transcritical and the other is pitchfork.*

Proof. Observe that equilibria of the coupled cell systems associated with network 3 also have the skew-product form:

$$f(x_1, x_1, x_1, \lambda) = 0,$$

$$f(x_2, x_1, x_1, \lambda) = 0,$$

$$f(x_3, \overline{x_1, x_2}, \lambda) = 0.$$

This form allows us to find equilibria branches by solving the equations in turn. □

The analysis of the stability of solutions proceeds in a way entirely analogous to that in network 3. See remark 4.7. The results are recorded in table 7.

Network 27: critical eigenvalue $f_u(0)$; eigenvector $(0, 0, 1)$. The identification of the generic synchrony-breaking bifurcations that can occur in network 27 can be described as follows. Observe that the equations for equilibria in cells 1 and 2 are of the form shown in (4.21). Thus, as described for network 7 in section 4.3, generically the first equations will undergo a pitchfork bifurcation leading to one trivial branch and one parabolic pitchfork branch of solutions. Each of these branches can be entered into the third equation obtaining two nontrivial branches bifurcating from the trivial solution. We will show that when viewed in cell 3 one is transcritical and the other has a rate of growth of order $\lambda^{1/4}$.

Synchrony-breaking bifurcations occur when $f_u(0) = 0$. Assume the following nondegeneracy conditions:

$$f_v \neq 0 \qquad f_{u\lambda} \neq 0 \qquad f_{uu} \neq 0 \qquad A \neq 0 \qquad B \neq 0, \tag{4.28}$$

where A and B are given in (4.23).

Theorem 4.9. *Consider the coupled cell system associated with network 27 satisfying the nondegeneracy conditions (4.28). Then, two branches of equilibria bifurcate from the trivial solution: when viewed in cell* 3 *one branch is transcritical and the other grows as* $\lambda^{1/4}$.

In table 8 we also calculate the sign of the eigenvalues (at lowest nonzero order) along the branches of equilibria.

Proof. On substituting the trivial solution $x_1 = x_2 = 0$ into the cell 3 equation we obtain a transcritical bifurcation if $f_{uu}(0) \neq 0$ and $f_{u\lambda}(0) \neq 0$. Both of these conditions are assumed valid in (4.28). We denote the transcritical branch by $x_3 = X_3(\lambda)$, where $X_3(0) = 0$ and $X_3'(0) \equiv -2f_{u\lambda}(0)/f_{uu}(0)$. In this way we obtain the first two branches of equilibria.

Table 8. Branches of equilibria and (to lowest nonzero order) eigenvalues along solution branches for network 27. Derivatives of f are evaluated at 0; B is defined in (4.23).

Branches of equilibria	Type	sgn(Eigenvalues)		
$(0, 0, 0, \lambda)$	trivial	f_v	$f_{u\lambda}\lambda$	$f_{u\lambda}\lambda$
$(0, 0, X_3(\lambda), \lambda)$	transcritical	f_v	$f_{u\lambda}\lambda$	$-f_{u\lambda}\lambda$
$(X_1(x_3), X_2(X_1(x_3)), x_3, \Lambda(X_1(x_3)))$		f_v	$f_v f_{uu} B$	$f_{uu}x_3$

Next we assume the pitchfork branch in the equations for cells 1 and 2 is $(x_1, X_2(x_1), \Lambda(x_1))$, where $X_2(0) = \Lambda(0) = \Lambda'(0) = 0$, $X_2'(0) = -1$ and $\Lambda''(0)$ is defined in (4.24). The equation for cell 3 is

$$g(x_3, x_1) \equiv f(x_3, \overline{x_1, x_1}, \Lambda(x_1)) = 0.$$

Straightforward calculations show that

$$g(0) = 0 \qquad g_{x_3}(0) = 0 \qquad g_{x_1}(0) = 2f_v(0) \neq 0 \qquad g_{x_3x_3}(0) = f_{uu}(0) \neq 0.$$

The implicit function theorem guarantees the existence of a unique solution $x_1 = X_1(x_3)$ satisfying $X_1(0) = 0$ and

$$f(x_3, X_1(x_3), X_1(x_3), \Lambda(X_1(x_3))) \equiv 0.$$

Implicit differentiation implies that $X_1'(0) = 0$ and

$$X_1''(0) = -\frac{f_{uu}(0)}{2f_v(0)} \neq 0.$$

Therefore, the first nonzero term in the Taylor expansion of $\Lambda(X_1(x_3))$ is x_3^4. It follows that when viewed in cell 3 the third branch of equilibria has a rate of growth of order $\lambda^{1/4}$. $\square$

Remark 4.10. In the generic bifurcation problem governed by theorem 4.9 with $k = 1$ the transcritical solutions are never stable, whereas one-half branch of the other nontrivial branch of solutions can be stable. This assertion is proved by expanding the eigenvalues of J (at lowest nonzero order) along each solution branch given in table 8. From this table we see that $f_v(0) < 0$ is needed in order to have stable equilibria on any of the bifurcating branches.

The Jacobian of the system at an arbitrary equilibrium is

$$\begin{bmatrix} (f_u + f_v)(x_1, \overline{x_1, x_2}, \lambda) & f_v(x_1, \overline{x_1, x_2}, \lambda) & 0 \\ f_v(x_2, \overline{x_2, x_1}, \lambda) & (f_u + f_v)(x_2, \overline{x_2, x_1}, \lambda) & 0 \\ - & - & f_u(x_3, \overline{x_1, x_1}, \lambda) \end{bmatrix}.$$

The calculation of the signs of the eigenvalues in table 8 are similar to the ones described in remark 4.5. $\Diamond$

Network 6: critical eigenvalue $f_u(0)$*; eigenvector* $(1, -1, -1)$. When $k = 1$ the steady-state equations associated with network 6 have the form

$$\begin{aligned} f(x_1, \overline{x_1, x_3}, \lambda) &= 0, \\ f(x_2, \overline{x_2, x_1}, \lambda) &= 0, \\ f(x_3, \overline{x_1, x_2}, \lambda) &= 0. \end{aligned} \tag{4.29}$$

At criticality for this S3 bifurcation the eigenvalues of the Jacobian matrix are $2f_v(0)$ and 0 with multiplicity two. Thus, in order for any equilibria to be stable near the bifurcation we must have $f_v(0) < 0$, which we now assume.

Theorem 4.11. *Assume that the following nondegeneracy conditions for (4.29) are valid*

$$f_v(0) < 0 \qquad f_{u\lambda}(0) > 0 \qquad f_{uu}(0) \neq 0 \qquad A \neq 0, \tag{4.30}$$

where A is given in (4.23). Then there are two nontrivial branches of equilibria bifurcating from the trivial solution in (4.29): a pitchfork branch in the plane $x_2 = x_3$ and a transcritical branch. The trivial solution is stable subcritically and the transcritical branch consists of saddles. The pitchfork branch is supercritical if $A < 0$ and then one-half of the branch is stable.

This cell system has a special structure: the restriction of (4.29) to the flow-invariant subspace $x_2 = x_3$ has a symmetry $(x_1, x_2) \mapsto (x_2, x_1)$. We begin by showing that this symmetry can be preserved under centre manifold reduction.

Lemma 4.12. *Let the vector field $F : \boldsymbol{R}^N \to \boldsymbol{R}^N$ have a flow invariant subspace V with an equilibrium at $X_0 \in V$. Let E^c be the centre subspace at X_0. Then a centre manifold reduction $f : E^c \to E^c$ can be chosen so that the subspace $E^c \cap V$ is flow invariant for f. Moreover, if $\sigma : V \to V$ is a symmetry of $F|V$ that leaves $E^c \cap V$ invariant, then the centre manifold reduction f may be chosen so that $\sigma|E^c \cap V$ is a symmetry for $f|E^c \cap V$.*

Proof. Although centre manifolds are in general not unique, they are unique if a certain cutoff function on a fixed small domain is chosen. This fact is the basis for proving that centre manifold reductions inherit symmetry [18]. We now fix a domain and a cutoff function that is σ invariant.

We consider two centre manifolds. Let M be the centre manifold for F at X_0 and let M_V be the centre manifold for $F|V$ at X_0. By the uniqueness of centre manifolds with a fixed cutoff function $M_V = M \cap V$, M_V is σ-invariant and $F|M_V$ is σ-equivariant.

Let $g : E^c \to E^c$ be a centre manifold reduction of F and let $g_V : E^c \cap V \to E^c \cap V$ be a centre manifold reduction of $F|M_V$. Note that E^c is the tangent space of M at X_0 and that $E^c \cap V$ is the tangent space of M_V at X_0. Since $E^c \cap V$ is σ-invariant, it follows that we can assume that g_V is σ-equivariant. □

Next we apply lemma 4.12. The centre subspace for this S3 bifurcation is

$$E^c = \{y(1, -1, -1) + z(1, -2, 0) : x, y \in \boldsymbol{R}\}.$$

The subspace $V = \{x_1(1, 0, 0) + x_2(0, 1, 1) : x_1, x_2 \in \boldsymbol{R}\}$ is flow-invariant and

$$E^c \cap V = \{y(1, -1, -1) : y \in \boldsymbol{R}\}.$$

The coupled cell vector field F restricted to V commutes with the symmetry $\sigma(x_1, x_2) = (x_2, x_1)$. Note that $\sigma|E^c \cap V$ maps y to $-y$. Lemma 4.12 implies that we may assume that the centre manifold reduction $g(y, z, \lambda)$ leaves the subspace $z = 0$ invariant and the restriction of $g(y, 0, \lambda)$ is an odd function. Moreover, we can assume that g has a trivial equilibrium; that is, g has the form

$$\begin{aligned} \dot{y} &= a(y, \lambda)y + b(y, z, \lambda)z, \\ \dot{z} &= c(y, z, \lambda)z, \end{aligned} \tag{4.31}$$

where a is even in y. Moreover, the linearization of g along the trivial solution is

$$J(0, 0, \lambda) = \begin{bmatrix} a(0, \lambda) & b(0, 0, \lambda) \\ 0 & c(0, 0, \lambda) \end{bmatrix}.$$

The nilpotence of J implies that $a(0, \lambda) = c(0, 0, \lambda)$ and that $b(0) \neq 0$.

Proposition 4.13. *Assume that the following nondegeneracy conditions for (4.31) are valid*

$$a_\lambda(0) > 0 \qquad a_{yy}(0) \neq 0 \qquad b(0) \neq 0 \qquad c_y(0) \neq 0. \tag{4.32}$$

Then there are two nontrivial branches of equilibria bifurcating from the trivial solution in (4.31): a pitchfork branch in the line $z = 0$ and a transcritical branch. The trivial solution is stable subcritically and the transcritical branch consists of saddles. The pitchfork branch is supercritical if $a_{yy} < 0$ and then one-half of the branch is stable.

Proof. From (4.31) we see that nontrivial branches of equilibria satisfy either $z = 0$ or the system

$$\begin{aligned} &a(y,\lambda)y + b(y,z,\lambda)z = 0,\\ &c(y,z,\lambda) = 0. \end{aligned} \tag{4.33}$$

In the first case, since a is even in y, generically there is a pitchfork bifurcation defined by $\lambda = \Lambda(y^2)$, where $\Lambda(0) = 0$ and

$$\Lambda'(0) = -\frac{a_{yy}(0)}{2a_\lambda(0)}.$$

The nondegeneracy condition is $a_{yy}(0) \neq 0$.

In the second case, we note that the Jacobian of (4.33) at the origin

$$J_1 = \begin{bmatrix} 0 & b(0) \\ c_y(0) & c_z(0) \end{bmatrix}$$

is invertible if $c_y(0) \neq 0$, which we assume. The implicit function theorem implies that there is a unique solution branch to (4.33) parametrized by λ and denoted by $(Y(\lambda), Z(\lambda), \lambda)$. We claim that $Y'(0) \neq 0$ so that the branch is transcritical and that $Z'(0) = 0$. Implicit differentiation of (4.33) along this branch of solutions and evaluated at the origin yields

$$\begin{aligned} &b(0)Z'(0) = 0,\\ &c_y(0)Y'(0) + c_z(0)Z'(0) + c_\lambda(0) = 0. \end{aligned}$$

Since $b(0) \neq 0$, we see that $Z'(0) = 0$. Since $c_y(0) \neq 0$ and $c_\lambda(0) = a_\lambda(0) \neq 0$ we see that

$$Y'(0) = -\frac{a_\lambda(0)}{c_y(0)} \neq 0 \tag{4.34}$$

as claimed.

Next we consider the stability of solutions on the pitchfork and the transcritical branches. The Jacobian matrix of the system (4.31) is

$$J(y,z,\lambda) = \begin{bmatrix} a + ya_y + zb_y & b + zb_z \\ zc_y & c + zc_z \end{bmatrix}. \tag{4.35}$$

Along the pitchfork branch

$$J(y,0,\lambda) = \begin{bmatrix} ya_y & b \\ 0 & c \end{bmatrix}. \tag{4.36}$$

The eigenvalues are $ya_y(y, \Lambda(y)) = a_{yy}(0)y^2 + O(y^3)$ and $c(y, 0, \Lambda(y^2)) = c_y(0)y + O(y^2)$. The signs of the eigenvalues are $\mathrm{sgn}(a_{yy}(0))$ and $\mathrm{sgn}(c_y(0)y)$. If the pitchfork branch is supercritical, the first eigenvalue is negative; the second eigenvalue is also negative on half the branch since $c_y(0) \neq 0$.

Along the transcritical branch the Jacobian matrix is

$$J(Y(\lambda), Z(\lambda), \lambda) = \begin{bmatrix} a + Ya_y + Zb_y & b + Zb_z \\ Zc_y & Zc_z \end{bmatrix}.$$

In fact,

$$J(Y(\lambda), Z(\lambda), \lambda) = \begin{bmatrix} O(\lambda) & b(0) + O(\lambda) \\ c_y(0)\frac{Z''(0)}{2}\lambda^2 & O(\lambda^2) \end{bmatrix}.$$

It follows that

$$\det J(Y(\lambda), Z(\lambda), \lambda) = -b(0)c_y(0)\frac{Z''(0)}{2}\lambda^2 + O(\lambda^3).$$

To compute $Z''(0)$ expand the first equation in (4.33) to second order in λ along the transcritical branch and use (4.34) to obtain

$$Z''(0) = \frac{2}{b(0)}\frac{a_\lambda(0)}{c_y(0)}^2(c_y(0) - a_y(0)).$$

Hence

$$\det J(Y(\lambda), Z(\lambda), \lambda) = -a_\lambda(0)^2\left(1 - \frac{a_y(0)}{c_y(0)}\right)\lambda^2 + O(\lambda^3). \tag{4.37}$$

Finally, we use the fact that a is even in y (which implies that $a_y(0) = 0$) to conclude that $\det J$ along the transcritical branch is negative, and hence that the transcritical branch contains only saddles near bifurcation. □

To complete the proof of theorem 4.11 it remains to show that the validity of the nondegeneracy conditions (propnd) follow from (4.30). A calculation similar to the one given in section 4.2 shows that $A = a_{yy}(0)$. Similarly, $c_y(0) = -2f_{uu}(0)$.

Network 11: critical eigenvalue $f_u(0) - f_v(0)$*; eigenvector* $(2, -1, -1)$. The steady-state equations associated with network 11 have the form

$$\begin{aligned} f(x_1, \overline{x_2, x_2}, \lambda) &= 0 \\ f(x_2, \overline{x_1, x_3}, \lambda) &= 0 \\ f(x_3, \overline{x_1, x_2}, \lambda) &= 0 \end{aligned} \tag{4.38}$$

where $x_j \in \boldsymbol{R}^k$. Synchrony-breaking steady-state bifurcation occurs when an eigenvalue of the $k \times k$ matrix $f_u(0) - f_v(0)$ is zero. The analysis of this system is similar to that of network 6, but the results are different: the two nontrivial branches of solutions are both transcritical and when $k = 1$ they are saddles. The form of the centre manifold vector field suggests that solutions on these nontrivial branches can be stable when $k > 1$. We now restrict to $k = 1$ and return to the general case below.

The eigenvalues of the Jacobian matrix at the origin are $f_u(0) + 2f_v(0)$ and $f_u(0) - f_v(0)$ with multiplicity two. At criticality for this S3 bifurcation the eigenvalues are $3f_v(0)$ and 0 twice. Note that in order for any equilibria to be stable near bifurcation, we must have $f_v(0) < 0$.

Theorem 4.14. *Let $k = 1$ and assume that the following nondegeneracy conditions for (4.38) are valid*

$$f_v(0) \neq 0 \qquad f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0 \qquad A \neq 0, \tag{4.39}$$

where

$$A = f_{uu}(0) - 2f_{uv}(0) - f_{vv}(0) + 2f_{vw}(0). \tag{4.40}$$

Then there are two transcritical branches of equilibria bifurcating from the trivial solution in (4.38), one of which occurs in the plane $x_2 = x_3$. The solutions on both transcritical branches are unstable saddles.

The plane $x_2 = x_3$ is flow-invariant for this cell system. Thus, we can apply lemma 4.12 to get a centre manifold reduction of the form (4.31), where in this case a is not constrained to be even in y. The centre subspace for this S3 bifurcation is

$$E^c = \{y(2, -1, -1) + z(0, 1, -2) : y, z \in \boldsymbol{R}\}.$$

The subspace $V = \{x_1(1, 0, 0) + x_2(0, 1, 1) : x_1, x_2 \in \boldsymbol{R}\}$ is flow-invariant and

$$E^c \cap V = \{y(2, -1, -1) : y \in \boldsymbol{R}\}.$$

Lemma 4.12 implies that we may assume that the centre manifold reduction leaves the subspace $z = 0$ invariant. Moreover, we can assume that this reduction has a trivial equilibrium. For easy reference the centre manifold equations are

$$\begin{aligned} \dot{y} &= a(y, \lambda)y + b(y, z, \lambda)z, \\ \dot{z} &= c(y, z, \lambda)z, \end{aligned} \tag{4.41}$$

where $a(0, \lambda) = c(0, 0, \lambda)$ and $a(0) = 0$.

Proposition 4.15. *Assume that the following nondegeneracy conditions for (4.41) are valid*

$$a_\lambda(0) > 0 \qquad a_y(0) \neq 0 \qquad b(0) \neq 0 \qquad c_y(0) \neq 0. \tag{4.42}$$

Then there are two nontrivial branches of equilibria bifurcating from the trivial solution in (4.41): a synchronous transcritical branch in the line $z = 0$ and an asynchronous transcritical branch off this line.

The synchronous branch can be stable if

$$\frac{a_y(0)}{c_y(0)} > 0. \tag{4.43}$$

The supercritical part is stable if $a_y(0) < 0$ and the subcritical part is stable if $a_y(0) > 0$. The asynchronous branch can be stable if

$$\frac{a_y(0)}{c_y(0)} > 1. \tag{4.44}$$

In this case the solutions transition from nodal sources to nodal sinks as λ increases through zero. If the inequality in (4.44) is reversed the solutions are saddles.

Proof. From the second equation in (4.41) we see that equilibria satisfy either $z = 0$ or $c = 0$. In the first case solving $a(y, \lambda) = 0$ yields a synchronous transcritical branch $Y_s(\lambda)$, since $a_\lambda(0)$ and $a_y(0)$ are assumed nonzero. The second case also yields a transcritical branch $(X(\lambda), Y(\lambda), \lambda)$; the proof is identical to the one in the proof of proposition 4.13. Moreover, the values $Y'(0) = 0$, $Z'(0)$, and $Z''(0)$ are also identical to the values calculated in that proposition.

Using (4.36) we see that the signs of the eigenvalues of the Jacobian along the synchronous transcritical branch are given by $\mathrm{sgn}(a_y(0)y)$ and $\mathrm{sgn}(c_y(0)y)$. Thus, this branch can be stable only if (4.43) is satisfied.

At this point the calculation of the stability of the asynchronous transcritical solutions deviates from the corresponding calculation in the proof of proposition 4.13. Although the calculation of the determinant of the Jacobian along this branch in (4.37) is still valid, the fact that $a_y(0)$ is nonzero permits the determinant to be of either sign. To determine the eigenvalues we need to compute the trace of that Jacobian (at least to lowest order in λ). It is a straightforward calculation from (4.35) to see that

$$\mathrm{tr}\ J(Y(\lambda), Z(\lambda), \lambda) = (a_\lambda(0) + 2a_y Y'(0))\lambda + O(\lambda^2) = a_\lambda(0)\left(1 - 2\frac{a_y(0)}{c_y(0)}\right)\lambda + O(\lambda^2). \tag{4.45}$$

It follows from (4.37) and (4.45) that the eigenvalues of $J(Y(\lambda), Z(\lambda), \lambda)$ are real (near the origin). Moreover, the asynchronous transcritical solutions are saddles if the determinant of J is negative; that is if the inequality (4.44) is reversed (which includes the case $a_y(0) = 0$ in network 6) and transitions from nodal sources to nodal sinks if (4.44) is valid (since determinant positive implies that the eigenvalues have the same sign and the trace changes sign). □

At this point, for $k = 1$, the calculation of the stability of the asynchronous transcritical solutions deviates from the corresponding calculation in the proof of proposition 4.13. We claim that

$$c_y(0) = -2a_y(0). \tag{4.46}$$

Hence, proposition 4.15 implies that the solutions on this branch are saddles. We do not assert that (4.46) when $k > 1$.

It may seem surprising that an identity between coefficients of quadratic terms in (4.41) like (4.46) could be valid for all admissible vector fields. However, note that when $u, v, w \in \boldsymbol{R}$ the function $f(u, \overline{v, w})$ has only four independent quadratic terms u^2, $u(v + w)$, $v^2 + w^2$, vw, whereas there are five linearly independent quadratic terms in the centre manifold vector field (4.41). Thus, because $k = 1$, there must be at least one relationship like (4.46) among the coefficients of the quadratic terms in (4.41). When $k > 1$, there are many linearly independent quadratic terms in f and the restriction (4.46) is unlikely to hold. We have not verified this point.

To complete the proof we show that (4.46) holds. To find $a_y(0)$ and $c_y(0)$ we calculate the centre manifold reduction of (4.38) (see [19]) in order to obtain (4.41). We start with a linear change in coordinates $X = SY$, where $Y = (y_1, y_2, y_3)$, $X = (x_1, x_2, x_3)$ and $S = [v_1\ v_2\ v_3]$ with $v_1 = (1, 1, 1)^t$, $v_2 = (2, -1 - 1)^t$ and $v_3 = (0, 1, -2)^t$. It follows that

$$x_1 = y_1 + 2y_2,$$

$$x_2 = y_1 - y_2 + y_3,$$

$$x_3 = y_1 - y_2 - 2y_3.$$

Observe that

$$S^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 4 & 2 \\ 3 & -2 & -1 \\ 0 & 3 & 3 \end{bmatrix}.$$

This change in coordinates gives

$$\begin{aligned}
9\dot{y}_1 &= 3f(y_1+2y_2, y_1-y_2+y_3, y_1-y_2+y_3, \lambda) \\
&\quad + 4f(y_1-y_2+y_3, y_1+2y_2, y_1-y_2-2y_3, \lambda) \\
&\quad + 2f(y_1-y_2-2y_3, y_1+2y_2, y_1-y_2+y_3, \lambda), \\
9\dot{y}_2 &= 3f(y_1+2y_2, y_1-y_2+y_3, y_1-y_2+y_3, \lambda) \\
&\quad - 2f(y_1-y_2+y_3, y_1+2y_2, y_1-y_2-2y_3, \lambda) \\
&\quad - f(y_1-y_2-2y_3, y_1+2y_2, y_1-y_2+y_3, \lambda), \\
9\dot{y}_3 &= 3f(y_1-y_2+y_3, y_1+2y_2, y_1-y_2-2y_3, \lambda) \\
&\quad - 3f(y_1-y_2-2y_3, y_1+2y_2, y_1-y_2+y_3, \lambda).
\end{aligned} \tag{4.47}$$

Since the double eigenvalue $f_u(0) - f_v(0)$ is critical, the centre manifold is given by $W^c = \{(y_1, y_2, y_3, \lambda) \in \boldsymbol{R}^3 \times \boldsymbol{R} : \ y_1 = Y_1(y_2, y_3, \lambda), Y_1(0) = DY_1(0) = 0\}$.

To find $a_y(0)$ we begin by observing that $y_3 = 0$ is flow invariant. The first equation in (4.41) and the second equation in (4.47) imply that the centre manifold reduction on this invariant subspace is

$$3a(y_2, 0)y_2 = f(Y_1+2y_2, Y_1-y_2, Y_1-y_2, 0) - f(Y_1-y_2, Y_1+2y_2, Y_1-y_2, 0). \tag{4.48}$$

Recall that $Y_1(y_2, 0, 0) = O(y_2^2)$. Hence, differentiating both sides twice with respect to y_2 and evaluating at the origin yields

$$a_y(0) = \tfrac{1}{2}A.$$

Observe that the terms involving $(Y_1)_{y_2y_2}(0)$ all cancel.

Similarly, the second equation in (4.41) and the third equation in (4.47) imply

$$\begin{aligned}
3c(y_2, y_3, 0)y_3 &= f(Y_1-y_2+y_3, Y_1+2y_2, Y_1-y_2-2y_3, 0) \\
&\quad - f(Y_1-y_2-2y_3, Y_1+2y_2, Y_1-y_2+y_3, 0)
\end{aligned} \tag{4.49}$$

Differentiating both sides of (4.49) with respect to y_2 and y_3 and evaluating at the origin leads to

$$c_y(0) = -A,$$

which verifies (4.46). Here also the terms involving $(Y_1)_{y_2y_3}(0)$ cancel.

To complete the proof of theorem 4.14 observe that the nondegeneracy conditions (4.42) follow from (4.39).

4.5. S4: double eigenvalue, one synchrony eigenvector

Network 12 is the only network in figure 5 that can exhibit an S4 bifurcation (see section 3). Steady-state solutions to the coupled cell system associated with network 11 are found by solving

$$\begin{aligned}
&\text{(a)} \quad f(x_1, \overline{x_1, x_1}, \lambda) = 0, \\
&\text{(b)} \quad f(x_2, \overline{x_1, x_3}, \lambda) = 0, \\
&\text{(c)} \quad f(x_3, \overline{x_3, x_3}, \lambda) = 0.
\end{aligned} \tag{4.50}$$

This system has a skew-product form, where (4.50)(a) and (4.50)(c) are decoupled and identical. So we may solve either the cell 1 equation or cell 3 equation for zeros. Then we use this information to find the zeros for (4.50)(b).

Write $f = f(u, v, w)$. Let

$$A = f_{uu} + f_{vv} + f_{ww} + 2(f_{uv} + f_{uw} + f_{vw}), \tag{4.51}$$

where all derivatives are evaluated at the origin.

Theorem 4.16. *Assume the defining condition $f_u(0) + 2f_v(0) = 0$ and the nondegeneracy conditions $A \neq 0$, $f_\lambda \neq 0$, and $f_u(0) \neq 0$. Then there exists a saddle node of synchronous solutions and a saddle node of asynchronous solutions bifurcating from the origin. Moreover, either both branches are subcritical or both branches are supercritical.*

Proof. Observe that

$$g(x_1, \lambda) = f(x_1, \overline{x_1, x_1}, \lambda)$$

has a saddle node bifurcation if $g_\lambda(0) \neq 0$ and $g_{x_1,x_1}(0) \neq 0$. Equivalently, we assume $f_\lambda(0) \neq 0$ and $A \neq 0$. Under these assumptions we can solve (4.50)(a) for a function $\Lambda(x_1)$ so that $g(x_1, \Lambda(x_1)) \equiv 0$, where $\Lambda(0) = \Lambda'(0) = 0$ and $\Lambda''(0) = -A \neq 0$. Equivalently,

$$f(x_1, \overline{x_1, x_1}, \Lambda(x_1)) \equiv 0.$$

Equation (4.50)(c) is solved similarly.

Because $\Lambda(x) = cx^2 + \cdots$ is parabola-like near the origin, there is a locally defined diffeomorphism $\varphi : \boldsymbol{R} \to \boldsymbol{R}$ with $\varphi(0) = 0$ and $\varphi'(0) = -1$ satisfying $\Lambda(\varphi(x)) = \Lambda(x)$. It follows that there are two types of simultaneous solutions to equations (4.50)(a) and (4.50)(c), namely, $x_3 = x_1$ and $x_3 = \varphi(x_1)$.

Solving (4.50)(b) when $x_3 = x_1$ is straightforward. We can solve

$$f(x_2, \overline{x_1, x_1}, \Lambda(x_1)) = 0$$

uniquely for $x_2 = X_2(x_1)$ by the implicit function theorem since $f_u(0) \neq 0$. However, there is a saddle node bifurcation for this system in the synchrony subspace $\triangle$, from which it follows that $X_2(x_1) = x_1$. So the synchronous saddle node has the form $(x_1, x_1, x_1, \Lambda(x_1))$.

Solving (4.50)(b) when $x_3 = \varphi(x_1)$ is similar. We can solve

$$f(x_2, \overline{x_1, \varphi(x_1)}, \Lambda(x_1)) = 0$$

for $x_2 = Y_2(x_1)$ by the implicit function theorem since again $f_u(0) \neq 0$. Note that $Y_2'(0) = 0$ since

$$\frac{\partial}{\partial x_1} f(x_2, \overline{x_1, \varphi(x_1)}, \Lambda(x_1))\bigg|_{x_1=x_2=0} = f_v(0) - f_w(0) = 0.$$

It follows that there is a second branch of saddle node solutions of the form

$$(x_1, Y_2(x_1), \varphi(x_1), \Lambda(x_1)).$$

Whether either branch is super or subcritical is determined by the sign of $\Lambda''(0)$. □

Remark 4.17. In the generic bifurcation problem governed by theorem 4.16, only the synchronous equilibria can be stable near the origin. The eigenvalues of the Jacobian matrix at the asynchronous solutions have the form

$$\begin{aligned}
&(f_u + 2f_v)(x_1, x_1, x_1, \Lambda(x_1)) = Ax_1 + \cdots, \\
&f_u(Y_2(x_1), x_1, \varphi(x_1), \Lambda(x_1)) = O(x_1^2), \\
&(f_u + 2f_v)(\varphi(x_1), \varphi(x_1), \varphi(x_1), \Lambda(x_1)) = -Ax_1 + \cdots.
\end{aligned}$$

The instability of these solutions follows from the nondegeneracy condition $A \neq 0$. ◊

5. Classification of codimension one Hopf bifurcations

In this section we classify the synchrony-breaking codimension one Hopf bifurcations from a synchronous equilibrium that occur in coupled cell systems associated with the 34 networks listed in figure 5. Note that the generic synchrony-preserving Hopf bifurcation occurs at simple eigenvalues and the bifurcations reduce to a standard Hopf bifurcation on the synchrony subspace Δ. Unlike in the classification of steady-state bifurcations in section 4, we do not discuss the asymptotic stability of solutions.

More precisely, we classify the synchrony-breaking Hopf bifurcations for each bifurcation type identified in section 3 (also see proposition 3.3), which includes

- H1: simple critical eigenvalues that are forced by network architecture to be complex,
- H2: simple critical eigenvalues with eigenvectors not in Δ,
- H3: double critical eigenvalues with a complete set of eigenvectors not in Δ.
- H4: double critical eigenvalues with an incomplete set of eigenvectors,
- H5: double critical eigenvalues with a complete set of eigenvectors some of them in Δ.

For each bifurcation type, we find the number of branches of periodic solutions. We also determine the approximate patterns (in symmetric networks the patterns are in fact exact) of oscillations associated with solutions in each branch; these characteristics of solutions are network invariants.

In sections 5.1, 5.2, 5.3 and 5.5 we classify H1, H2, H3 and H5, respectively. The analysis of H4 bifurcations is contained in Elmhirst and Golubitsky [6] and their results are reviewed in section 5.4. Note that H1 bifurcations can occur when $k = 1$, whereas the others require $k \geqslant 2$. Note also that admissible vector fields with internal dimension k can be embedded in admissible vector fields with internal dimension ℓ when $\ell > k$. Based on that observation, it can be shown that the classes of codimension one bifurcations that can occur when $\ell > k$ are identical to those that can occur for $k = 1$ (H1) and $k = 2$ (all other Hopf bifurcations).

5.1. H1: simple complex eigenvalues

H1 bifurcations can occur in networks 2, 14, 18, 19 and 24 (see section 3). Because the critical eigenvalues are simple the standard Hopf bifurcation theorem applies. Moreover, because the eigenvalues of the adjacency matrix are complex, Hopf bifurcation can occur when the dimension of the internal dynamics is $k = 1$, as well as for larger k. When the critical eigenvalues cross the imaginary axis with nonzero speed, there exists a unique branch of small amplitude periodic solutions.

We show that H1 bifurcations lead to relationships, at lowest order in λ, between the projected amplitudes and phases of solutions in the three cells. This phenomenon is well known in the $\boldsymbol{Z}_3$ symmetric network 2, where the bifurcating solutions are all discrete rotating waves; that is, the wave forms in each cell are identical with exact one-third period phase shifts between adjacent cells cf [10]. Note that two different rotating waves can occur in network 2: either cell 2 lags cell 1 by one-third or by two-thirds of a period.

Patterns of oscillations of periodic solutions. Suppose that the Jacobian J of a coupled cell network at a synchronous equilibrium has a Hopf bifurcation with simple critical eigenvalues. Recall that the eigenvalues and eigenvectors of J can be computed using proposition 3.1. Let μ_1, μ_2, μ_3 be the eigenvalues of the adjacency matrix A. For H1 networks μ_1 is the valency of the network and $\mu_3 = \overline{\mu_2} \in \boldsymbol{C}$. The eigenvalues of J are the eigenvalues of one of the $k \times k$ submatrices $Q + \mu_j R$. H1 bifurcations occur when the matrix $Q + \mu_2 R$ has a purely imaginary eigenvalue $2\pi\omega\mathrm{i}$ (then the matrix $Q+\mu_3 R$ will have $-2\pi\omega\mathrm{i}$ as an eigenvalue). For the remainder of this discussion we fix μ_2.

Let

$$z = (z_1, z_2, z_3) \in \boldsymbol{C}^3$$

be an eigenvector of A corresponding to μ_2. Let ζ be the critical eigenvector, that is $J\zeta = 2\pi\omega\mathrm{i}\zeta$. It follows from the proof of proposition 3.1 that ζ has the form

$$\zeta = (z_1 x, z_2 x, z_3 x)$$

for some $x \in \boldsymbol{R}^k$. Moreover, for H1 bifurcations, we can assume $z_1 = 1$. It is straightforward to check that for these networks the eigenvalue crossing condition in Hopf bifurcation is satisfied if $f_{u\lambda}(0) \neq 0$. Under this assumption a unique branch of periodic solutions emanates from the bifurcation point, and the amplitude of the solutions on this branch grow at the rate $\lambda^{1/2}$. Indeed, to lowest order in λ, the bifurcating periodic solutions have the form

$$X(t) = \lambda^{1/2}\mathrm{Re}(\mathrm{e}^{tJ}\zeta) = \lambda^{1/2}\mathrm{Re}(\mathrm{e}^{2\pi\omega\mathrm{i}t}\zeta).$$

If we let

$$A_j = \lambda^{1/2} z_j x,$$

then $|A_j|$ is the maximum amplitude of the projection of the periodic solution in cell j and the ratios $|A_2|/|A_1| = |z_2|$ and $|A_3|/|A_1| = |z_3|$ are independent of λ. Moreover, since the form of the eigenvector ζ depends only on which eigenvalue in A is critical and not on the specific admissible vector field associated with Hopf bifurcation, these ratios are network invariants.

Next, we define the *phases relative to cell 1* as $\phi_1 = 0$, $z_2 = \mathrm{e}^{2\pi\mathrm{i}\phi_2} y_2$ and $z_3 = \mathrm{e}^{2\pi\mathrm{i}\phi_3} y_3$ where $y_2, y_3 \geqslant 0$ and $0 \leqslant \phi_2, \phi_3 < 1$. Then, to lowest order in λ, the branch of periodic solutions has the form

$$X(t) \approx (\mathrm{Re}(\mathrm{e}^{2\pi\omega\mathrm{i}(t+\phi_1)} A_1), \mathrm{Re}(\mathrm{e}^{2\pi\omega\mathrm{i}(t+\phi_2)} A_2), \mathrm{Re}(\mathrm{e}^{2\pi\omega\mathrm{i}(t+\phi_3)} A_3)). \tag{5.1}$$

Note that the relative phase shifts $\phi_2 - \phi_1$ and $\phi_3 - \phi_1$ between cell coordinate projections of these periodic solutions actually depend on the sign of ω. If $\omega > 0$, cell 2 lags cell 1 by ϕ_2, whereas if $\omega < 0$, then the reverse is true. Since we are normalizing phases by cell 1, we say that in the second case cell 2 lags cell 1 by $1 - \phi_2$.

We have proved the following theorem.

Theorem 5.1. *Consider a homogeneous three-cell network in which asynchronous periodic solutions are obtained by Hopf bifurcation with simple eigenvalues. There are two types of Hopf bifurcation. In each, at lowest order in the bifurcation parameter, the periodic solutions on the unique bifurcating branch have amplitude and phase relations between cells that depend only on which eigenvalue of the adjacency matrix of the network is critical and not on the choice of the vector field.*

For each network with an H1 bifurcation, we list in table 9 the eigenvalue μ_2, the critical eigenvector of $Q + \mu_2 R$ in block form and the relevant amplitude ratios and relative phases. The form of the admissible systems may be found in table 2.

The patterns of oscillation for networks 2, 14, 18, 19 and 24 are given in table 9. As an example we illustrate how to calculate the phase and amplitude relations for network 18. From Table 9 we see that the eigenvector of the adjacency matrix corresponding to the eigenvalue $\mu_2 = -\frac{1-\sqrt{3}\mathrm{i}}{2}$ is $z = \left(1, \frac{-3+\sqrt{3}\mathrm{i}}{2}, -1-\sqrt{3}\mathrm{i}\right)$. Therefore,

$$\frac{|A_2|}{|A_1|} = \sqrt{3} \qquad \frac{|A_3|}{|A_1|} = 2 \qquad \phi_2 = \frac{5}{12} \approx 0.42 \qquad \phi_3 = \frac{2}{3} \approx 0.67.$$

Table 9. H1 bifurcations: $2\pi\omega\mathrm{i}$ is critical eigenvalue of $Q + \mu_2 R$. Critical eigenvector is in block form ($x \in \boldsymbol{R}^k$).

Net	μ_2	Eigenvector	$\frac{\|A_2\|}{\|A_1\|}$	$\frac{\|A_3\|}{\|A_1\|}$	sgn(ω)	$\phi_2 - \phi_1$	$\phi_3 - \phi_1$
2	$-\frac{1-\sqrt{3}\mathrm{i}}{2}$	$\left(x, -\frac{1+\sqrt{3}\mathrm{i}}{2}x, -\frac{1-\sqrt{3}\mathrm{i}}{2}x\right)$	1	1	+	2/3	1/3
					−	1/3	2/3
14	$-1+\mathrm{i}$	$\left(x, -(1+\mathrm{i})x, -\frac{1-\mathrm{i}}{2}x\right)$	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	+	5/8	3/8
					−	3/8	5/8
18	$-\frac{1-\sqrt{3}\mathrm{i}}{2}$	$\left(x, -\frac{3-\sqrt{3}\mathrm{i}}{2}x, -(1+\sqrt{3}\mathrm{i})x\right)$	$\sqrt{3}$	2	+	5/12	2/3
					−	7/12	1/3
19	$-\frac{1-\sqrt{7}\mathrm{i}}{2}$	$\left(x, -\frac{3-\sqrt{7}\mathrm{i}}{2}x, -\frac{1+\sqrt{7}\mathrm{i}}{2}x\right)$	2	$\sqrt{2}$	+	0.38…	0.19…
					−	0.61…	0.80…
24	i	$\left(x, -\frac{1+\mathrm{i}}{2}x, -(1-\mathrm{i})x\right)$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	+	5/8	3/8
					−	3/8	5/8

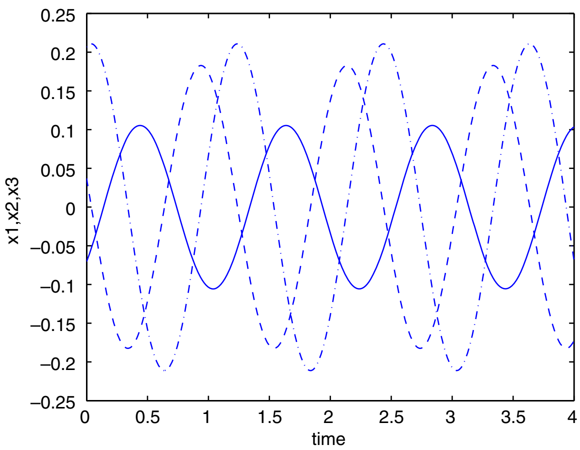

Figure 9. Time series from networks 18: superimposed cells. The solid, dashed and dashed–dot lines correspond to x_1, x_2 and x_3, respectively. The amplitude and phase relations between cells that are obtained numerically are $\frac{|A_2|}{|A_1|} \approx 1.73$, $\frac{|A_3|}{|A_1|} \approx 1.91$, $\phi_2 - \phi_1 \approx 0.33$ and $\phi_3 - \phi_1 \approx 0.60$.

(This figure is in colour only in the electronic version)

A time series from integrating an admissible system of ODE is shown in figure 9. It follows that the bifurcation corresponding $\omega < 0$ satisfies

$$\frac{|A_2|}{|A_1|} = \sqrt{3} \qquad \frac{|A_3|}{|A_1|} = 2 \qquad \phi_2 = \frac{7}{12} \qquad \phi_3 = \frac{1}{3}.$$

5.2. H2: simple real eigenvalues

From section 3 it follows that the 21 networks 1, 5, 9, 10, 11, 13–16, 17, 20, 21–23, 25, 26, 29–34 can exhibit H2 bifurcations. These bifurcations can only occur when the dimension of the internal dynamics in each cell satisfies $k \geqslant 2$. The coupled cell systems associated with these networks are listed in table 2.

Patterns in periodic solutions. There are similarities and differences between the H2 and H1 Hopf bifurcations. In both case the critical eigenvalues are simple and after a rescaling of time lead to a unique branch of small amplitude periodic solutions via the standard Hopf bifurcation theorem. The rate of amplitude growth is in both cases $\lambda^{1/2}$. At lowest order in the bifurcation parameter λ, the solutions have the form (5.1), where the amplitudes $A_j = \lambda^{1/2} z_j x$ are formed from a complex vector $x \in \boldsymbol{C}^k$ and (unlike the H1 bifurcations) a real eigenvector $(z_1, z_2, z_3) \in \boldsymbol{R}^3$ of the network adjacency matrix A.

Suppose that all z_j are nonzero. Then we can assume that $z_1 = 1$ and the amplitude ratios $|A_2|/|A_1| = |z_2|$ and $|A_3|/|A_1| = |z_3|$ are independent of λ and the specific admissible vector field. Moreover, as before, we can write the relative phases as $\phi_1 = 0$, $z_2 = e^{2\pi i \phi_2} y_2$ and $z_3 = e^{2\pi i \phi_3} y_3$ where y_2, y_3 are real and nonnegative and here ϕ_2, ϕ_3 can equal only 0 or $\frac{1}{2}$. That is, at lowest order, the periodic states are either synchronous or a half-period out of phase.

There are two additional noteworthy features in these networks. First, one or more of the z_j can equal zero so that at lowest order in λ the periodic solution is zero. In some of these cases the periodic solution will project to be identically zero in certain cells; whereas, in other networks the periodic solution is constant only up to order $\lambda^{1/2}$. Second, in some of these networks the periodic solution will project to be identically synchronous in two cells; whereas in other networks the synchrony is only up to order $\lambda^{1/2}$. Note that networks with critical eigenvectors with two cell components equal will be synchronous to lowest order in λ, but exact synchrony only follows for symmetry or interior symmetry in these networks.

In lemma 5.2 we show that periodic solutions in coupled cell systems with skew-product form can have cells that are constant for all time.

Lemma 5.2. *Consider a homogeneous three-cell network with valency $n = 1, 2$ in which H2 bifurcation can occur. Assume that cell i receives n self-couplings inputs. Then the coordinate of the periodic solution emanating from this H2 bifurcation satisfies $x_i(t) \equiv 0$ for all t, where x_i is the coordinate on cell i.*

Proof. Without loss of generality we may assume that cell 1 has n self-coupling inputs. The differential equation associated with cell 1 is

$$\dot{x}_1 = f(x_1, x_1, x_1, \lambda) \qquad \text{when } n = 2,$$

$$\dot{x}_1 = f(x_1, x_1, \lambda) \qquad \text{when } n = 1.$$

Since either $f(0, 0, 0, \lambda) = 0$ or $f(0, 0, \lambda) = 0$ (see section 4.2), it follows that the subspace $V_1 = \{X \in (\boldsymbol{R}^k)^3 : x_1 = 0\}$ is flow-invariant. The Jacobian of the coupled cell system has the form

$$J = \begin{bmatrix} Q + nR & 0 \\ * & J_2 \end{bmatrix},$$

where J_2 is a $2k \times 2k$ matrix. Since eigenvalues of $Q + nR$ correspond to synchrony preserving bifurcations, the centre subspace at an H2 bifurcation corresponds to eigenvalues of J_2 and has critical eigenvectors in V_1. It follows that the centre subspace and the centre manifold of an H2 bifurcation are contained in V_1. Hence $x_1(t) \equiv 0$ for any periodic solution emanating from an H2 bifurcation. $\square$

The argument in this lemma can iterate. If the coordinate of cell i is 0, all of the couplings to cell j are either self-coupling or from cell i, and the coordinate of the critical eigenvector on cell j is 0, then the conclusion that cell j has coordinate $x_j(t) \equiv 0$ holds for all t.

The networks discussed in lemma 5.2 all have a feed-forward structure. There are eight networks satisfying the hypothesis of this lemma that also have H2 bifurcations; they are 9, 12 and 29–34. There are another 11 networks that have an H2 bifurcation with a critical eigenvector with some cell components equal 0. In these networks, the bifurcating periodic solutions have cells that are constant to lowest order in λ, but nonzero at higher order. These networks are: 1, 10, 13, 15, 16, 17, 20, 21, 22, 25 and 26. This phenomenon is known in the $\mathbf{Z}_2$-symmetric network 10, where Hopf bifurcation leads to solutions where cells 1 and 3 are a half-period out of phase and cell 2 oscillates at twice the frequency. The twice frequency cell projection is known to be of small amplitude in this bifurcation [10, 11] and can also be found in networks 13, 15 and for the same reasons.

Exact synchrony can be observed either by direct observation or by balanced colourings [13] or by symmetry (fixed-point subspaces [10]) or by interior symmetry [8]. Exact phase shift synchrony is only a consequence of symmetry. We indicate exact synchrony and exact phase shift synchrony of bifurcating solutions in the last column of table 10. A blank entry in that column indicates that any synchrony or phase shift synchrony that may exist (according to the critical eigenvectors) is only valid to lowest order in λ.

A remark on stability.

Remark 5.3. In networks in which H2 bifurcation can occur in two ways and when $k = 2$, there is one bifurcation for which the periodic solutions on the bifurcating branch can be asymptotically stable near the bifurcation and one for which the solutions on the bifurcating branch are always unstable. In network 11 the periodic solutions on the bifurcating branch can be asymptotically stable near the origin. $\lozenge$

We discuss remark 5.3 considering networks 5 and 11. With network 5 we illustrate how the result follows for networks in which H2 can occur in two ways.

Consider network 5. The eigenvalues of J are eigenvalues of $B_1 = Q+2R$, Q and $Q-R$. We assume Q has a pair of purely imaginary eigenvalues. This implies that $\mathrm{tr}(Q) = 0$. The stability of solutions is determined by the six eigenvalues of J. The noncritical eigenvalues must have negative real part in order to have asymptotically stable periodic solutions near bifurcation. That is, the following conditions must hold simultaneously:

$$\mathrm{tr}(Q+2R) < 0, \qquad \mathrm{tr}(Q-R) < 0, \qquad \det(Q+2R) > 0 \text{ and } \det(Q-R) > 0.$$

Since $\mathrm{tr}(Q) = 0$, it follows that $\mathrm{tr}(Q+2R) = 2\mathrm{tr}(R) < 0$ and $\mathrm{tr}(Q-R) = -\mathrm{tr}(R) < 0$. These two conditions are not compatible. Therefore, the branch of periodic solutions on the branch emanating from this bifurcation is always unstable. Next, we consider the bifurcation for which $Q - R$ has critical eigenvalues. A similar argument shows that the noncritical eigenvalues have negative real part if the following condition holds:

$$\mathrm{tr}(Q) = \mathrm{tr}(R) < 0 \qquad \text{and} \qquad \det(Q+2R) > 0 \qquad \text{and} \qquad \det(Q) > 0.$$

Since there is a choice of Q and R for which these conditions are satisfied, the branch of periodic solutions emanating from this bifurcation can be asymptotically stable near the bifurcation and the stability is determined by the standard exchange of stability for Hopf bifurcation.

Table 10. Patterns of oscillations associated with the 21 networks where H2 bifurcations can occur. Generically, periodic solutions grow with the standard rate of $\lambda^{1/2}$. If the eigenvector has a 0 component and the corresponding cell is identically 0, then that component is denoted by $0^\star$. Otherwise the component amplitude grows with rate λ. Ratios of nonzero components in the eigenvector give ratio of maximum amplitudes in the cell coordinates of the bifurcating solution. The oscillations of two components of a solution are in phase if the constants are of the same sign and a half-period out of phase if constants are of opposite sign. Exact synchrony and exact phase synchrony are listed in the last column.

Network	Matrices with critical Ev's	Patterns of oscillation	Synchrony
1	Q	$(0, 0, 1)$	$x_1(t) = x_2(t)$
	$Q - R$	$(1, -1, 1)$	—
5	Q	$(1, -1, 1)$	—
	$Q - R$	$(1, -2, 1)$	—
9	$Q + \frac{1+\sqrt{5}}{2}R$	$\left(0^\star, 1, \frac{-1+\sqrt{5}}{2}\right)$	—
	$Q + \frac{1-\sqrt{5}}{2}R$	$\left(0^\star, 1, \frac{1-\sqrt{5}}{2}\right)$	—
10	$Q - R$	$(1, 2, 1)$	$x_1(t) = x_3(t)$
	$Q + R$	$(1, 0, -1)$	$x_1(t) = x_3\left(t + \frac{1}{2}\right)$
12	Q	$(0^\star, 1, 0^\star)$	$x_1(t) = x_3(t)$
13	Q	$(0, 0, 1)$	$x_1(t) = x_2(t)$
	$Q - 2R$	$(1, -1, 0)$	$x_1(t) = x_2\left(t + \frac{1}{2}\right)$
15	Q	$(1, -1, 0)$	$x_1(t) = x_2\left(t + \frac{1}{2}\right)$
	$Q - 2R$	$(1, 1, -1)$	$x_1(t) = x_2(t)$
16	Q	$(0, 1, 0)$	—
	$Q - R$	$(1, 1, -2)$	—
17	Q	$(1, -1, 0)$	—
	$Q - R$	$(1, -2, 1)$	—
20	$Q + R$	$(1, 0, 0)$	$x_3(t) = x_2(t)$
	$Q - 2R$	$(1, -3, 3)$	
21	Q	$(0, 1, 0)$	—
	$Q - R$	$(1, 4, -2)$	—
22	Q	$(0, 1, 0)$	—
	$Q - R$	$(1, -2, -2)$	—
23	$Q + \frac{-1+\sqrt{5}}{2}R$	$\left(1, \frac{-3+\sqrt{5}}{2}, 2\frac{3-\sqrt{5}}{1-\sqrt{5}}\right)$	—
	$Q - \frac{1+\sqrt{5}}{2}R$	$\left(1, \frac{-3+\sqrt{5}}{2}, 2\frac{3+\sqrt{5}}{1+\sqrt{5}}\right)$	—
25	$Q - R$	$(1, 1, -2)$	—
	$Q + R$	$(0, 1, 0)$	—
26	$Q - R$	$\left(1, -\frac{1}{2}, -2\right)$	
	$Q + R$	$(0, 1, 0)$	—
29	$Q + R$	$(0^\star, 1, 1)$	$x_3(t) = x_2(t)$
	$Q - R$	$(0^\star, 1, -1)$	$x_3(t) = x_2\left(t + \frac{1}{2}\right)$
30	$Q + \sqrt{2}R$	$(0^\star, 1, \sqrt{2})$	—

Table 10. Continued.

Network	Matrices with critical Ev's	Patterns of oscillation	Synchrony
	$Q-\sqrt{2}R$	$(0^\star, 1, -\sqrt{2})$	—
31	$Q+R$	$(0^\star, 1, 1)$	—
	Q	$(0^\star, 0^\star, 1)$	—
32	$Q+R$	$(0^\star, 1, 2)$	—
	Q	$(0^\star, 0^\star, 1)$	—
33	$Q+R$	$(0^\star, 1, 0^\star)$	—
	Q	$(0^\star, 0^\star, 1)$	—
34	$Q+R$	$(0^\star, 1, 0)$	—
	Q	$(0^\star, 1, -1)$	—

Consider network 12. The stability is determined by the eigenvalues of J that are close to the eigenvalues of $B_1 = Q+2R$ (repeated twice) and of Q. By definition of H2 bifurcation type, the noncritical eigenvalues of J are eigenvalues of $Q+2R$, which must have negative real part in order to have stable periodic solutions near the bifurcation. So, to find bifurcations where asymptotically periodic solutions can appear, we must assume that the real part of eigenvalues of B_1 is negative. Hence, asymptotically stable periodic solutions can occur near the bifurcation and stability is determined by standard exchange of stability for Hopf bifurcation.

5.3. H3: double eigenvalues with two eigenvectors

From section 3, networks 4, 7 and 8 can have H3 bifurcations if $k \geqslant 2$. So we assume $k = 2$, and we discuss the bifurcations first for network 8, then for network 4 and finally for network 7.

Network 8. Network 8 is the well-known bidirectional ring with $\boldsymbol{S_3}$-symmetry. The associated periodic solutions that can be obtained via synchrony-breaking bifurcations (more precisely, via symmetry-breaking bifurcations) are completly described in [11, Section 3.4]. There are three types of periodic solutions: discrete rotating waves (each cell is one-third of a period out of phase with the next, two cells in phase and two cells one half-period out of phase with the third cell oscillating at double frequency).

Network 4. Consider the coupled cell system associated with network 4 (see table 2):

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, \lambda), \\ \dot{x}_2 &= f(x_2, x_2, \lambda), \\ \dot{x}_3 &= f(x_3, x_2, \lambda). \end{aligned} \tag{5.2}$$

We show that H3 synchrony breaking bifurcations in this network lead to a unique branch of 2-tori, each foliated by periodic solutions. The four-dimensional centre subspace is

$$E^c = \{(x_1, x_2, x_3) \in (\boldsymbol{R}^2)^3 : x_2 = 0\}.$$

Moreover, the assumption of a trivial solution shows by inspection that E^c is flow-invariant for the nonlinear system and hence is the centre manifold. (This observation could also follow from lemma 5.2.) Equations (5.2) restricted to the centre manifold E^c have the decoupled form

$$\begin{aligned} \dot{x}_1 &= f(x_1, 0, \lambda), \\ \dot{x}_3 &= f(x_3, 0, \lambda). \end{aligned} \tag{5.3}$$

Generically, the first equation in (5.3) undergoes a nondegenerate Hopf bifurcation with simple eigenvalues leading to a family of periodic solutions $x_1^\lambda(t)$ whose amplitude growth is $\lambda^{1/2}$ and whose stability (in the x_1 directions) is given by exchange of stability. Of course, the second equation in (5.3) is identical to the first, so this equation undergoes the same bifurcation leading to a family of solutions $x_3^\lambda(t)$. The second family of solutions is identical to the first except for an arbitrary phase shift; that is,

$$x_3^\lambda(t) = x_1^\lambda(t + t_0^\lambda).$$

We thus get a unique family of tori as solutions to (5.2).

Observe that when these tori are stable nearby solutions will limit on a single periodic orbit exhibiting a fixed phase shift t_0^λ between x_1 and x_3.

Network 7. The coupled cell system associated with network 7 is

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1, x_2, \lambda), \\ \dot{x}_2 &= f(x_2, x_2, x_1, \lambda), \\ \dot{x}_3 &= f(x_3, x_1, x_2, \lambda). \end{aligned} \tag{5.4}$$

We claim that generically there are two branches of periodic solutions that emanate from an H3 bifurcation in network 7 and that these branches are independently either super or subcritical.

First, observe that the centre subspace of (5.4) corresponding to an H3 bifurcation at the origin is the four-dimensional subspace

$$E^c = \{(z_1, -z_1, z_2) : z_1, z_2 \in \boldsymbol{C}\}.$$

Second, note that (5.4) decouples so that the first two equations

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1, x_2, \lambda), \\ \dot{x}_2 &= f(x_2, x_2, x_1, \lambda) \end{aligned} \tag{5.5}$$

are independent of the third. Generically, there are two branches of solutions to (5.5): the trivial equilibrium $x_1 = x_2 = 0$ and a unique branch of periodic solutions where $x_1(t)$ and $x_2(t)$ are a half-period out of phase and the amplitude of x_1 grows at rate $\lambda^{1/2}$. This second branch is obtained by a standard symmetry-breaking Hopf bifurcation using the fact that $\sigma(x_1, x_2) = (x_2, x_1)$ is a symmetry of (5.5).

Third, when $x_1 = x_2 = 0$, the third equation in (5.4) has the form

$$\dot{x}_3 = f(x_3, 0, 0, \lambda). \tag{5.6}$$

Under the assumption of an H3 bifurcation, the standard Hopf bifurcation theorems imply the existence of a unique branch of periodic solutions. Note that whether this branch of solutions is super or subcritical is independent of whether the branch of half-period out of phase solutions to (5.5) is super or subcritical.

Finally, we consider the third equation when $(x_1(t), x_2(t))$ is a half-period out of phase solution. In this case we can view the third equation as a periodically forced equation

$$\dot{x}_3 = f(x_3, x_1(t), x_2(t), \lambda). \tag{5.7}$$

We prove that under these assumptions $x_3(t)$ is periodic in t with the same frequency as $x_1(t)$ and whose amplitude also grows (generically) at the rate $\lambda^{1/2}$. We complete this last step by applying the Liapunov–Schmidt approach to Hopf bifurcation directly to (5.4).

Proof. As usual with the Liapunov–Schmidt proof, we look for small amplitude near 1-periodic solutions by converting (5.4) to an operator on continuously differentiable 1-periodic functions and introducing a perturbed period parameter τ. We note that the linearization of this operator at the bifurcation point has a four-dimensional kernel and cokernel given by E^c and that the linearized operator commutes with $\boldsymbol{S}^1$ where the action of $\boldsymbol{S}^1$ on E^c is defined by

$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

Liapunov–Schmidt reduction implies that solving the operator equation for 1-periodic solutions is equivalent to solving the reduced system

$$\begin{aligned} h_1(z_1, \lambda, \tau) &= 0, \\ h_2(z_1, z_2, \lambda, \tau) &= 0, \end{aligned} \tag{5.8}$$

where $h_1 : \boldsymbol{C} \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{C}$, $h_2 : \boldsymbol{C}^2 \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{C}$ and $h = (h_1, h_2)$ is $\boldsymbol{S}^1$-equivariant. Observe that h_1 is independent of z_2 and this follows from the feed-forward structure of network 7; that is, (5.5) is independent of x_3.

The $\boldsymbol{S}^1$ equivariance implies that (5.8) may be written uniquely in the form

$$\begin{aligned} &P(|z_1|^2, \lambda, \tau) z_1 = 0, \\ &Q(|z_1|^2, |z_2|^2, z_1\overline{z_2}, \lambda, \tau) z_1 + R(|z_1|^2, |z_2|^2, z_2\overline{z_1}, \lambda, \tau) z_2 = 0, \end{aligned} \tag{5.9}$$

where P, Q, R are complex-valued. This is a standard equivariant theory calculation as may be found in [10]. The first equation leads to $z_1 = 0$ or $P = 0$. If $z_1 = 0$, then the second equation becomes $R(0, |z_2|^2, 0, \lambda, \tau) z_2 = 0$. This equation yields the trivial equilibrium and the branch of 1-periodic solutions of the form $x_1 = x_2 = 0$ discussed previously.

Using $\boldsymbol{S}^1$ symmetry we may assume that $z_1 > 0$ and $P = 0$. Assuming the eigenvalue crossing condition, we can solve $P = 0$ for $\tau = \tau(z_1^2)$ and $\lambda(z_1^2)$ as in the standard proof of simple eigenvalue Hopf bifurcation. Moreover, generically $\lambda'(0) \neq 0$ which yields the super or subcriticality of the half-period out of phase solution to (5.5) discussed previously.

The last step in this proof shows that

$$Q(z_1^2, |z_2|^2, z_1\overline{z_2}, \lambda(z_1^2), \tau(z_1^2)) z_1 + R(z_1^2, |z_2|^2, z_2, z_1, \lambda(z_1^2), \tau(z_1^2)) z_2 = 0 \tag{5.10}$$

can be solved uniquely for z_2 as a function of z_1. Since $z_1 \neq 0$ we can write $z_2 = y z_1$ for $y \in \boldsymbol{C}$. Then, after dividing by z_1, (5.10) becomes

$$Q(z_1^2, |y|^2 z_1^2, z_1^2 \overline{y}, \lambda(z_1^2), \tau(z_1^2)) + R(z_1^2, |y|^2 z_1^2, z_1^2 y, \lambda(z_1^2), \tau(z_1^2)) y = 0. \tag{5.11}$$

Since $\lambda(0) = \tau(0) = 0$ and $Q(0) = R(0) = 0$, it follows that we can write

$$Q = S(z_1^2, y) z_1^2 \text{ and } R = T(z_1^2, y) z_1^2.$$

On dividing by z_1^2, (5.11) becomes $S(z_1^2, y) + T(z_1^2, y) y = 0$. The implicit function theorem completes the proof if we can show that $S(0) \neq 0$ and $T(0) \neq 0$. However,

$$\begin{aligned} S(0) &= Q_1(0) + Q_4(0)\lambda'(0) + Q_5(0)\tau'(0), \\ T(0) &= R_1(0) + R_4(0)\lambda'(0) + R_5(0)\tau'(0), \end{aligned}$$

where the subscripts denote partial derivatives. Generically $S(0)$ and $T(0)$ are nonzero. □

5.4. H4: double eigenvalue with an incomplete set of eigenvectors

In this section we state the results obtained on H4 bifurcations studied in [6]. These results are the following.

(a) In networks 3 and 28 Hopf bifurcations generically leads to two branches of periodic solutions with amplitude growth at rates of $\lambda^{1/6}$ and $\lambda^{1/2}$. The second branch is unstable near transition.

(b) In networks 6 and 11 Hopf bifurcations generically leads to two or four branches of periodic solutions with amplitude growth of order $\lambda^{1/2}$.

(c) In network 27 Hopf bifurcations generically leads to a branch of periodic solutions that grows at rate $\lambda^{1/2}$ in cells 1 and 2. However, the amplitude of oscillations in cell 3 grows at rate $\lambda^{1/6}$. The periodic solutions are characterized by cells 1 and 2 oscillating with the same frequency, but forced by (interior) symmetry to be a half-period out of phase.

5.5. H5: double eigenvalue with eigenvector in synchrony subspace

Only network 11 can have an H5 bifurcation. The coupled cell system for this network is:

$$\begin{aligned}
\dot{x}_1 &= f(x_1, x_1, x_1, \lambda), \\
\dot{x}_2 &= f(x_2, \overline{x_3, x_1}, \lambda), \\
\dot{x}_3 &= f(x_3, x_3, x_3, \lambda).
\end{aligned} \tag{5.12}$$

We claim that generically this bifurcation yields two branches of periodic solutions and a branch of tori foliated by periodic solutions. To see this observe that the first and third equations decouple from (5.12) and are identical in form. So each equation individually yields a branch of trivial equilibria and a branch of periodic solutions. If $x_1 = x_3 = 0$, then the second equation has the form

$$\dot{x}_2 = f(x_2, 0, 0, \lambda). \tag{5.13}$$

Observe that $x_2 = 0$ is a hyperbolic equilibrium for (5.13). Hence, $x_2 = 0$ is the only 'periodic' solution near the origin and that periodic solution is the trivial solution.

Suppose that $x_1(t)$ and $x_3(t)$ are periodic solutions (one of which may be zero). Then the second equation in (5.12) is a periodically forced equation (5.13). It follows from the standard theory of forced equations that the solution $x_2(t)$ is periodic with the same frequency as (x_1, x_3) and with the same order of amplitude in λ. In particular, if $x_1 = 0$ and $x_3(t) \neq 0$ (or conversely), we obtain a periodic solution to (5.12). This yields the two isolated families noted above, since the third equation in (5.12) can undergo a Hopf bifurcation.

Similarly, the standard simple eigenvalue Hopf theorems imply the existence of a unique branch of periodic solutions $(x_1(t), x_3(t))$ (both coordinates of which are nonzero) with amplitude growth rate of $\lambda^{1/2}$. Moreover, since the two equations are identical in form the trajectories x_3 and x_1 must be identical. It follows that $x_3(t) = x_1(t + \theta)$ for some choice of θ (depending on λ). Again treating the second equation in (5.12) as a periodically forced perturbation of (5.13) shows that the second equation can be solved uniquely for a periodic solution $x_2(t)$ whose frequency equals that of $x_1(t)$. It follows that this H5 bifurcation leads to a branch of tori foliated by periodic solutions.

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