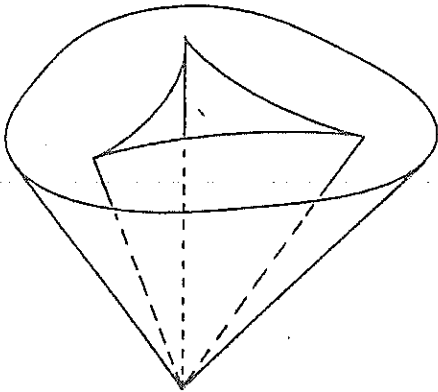


M. Roberts I. Stewart (Eds.)

# Singularity Theory and its Applications

Warwick 1989, Part II



Springer-Verlag

# Boundary Conditions as Symmetry Constraints

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## ABSTRACT

Fujii, Mimura, and Nishiura [1985] and Armbruster and Dangelmayr [1986, 1987] have observed that reaction-diffusion equations on the interval with Neumann boundary conditions can be viewed as restrictions of similar problems with periodic boundary conditions; and that this extension reveals the presence of additional symmetry constraints which affect the generic bifurcation phenomena. We show that, more generally, similar observations hold for multi-dimensional rectangular domains with either Neumann or Dirichlet boundary conditions, and analyse the group-theoretic restrictions that this structure imposes upon bifurcations. We discuss a number of examples of these phenomena that arise in applications, including the Taylor-Couette experiment, Rayleigh-Bénard convection, and the Faraday experiment.

## 0 Introduction

Let  $\mathcal{P}(u)$  denote a reaction-diffusion equation on the line. Then  $\mathcal{P}(u)$  is invariant under translations and reflections. It is well known that a solution  $u(x)$  to  $\mathcal{P}(u) = 0$  on the interval  $[0, \pi]$  with Neumann boundary conditions (NBC) may be extended to a solution of the same PDE on the whole line that satisfies periodic boundary conditions (PBC) on the interval  $[-\pi, \pi]$ . This extension is accomplished by reflection across the boundaries, that is, by defining

$$u(-x) = u(x) \quad \text{for } x \in [-\pi, 0]$$

and then extending  $u$  to be  $2\pi$ -periodic on  $\mathbb{R}$ . By using Euclidean invariance, the Neumann boundary conditions, and the second order structure of the PDE, it is not hard to show that this extension procedure preserves regularity of the solutions: for example  $C^\infty$  solutions remain  $C^\infty$ , provided the operator  $\mathcal{P}$  is itself  $C^\infty$ .

Fujii, Mimura, and Nishiura [1985] and Armbruster and Dangelmayr [1986, 1987] observe that this extension property changes, in a subtle way, the *generic* behaviour of codimension two steady-state mode interactions. In this note we give a straightforward group-theoretic description of why genericity is affected by this extension property: simply put, NBC can be thought of as a symmetry constraint on the PBC problem.

Using this general construction we indicate several ways in which this idea may be extended. In particular, we show how the same general construction can be applied to Dirichlet boundary conditions (DBC) and to Euclidean-invariant PDEs in several spatial variables, defined on generalized rectangles. We also remark on several physical systems whose analyses illustrate these ideas. These include the Taylor-

Couette experiment, Rayleigh-Bénard convection, and the Faraday experiment.

This note was inspired by several conversations and seminars held at the week-long Workshop on Dynamics, Bifurcation, and Singularity Theory during the 1988-89 Warwick Symposium on Singularity Theory and its Applications. During the workshop it became clear that the same issues concerning boundary conditions were appearing in a variety of applications, and it was felt that a short note for the conference proceedings, exploring these issues, would be both appropriate and of general interest. We wish to thank the rather large group of participants, in particular Andrew Cliffe and Tom Mullin, who joined in those conversations and who helped formulate the ideas presented here.

## 1 Neumann Boundary Conditions and Symmetry

As before, let  $\mathcal{P}(u)$  denote a reaction-diffusion equation on the line, which is invariant under translations and reflections.

**Lemma 1.1** Solutions to  $\mathcal{P}(u) = 0$  satisfying Neumann boundary conditions on  $[0, \pi]$  are in 1:1 correspondence with solutions satisfying periodic boundary conditions on  $[-\pi, \pi]$  having the symmetry

$$u(-x) = u(x). \quad (1.1)$$

**Remark** Define the two-element group

$$B_N = \{1, R\} \quad (1.2)$$

where  $R$  is the reflection

$$Rx = -x. \quad (1.3)$$

Then (1.1) consists of those functions fixed by  $B_N$ .

**Proof** We showed in the Introduction that solutions satisfying NBC lead to the desired type of solution satisfying PBC. It remains to prove the converse.

Let  $u(x)$  be a smooth  $2\pi$ -periodic solution to  $\mathcal{P}(u) = 0$  satisfying (1.1). Differentiating (1.1) implies that

$$u'(0) = 0 \quad \text{and} \quad u'(-\pi) = -u'(\pi).$$

Periodicity implies that  $u'(\pi) = u'(-\pi)$ , so that  $u'(\pi) = 0$ , whence  $u$  satisfies NBC on  $[0, \pi]$ .  $\square$

As observed by Dangelmayr and Armbruster [1986, 1987], there are consequences of Lemma 1.1 for the generic behaviour of bifurcations of PDEs with Neumann boundary conditions. This change stems from the fact that the bifurcation problem with PBC has  $O(2)$  symmetry generated by translations modulo  $2\pi$  and reflection.

More precisely, the change in generic behaviour occurs as follows. Instead of studying the NBC problem, one first studies the PBC bifurcation problem using  $O(2)$  symmetry, and then restricts the result to the fixed-point space  $\text{Fix}(B_N)$  to recover the answer for NBC. The essential reason for the change in genericity is that the general  $O(2)$ -equivariant bifurcation problem obtained when analyzing the PBC case may *not* restrict to a general bifurcation problem on  $\text{Fix}(B_N)$  with the symmetry of NBC. In cases where this restricted problem has special features we get a change in genericity.

We illustrate this point in the simplest instance, steady-state bifurcation. Assume that  $\mathcal{P}$  depends on a bifurcation parameter  $\lambda$  and that  $\mathcal{P}$  has a trivial translation-invariant solution at  $\lambda = 0$  which, without loss of generality, we may assume to be  $u = 0$ . Finally, assume that this trivial solution undergoes a steady-state bifurcation at  $\lambda = 0$ . Observe that the only symmetry of NBC is the reflection

$$\tau: x \mapsto \pi - x. \quad (1.4)$$

**Proposition 1.2** Under the above hypotheses on  $\mathcal{P}(u) = 0$ , satisfying Neumann boundary conditions, we have:

- (a) Bifurcating solutions have a well-defined non-negative integer mode number  $m$ .
- (b) Generically, when  $m > 0$ , the bifurcation is a pitchfork.

#### Remarks

- (a) The mode number is associated with 'pattern' formation and can be observed in experiments.
- (b) For many operators the natural modes are obtained by separation of variables, leading to a spatial variation with eigenfunctions like  $\cos(mx)$ . For a general  $Z_2$ -equivariant bifurcation it would be a surprise for this pure mode to occur as an eigenfunction. In general one would expect the eigenfunctions to be (perhaps infinite) linear combinations  $\sum a_k \cos(kx)$ .
- (c) The pitchfork bifurcation occurs for  $m$  even, even though when  $m$  is even (1.4) does not force this type of bifurcation in the NBC model.

**Proof** By Lemma 1.1 there is a bifurcation at  $\lambda = 0$  in equilibrium solutions of  $\mathcal{P}(u) = 0$  with PBC on  $[-\pi, \pi]$ . The group of symmetries of this bifurcation problem is  $O(2)$ .

- (a) Let  $L = d\mathcal{P}$  denote the linearized equations about  $u = 0$  at  $\lambda = 0$  and let  $K = \ker L$ .

By  $O(2)$  symmetry we expect  $K$  to be either 1- or 2-dimensional, since irreducible representations of  $O(2)$  have those dimensions, Golubitsky, Stewart, and Schaeffer [1988] p. 330. We may write the action of  $SO(2)$  on  $K$  as

$$z \mapsto e^{im\theta} z, \quad (1.5)$$

where  $m = 0$  in the simple eigenvalue case and  $m > 0$  in the double eigenvalue case. The integer  $m$  defined in (1.5) is the *mode number*.

Let  $\Sigma'$  denote the kernel of the representation of  $O(2)$  on  $K$ . Then

$$\Sigma' = \begin{cases} O(2) \text{ or } SO(2) & \text{when } m = 0 \\ Z_m & \text{when } m > 0 \end{cases} \quad (1.6)$$

The isotropy subgroup  $\Sigma$  of any bifurcating solution will contain  $\Sigma'$ . Therefore bifurcating solutions will be translation-invariant when  $m = 0$  (and hence constant), and invariant under translation by  $x \mapsto x + 2\pi/m$  when  $m > 0$ . This translation-invariance is what gives the bifurcating solution a 'pattern'.

When  $m > 0$ ,  $\Sigma$  is actually isomorphic to  $D_m$ , generated by  $\Sigma'$  and a reflection  $x \mapsto x_0 - x$  for some  $x_0$ .

(b) Next we discuss the expected type of bifurcation in the PBC case. When  $m > 0$ , the effective action of  $O(2)$  on the bifurcation is by  $O(2)/\Sigma' = O(2)/Z_m \cong O(2)$ . Hence, generically we expect a pitchfork of revolution. When  $m = 0$  and  $\Sigma' = O(2)$ , the group  $O(2)/\Sigma'$  is trivial and generically we expect a limit point bifurcation (in the branch of constant solutions). On the other hand when  $\Sigma' = SO(2)$  we have  $O(2)/\Sigma' \cong Z_2$ , and generically we expect a pitchfork bifurcation.

The solutions to  $\mathcal{P}(u) = 0$  satisfying NBC are found in

$$\text{Fix}(B_N) = \{z \in \ker d\mathcal{P} : Rz = z\}.$$

Thus, when  $m > 0$ , NBC picks out those solutions in the pitchfork of revolution that are invariant under the reflection  $Rx = -x$ , rather than a general reflection  $x \mapsto x_0 - x$ .

These solutions form a pitchfork, for the following reason. Let

$$T(x) = x + 2\pi/2m. \quad (1.7)$$

The translation  $T$  lies in the normalizer of the isotropy subgroup

$$D_m = \langle \Sigma', R \rangle,$$

the isotropy subgroup of solutions satisfying NBC. Such solutions are found in the 1-dimensional space  $\text{Fix}(D_m)$ , and  $T$  acts as  $-I$  on that subspace. Thus the two half-branches of the pitchfork are identified by  $T$ .

Observe that this translation  $T$ , which drives the pitchfork bifurcation with NBC, is *not* a symmetry of the original equations satisfying NBC.  $\square$

Fig. 1 and 2 illustrate these results. Armbruster and Dangelmayr [1986, 1987] use these ideas to study steady-state mode interactions with NBC. Their arguments depend on somewhat subtler observations concerning restrictions of  $O(2)$ -equivariant bifurcation problems to the NBC case. These will be discussed in more detail in the next section; here we use Proposition 1.2 to indicate one of the effects of  $O(2)$  symmetry on the linear terms at mode interactions.

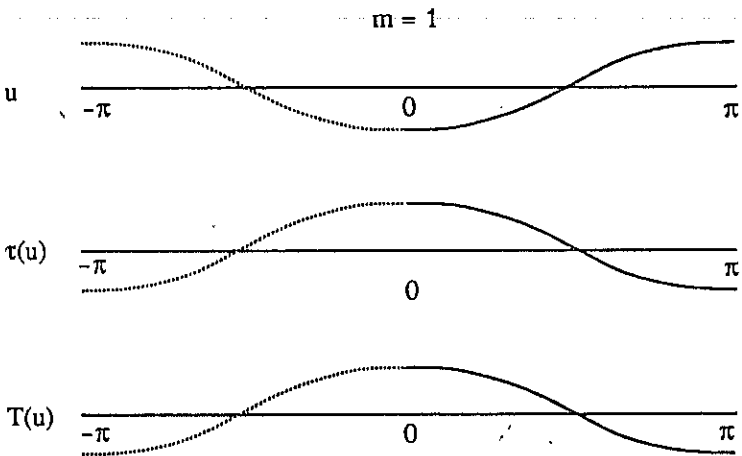


Fig. 1 (a) An NBC mode with  $m$  odd, defined on  $-\pi \leq x \leq \pi$ . (b) Its form under the reflection  $\tau: x \mapsto \pi - x$ . (c) Its form under the translation  $T: x \mapsto x + \pi/2$ . Both  $\tau$  and  $T$  act by  $-I$ .

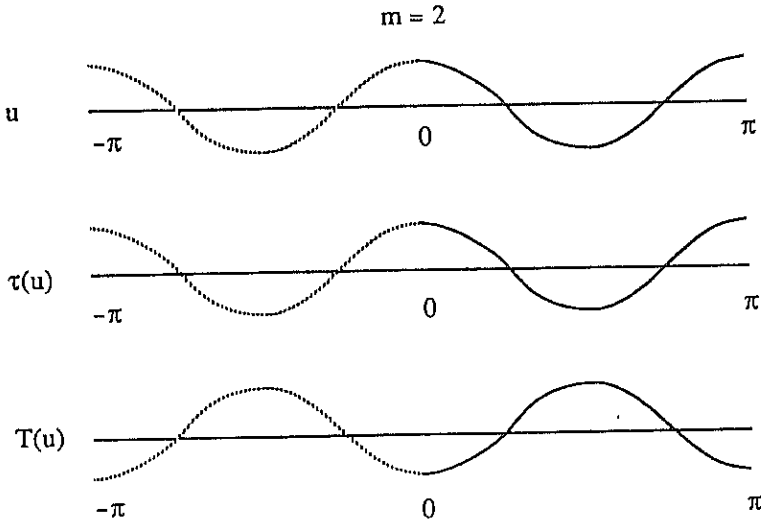


Fig. 2 As for Fig. 1 but with  $m$  even. Now only  $T$  acts by  $-I$ .

Assume that  $\mathcal{P}(u)$  depends on two parameters and that when these parameters are set to zero the linearized NBC problem has a double zero eigenvalue and no other eigenvalues on the imaginary axis. Then  $0$  lies in the intersection of two curves in parameter space along which  $Lu = 0$  has a simple eigenvalue. Let  $m$  and  $n$  be the mode numbers along these curves, guaranteed by Proposition 1.2.

**Corollary 1.3** Suppose that  $m \neq n$ . Then  $\dim \ker L = 2$ .

**Remark** Without the effect of  $O(2)$  symmetry, we might have expected a Takens-Bogdanov type singularity at  $\lambda = 0$ . The distinct mode numbers rule this out.

**Proof** Extend to PBC and let

$$\dot{z} = Mz + \dots \quad (1.8)$$

be the  $O(2)$ -equivariant vector field on  $\mathbb{R}^4 \cong \mathbb{R}^2 \times \mathbb{R}^2$  obtained by centre manifold reduction. Since  $m \neq n$ ,  $O(2)$  acts by distinct irreducible (indeed absolutely irreducible) representations on each copy of  $\mathbb{R}^2$ . Hence

$$M = \begin{pmatrix} c_1 I_2 & 0 \\ 0 & c_2 I_2 \end{pmatrix}.$$

The assumption of a double zero eigenvalue in NBC implies that  $c_1 = c_2 = 0$ . Thus  $M \equiv 0$ . Restricting (1.8) to NBC proves the result.  $\square$

## 2 Generalizations

In this section we consider four types of generalization of the discussion above:

- Dirichlet boundary conditions
- mode interactions
- equations involving many space variables
- more general types of PDE.

### (a) Dirichlet Boundary Conditions and Symmetry

Consider  $u(x)$  satisfying Dirichlet boundary conditions (DBC) on  $[0, \pi]$  for the linearized equation  $Lu = 0$ . Extend  $u$  to  $[-\pi, \pi]$  by

$$u(-x) = -u(x) \quad \text{on } [-\pi, 0] \quad (2.1)$$

and to the whole line by  $2\pi$ -periodicity. Now  $u$  satisfies  $Lu = 0$  and PBC on  $[-\pi, \pi]$ , but here one must appeal to the linearity of  $L$ , namely

$$L(-u) = -Lu.$$

For the extended  $u$  to satisfy the nonlinear equation  $\mathcal{P}(u) = 0$  we must assume that

$$\mathcal{P}(-u) = -\mathcal{P}(u) \quad (2.2)$$

in addition to Euclidean invariance. Again it is not hard to show that the extension procedure (2.1) preserves regularity. When (2.2) holds, remarks similar to those made for NBC apply to DBC. In particular, the assignment of mode numbers and the occurrence of pitchfork bifurcation may be expected generically.

The only change in the analysis is in specifying the symmetry of DBC. Define the reflection  $S$  by

$$(Su)(x) = -u(-x), \quad (2.3)$$

and define the two-element group

$$B_D = \{1, S\}.$$

Then solutions to  $\mathcal{P}(u) = 0$  satisfying DBC are found by solving the equations with PBC and restricting to

$$\text{Fix}(B_D) = \{u(x) : u(-x) = -u(x)\}. \quad (2.4)$$

Note that bifurcation problems arising from PBC now have  $O(2) \times Z_2$  symmetry where  $Z_2 = \{\pm 1\}$ . The isotropy subgroup  $\Sigma$  of given solutions will change slightly from the NBC case because of the extra  $Z_2$  symmetry, but not the general structure. In particular, the translation  $T$  will still identify the two half-branches of the pitchfork.

### (b) Mode Interactions

Armbruster and Dangelmayr [1986, 1987] consider steady-state mode interactions of two nontrivial modes ( $m, n > 0, m \neq n$ ) in reaction-diffusion equations with NBC. When extended to PBC, the kernel  $K$  of  $d\mathcal{P}$  is 4-dimensional, and may be identified with  $\mathbb{C}^2$ . The action of  $O(2)$  on  $\mathbb{C}^2$  is generated by

$$\begin{aligned} \theta(z, w) &= (e^{mi\theta} z, e^{ni\theta} w) \\ \kappa(z, w) &= (\bar{z}, \bar{w}). \end{aligned} \quad (2.5)$$

(Note: For the group theory and invariant theory we may without loss of generality assume that  $m$  and  $n$  are relatively prime by factoring out the kernel of this action. This kernel must be restored when interpreting the results.)

Let

$$f: \mathbb{C}^2 \times \mathbb{R} \rightarrow \mathbb{C}^2$$

be the reduced bifurcation equations for PBC obtained by a Liapunov-Schmidt reduction. Then  $f$  is  $O(2)$ -equivariant under the action (2.5). In these coordinates the action of the group  $B_N$  is just  $Z_2(\kappa)$ . Since  $\text{Fix}(B_N) = \text{Fix}(Z_2(\kappa)) = \mathbb{R}^2$ , the bifurcation equations corresponding to NBC are just

$$g = f | \mathbb{R}^2 \times \mathbb{R} : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2.$$

If  $H$  is a subgroup of a group  $G$ , denote the normalizer of  $H$  in  $G$  by  $N_G(H)$ . We know that  $N_{O(2)}(Z_2(\kappa))/Z_2(\kappa)$  acts nontrivially on  $\mathbb{R}^2$  and provides symmetry constraints on  $g$ . If  $\theta \in SO(2)$  then  $(-\theta)\kappa(\theta) = (-2\theta)$ , so this group has two elements and is generated by the true symmetry of NBC

$$\tau: x \mapsto \pi - x \quad (2.6)$$

whose action on  $\mathbb{C}^2$  depends on the parity of  $m$  and  $n$ . Since they are coprime, one, at least, is odd. So we assume  $m$  is odd. Then (2.6) leads to the action on  $\mathbb{C}^2$  given by

$$(z, w) \mapsto (-\bar{z}, (-1)^n \bar{w}),$$

and hence by restriction on  $(r, s) \in \mathbb{R}^2$  as

$$(r, s) \mapsto (-r, (-1)^n s). \quad (2.7)$$

If one does not consider the extension to PBC, then one will still find the symmetry (2.6) since it is generated by a symmetry of the domain  $[0, \pi]$ . Thus we would still know that the restriction  $g$  commutes with (2.7), but what is perhaps surprising is that the form of  $g$  is further constrained, just by knowing that  $g$  is the restriction of an  $O(2)$ -equivariant  $f$ . This is the main point of Armbruster and Dangelmayr [1986, 1987].

Indeed part, but only part, of the constraints on  $g$  can be understood group-theoretically. Suppose that  $n \equiv 2 \pmod{4}$ , so that (2.7) becomes

$$(r, s) \mapsto (-r, s). \quad (2.8)$$

Note that the  $s$ -axis is invariant since  $\text{Fix}(2.7) = (0, s)$ . Consider the action of translation by a quarter period,

$$x \mapsto x + \pi/2,$$

which acts on  $\mathbb{C}^2$  by

$$(z, w) \mapsto (\pm iz, -w). \quad (2.9)$$

By (2.9)  $g(0, w)$  must commute with  $w \mapsto -w$ , a constraint not generated by any symmetry of the domain  $[0, \pi]$ . Thus the bifurcation along the  $s$ -axis is a pitchfork, which might otherwise have been unexpected; algebraically  $g(0, s)$  consists only of odd degree terms. Such symmetries on subspaces were first noted by Hunt [1982] and formalized in Golubitsky, Marsden, and Schaeffer [1984]. The results of Armbruster and Dangelmayr [1986, 1987] are more extensive, depending precisely on the values of  $(m, n)$ , and we shall not reproduce them here.

The extension to PBC has a small effect on NBC when  $m = n > 0$ . Here the linearized problem may have a nilpotent part, as in the Takens-Bogdanov bifurcation. Dangelmayr and Knobloch [1987] have discussed the Takens-Bogdanov singularity with  $O(2)$  symmetry. Using their results and restricting to  $\text{Fix}(B_N)$  it can be shown that the Takens-Bogdanov singularity with NBC always has the symmetry  $(x, y) \mapsto (-x, -y)$ . When  $m$  is odd this is not surprising since the symmetry is just the NBC symmetry (2.6). When  $m$  is even, however, this symmetry is generated by the phase shift (1.7). So in all cases, one expects a symmetric Takens-Bogdanov bifurcation when NBC are used.



### (c) Higher-Dimensional Domains

The remarks made previously extend to higher dimensions - provided the domain is suitable. For example, consider a reaction-diffusion equation defined on a rectangle in  $\mathbb{R}^2$  with NBC. Bifurcation problems for such equations can be embedded in problems with PBC in both directions. The periodic boundary conditions lead to bifurcation with symmetry  $D_2 \dot{+} T^2$ , where  $T^2$  is the 2-torus generated by planar translations modulo periodicity in both directions.

If the domain is a square, then PBC will lead to  $D_4 \dot{+} T^2$  symmetry. Investigation of the resulting bifurcation restrictions is being pursued by Gomes [1989]. Preliminary results suggest that not much, beyond what is noted in §1, will change in bifurcations with NBC for rectangles, but definite restrictions will appear in mode interactions on squares.

The analysis of Armbruster and Dangelmayr [1986, 1987] must be worked out for DBC - the results should be slightly different than for NBC, due to the extra reflectional symmetry. Details will appear in Gomes [1989].

Similarly one can imagine equations in the plane where NBC are imposed on sides parallel to the  $y$ -axis and DBC on sides parallel to the  $x$ -axis; and similar extension arguments apply, subject to a mild symmetry restriction on  $\mathcal{P}$  for the Dirichlet direction. Once the general idea is understood, the analyses in this or in higher-dimensional cases can be worked out when needed.

The general idea discussed here is the classical observation that solutions to PDEs can sometimes be extended, by reflection or similar methods, across boundaries. The implication is that this procedure can enlarge the effective symmetries of the equations, due to new symmetries of the extended domain.

Another possible extension concerns the sphere  $S^2$ . Imagine posing a PDE on the upper hemisphere of  $S^2$  with NBC on the equator. Solutions can be extended to the lower hemisphere by reflection. Supposing that the resulting operator is  $SO(3)$ - or  $O(3)$ -equivariant, we obtain a new notion of genericity for the original bifurcation problem on the hemisphere. These results will be published in Field, Golubitsky, and Stewart [1990]. It is not hard to envisage more elaborate variations on this theme.

### (d) Other types of PDE

We have chosen the case of reaction-diffusion equations because these automatically possess Euclidean invariance and provide the simplest setting for our observations. The ideas generalize directly to second order PDEs with Euclidean invariance, for example the Navier-Stokes equations. In many applications  $u$  is vector-valued and the boundary conditions are mixed, either NBC or DBC depending on the component of  $u$ ; the methods extend easily to this case. The approach also applies to suitable PDEs of order higher than 2, for example the von Kármán equations; we require boundary conditions that force odd order partial derivatives to zero (for NBC) or even order partial derivatives to zero (for DBC).

Rather than formulate a general theorem to cover these disparate cases, we describe typical examples in the remaining sections. Until now we have worked abstractly and concentrated on group-theoretic restrictions. We now consider specific applications whose analysis is aided by these ideas.

### 3 The Couette-Taylor Experiment

The Couette-Taylor apparatus consists of a fluid contained between two independently rotating coaxial circular cylinders. Depending on the speeds of rotation of the cylinders, a great variety of flow patterns can form. In his analysis of what are now called Taylor vortices, Taylor [1923] assumes PBC on the flow within a vortex pair, even though physically this assumption is questionable. His theory nevertheless agrees remarkably well with experiment.

More recently many patterned states have been catalogued, see Andereck, Liu, and Swinney [1986], and theorists have been busy trying to explain how they arise. One fruitful approach to describing the many states discovered since Taylor's pioneering work is to retain PBC and focus on the  $O(2)$  symmetry thereby introduced. See Golubitsky and Stewart [1986], Iooss [1986], Chossat, Demay and Iooss [1987], and Golubitsky and Langford [1988].

One of the basic consequences of this symmetry is that the transition from the laminar (unpatterned) Couette flow to Taylor vortices is the expected  $O(2)$ -symmetric pitchfork of revolution. In his talk at this Workshop, Tom Mullin described his experiments and supporting numerical computations of Andrew Cliffe, which show that even for cylinders of moderate length the initial transition from laminar flow to vortices does *not* occur by a pitchfork of revolution, but by a perturbed pitchfork. Benjamin [1978] and Mullin [1982] previously made similar observations in experiments with short cylinders. The work of Benjamin [1978], together with unpublished results of Mullin and Cliffe which are briefly described below, cast doubt on the assumption of PBC. In this section we interpret their work in the light of symmetry, taking into account the effect of boundary conditions discussed previously.

In a real cylinder there are two types of Taylor vortex flow: regular and anomalous. In both cases the Taylor vortices occur in pairs with the flow oriented inward along the mid-plane of the pair. In the regular case an integer number of such pairs fits into the cylinder, while in the anomalous case a half-pair occurs at each end. In other words, the direction of flow at the ends is different in the two cases. (States with a single half-pair at one end, that is, an odd number of vortex cells altogether, can also occur, but we ignore them here.) In Mullin's experiments the outer cylinder is

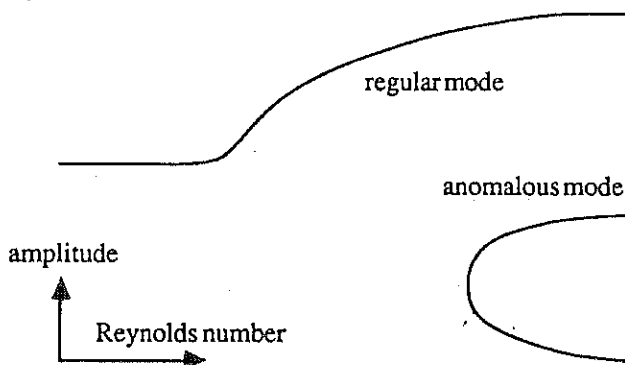


Fig. 3 A perturbed pitchfork as found in Mullin's experiment (schematic).

held fixed and the speed of the inner cylinder is increased quasistatically. A schematic of his results is shown in Fig. 3.

There are two interesting features of Fig. 3. First, there is a range of speeds for which no laminar-like flow exists; secondly, there are two stable half-branches correspond to a regular mode and an anomalous mode, related by a shift of one vortex along the axis of the cylinder.

Cliffe obtains similar results numerically, as follows. Following Schaeffer [1980] he includes a homotopy parameter  $\nu$  in the boundary conditions at the ends of the cylinder, such that the boundary conditions are Neumann for  $\nu = 0$  and physically realistic for  $\nu = 1$ . When  $\nu = 0$  Cliffe finds a pitchfork bifurcation which, as  $\nu$  is turned on, breaks to a perturbed pitchfork as in Fig. 3.

The discussion in §2 leads to the following observations. The  $\nu = 0$  model with NBC should indeed lead to a pitchfork, via the introduction of PBC and  $O(2)$ -symmetry. Moreover, the two half-branches of the pitchfork are related by the translation (1.7). That is, one *expects* to find one half-branch of regular modes and one half-branch of anomalous modes. Note that this conclusion can only be reached by making the extension to PBC. Note also that the only genuine axial symmetry in the apparatus, reflection in the midplane, acts *trivially* in this pitchfork bifurcation, because the number of vortices is even. Thus it is not surprising that when NBC are violated ( $\nu \neq 0$ ) the pitchfork becomes imperfect, as in Fig. 3.

In this instance the experiments of Mullin are in good agreement with the numerics of Cliffe, and they both agree for perturbed NBC as in Schaeffer's approach. It should be remembered, however, that there are numerous fluid states - such as wavy vortices and spirals - that are *not* consistent with NBC, but are admitted by PBC. The work cited previously leads to equally good predictions concerning these other PBC-based patterns, several of which have been verified by experiment. The situation appears to be that neither NBC nor PBC provides a fully adequate model, but that each works surprisingly well for an appropriate range of flow patterns.

#### 4 Rayleigh-Bénard Convection

Next we discuss Rayleigh-Bénard convection in a box. Consider the onset of the convective instability in a 2-dimensional problem in  $\{0 \leq x \leq \pi, 0 \leq y \leq \pi\}$  with  $(x,y)$  denoting horizontal and vertical directions respectively. The problem is more complex than reaction-diffusion equations on a line because boundary conditions must be imposed in both  $x$  and  $y$ . We consider here the case in which the boundary conditions on the horizontal surfaces  $y = 0, \pi$  are homogeneous and distinct. For example, a Robin-type boundary condition applies to the temperature at the top if the top surface radiates heat according to Newton's law of cooling.

In this case there are no symmetries associated with the boundary conditions in  $y$ , and there is no modal structure in  $y$ . In the absence of the vertical sidewalls the equations of motion are invariant under translation  $x \mapsto x+\ell$  and reflections  $x \mapsto x_0-x$ . When sidewalls are present we may take the boundary conditions to be

$$u(x,y) = \frac{\partial v}{\partial x}(x,y) = \frac{\partial \theta}{\partial x}(x,y) = 0 \text{ on } x = 0, \pi \quad (4.1a)$$

or

$$u(x,y) = v(x,y) = \frac{\partial \theta}{\partial x}(x,y) = 0 \text{ on } x = 0, \pi, \quad (4.1b)$$

where  $(u,v)$  are the  $(x,y)$ -components of the velocity, and  $\theta$  is the temperature departure from pure conduction. In both cases the boundary conditions at the sides are identical, and the problem therefore has  $Z_2$  symmetry  $\tau: x \mapsto \pi - x$ . The boundary conditions (4.1a) describe free-slip perfectly insulating boundaries, and extend to PBC on  $-\pi \leq x \leq \pi$  with

$$\begin{aligned} u(-x,y) &= -u(x,y) \\ v(-x,y) &= v(x,y) \\ \theta(-x,y) &= \theta(x,y). \end{aligned} \quad (4.2)$$

Consequently there is a well-defined mode number  $m$ . Consider now the action of the reflection  $\tau$ . Since  $(u,v)$  are components of a vector we know that

$$\tau(u,v) = (-u,v). \quad (4.3)$$

Therefore  $\tau$  acts on mode  $m$  by

$$\begin{aligned} \tau(u_m, v_m, \theta_m) &= (-(-1)^{m+1}u_m, (-1)^m v_m, (-1)^m \theta_m) \\ &= (-1)^m (u_m, v_m, \theta_m), \end{aligned} \quad (4.4)$$

and the reflection symmetry acts nontrivially on the odd modes and trivially on the even modes. The odd modes therefore automatically undergo a pitchfork bifurcation. The even modes also undergo a pitchfork, but only because the horizontal translation (1.7) acts by  $-I$  on both even and odd modes. Thus the pitchfork bifurcation in the even modes is a consequence of the translation symmetry of PBC.

Case (4.1b) corresponds to no-slip, thermally insulating boundaries. Since  $u(x,y) = v(x,y) = 0$  on  $x = 0, \pi$  one might try to extend the solution to  $-\pi \leq x \leq \pi$  by

$$\begin{aligned} u(-x,y) &= -u(x,y) \\ v(-x,y) &= -v(x,y) \end{aligned} \quad (4.5)$$

as in the scalar case (2.1). But since this violates (4.3) this problem cannot be extended to PBC on  $-\pi \leq x \leq \pi$ , and hence there is no mode structure of the form (1.5).— Indeed, explicit calculation shows that the eigenfunctions are sums of trigonometric and hyperbolic functions, Drazin [1975]. These, nonetheless, divide into two classes, odd and even with respect to  $x$ . The odd eigenfunctions break  $\tau$  and bifurcate in pitchforks. Since there is no translational symmetry we do *not* expect the even modes to bifurcate in pitchforks.

An additional reflectional symmetry  $\bar{\tau}$  is present if the boundary conditions on top and bottom are identical. In the special case

$$\frac{\partial u}{\partial z}(x,y) = v(x,y) = \theta(x,y) = 0 \quad \text{on } y = 0, \pi \quad (4.6)$$

the boundary conditions extend to PBC on  $-\pi \leq y \leq \pi$  under

$$\begin{aligned} u(x,-y) &= u(x,y) \\ v(x,-y) &= -v(x,y) \\ \theta(x,-y) &= -\theta(x,y) \end{aligned} \quad (4.7)$$

and a mode structure exists in the vertical direction. Since  $\bar{\tau}: y \mapsto \pi - y$  acts by

$$\bar{\tau}(u,v) = (u,-v), \quad \bar{\tau}(\theta) = -\theta. \quad (4.8)$$

it acts on mode  $n$  by

$$\bar{\tau}(u_n, v_n, \theta_n) = (-1)^n (u_n, v_n, \theta_n). \quad (4.9)$$

Hence odd modes in  $y$  bifurcate in a pitchfork because they break  $\bar{\kappa}$ , while the even modes do so because the translation analogous to (1.7),  $y \mapsto y + 2\pi/2n$ , acts by  $-I$ . If the boundary conditions in  $y$  do not extend to PBC we expect only the odd modes to undergo a pitchfork bifurcation. These results explain why Hall and Walton [1977] find a pitchfork in the Rayleigh-Bénard problem with the boundary conditions (4.1b) and (4.6) for both odd *and* even eigenfunctions in the horizontal direction.

## 5 The Faraday Experiment

In the Faraday experiment a fluid layer is subjected to a vertical oscillation at frequency  $\omega$ . When the forcing amplitude  $A$  is small, the fluid surface remains essentially flat, but waves are parametrically excited when the amplitude is increased. Indeed, in careful experiments, for most frequencies of vibration the initial transition from the flat surface is to a standing wave at  $\omega/2$ , half the driving frequency.

In this section we discuss results of Gollub and coworkers using containers with differing geometry, and hence different symmetries; and we explain how these symmetries, when coupled with boundary conditions, affect the analysis of these parametric instabilities. We focus on the experiments of Ciliberto and Gollub [1985a] and Gollub and Simonelli [1989] which employ containers of circular and square cross-section, respectively. The experiments were performed by fixing the forcing frequency  $\omega$ , slowly varying the amplitude  $A$ , and observing the asymptotic behaviour of the surface.

For a circular vessel, Ciliberto and Gollub [1985a] find that for most frequencies the initial transition is to a standing wave with azimuthal mode number  $m$ ; that is, the spatial pattern is invariant under rotations by  $2\pi/m$ . There is also a radial index for the number of radial modes, but the radial structure does not play a significant role. The existence of well-defined modes is not surprising, given the  $O(2)$  symmetry of the apparatus. Further, the experiments show that different choices of  $\omega$  lead to standing waves with different azimuthal mode numbers, and hence that there exist isolated values of  $\omega$  at which the primary transition from a flat surface occurs by the simultaneous instability of two modes with unequal azimuthal mode numbers  $m$  and  $n$ . Ciliberto and Gollub [1985a] studied such a codimension two instability for modes with  $m = 4$  and  $n = 7$ . Near the point of multiple instability they observed complicated dynamics, including quasiperiodic and chaotic motion.

The multiple instability has been analysed by numerous authors using a variety of approximation techniques; see Ciliberto and Gollub [1985b], Meron and Procaccia [1986], and Umeki and Kambe [1989]. We focus here on the approach of Crawford, Knobloch, and Riecke [1989]. Since the forcing is periodic, it is natural to consider the stroboscopic map  $S$  which takes the fluid state at time  $t$  to its state one period later, at time  $t + 2\pi/\omega$ . Indeed the experimental measurements provide essentially a reconstruction of the dynamics of  $S^2$ , the twice iterated map. The construction of  $S$  from 'first principles' would require integrating the Navier-Stokes equations. The idea of Crawford et al. [1989] is to develop a description of  $S$  by appealing to symmetry and genericity. First, note that the flat surface  $F$  is a fixed point of  $S$  and that the  $\omega/2$  standing wave is a 2-cycle; hence the parametric instability can be identified as a period-doubling bifurcation for  $S$ . At the period-doubling bifurcation point the linearization

$(dS)_F$  of  $S$  at  $F$  has an eigenvalue  $-1$ . This conclusion is supported by the linear analysis of Benjamin and Ursell [1954]. Any model of this experiment will be  $O(2)$ -symmetric; hence the generalized eigenspace  $V$  of  $(dS)_F$  for eigenvalue  $-1$  is  $O(2)$ -invariant.

Now we impose genericity. For a nonzero azimuthal mode number we expect  $V$  to be 2-dimensional. The reason is that generically the action of  $O(2)$  is irreducible, hence of dimension  $\leq 2$ ; but if the dimension is 1 then the period-doubled state has rotational symmetry, and hence has azimuthal mode number zero. So we may identify  $V$  with  $\mathbb{C}$  and write the action of  $\theta \in SO(2)$  on  $V$  as

$$z \mapsto e^{mi\theta} z. \quad (5.1)$$

Crawford *et al.* [1989] analyse the interactions of modes with mode numbers  $n > m \geq 1$  as follows. They assume that a centre manifold reduction has been performed to yield an  $O(2)$ -equivariant mapping

$$S: \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad S(0) = 0, \quad (dS)_0 = -I \quad (5.2)$$

whose asymptotic dynamics is equivalent to that of  $S$ . The action of  $O(2)$  on  $\mathbb{C}^2$  is determined by the mode numbers and is

$$\begin{aligned} \theta.(z_1, z_2) &= (e^{mi\theta} z_1, e^{ni\theta} z_2) \\ \kappa(z_1, z_2) &= (\bar{z}_1, \bar{z}_2). \end{aligned} \quad (5.3)$$

Since  $m \neq n$  the representations of  $O(2)$  on the two coordinates  $z_j$  are distinct. Thus  $(dS)_0$  cannot be nilpotent. Crawford *et al.* show that there exist appropriate choices for the low order terms of  $S$  for which the dynamics of  $S$  corresponds approximately to that of  $S$  observed in experiments. Some questions remain open, but their resolution requires further experiments and will not be discussed here.

In contrast, the experiments of Simonelli and Gollub [1989] are performed in a square container. Here mode numbers  $m$  and  $n$  corresponding to the two horizontal directions  $x$  and  $y$  are also observed. Not surprisingly, whenever mode  $(m, n)$  is observed the surface can be perturbed to a new surface with mode numbers  $(n, m)$ , corresponding to the reflectional symmetry of the square about a diagonal. When  $m \neq n$  several authors have analysed the initial transition. In particular Silber and Knobloch [1989] describe the stroboscopic map, but with the symmetry modified from  $O(2)$  to  $D_4$  to correspond to the symmetry of the container. Feng and Sethna [1989] perform an asymptotic analysis of the Navier-Stokes equations to study the nonlinear behaviour of the standing waves in a nearly square cross-section.

Boundary conditions play a more subtle role in the square case. Let the fluid state be specified by the surface deformation  $\zeta(x, y)$  and the fluid velocity field  $u(x, y, z)$ , where  $0 \leq x, y \leq \pi$  are the horizontal coordinates and  $z$  is the vertical coordinate. The realistic no-slip boundary conditions at the sidewalls require

$$\begin{aligned} u(0, y, z) = u(\pi, y, z) &= 0, \\ u(x, 0, z) = u(x, \pi, z) &= 0; \end{aligned}$$

while the  $\zeta$  field may satisfy either NBC or DBC depending on the experimental arrangement:

$$\frac{\partial \zeta}{\partial x}(0, y) = \frac{\partial \zeta}{\partial x}(\pi, y) = \frac{\partial \zeta}{\partial y}(x, 0) = \frac{\partial \zeta}{\partial y}(x, \pi) = 0,$$

or

$$\zeta(0, y) = \zeta(\pi, y) = \zeta(x, 0) = \zeta(x, \pi) = 0.$$

It has also been suggested that contaminants on the surface of the fluid might lead to Robin boundary conditions, see Hocking [1987].

The extension to  $-\pi \leq x, y \leq \pi$  with PBC is straightforward for  $u$ . Let  $u = (u, v, w)$ ; then

$$(u, v, w)(x, -y, z) = (u, -v, w)(x, y, z) \quad 0 \leq x \leq \pi, -\pi \leq y \leq \pi$$

and

$$(u, v, w)(-x, y, z) = (-u, v, w)(x, y, z) \quad -\pi \leq x \leq \pi, -\pi \leq y \leq \pi.$$

The extension for  $\zeta(x, y)$  depends as usual on whether NBC or DBC are selected:

$$\text{NBC:} \quad \zeta(x, -y) = \zeta(x, y) \quad 0 \leq x \leq \pi, -\pi \leq y \leq \pi$$

$$\zeta(-x, y) = \zeta(x, y) \quad -\pi \leq x \leq \pi, -\pi \leq y \leq \pi;$$

$$\text{DBC:} \quad \zeta(x, -y) = -\zeta(x, y) \quad 0 \leq x \leq \pi, -\pi \leq y \leq \pi$$

$$\zeta(-x, y) = -\zeta(x, y) \quad -\pi \leq x \leq \pi, -\pi \leq y \leq \pi.$$

In either case the extended problem has  $D_4 + T^2$  symmetry, and one naturally expects the eigenfunctions of  $(dS)_F$  to have well-defined mode numbers, for example,  $\zeta \sim \cos(mx)\cos(ny)$  for NBC. By contrast a Robin boundary condition would force the eigenfunctions to be mixtures of these pure modes.

The  $D_4 + T^2$ -action on  $-\pi \leq x, y \leq \pi$  is generated by

$$D_4: \quad \kappa_1: (x, y) \mapsto (-x, y)$$

$$\kappa_2: (x, y) \mapsto (y, x)$$

$$T^2: \quad (\varphi_1, \varphi_2): (x, y) \mapsto (x + \varphi_1, y + \varphi_2).$$

If, for given  $(m, n)$  with  $m \neq n$ , we choose

$$z_1 e^{i(mx+ny)} + z_2 e^{i(mx-ny)} + z_3 e^{i(nx+my)} + z_4 e^{i(nx-my)}$$

as a basis for the eigenspace, then the coordinates  $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$  are transformed by

$$\kappa_1: (z_1, z_2, z_3, z_4) \mapsto (\bar{z}_2, \bar{z}_1, \bar{z}_4, \bar{z}_3)$$

$$\kappa_2: (z_1, z_2, z_3, z_4) \mapsto (z_3, \bar{z}_4, z_1, \bar{z}_2)$$

$$(\varphi_1, \varphi_2): (z_1, z_2, z_3, z_4) \mapsto$$

$$(e^{i(m\varphi_1+n\varphi_2)} z_1, e^{i(m\varphi_1-n\varphi_2)} z_2, e^{i(n\varphi_1+m\varphi_2)} z_3, e^{i(n\varphi_1-m\varphi_2)} z_4).$$

In this case NBC are selected by invariance under the group

$$B_N = \{\kappa_1, \kappa_3\}$$

generated by

$$\kappa_1: (x, y) \mapsto (-x, y)$$

and

$$\kappa_3: (x, y) \mapsto (x, -y).$$

Note that  $\kappa_3 = \kappa_2 \kappa_1 \kappa_2$ .

For DBC we have the additional symmetry operation

$$(\sigma, \zeta)(x, y, z) = -\zeta(-x, y, z)$$

which acts on the eigenspace by

$$\sigma: (z_1, z_2, z_3, z_4) \mapsto (-\bar{z}_2, -\bar{z}_1, -\bar{z}_4, -\bar{z}_3).$$

Now DBC are selected by invariance with respect to the subgroup

$$B_D = \{\sigma, \kappa_2 \sigma \kappa_2\}.$$

The bifurcation problems relevant to the models of the Faraday experiment occur on  $\text{Fix}(B_N)$  and  $\text{Fix}(B_D)$ . Interestingly, the effective symmetry group occurring in the bifurcation of mode  $(m, n)$  can be smaller than the  $D_4$  symmetry suggested by the

experimental geometry. 'Upper bounds' on the resulting constraints can be determined by calculating the relevant normalizer in each case, but additional constraints may also arise, not inherited in this way from symmetries of the original problem, a point that we do not discuss further here. For NBC the normalizer constraints depend on the parities of the mode numbers as follows:

$N_{D_4+T^2(B_N)/B_N}$	mode (m,n)
$D_4$	m+n odd
$D_2$	m+n even; m, n each odd
$Z_2$	m+n even, m, n each even

Simonelli and Gollub [1989] also demonstrate the existence of codimension two mode interactions between different standing waves, as had previously been obtained in the circular case. Although the dynamics near such points of multiple instability have not been studied in detail either experimentally or theoretically, it is clear that a wider variety of possibilities arises than in the circular problem. Most significant is the role of boundary conditions. For NBC or DBC the primary standing waves are pure modes, and the linearization  $(dS)_0$  of the centre manifold map is diagonal, that is,  $(dS)_0 = -I$ , as in the  $O(2)$  mode interaction. If, however, the experimental conditions require Robin boundary conditions, then  $(dS)_0$  should have a nilpotent part. The normal forms selected by these two linearizations involve quite different nonlinear terms, and presumably quite different dynamics.

For example, a stability analysis of 2-cycle solutions indicates that in the nilpotent mode interaction the primary modes can undergo a secondary Hopf bifurcation, while in the diagonal interaction the first possibility for Hopf bifurcation occurs only along a secondary mixed mode branch. Also, preliminary numerical results suggest that chaotic behaviour is readily found for the nilpotent mode interaction, whereas for the diagonal case chaos is likely to occur only for parameters in a very thin region, as in the  $O(2)$  mode interaction. In addition, when  $(dS)_0 = -I$ , the dependence of the normalizer on the mode numbers indicates that the 'parity' of the modes influences the dynamics. A detailed discussion of these issues will be given in Crawford, Golubitsky and Knobloch [1989].

### Acknowledgements

Much of this paper was written during the 1988-89 Warwick Symposium on Singularity Theory and its Applications, which was supported by a grant from SERC. The work of individual authors was partially supported by the following grants: (JDC) DARPA ACM Program; (MG) DARPA/NSF (DMS-8700897), NASA-Ames (2-432), and the Texas Advanced Research Program (ARP-1100); (MGMG) JNICT; (EK) DARPA/NSF (DMS-8814702); (INS) SERC.



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