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STABILITY COMPUTATIONS FOR NILPOTENT HOPF BIFURCATIONS IN COUPLED CELL SYSTEMS

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Vanderbauwhede and van Gils, Krupa, and Langford studied unfoldings of bifurcations with purely imaginary eigenvalues and a nonsemisimple linearization, which generically occurs in codimension three. In networks of identical coupled ODE these nilpotent Hopf bifurcations can occur in codimension one. Elmhirst and Golubitsky showed that these bifurcations can lead to surprising branching patterns of periodic solutions, where the type of bifurcation depends in part on the existence of an invariant subspace corresponding to partial synchrony. We study the stability of some of these bifurcating solutions. In the absence of partial synchrony the problem is similar to the generic codimension three problem. In this case we show that the bifurcating branches are generically unstable. When a synchrony subspace is present we obtain partial stability results by using only those near identity transformations that leave this subspace invariant.

Keywords: Networks of coupled cells; Hopf bifurcation; normal form; stability.

1. Introduction

Networks of coupled systems of differential equations arise naturally in a variety of contexts (see [Stewart, 2004]) from discretization of partial differential equations to models of locomotor central pattern generators, [Kopell & Ermentrout, 1988]. The dynamics occurring in such networks often have properties that are nongeneric if the special structure of the network is absent. One such property is the occurrence of Hopf bifurcation with nonsemisimple eigenvalues in codimension one bifurcations. An example of a network with this property was given by [Golubitsky *et al.*, 2004a]. Building on the work of Golubitsky *et al.* [2004a], Elmhirst and Golubitsky [2006] analyzed branching for nilpotent Hopf bifurcations occurring in three different types of networks, finding branching patterns of periodic solutions very different from the single nontrivial branch in generic Hopf bifurcation.

Nilpotent Hopf bifurcations occur when an equilibrium has a purely imaginary eigenvalue of algebraic multiplicity two and a nonsemisimple linearization. Such bifurcations are of codimension three in the absence of special structure. Branching patterns were studied by Vanderbauwhede [1986] using the Liapunov–Schmidt method and by van Gils *et al.* [1990] using the normal form approach. In the context of coupled cells there are degeneracies both in the linear and nonlinear parts of the vector field. Consequently, the results of either Vanderbauwhede [1986] or van Gils *et al.* [1990]

have limited direct application to the coupled cell problem.

As shown by Cesari and Hale, Liapunov–Schmidt reduction can be used to study Hopf bifurcations (see [Golubitsky & Schaeffer, 1984]) and is the main tool used in [Elmhirst & Golubitsky, 2006]. This method has the advantage that branches of periodic solutions can be found by solving for the zeros of a "reduced" mapping and the disadvantage that the stability of these solutions is not necessarily preserved in the reduction. The alternative reduction method, center manifold plus normal form (see [Vanderbauwhede, 1989]) has the advantage of preserving stability of solutions, but is often more difficult to compute. A point of particular importance for coupled cell networks is that the network structure leads to degeneracies in the bifurcation equations obtained using either reduction method and unlike symmetry these degeneracies are not easy to keep track of. In either case the relationship between coefficients in the reduced equations and coefficients in the original coupled cell vector field is not easy to establish.

Our goal is to use the normal form approach to study nilpotent Hopf bifurcations in coupled cell networks and thereby to compute the stability of periodic solutions emanating from these bifurcations. In this work we consider specifically two of the networks studied in [Elmhirst & Golubitsky, 2006], the three-cell and five-cell networks shown in Fig. 1. Using normal form theory we are able to recover the bifurcation equations derived in [Elmhirst & Golubitsky, 2006] for both of these networks and obtain complete stability information for the five-cell network (all periodic solutions are unstable). We obtain only partial stability results for the three-cell network.

In the theory developed in [Stewart *et al.*, 2003] and [Golubitsky *et al.*, 2004b], graphs such as those in Fig. 1 correspond to classes of systems of differential equations. The three-cell network corresponds to coupled systems of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_1, x_3}, \lambda) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_3}, \lambda) \\ \dot{x}_3 &= f(x_3, \overline{x_2, x_3}, \lambda) \end{aligned} \tag{1}$$

where $x_1, x_2, x_3 \in \mathbf{R}^k$, $\lambda \in \mathbf{R}$ is a bifurcation parameter, and the overbar indicates that $f : \mathbf{R}^k \times \mathbf{R}^{2k} \times \mathbf{R} \to \mathbf{R}^k$ satisfies $f(a, b, c, \lambda) = f(a, c, b, \lambda)$. The five-cell network corresponds to systems of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_1, x_4, x_4}, \lambda) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_2, x_5}, \lambda) \\ \dot{x}_3 &= f(x_3, \overline{x_2, x_4, x_4}, \lambda) \\ \dot{x}_4 &= f(x_4, \overline{x_2, x_4, x_5}, \lambda) \\ \dot{x}_5 &= f(x_5, \overline{x_1, x_2, x_3}, \lambda) \end{aligned} \tag{2}$$

where $x_j \in \mathbf{R}^k$, $f : \mathbf{R}^k \times \mathbf{R}^{3k} \times \mathbf{R} \to \mathbf{R}^k$, and the overbar indicates that $f(a, b, c, d, \lambda)$ is invariant under permutation of b, c, d.

By the results of Elmhirst and Golubitsky [2006] the five-cell network has branches of periodic orbits of opposite criticality whose amplitudes grow at order λ. We prove that these solutions are generically (within the class (2)) unstable. For the three-cell network, Elmhirst and Golubitsky [2006] proved the existence of two or four branches of periodic solutions, each of which grows at order $\lambda^{1/2}$. One of these solutions has partial synchrony (since $x_1 = x_2$ is a flow-invariant subspace). We obtain complete stability information for this partially synchronous solution. For the other solutions we outline a possible approach to the stability problem, noting that, unless there is a new idea, only partial results can be expected.

The structure of coupled cell networks may impose restrictions on both the linear and the nonlinear parts of the normal form. The restrictions on the linear level are present in most networks, but

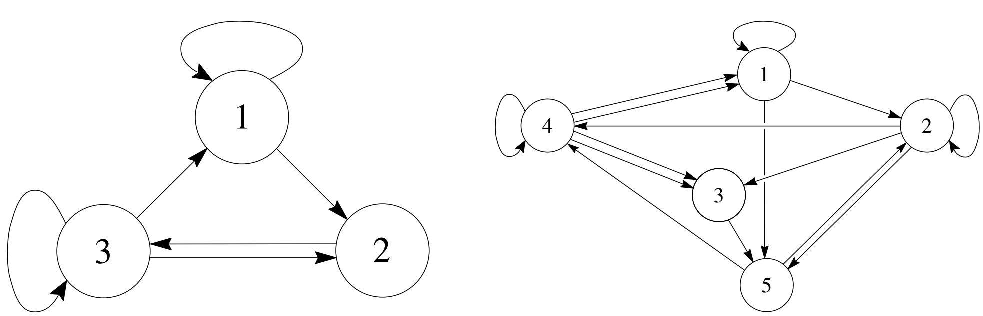

Fig. 1. Three-cell and five-cell networks with nilpotent linear parts.

the degeneracies at nonlinear level in the normal form may or may not be present. We conjecture that network architecture must force the presence of an invariant space of partial synchrony with a nontrivial intersection with the center subspace in order for restrictions on the nonlinear terms in the normal form to be present in a robust way. The five-cell network in [Elmhirst & Golubitsky, 2006] (Fig. 1 (right)) is an example of a network where there are no restrictions on the nonlinear level, whereas the three-cell network in [Elmhirst & Golubitsky, 2006] (Fig. 1 (left)) gives an example of a network with an invariant space of partial synchrony, thus providing an example where restrictions on cubic terms in the normal form are present. Section 2 of this paper contains the analysis of systems with no restrictions on the nonlinear level. In Sec. 3 we treat systems with a flow-invariant synchrony subspace.

2. No Restrictions on Reduction at Cubic Level

In this section we consider the five-cell network (2) analyzed by Elmhirst and Golubitsky [2006], who proved using the Liapunov–Schmidt method that there exist two branches of solutions with amplitude $O(\lambda)$, one supercritical and one subcritical. Here, we reprove the result of Elmhirst and Golubitsky [2006] using normal form theory and we show that both of the bifurcating branches are unstable.

Recall that the Liapunov–Schmidt reduction preserves the phase shift symmetry, which leads naturally to $\mathbf{S}^1$ symmetry in the bifurcation equation. The normal form is not unique, but it can be chosen to be $\mathbf{S}^1$ symmetric up to any finite order. We use the following $\mathbf{S}^1$ equivariant normal form (derived in [Elphick *et al.*, 1987]).

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix} = \begin{pmatrix} i+\lambda & 1\\ \mu & i+\lambda\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \Phi_1\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \Phi_2\begin{pmatrix}0\\ z_1\end{pmatrix}, \tag{3}$$

with $z_i \in \mathbf{C}$, $\lambda \in \mathbf{R}$, $\mu \in \mathbf{C}$ and

$$\Phi_k \equiv \Phi_k(z_1\overline{z}_1, \mathrm{Im}\,(\overline{z}_1 z_2), \lambda, \mu).$$

The $\mathbf{S}^1$ action is given by

$$\theta(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2).$$

Since (3) is a normal form for a generic nilpotent Hopf bifurcation it has one real parameter λ and one complex parameter μ, which unfold the linear part. For a certain class of networks, including all the networks studied by Elmhirst and Golubitsky [2006], the network architecture forces the existence of a nontrivial Jordan block, which implies that $\mu = 0$. Indeed, since we study only codimension one coupled cell bifurcations, we may assume that Φ_k is independent of μ.

The network architecture can also impose constraints on the nonlinear terms in the normal form. In this section we make a nondegeneracy assumption that holds for (2) but is violated for any network with an invariant subspace of partial synchrony which has a nontrivial intersection with the center subspace. Networks of this type will be considered in Sec. 3. Let $a + ib$ be defined as follows

$$a + ib = \frac{\partial \Phi_2}{\partial z_1\overline{z}_1}(0,0,0).$$

The nondegeneracy assumption is

$$b \neq 0. \tag{4}$$

The following result is proved using MAPLE.

Proposition 2.1. *The condition (4) holds generically for the five-cell network (2).*

In this section we prove the following results.

Proposition 2.2. *If $b \neq 0$, then there exist two branches of periodic solutions with amplitude $O(\lambda)$; one of the branches is supercritical and one is subcritical.*

Proposition 2.3. *Periodic solutions on both branches are unstable.*

Proposition 2.2 is proved in Sec. 2.1 and Proposition 2.3 is proved in Sec. 2.2.

2.1. *Proof of Proposition 2.2*

Recall that periodic orbits near a Hopf bifurcation correspond to relative equilibria of the $\mathbf{S}^1$ symmetric normal form. We define new variables (w_1, w_2), suitable for the analysis of relative equilibria, as follows:

$$z_1 = w_1 e^{i(1-\tau)t}, \quad z_2 = w_2 e^{i(1-\tau)t}. \tag{5}$$

After transforming to the new variables, we observe that in both equations there is a factor $e^{i(1-\tau)t}$. We cancel this factor and obtain the following

$$\begin{aligned}\dot{w}_1 &= (\lambda + i\tau)w_1 + w_2 + \Phi_1 w_1\\ \dot{w}_2 &= (\lambda + i\tau)w_2 + \Phi_2 w_1 + \Phi_1 w_2.\end{aligned} \tag{6}$$

We will carry out the analysis for (6) truncated at lowest order:

$$\begin{aligned}\dot{w}_1 &= (\lambda + i\tau)w_1 + w_2 \\ \dot{w}_2 &= (\lambda + i\tau)w_2 + (a+bi)w_1^2\overline{w}_1.\end{aligned} \tag{7}$$

By finding the equilibria of (7) near 0 with λ and τ small we obtain all the first order approximations of small amplitude periodic solutions of the original problem. The result can be extended to (6) by standard perturbation theory.

We need to solve:

$$\begin{aligned}0 &= (\lambda + i\tau)w_1 + w_2 \\ 0 &= (\lambda + i\tau)w_2 + (a+ib)w_1^2\overline{w}_1.\end{aligned} \tag{8}$$

We solve the first equation in (8) for w_2 as function of w_1 and substitute the result into the second equation. Subsequently we cancel the factor w_1, obtaining the bifurcation equation:

$$0 = -(\lambda + i\tau)^2 + (a+ib)w_1\overline{w}_1. \tag{9}$$

Equation (9) is S^1 invariant, so when finding solutions we can assume that $w_1 = x > 0$. We substitute this choice into (9) and replace the complex equation by two real ones, obtaining:

$$\begin{aligned}0 &= \tau^2 - \lambda^2 + ax^2 \\ 0 &= -2\tau\lambda + bx^2.\end{aligned} \tag{10}$$

We now prove a result which implies Proposition 2.2.

Proposition 2.4. *Assume $b \neq 0$ and let $\Delta = \sqrt{a^2+b^2} - a$. Then the solutions of (10) are given by*

$$\tau = \frac{\lambda}{b}\Delta, \quad x = \sqrt{2\left(\frac{\lambda}{b}\right)^2\Delta}. \tag{11}$$

Proof. Solving the second equation in (10) for x^2 we obtain

$$x^2 = \frac{2\tau\lambda}{b}. \tag{12}$$

Consequently we reduce (10) to the quadratic equation:

$$b\tau^2 + 2a\lambda\tau - b\lambda^2 = 0.$$

One choice of the solution is

$$\tau = \frac{\lambda}{b}\Delta. \tag{13}$$

Combining with (12) we conclude that

$$x^2 = 2\left(\frac{\lambda}{b}\right)^2\Delta,$$

which is the positive solution. The other choice of the solution for τ gives $x^2 < 0$, which is impossible. ∎

2.2. *Proof of Proposition 2.3*

We first note that the Floquet exponents of the small amplitude periodic solutions of the normal form system (3) are just the eigenvalues of the linearization of (6) at zeroes of that system.

The proof of Proposition 2.3 proceeds as follows. We compute the eigenvalues of the linearization of the truncated system (7) at the solutions (10). The characteristic polynomial is of degree four, but it has a 0 root corresponding to the eigenvector along the orbit of the normal form $\mathbf{S}^1$ symmetry. We show below that the three nonzero roots of this characteristic polynomial correspond to the roots of

$$\hat{q}(\mu) = \hat{\mu}^3 - 4\hat{\mu}^2 + 2(1+3K)\hat{\mu} + 4(1+K) \tag{14}$$

in the scaled variable $\hat{\mu} = \mu/\lambda$, where $K = \Delta^2/b^2 > 0$ is a constant depending on a and b. More precisely, we show that a root $\hat{\mu}$ of (14) corresponds to an eigenvalue of the linearization of the truncated system (7) at a solution for some $\lambda \neq 0$, where $\mu = \lambda\hat{\mu}$ and $\tau = \sqrt{K}\lambda$.

Rather than trying to find the roots of $\hat{q}$ explicitly, which would result in lengthy formulas, we prove that $\hat{q}$ always has at least one root with positive real part and at least one root with negative real part, which proves Proposition 2.3 covering both the cases of $\lambda > 0$ and $\lambda < 0$. The proof divides into two parts: the derivation of (14) and the existence of a positive real part eigenvalue.

Derivation of $\hat{q}$. We begin by obtaining the linearization and its determinant. We transform (7) to real coordinates:

$$w_1 = x_1 + iy_1, \quad w_2 = x_2 + iy_2.$$

The system (7) in the real coordinates becomes:

$$\begin{aligned}\dot{x}_1 &= \lambda x_1 - \tau y_1 + x_2 \\ \dot{y}_1 &= \tau x_1 + \lambda y_1 + y_2 \\ \dot{x}_2 &= \lambda x_2 - \tau y_2 + (x_1^2 + y_1^2)(ax_1 - by_1) \\ \dot{y}_2 &= \tau x_2 + \lambda y_2 + (x_1^2 + y_1^2)(bx_1 + ay_1)\end{aligned} \tag{15}$$

Recall that we have found equilibria of the form $(x, 0)$, where x satisfies (10). Linearizing (15) at such an equilibrium we get the matrix:

$$A = \begin{pmatrix} \lambda & -\tau & 1 & 0 \\ \tau & \lambda & 0 & 1 \\ 3ax^2 & -bx^2 & \lambda & -\tau \\ 3bx^2 & ax^2 & \tau & \lambda \end{pmatrix}$$

Let

$$B = \begin{pmatrix} \lambda & -\tau \\ \tau & \lambda \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 3ax^2 & -bx^2 \\ 3bx^2 & ax^2 \end{pmatrix}.$$

Note that $\det A = \det(B^2 - C)$ and similarly

$$p(\mu) = \det(A - \mu I) = \det((B - \mu I)^2 - C).$$

Using (10) it follows that

$$(B - \mu I)^2 - C = \begin{pmatrix} (\lambda - \mu)^2 - \tau^2 - 3ax^2 & -2(\lambda - \mu)\tau + bx^2 \\ 2(\lambda - \mu)\tau - 3bx^2 & (\lambda - \mu)^2 - \tau^2 - ax^2 \end{pmatrix} = \begin{pmatrix} \mu^2 - 2\lambda\mu - 2ax^2 & 2\tau\mu \\ -2\tau\mu - 2bx^2 & \mu^2 - 2\lambda\mu \end{pmatrix}$$

Note that when $\mu = 0$ we see that $\det A = 0$, as expected. We now have $p(\mu) = \mu q(\mu)$, where:

$$\begin{aligned} q(\mu) &= \frac{\det((B - \mu I)^2 - C)}{\mu} \\ &= \mu^3 - 4\lambda\mu^2 + (2\lambda^2 + 6\tau^2)\mu + 4\lambda(\lambda^2 + \tau^2). \end{aligned}$$

Next recall from (11) that $\tau = \lambda\Delta/b$. Then

$$q(\mu) = \mu^3 - 4\lambda\mu^2 + 2(1 + 3K)\lambda^2\mu + 4(1 + K)\lambda^3.$$

Finally, we divide q by λ^3 and obtain the polynomial (14).

Verification of a real part positive root. As mentioned at the beginning of the section, the roots of q cannot be expressed in simple form for general a and b. We can however show that the roots of q are never purely imaginary, and then prove Proposition 2.3 by verifying the root structure of q for a single choice of a and b. The basic tool that we use is the fact that a cubic polynomial

$$\mu^3 + a_2\mu^2 + a_1\mu + a_0$$

has purely imaginary roots if and only if

$$a_0 - a_1a_2 = 0. \tag{16}$$

Clearly $\hat{q}$ cannot have a 0 root. We also prove that it can never have a root on the imaginary axis. In the case of $\hat{q}$, we have

$$a_0 - a_1a_2 = 4(3 + 7K),$$

which is nonzero for any choice of a and $b \neq 0$. We can easily find a specific choice of a and b (for example $K = 1$) for which $\hat{q}$ has both a root with positive real part and a root with negative real part. The result must hold by continuous dependence of the roots of $\hat{q}$ on a and b.

3. Equations with Synchrony Subspace

Consider a three-cell system of the form (1). Then there is a synchrony subspace defined by $x_1 = x_2$. We assume for simplicity that the internal dynamics is two-dimensional, which implies that the synchrony subspace is four-dimensional. We assume that there is a Hopf bifurcation from a synchronous equilibrium and that the generalized eigenspace of the critical eigenvalue is not contained in the synchrony subspace. The following is proved in [Elmhirst & Golubitsky, 2006].

Lemma 3.1. *The eigenspace of the critical eigenvalue is contained in the synchrony subspace. The intersection of the center subspace with the synchrony subspace is the eigenspace.*

We now perform a center manifold reduction to a four-dimensional center manifold. Clearly the intersection of the center manifold with the synchrony subspace is a two-dimensional invariant manifold tangent to the critical eigenspace. Moreover, [Leite & Golubitsky, 2006, Lemma 4.12] lets us choose coordinates (z_1, z_2) on the center manifold in such a way that the two-dimensional plane $z_2 = 0$ is both the critical eigenspace and is flow invariant. General theory of Elphick *et al.* [1987] tells us that there is an $\mathbf{S}^1$ symmetric normal form. We prove that there is such a normal form for which the space $z_2 = 0$ is flow invariant. The question centers on how the invariant space $z_2 = 0$ transforms when the system is brought to normal form.

More precisely, consider

$$\begin{aligned} \dot{z}_1 &= iz_1 + z_2 + f_1(z_1, z_2) \\ \dot{z}_2 &= iz_2 + f_2(z_1, z_2). \end{aligned} \tag{17}$$

We assume that f and g are C^∞ smooth functions whose constant and linear terms vanish. We prove the following.

Theorem 3.2. *Assume $z_2 = 0$ is an invariant space for (17), that is, $f_2(z_1, 0) = 0$. Then, for any $N > 0$,* *(17) can be transformed to the normal form*

$$\begin{aligned}\dot{z}_1 &= iz_1 + z_2 + g_1(z_1, z_2) + O(|z|^{N+1})\\ \dot{z}_2 &= iz_2 + g_2(z_1, z_2) + O(|z|^{N+1})\end{aligned} \tag{18}$$

where g_1 and g_2 are $\mathbf{S}^1$-equivariant polynomial mappings of degree N, and (18) has an invariant space $z_2 = 0$.

The next subsection is devoted to the proof of Theorem 3.2. Note that $\mathbf{S}^1$ symmetric normal forms are not unique and that Theorem 3.2 does not hold for every such normal form, including the normal form of Elphick *et al.* [1987].

In the sequel we consider the truncated and unfolded normal forms:

$$\begin{aligned}\dot{z}_1 &= (\lambda + i)z_1 + z_2 + g_1(z_1, z_2)\\ \dot{z}_2 &= (\lambda + i)z_2 + g_2(z_1, z_2)\end{aligned} \tag{19}$$

where g_1, g_2 are $\mathbf{S}^1$-equivariant polynomial mappings homogeneous of degree 3, and λ is a bifurcation parameter.

3.1. *Proof of Theorem 3.2*

Suppose that a vector field F has a flow-invariant subspace W and that φ is a diffeomorphism that leaves W invariant. Then the change of coordinates of F by φ is a vector field that leaves W invariant. For each $2 \leq n \leq N$ we carry out the proof of Theorem 3.2 in two steps, each of which preserves the invariant plane $z_2 = 0$.

Step 1. Use a transformation of the form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 + q(w_1, w_2) \end{pmatrix}, \tag{20}$$

where q is a homogeneous polynomial of degree n satisfying $q(w_1, 0) = 0$. We show that we can choose q so that the terms of order n in the resulting $\dot{w}_2$ equation are $\mathbf{S}^1$-equivariant and the space $w_2 = 0$ remains flow-invariant.

Step 2. We use the transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 + p(w_1, w_2) \\ w_2 \end{pmatrix} \tag{21}$$

so that the terms of order n in the resulting $\dot{w}_1$ equation are $\mathbf{S}^1$-equivariant, the RHS of the w_2 equation remains unchanged up to order n, and the space $w_2 = 0$ remains flow-invariant.

Let

$$L = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}.$$

Define the *adjoint operator ad_L* by

$$\begin{aligned}ad_L(q) = iq - (&q_{z_1}(iz_1 + z_2) + q_{z_2}(iz_2)\\ &- q_{\bar{z}_1}(i\bar{z}_1 - \bar{z}_2) - q_{\bar{z}_2}(i\bar{z}_2)).\end{aligned} \tag{22}$$

Let $\mathcal{P}_n$ be the space of homogeneous polynomials of degree n in $z_1, \bar{z}_1, z_2, \bar{z}_2$. It follows that, for every $n > 1$, ad_L is a linear transformation of $\mathcal{P}_n$ into itself.

Suppose a change of coordinates of the form (20) is applied to (17), with $q \in \mathcal{P}_n$. Then the transformed system has the form

$$\begin{aligned}\dot{w}_1 &= iw_1 + w_2 + q + f_1(w_1, w_2) + O(|z|^{n+1})\\ \dot{w}_2 &= iw_2 + f_2(w_1, w_2) + ad_L(q) + O(|z|^{n+1})\end{aligned} \tag{23}$$

Note that for any $n > 1$, ad_L can be used to remove terms of order n from the RHS of (17). The following two lemmas specify which terms can be removed.

Lemma 3.3. *The subspace $\mathcal{P}_n$ can be decomposed into ad_L invariant subspaces V_k, where $k = 0, \ldots, n$, with*

$$\begin{aligned}V_k = \text{span}\{&z_1^{k_1}\bar{z}_1^{l_1}z_2^{k_2}\bar{z}_2^{l_2} : k_1 + k_2 = k;\\ &l_1 + l_2 = n - k\}\end{aligned}$$

Moreover, the spaces $V_{k,0} \subset V_k$ defined by

$$\begin{aligned}V_{k,0} = \text{span}\{&z_1^{k_1}\bar{z}_1^{l_1}z_2^{k_2}\bar{z}_2^{l_2} : k_2 + l_2 > 0,\ k_1 + k_2 = k;\\ &l_1 + l_2 = n - k\}\end{aligned}$$

are also invariant for ad_L.

Proof. Follows by inspection from the definition of ad_L. ■

Lemma 3.4. *If $k \neq (n + 1)/2$ then ad_L restricted to either V_k or $V_{k,0}$ is an isomorphism.*

Proof. Suppose $z_1^{k_1}\bar{z}_1^{l_1}z_2^{k_2}\bar{z}_2^{l_2}$ is an element of $\mathcal{P}_n$. Then

$$\begin{aligned}ad_L(z_1^{k_1}\bar{z}_1^{l_1}z_2^{k_2}\bar{z}_2^{l_2}) = i(n + 1 - 2k)&z_1^{k_1}\bar{z}_1^{l_1}z_2^{k_2}\bar{z}_2^{l_2}\\ &+ k_1 z_1^{k_1-1}\bar{z}_1^{l_1}z_2^{k_2+1}\bar{z}_2^{l_2}\\ &+ l_1 z_1^{k_1}\bar{z}_1^{l_1-1}z_2^{k_2}\bar{z}_2^{l_2+1}\end{aligned}$$

It follows that

$$ad_L = (n + 1 - 2k)I + N$$

where N is nilpotent. The claim about V_k follows. The claim about $V_{k,0}$ is proved analogously. ■

Corollary 3.5. *Terms in V_k can be removed provided that $k \neq (n+1)/2$. When n is odd and $k = (n+1)/2$ the terms in V_k are $\mathbf{S}^1$-equivariant.*

Proof of Theorem 3.2. Suppose we apply a transformation of the form (21), with p homogeneous of order n, to a system of the form (17). Then the transformed system has the form

$$\begin{aligned}\dot{w}_1 &= iw_1 + f_1(w_1, w_2) + ad_L(p) + O(|z|^{n+1})\\ \dot{w}_2 &= iw_2 + f_2(w_1 + p, w_2).\end{aligned} \tag{24}$$

Note that (24) still has an invariant space $w_2 = 0$, regardless of the form of p, since $f_2(w_1 + p, 0) = 0$. Also, the w_2 equation in (24) can be rewritten in the form

$$\dot{w}_2 = iw_2 + f_2(w_1, w_2) + O(|z|^{n+1})$$

which means that the equivariance of f_2 up to order n is preserved. It now follows that we can bring (17) to the form (18) by successively applying transformations of the form (20) and (21). At any order $n \geq 2$ we first carry out (20) and then (21), bringing the system to the normal form of order n and preserving the invariance of $w_2 = 0$. ■

3.2. *Branching*

Suppose a normal form system of equations of the form (19) is given. Transforming to the variables (5) we obtain the equations

$$\begin{aligned}\dot{w}_1 &= (\lambda + \tau i)w_1 + w_2 + g_1(w_1, w_2)\\ \dot{w}_2 &= (\lambda + \tau i)w_2 + g_2(w_1, w_2)\end{aligned} \tag{25}$$

where τ is a small parameter corresponding to the deviation of the period from 2π. Recall that periodic orbits of (1) near a Hopf bifurcation correspond to equilibria of (25). We look for synchronous periodic orbits which correspond to equilibria of (25) with $w_2 = 0$. We can assume, due to $\mathbf{S}^1$ symmetry, that $w_1 = x \in \mathbf{R}$. Consequently, we obtain the branching equations:

$$\begin{aligned}\lambda + p(x^2) &= 0\\ \tau + q(x^2) &= 0.\end{aligned} \tag{26}$$

where p and q are such that $g_1(x, 0) = x(p(x^2) + iq(x^2))$. Expanding:

$$p(x^2) = A_R x^2 + O(x^4), \quad q(x^2) = A_I x^2 + O(x^4)$$

we obtain that (26) at lowest order have the form:

$$\begin{aligned}\lambda + A_R x^2 &= 0\\ \tau + A_I x^2 &= 0.\end{aligned} \tag{27}$$

3.3. *Stability of the synchronous solution*

We assume that x is a solution of (26). We now compute the Jacobian of (25) at the solution $w_1 = x$, $w_2 = 0$. The Jacobian of (25) is given in 2×2 block form as

$$J = \begin{pmatrix} D_{w_1}g_1(x,0) & D_{w_2}g_1(x,0)\\ \mathbf{0} & D_{w_2}g_2(x,0)\end{pmatrix},$$

where $\mathbf{0}$ denotes the 2×2 zero matrix and $D_{w_i}g_j$ are 2×2 real matrices. Using complex notation

$$(D_w g)_p(u) = \frac{\partial g}{\partial w}(p)u + \frac{\partial g}{\partial \bar{w}}(p)\bar{u}.$$

Since J is upper block diagonal, its eigenvalues are given by the eigenvalues of $D_{w_1}g_1(x,0)$ and the eigenvalues of $D_{w_2}g_2(x,0)$. It follows also that $D_{w_1}g_1(x,0)$ has an eigenvalue 0, corresponding to translation along the $\mathbf{S}^1$ group orbit, and an eigenvalue $2p(x^2)$ (for the definition of p see the sentence following (26)). Hence the branch can only be stable if it is supercritical.

The condition for the other two eigenvalues to both have negative real parts is

$$\operatorname{tr} D_{w_2}g_2(x,0) < 0 \quad \text{and} \quad \det D_{w_2}g_2(x,0) > 0,$$

which is equivalent to

$$\begin{gathered}\operatorname{Re}\frac{\partial g_2}{\partial w_2}(x,0) < 0 \quad \text{and}\\ \left|\frac{\partial g_2}{\partial w_2}(x,0)\right|^2 - \left|\frac{\partial g_2}{\partial \bar{w}_2}(x,0)\right|^2 > 0.\end{gathered} \tag{28}$$

Writing:

$$\begin{gathered}\frac{\partial g_2}{\partial w_2}(x,0) = Bx^2 + O(x^4) \quad \text{and}\\ \frac{\partial g_2}{\partial \bar{w}_2}(x,0) = Cx^2 + O(x^4)\end{gathered}$$

we obtain that (28) is given, up to $O(x^2)$, by

$$\operatorname{Re} B < 0 \quad \text{and} \quad |B|^2 - |C|^2 > 0. \tag{29}$$

It is clear that conditions (29) can be satisfied. Figure 2 shows a stable synchronous solution, computed for a network of type (1) with f given as

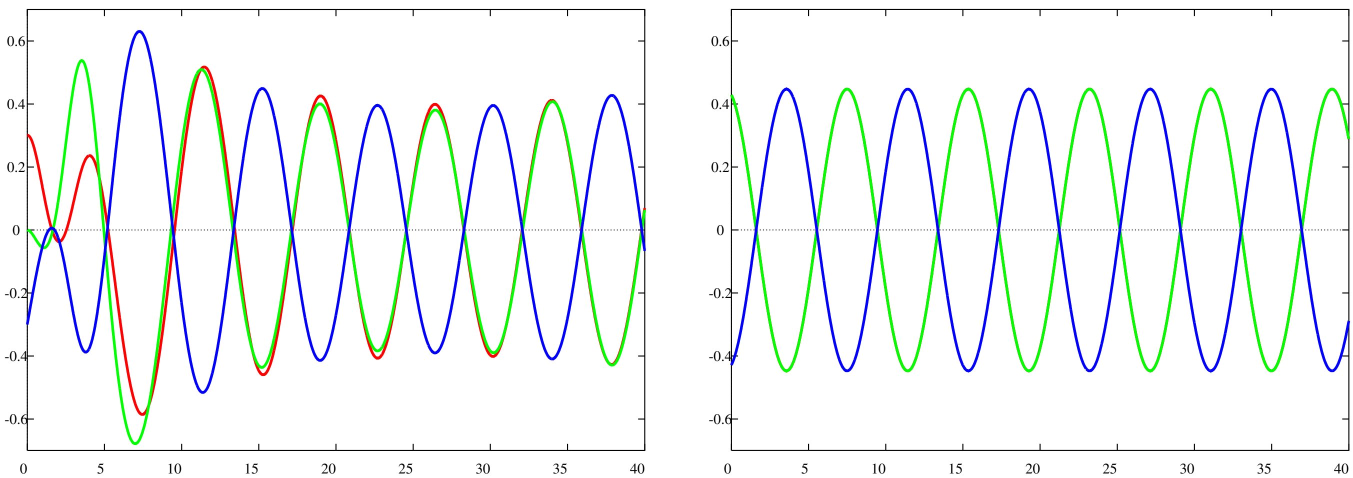

Fig. 2. Synchronous periodic solution — transient on left and no transient on right. Simulation with f in (30), $\lambda = 0.1$, $\alpha_R = -0.5$, $\beta_R = -0.5$, $\alpha_I = -1$ and $\beta_I = 0.75$.

follows:

$$
\begin{aligned}
f(a,b,c) = \lambda a + Ka - b - c + \frac{1}{2}|a|^2(\beta_R a + \beta_I Ka) \\
+ \frac{1}{4}(\beta_R La - \beta_I LKa)(b_1^2 - b_2^2 + c_1^2 - c_2^2) \\
- \frac{1}{2}(\beta_R LKa + \beta_I La)(b_1 b_2 + c_1 c_2) \\
+ \frac{1}{2}(|b|^2 + |c|^2)(\alpha_R a + \alpha_I Ka) \qquad (30)
\end{aligned}
$$

where $\lambda \in \mathbf{R}$, $\alpha, \beta \in \mathbf{C}$,

$$
K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

3.4. *Remarks on stability of asynchronous solutions*

In this section we consider a truncation of (25) of the form

$$
\begin{aligned}
\dot{w}_1 &= (\lambda + i\tau)w_1 + w_2 + Aw_1^2\overline{w}_1 \\
\dot{w}_2 &= (\lambda + i\tau)w_2 + Bw_2 w_1 \overline{w}_1
\end{aligned} \qquad (31)
$$

For (31) we can easily find explicit expression for an asynchronous solution and we can derive a sufficient condition for stability, which can be satisfied. Thus we prove that asynchronous solutions can be stable.

The branching equations for (31) are

$$
\begin{aligned}
0 &= (\lambda + i\tau)w_1 + w_2 + Aw_1^2\overline{w}_1 \\
0 &= (\lambda + i\tau)w_2 + Bw_2 w_1 \overline{w}_1
\end{aligned} \qquad (32)
$$

Due to the $\mathbf{S}^1$ action there exists a solution of (32) of the form (x, w_2), where x is real. We can now easily solve (32), obtaining the branching equations

$$
\begin{aligned}
\lambda + B_R x^2 &= 0 \\
\tau + B_I x^2 &= 0 \\
w_2 &= (B - A)x^3.
\end{aligned} \qquad (33)
$$

Remark. By the results of Elmhirst and Golubitsky [2006] a network of the form (1) has two or four branches of solutions. For $C = 0$ we are in the case of two branches of solutions, rather than four.

We prove the following result on stability of solutions given by (33).

Proposition 3.6. *The branch given by* (*33*) *can only be stable if it is supercritical. Moreover, there exists an open region in the space of coefficients* (A, B), *where this solution is stable.*

Proof. Rewriting (31) in real form we obtain

$$
\begin{aligned}
\dot{x}_1 &= \lambda x_1 - \tau y_1 + x_2 + (x_1^2 + y_1^2)(A_R x_1 - A_I y_1) \\
\dot{y}_1 &= \tau x_1 + \lambda y_1 + y_2 + (x_1^2 + y_1^2)(A_I x_1 + A_R y_1) \\
\dot{x}_2 &= \lambda x_2 - \tau y_2 + (B_R x_2 - B_I y_2)(x_1^2 + y_1^2) \\
\dot{y}_2 &= \tau x_2 + \lambda y_2 + (B_R y_2 + B_I x_2)(x_1^2 + y_1^2)
\end{aligned} \qquad (34)
$$

Linearizing (34) about the solution (33) yields the following Jacobian matrix:

$$\begin{pmatrix} (3A_R - B_R)x^2 & (B_I - A_I)x^2 & 1 & 0 \\ (3A_I - B_I)x^2 & (A_R - B_R)x^2 & 0 & 1 \\ 2x^4(B_R(B_R - A_R) - B_I(B_I - A_I)) & 0 & 0 & 0 \\ 2x^4(B_R(B_I - A_R) + B_I(B_R - A_R)) & 0 & 0 & 0 \end{pmatrix}, \tag{35}$$

where x^2 is determined by the first equation in (33). The characteristic polynomial for (35) has the form

$$p(\mu) = \mu(a_0 x^6 + a_1 x^4 \mu + a_2 x^2 \mu^2 + \mu^3)$$

with

$$\begin{aligned} a_0 &= -2B_R((B_I - A_I)^2 + (B_R - A_R)^2) \\ a_1 &= (A_R - B_R)^2 + 2(A_R + B_R)(A_R - B_R) \\ &\quad + 3(A_I - B_I)^2 \\ a_2 &= 4A_R - 2B_R \end{aligned}$$

We can now introduce a polynomial q:

$$q(\nu) = a_0 + a_1\nu + a_2\nu^2 + \nu^3.$$

The roots of p and q are in 1-1 correspondence (if $x \neq 0$), namely if ν_0 is a root of q then $\nu_0 x^2$ is a root of p. The coefficients a_0 and a_2 correspond to the product and the sum of the roots, respectively, so $a_0 < 0$ and $a_2 < 0$ are necessary for stability. Since $a_0 < 0$ is equivalent to $B_R < 0$, $A_I \neq B_I$ and $A_R \neq B_R$ the branch can only be stable if it is supercritical. If $a_0 = 0$, $a_1 > 0$ and $a_2 < 0$ then q has a 0 eigenvalue and two eigenvalues with negative real parts. It follows that the conditions $a_0 < 0$, $a_1 > 0$ and $a_2 < 0$ imply stability provided that a_0 is sufficiently close to 0. ■

Figure 3 shows a stable asynchronous solution computed for a network of type (1).

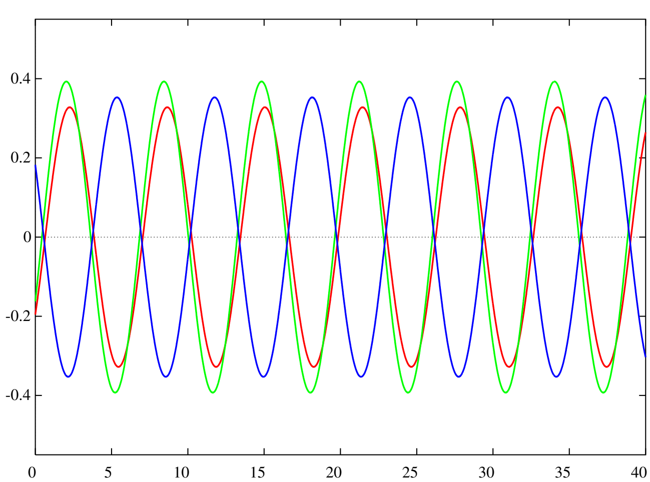

Fig. 3. Asynchronous periodic solution obtained by simulation with f in (30), $\lambda = 0.1$, $\alpha_R = -1.5$, $\beta_R = 0.5$, $\alpha_I = -1$, and $\beta_I = 0.75$.

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