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Patterns in Square Arrays of Coupled Cells

David Gillis

Department of Mathematics, University of Houston, Houston, Texas 77204-3476

and

Martin Golubitsky

Department of Mathematics, University of Houston, Houston, Texas 77204-3476

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1. INTRODUCTION

In recent years it has been observed that reaction–diffusion equations with Neumann boundary conditions (as well as other classes of PDEs) possess more symmetry than that which may be expected, and these "hidden" symmetries affect the generic types of bifurcation which occur (see Golubitsky *et al.* [6], Field *et al.* [5], Armbruster and Dangelmayr [1], Crawford *et al.* [2], Gomes and Stewart [8, 9], and others). In addition, equilibria with more highly developed patterns may exist for these equations than might otherwise be expected. Epstein and Golubitsky [4] show that these symmetries also affect the discretizations of reaction–diffusion equations on an interval. In particular, equilibria of such systems may have well-defined patterns, which may be considered as a discrete analog of Turing patterns.

In this paper, we use an idea similar to the one in [4] to show that the same phenomena occurs in discretizations of reaction–diffusion equations on a square satisfying Neumann boundary conditions. Such discretizations lead to $n \times n$ square arrays of identically coupled cells. By embedding the original $n \times n$ array into a new $2n \times 2n$ array, we can embed the Neumann boundary condition discretization in a periodic boundary condition discretization and increase the symmetry group of the equations from

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square symmetry to the symmetry group $\Gamma = \mathbf{D}_4 \dot{+} (HZ_{2n})^2$, which includes the discrete translation symmetries $(\mathbf{Z}_{2n})^2$.

This extra translation symmetry gives rise (generically) to branches of equilibria which restrict to the original $n \times n$ array yielding equilibria of the equations with Neumann boundary conditions. As shown in Figs. 3–7 of Section 4, these equilibria may take the form of rolls or quiltlike solutions, which are comprised of square blocks of cells symmetrically arranged within the array, with each block containing its own internal symmetries. The notation used in the figures is explained in Section 3.

This paper is structured as follows. In Section 2 we discuss the embedding and the corresponding symmetry group Γ. In Section 3 we list the irreducible representations of Γ, and use the equivariant branching lemma to produce our solutions. The proofs of the results listed in this section are given in Section 6. The analysis of steady-state bifurcations on the large array of cells is a "discretized" version of the analysis in Dionne and Golubitsky [3] of bifurcations in planar reaction–diffusion equations satisfying periodic boundary conditions on a square lattice. In Section 4 we discuss the types of pattern that arise from these solutions. Finally, in Section 5 we show how to compute the eigenvalues of the Jacobian at a trivial equilibrium for a discretized system of differential equations.

2. THE EMBEDDING

We begin by considering a square system of n^2 identical cells with identical nearest neighbor coupling (Fig. 1). Let $x_{l,m} \in \mathbf{R}^k$ be the state variables of the (l, m)th cell. (The number k is the number of equations in the original PDE system.) The general form of the cell system may then be written as

$$\dot{x}_{l,m} = f(x_{l-1,m}, x_{l,m-1}, x_{l,m}, x_{l,m+1}, x_{l+1,m}), \qquad 1 \leq l, m \leq n, \quad (2.1)$$

where f: $(\mathbf{R}^k)^5 \to \mathbf{R}^k$ satisfies

$$\begin{aligned} f(u, v, w, x, y) &= f(u, x, w, v, y) = f(y, v, w, x, u) \\ &= f(x, u, w, y, v). \end{aligned} \quad (2.2)$$

In (2.1) we use the convention that the indices l and m satisfy $l - 1 = 1$ when $l = 1$ and $l + 1 = n$ when $l = n$. This accounts for Neumann boundary conditions. The equalities (2.2) just reflect the fact that the coupling is identical in all directions. In general (2.1) has precisely square symmetry where the symmetries consist of permutations of the cells induced by the eight rotations and reflections of the square.

We embed (2.1) in a larger dimensional system by first folding the square over the right and bottom edges, and then folding it over again to

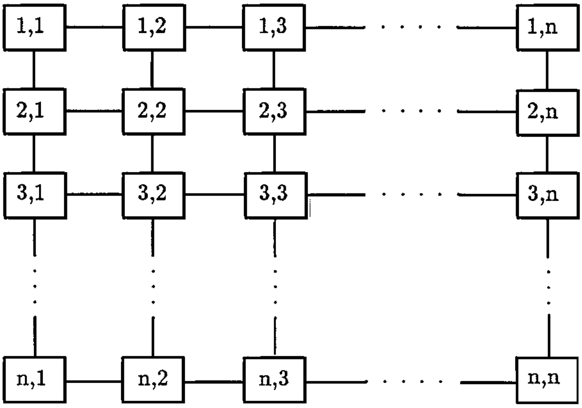

FIG. 1. Square array of n^2 identical cells.

fill out the large square. This leads to a $2n \times 2n$ square array, containing the original system in the upper left quadrant. In this larger system we impose periodic boundary conditions by coupling cells on opposite sides of the boundary (Fig. 2). This leads to the system of equations

$$\dot{x}_{l,m} = f(x_{l-1,m}, x_{l,m-1}, x_{l,m}, x_{l,m+1}, x_{l+1,m}), \qquad 1 \leq l, m \leq 2n, \quad (2.3)$$

where $l - 1 = 2n$ when $l = 1$ and $l + 1 = 1$ when $l = 2n$.

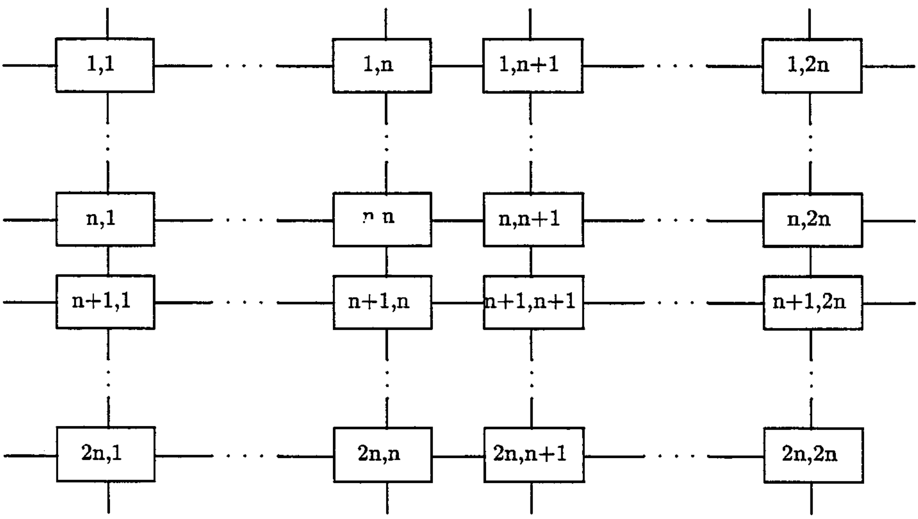

FIG. 2. Large array of $4n^2$ identical cells.

The new system still has square symmetry consisting of permutations on the cells induced from the rotations and reflections of the large square array. We denote this symmetry group by $\mathbf{D}_4$. Due to periodic boundary conditions this system is also symmetric with respect to translations of the form (r, s), which shifts the (l, m)th cell r cells down and s cells to the right. These translations form the group $(\mathbf{Z}_{2n})^2$ which does not commute with the action of $\mathbf{D}_4$. The group $\mathbf{D}_4$ does, however, normalize $(\mathbf{Z}_{2n})^2$ so that the new system has the semidirect product $\Gamma = \mathbf{D}_4 \dot{+} (\mathbf{Z}_{2n})^2$ as its group of symmetries.

Note that any solution $(x_{l,m}) \in (\mathbf{R}^{4n^2})^k$ to (2.3) that satisfies

$$x_{l,m} = x_{l,2n+1-m} = x_{2n+1-l,m}, \qquad 1 \le l, m \le 2n, \tag{2.4}$$

restricts to a solution of (2.1). In terms of group theory, let $\Delta \subset \mathbf{D}_4$ be the subgroup generated by the horizontal flip about the midline κ and the vertical flip about the midline η, where

$$\kappa \cdot (x_{l,m}) = (x_{2n+1-l,m}), \tag{2.5}$$

$$\eta \cdot (x_{l,m}) = (x_{l,2n+1-m}). \tag{2.6}$$

Define the fixed-point subspace

$$\mathrm{Fix}(\Delta) = \{(x_{l,m}) : \kappa \cdot (x_{l,m}) = (x_{l,m}) = \eta \cdot (x_{l,m})\}.$$

Points in Fix(Δ) are precisely those points that satisfy (2.4). Thus (2.4) implies that equilibrium solutions to (2.1) can be found by finding equilibria to (2.3) in Fix(Δ). This is the observation that allows us to determine patterned solutions to (2.1), as solutions to (2.3) inside Fix(Δ) may have larger symmetry groups than would be apparent by solving (2.1) for equilibria directly.

3. BIFURCATION THEORY

In this section we use equivariant bifurcation theory to explain how patterned solutions to the original $n \times n$ cell system may be expected to appear by bifurcation from a trivial group invariant equilibrium. From the point of view of pattern, we may assume that only one equation determines the dynamics of each individual cell; the general case reduces to this one after application of center manifold or Liapunov–Schmidt reduction techniques. That is, we may assume that $k = 1$ and that $(x_{l,m}) \in \mathbf{R}^{4n^2}$.

Suppose that f in (2.1) depends on a bifurcation parameter λ and that $X_0 \in \mathbf{R}^{n^2}$ is a group invariant equilibrium for (2.1) for all λ. Without loss of generality, we may assume that $X_0 = 0$ and hence that $f(0, \lambda) = 0$.

These assumptions then apply to the larger system (2.3) and we discuss our bifurcation analysis in terms of this larger system. We will use the equivariant branching lemma [7] to produce branches of equilibria to (2.3) which restrict to equilibria of (2.1).

We assume that the trivial equilibrium changes stability at a fixed value λ_0; we may assume, without loss of generality, that $\lambda_0 = 0$. Let $\hat{\mathbf{L}}$ and $\mathbf{L}$ be the linearizations of (2.1) and (2.3) at the trivial equilibrium 0 and at $\lambda = 0$. Let $\hat{\mathbf{V}}$ be the kernel of $\hat{\mathbf{L}}$ and let $\mathbf{V}$ be the kernel of $\mathbf{L}$. Our assumption on change of stability of the trivial solution implies that $\hat{\mathbf{V}} \neq \{0\}$. The same trick that allows us to extend the system of ODEs (2.1) to the system (2.3) allows us to embed $\hat{\mathbf{V}}$ as a subspace of $\mathbf{V}$, and hence to note that $\mathbf{V} \neq \{0\}$. Indeed, $\mathbf{V}$ is the smallest Γ-invariant subspace of $\mathbf{R}^{4n^2}$ that contains the image of $\hat{\mathbf{V}}$. We make the genericity assumption that $\mathbf{V}$ is Γ-irreducible (see [7]).

Our approach to finding patterned equilibria of (2.1) is to determine all Γ-irreducible subspaces $\mathbf{V}$ of $\mathbf{R}^{4n^2}$ which are generated by a nonzero subspace $\hat{\mathbf{V}}$ of $\mathbf{R}^{n^2}$. We then assume that $\ker L = \mathbf{V}$. The equivariant branching lemma [7] guarantees that for each axial subgroup of Γ acting on $\mathbf{V}$ there exists a branch of equilibria of (2.3) bifurcating from a trivial equilibrium. By an *axial* subgroup we mean an isotropy subgroup of Γ with a one-dimensional fixed-point subspace in $\mathbf{V}$. Finally, we determine which of these branches of equilibria restrict to equilibria of (2.1).

In the remainder of this section we list the distinct absolutely irreducible Γ-representations of $\mathbf{R}^{4n^2}$, and, for each of these representations, we list the axial subgroups which contain the subgroup Δ generated by the elements κ and η. By (2.4) these equilibria satisfy Neumann boundary conditions when restricted to (2.1). The proofs of these statements, which are summarized in Lemma 3.1, Lemma 3.2, and Theorem 3.3, are given in Section 6.

The $(\mathbf{Z}_{2n})^2$-Irreducible Subspaces

Note that any Γ-irreducible representation must be a direct sum of $(\mathbf{Z}_{2n})^2$-irreducible subspaces. Let $\mathbf{k} = (a, b) \in \mathbf{Z}^2$ and consider the space

$$\mathbf{V_k} = \left\{ \left(\mathrm{Re}\left(z \exp\left[\frac{\pi i}{n} (l, m) \cdot \mathbf{k} \right] \right) \right) \in \mathbf{R}^{4n^2} \colon z \in \mathbf{C} \text{ and } 1 \leq l, m \leq 2n \right\}. \tag{3.1}$$

Let I be the set of indices $\mathbf{k} = (a, b)$ listed in (1)–(8) of Table 1.

TABLE 1
Types of $\mathbf{Z}_{2n}^2$-irreducible representations in $\mathbf{V}$

Type	dim	$\mathbf{k}$	Restrictions
(1)	1	$(0,0)$	
(2)	1	(n,n)	
(3)	1	$(0,n)$	
(4)	1	$(n,0)$	
(5)	2	(a,a)	$1 \le a \le n-1$
(6)	2	$(a,0)$	$1 \le a \le n-1$
(7)	2	$(0,b)$	$1 \le b \le n-1$
(8)	2	(a,b)	$1 \le b < a \le 2n-1$

LEMMA 3.1.

(a) *Each* $\mathbf{V}_\mathbf{k}$ *is* $(\mathbf{Z}_{2n})^2$-*irreducible.*

(b) *The subspaces* $\mathbf{V}_\mathbf{k}$ *for* $\mathbf{k} \in I$ *are all distinct* $(\mathbf{Z}_{2n})^2$-*irreducible representations.*

(c) *The subspaces* $\mathbf{V}_\mathbf{k}$ *satisfy*

$$\bigoplus_{\mathbf{k} \in I} \mathbf{V}_\mathbf{k} = \mathbf{R}^{4n^2}. \tag{3.2}$$

Hence, every $(\mathbf{Z}_{2n})^2$-*irreducible representation is equal to* $\mathbf{V}_\mathbf{k}$ *for some* $\mathbf{k} \in I$.

The Γ-*Irreducible Subspaces*

We now enumerate the Γ-irreducible subspaces of $\mathbf{R}^{4n^2}$. The action of $\mathbf{D}_4$ permutes the $(\mathbf{Z}_{2n})^2$-irreducible representations $\mathbf{V}_\mathbf{k}$ since $(\mathbf{Z}_{2n})^2$ is a normal subgroup of $\Gamma = \mathbf{D}_4 \dot{+} (\mathbf{Z}_{2n})^2$. Hence the Γ-irreducible representations are just sums of the $(\mathbf{Z}_{2n})^2$-irreducibles $\mathbf{V}_\mathbf{k}$, where $\mathbf{k}$ is taken over a $\mathbf{D}_4$ group orbit. That is, each Γ-irreducible has the form

$$\mathbf{W}_\mathbf{k} = \sum_{\sigma \in \mathbf{D}_4} \mathbf{V}_{\sigma(\mathbf{k})}. \tag{3.3}$$

This sum is not direct as some of these spaces are redundant.

LEMMA 3.2. *The distinct* Γ-*irreducible representations of* $\mathbf{R}^{4n^2}$ *are those listed in Table* 2.

TABLE 2
Γ-irreducible subspaces

Type	dim	$\mathbf{W}_{(a,b)}$	Restrictions
T	1	$\mathbf{V}_{(0,0)}$	
I(a)	1	$\mathbf{V}_{(n,n)}$	
I(b)	4	$\mathbf{V}_{(a,a)} \oplus \mathbf{V}_{(a,2n-a)}$	$1 \leq a \leq n-1$
II(a)	2	$\mathbf{V}_{(n,0)} \oplus \mathbf{V}_{(0,n)}$	
II(b)	4	$\mathbf{V}_{(a,0)} \oplus \mathbf{V}_{(0,a)}$	$1 \leq a \leq n-1$
III	8	$\mathbf{V}_{(a,b)} \oplus \mathbf{V}_{(b,a)} \oplus \mathbf{V}_{(a,2n-b)} \oplus \mathbf{V}_{(b,2n-a)}$	$1 \leq b < n,\ b < a,\ a \neq n,\ a < 2n$

The Axial Subgroups

Our last step is to enumerate the axial subgroups for the irreducible representations given in Table 2. Note that bifurcations where the kernel of the linearization has a trivial Γ action cannot lead to patterned equilibria. Hence we do not need to consider type T bifurcations. Also note that any representation where $\mathrm{Fix}(\Delta) = 0$ cannot support axial solutions which restrict to the small square array satisfying Neumann boundary conditions. This happens in representations of Types I(a) and II(a); see Table 11 in Section 6.

Let d be the greatest common divisor

$$d = \begin{cases} \gcd(a, 2n), & \text{for types I(b) and II(b),} \\ \gcd(a, b, 2n), & \text{for type III,} \end{cases} \tag{3.4}$$

and define the subgroups

$$\mathbf{K}_1 = (\mathbf{Z}_d)^2, \tag{3.5}$$

$$\mathbf{K}_2 = \left\langle (\mathbf{Z}_d)^2, \left(\frac{n}{d}, \frac{n}{d} \right) \right\rangle, \tag{3.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the group generated by the elements within.

Let

$$\begin{aligned} \Sigma_1 &= \langle \delta, \kappa, \mathbf{K}_1 \rangle \\ \Sigma_2 &= \langle \delta, \kappa, \mathbf{K}_2 \rangle \\ \Omega_1 &= \left\langle \delta\left(\frac{n}{d}, \frac{n}{d} \right), \kappa, \mathbf{K}_1 \right\rangle \\ \Omega_2 &= \left\langle \delta\left(\frac{n}{d}, 0 \right), \kappa, \mathbf{K}_2 \right\rangle \\ R &= \langle \kappa, \eta, (0,1), \mathbf{K}_1 \rangle. \end{aligned} \tag{3.7}$$

THEOREM 3.3. *Up to conjugacy, the axial subgroups of* Γ *which contain the subgroup* Δ *are those given in Table* 3.

4. PATTERN FORMATION

As discussed in Section 1, the branches of equilibria found in Section 3 lead to interesting patterns appearing in the original system. Each equilibrium of the original discretized Neumann boundary condition cell system corresponds to an $n \times n$ array of vectors $x_{k,l}$. The isotropy subgroup of the equilibrium in the extended $2n \times 2n$ array forces certain of these vectors to be equal and hence forces a pattern to appear in the $n \times n$ array. If we represent each of the (possibly) different vectors by a different color, then the pattern of equalities in the $x_{k,l}$ will be immediately transparent.

In our analysis in Section 3 we showed that the kernel $\mathbf{K}$ always contains $\mathbf{Z}_d^2$; see (3.5) and (3.6). Consequently, each axial subgroup contains the translations $(2n/d, 0)$ and $(0, 2n/d)$. It follows that in each equilibrium corresponding to an axial subgroup, the vector $x_{k,l}$ has to be equal to the vector obtained by translating right, left, up, or down by $2n/d$ cells; that is,

$$x_{k,l} = x_{k+2n/d,l} = x_{k,l+2n/d}.$$

TABLE 3
Axial subgroups

Type	Parity	Restrictions	Axials
I(b)	$d \nmid n$		Σ_1
	$d \mid n$		Σ_2
II(b)	$d \nmid n$		Σ_1
	$d \mid n$		Σ_1
			Ω_1
			R
III	$d \nmid n$		Σ_1
	$d \mid n$	$\frac{a}{d}, \frac{b}{d}$ even	Σ_2
		$\frac{a}{d}, \frac{b}{d}$ odd	Σ_2
			Ω_2
		$\frac{a}{d} \neq \frac{b}{d} \bmod 2$	Σ_1
			Ω_1

Thus, the $2n \times 2n$ array of cells divides into $2n/d \times 2n/d$ blocks of cells, with corresponding cells in each block having identical values. We call the block $\mathscr{B}$ consisting of the upper left hand $2n/d \times 2n/d$ block of cells the *primary* block. If $(2n/d)|n$, then the original $n \times n$ array also divides into blocks identical to $\mathscr{B}$, while if $(2n/d) \nmid n$, then there will be an extra half block along the right and bottom of the $n \times n$ array. Only a half block can occur since $2n/d$ always divides $2n$.

Neumann boundary conditions also imply that the symmetries κ and η are in each axial subgroup (see (2.4)). These symmetries imply that the primary block $\mathscr{B}$ is symmetric when reflected across the horizontal and vertical midlines.

As an example, suppose $2n/d = 4$. If we imagine identical cells as being the same "color," we get the initial pattern for cells in a block as given in Table 4. In this table we represent colors as R for red, G for green, Y for yellow, and B for blue.

Suppose we let L be the subgroup generated by κ, η, and $\mathbf{Z}_d^2$. Then each of the five axial subgroups listed in (3.7) is generated by just one or two symmetries modulo L. Indeed

$$\begin{aligned}
\Sigma_1/L &= \langle \delta \rangle, \\
\Sigma_2/L &= \left\langle \delta, \left(\frac{n}{d}, \frac{n}{d}\right) \right\rangle, \\
\Omega_1/L &= \left\langle \delta\left(\frac{n}{d}, \frac{n}{d}\right) \right\rangle, \\
\Omega_2/L &= \left\langle \delta\left(\frac{n}{d}, 0\right), \left(\frac{n}{d}, \frac{n}{d}\right) \right\rangle, \\
R/L &= \langle (0,1) \rangle.
\end{aligned} \tag{4.1}$$

We now consider the axial subgroup Σ_1. The symmetry δ implies that the primary block is symmetric, that is, $\mathscr{B} = \mathscr{B}^t$. Thus $\mathscr{B}$ has square $\mathbf{D}_4$

TABLE 4
Initial pattern

R	Y	Y	R
G	B	B	G
G	B	B	G
R	Y	Y	R

TABLE 5
Final pattern for equilibria with symmetry Σ_1

R	G	G	R
G	B	B	G
G	B	B	G
R	G	G	R

symmetry, and the primary block has the pattern shown in Table 5. Figure 3 is an example of the pattern arising in an equilibrium with Σ_1 symmetry when $n = 8$, $a = 4$, and $2n/d = 4$.

On the R branch, the symmetry $(1, 0)$ forces cells in the same row to be identical. As shown in Fig. 4, these solutions lead to a roll-like pattern, comprised of horizontal stripes.

All the remaining branches of equilibria occur only when $d|n$, in which case the $n \times n$ array is exactly filled by copies of the primary block. Moreover, $2n/d$ is even and the primary block $\mathscr{B}$ has an even number of rows and columns. We can let $\mathscr{A}$ be the upper left hand $(n/d) \times (n/d)$ subblock of $\mathscr{B}$. Once the subblock $\mathscr{A}$ is determined, the block $\mathscr{B}$ is then determined by the horizontal and vertical flips κ and η.

Next we consider the pattern associated with the axial subgroup Σ_2. The subgroup Σ_2 is generated by the symmetries δ and $(n/d, n/d)$ over L. The symmetry δ forces the subblock $\mathscr{A}$ to be symmetric, that is, $\mathscr{A} = \mathscr{A}^t$. The symmetry $(n/d) \times (n/d)$ forces A to be invariant under a $180°$ rotation. It follows that when $n/d = 2$ the subblock

$$\mathscr{A} = \begin{pmatrix} R & G \\ G & R \end{pmatrix},$$

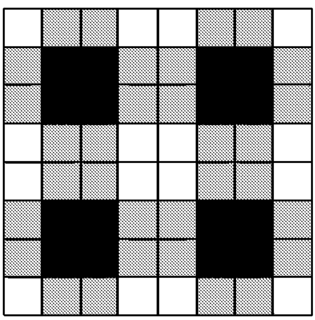

FIG. 3. Solution with Σ_1 symmetry: $n = 8$, $a = 4$, $d = 4$.

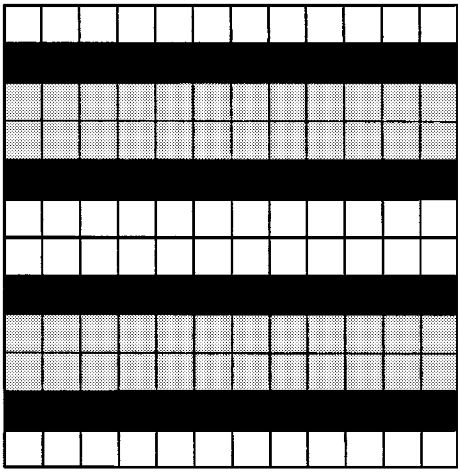

FIG. 4. Solution with R symmetry: $n = 14$, $a = 4$, $d = 4$.

and when $n/d = 3$ the subblock

$$\mathscr{A} = \begin{pmatrix} R & Y & G \\ Y & B & Y \\ G & Y & R \end{pmatrix}.$$

In Table 6 we picture the primary block when $n/d = 2$ and in Fig. 5 we picture the entire pattern when $n/d = 3$, $n = 12$, $a = 4$, and $d = 4$.

On the Ω_1 branch, as on the Σ_2 branch, we may again divide the primary block $\mathscr{B}$ into four subblocks determined by the upper left hand subblock $\mathscr{A}$. The $\delta(n/d, n/d)$ symmetry of Ω_1 forces $\mathscr{A}$ to equal the 180° rotation of itself. It follows that when $n/d = 2$ the subblock

$$\mathscr{A} = \begin{pmatrix} R & G \\ Y & R \end{pmatrix},$$

TABLE 6
Final pattern for equilibria with Σ_2 symmetry

R	G	G	R
G	R	R	G
G	R	R	G
R	G	G	R

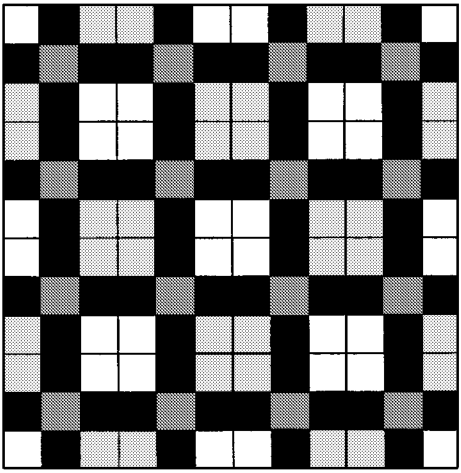

FIG. 5. Solution with Σ_2 symmetry: $n = 12$, $a = 4$, $d = 4$.

and when $n/d = 3$ the subblock

$$\mathscr{A} = \begin{pmatrix} R & Y & G \\ W & B & Y \\ P & W & R \end{pmatrix}.$$

In Table 7 we picture the primary block when $n/d = 2$, and in Fig. 6 we picture the entire pattern when $n/d = 3$, $n = 15$, $a = 5$, and $d = 5$.

The $(n/d, n/d)$ symmetry of the Ω_2 branch forces the subblock $\mathscr{A}$ to equal a 180° rotation of itself; that is,

$$a_{ij} = a_{n/d+1-i,\, n/d+1-j}.$$

The $\delta(n/d, 0)$ symmetry forces

$$a_{ij} = a_{n/d+1-j,\, i}.$$

Note that when $n/d = 2$ these conditions imply that all entries in $\mathscr{A}$ have the same color, thus yielding the constant pattern in Table 8. This is an

TABLE 7
Final pattern for equilibria with Ω_1 symmetry

R	G	G	R
Y	R	R	Y
Y	R	R	Y
R	G	G	R

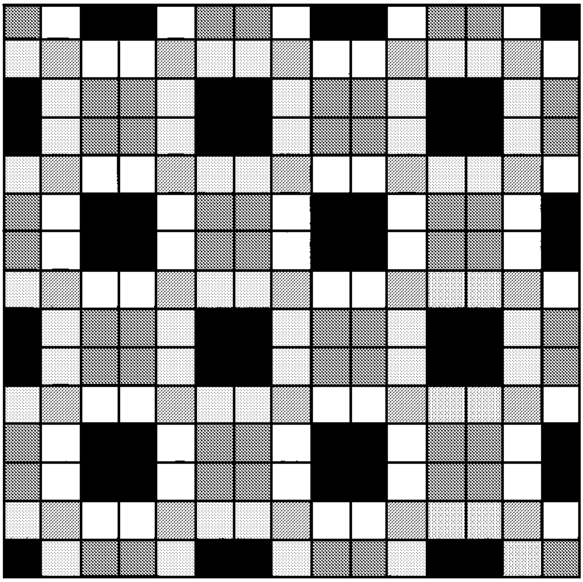

FIG. 6. Solution with Ω_1 symmetry: $n = 15$, $a = 5$, $d = 5$.

interesting example where symmetry breaking from an equilibrium yields a new equilibrium with the same symmetry as the original equilibrium. When $n/d \geq 3$ the pattern that appears is more complicated. An example of such a pattern is given in Fig. 7 with $n = 15$, $a = 3$, $b = 9$, $d = 3$, and $n/d = 5$.

5. LINEAR THEORY

In Sections 3 and 4 we discussed which types of pattern are possible in a general square array. The particular type of pattern one may expect to see in a particular bifurcation problem of course depends on the form of the kernel in the problem. In this section we discuss which irreducible repre-

TABLE 8
Final pattern for equilibria with Ω_2 symmetry

R	R	R	R
R	R	R	R
R	R	R	R
R	R	R	R

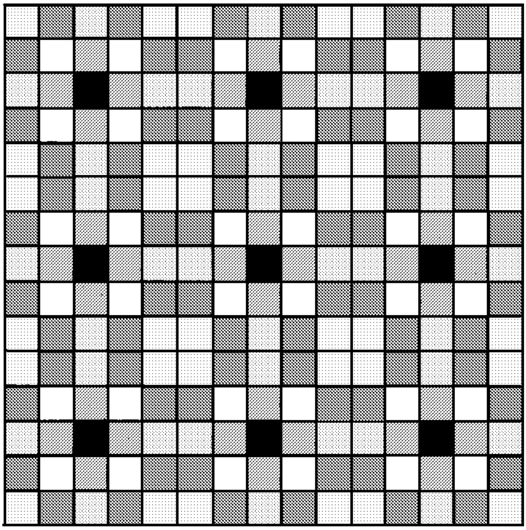

FIG. 7. Solution with Ω_2 symmetry: $n = 15$, $a = 3$, $b = 9$, $d = 3$.

sentations can be expected to arise for a given system of equations. In particular we compute the eigenvalues of the linearization of (2.3) along the trivial solution $(x_{l,m}) = 0$, and show that the eigenspaces at a zero eigenvalue correspond to the spaces $\mathbf{W_k}$ defined in (3.3). We also show that the eigenvalues of the linearization may be computed by computing the eigenvalues of $4n^2$ $k \times k$ matrices, a substantial reduction.

Let

$$f_{l,m} = f(x_{l-1,m}, x_{l,m-1}, x_{l,m}, x_{l,m+1}, x_{l+1,m}),$$

and set

$$F = (f_{1,1}, \ldots, f_{1,2n}, \ldots, f_{2n,1}, \ldots, f_{2n,2n}).$$

Note that $x_{i,j}$ and $x_{l,m}$ are nearest neighbors on the square array if $|i-l| + |j-m| = 1$ with appropriate exceptions made for indices on the boundary. Along the trivial branch we may use (2.2) to write

$$\begin{aligned} d_{x_{l,m}} f_{l,m}|_{x=0} &= p, & & \\ d_{x_{i,j}} f_{l,m}|_{x=0} &= q, & |i-l| + |j-m| &= 1, \\ d_{x_{i,j}} f_{l,m}|_{x=0} &= 0, & |i-l| + |j-m| &> 1, \end{aligned} \tag{5.1}$$

for certain $k \times k$ matrices p and q.

Let $L = (dF)_0$. We claim that the eigenvalues of L are eigenvalues of certain linear combinations of the matrices p and q. Recall that the spaces $W_{a,b}$ are distinct irreducible representations of the group Γ. Indeed, when $k = 1$, the state space $\mathbf{R}^{4n^2}$ decomposes into the direct sum of the $W_{a,b}$ and each of these is an invariant subspace for L. For general k, the state space $\mathbf{R}^{4n^2k}$ is the direct sum of the subspaces $W_{a,b}^k$ and each of these is L-invariant. Let

$$\ell_a = 2\cos\left(\frac{\pi a}{n}\right).$$

THEOREM 5.1. *The eigenvalues of L restricted to $W_{a,b}^k$ are the eigenvalues of the $k \times k$ matrix*

$$p + (\ell_a + \ell_b)q$$

each with multiplicity equal to $\dim W_{a,b}$.

Proof. Using (5.1) we can compute the Jacobian matrix L as follows. Let

$$y_l = (x_{l,1}, \dots, x_{l,2n})$$

denote the states of the lth row of cells, and let

$$x = (x_{1,1}, \dots, x_{1,2n}, \dots, x_{2n,1}, \dots, x_{2n,2n})$$

denote the states of all cells arranged one row at a time. Then

$$L = 2n\left\{\begin{bmatrix} P & Q & & & & Q \\ Q & P & Q & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & Q & P & Q \\ Q & & & & Q & P \end{bmatrix}\right., \tag{5.2}$$

where the $2nk \times 2nk$ matrices P and Q are defined by the y-coordinates. In the x-coordinates these matrices have the structure:

$$P = \begin{bmatrix} p & q & & & & q \\ q & p & q & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & q & p & q \\ q & & & & q & p \end{bmatrix} \quad \text{and}$$

$$Q = \begin{bmatrix} q & & & & & \\ & q & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & q & \\ & & & & & q \end{bmatrix}.$$

The circular structure of the matrices L and P come from the nearest neighbor coupling of our array.

The eigenvalue analysis for matrices of the form (5.2) is discussed in [7]. Let $\zeta = e^{\pi i/n}$ be a primitive $(2n)$th root of unity, and for $a = 0, \ldots, 2n-1$ define the complex space

$$\mathbf{Y}_a = \left\{ \left[v, \zeta^a v, \zeta^{2a} v, \ldots, \zeta^{(2n-1)a} v \right] : v \in \mathbf{R}^{2nk} \right\}. \tag{5.3}$$

A direct calculation shows

$$\begin{aligned} & L\left[v, \zeta^a v, \zeta^{2a} v, \ldots, \zeta^{(2n-1)a} v \right] \\ & \qquad = \left(P + \ell_a Q \right) \left[v, \zeta^a v, \zeta^{2a} v, \ldots, \zeta^{(2n-1)a} v \right], \end{aligned} \tag{5.4}$$

so that the eigenvalues of L are the same as the eigenvalues of the $2nk \times 2nk$ matrices $P + \ell_a Q$. There are $2n$ of these matrices. Each of these matrices can be written in terms of the $k \times k$ matrices p and q, as follows:

$$P + \ell_a Q = \begin{bmatrix} p + \ell_a q & q & & & & q \\ q & p + \ell_a q & q & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & q & p + \ell_a q & q \\ q & & & & q & p + \ell_a q \end{bmatrix},$$

$$a = 0, \ldots, 2n-1.$$

Note that these matrices have the same structure as (5.2). Define the $2nk$-vector

$$v_b = \{[w, \zeta^b w, \zeta^{2b} w, \dots, \zeta^{(2n-1)b} w] : w \in \mathbf{R}^k\}.$$

A similar calculation shows that

$$(P + \ell_a Q)[v_b] = (p + (\ell_a + \ell_b)q)[v_b]. \tag{5.5}$$

Substituting v_b for v in (5.4), it follows that the eigenvalues of L are the same as those of the $4n^2$ $k \times k$ matrices $L_{a,b}$, where

$$L_{a,b} = p + (\ell_a + \ell_b)q, \qquad a, b = 0, \dots, 2n-1,$$

with the corresponding eigenspace

$$\mathbf{Y}_{a,b} = \{[v_b, \zeta^a v_b, \zeta^{2a} v_b, \dots, \zeta^{(2n-1)a} v_b]\}.$$

Note that $(r, s) \in (\mathbf{Z}_{2n})^2$ acts on $\mathbf{Y}_{a,b}$ as rotations:

$$\begin{aligned}(r, s) \cdot & [v_b, \zeta^a v_b, \zeta^{2a} v_b, \dots, \zeta^{(2n-1)a} v_b] \\ &= \exp\left[-\frac{\pi i}{n}(r, s) \cdot (a, b)\right][v_b, \zeta^a v_b, \zeta^{2a} v_b, \dots, \zeta^{(2n-1)a} v_b].\end{aligned}$$

It follows that, as representations, the spaces $\mathbf{Y}_{a,b}$ and $\mathbf{W}^k_{a,b}$ defined in (3.3) are isomorphic.

Note that many of the $L_{a,b}$s are equal. In particular, since $\cos(-\theta) = \cos(\theta)$, we have $\ell_a = \ell_{2n-a}$ and $\ell_b = \ell_{2n-b}$. Furthermore, $\ell_a + \ell_b = \ell_b + \ell_a$. Thus

$$\begin{aligned}L_{a,b} &= L_{2n-a,b} = L_{a,2n-b} = L_{2n-a,2n-b} = L_{b,a} = L_{2n-b,a} \\ &= L_{b,2n-a} = L_{2n-b,2n-a}.\end{aligned}$$

Generically, each $L_{a,b}$ has only simple zero eigenvalues. It then follows that the dimension of $\ker L$ is given as in Table 9. It now follows that we may determine which of the representations listed in Table 2 may be expected to appear by calculating the eigenvalues of the $k \times k$ matrices $L_{a,b}$. ∎

The Discretized Brusselator

Following [4], we consider as an example a system of coupled Brusselators. As a notational convenience to indicate nearest neighbor coupling in a square array, let

$$(\Delta u)_{l,m} = u_{l-1,m} + u_{l,m-1} - 4u_{l,m} + u_{l,m+1} + u_{l+1,m}.$$

TABLE 9
Dimensions of kernel L

dim(ker L)	Applicable (a, b)
1	$(a, b) = (0, 0)$ or $(a, b) = (n, n)$
2	$(a, b) = (0, n)$ or $(a, b) = (n, 0)$
4	$0 < a = b < n$ or $0 = a < b < n$ or $0 = b < a < n$
8	$0 < a < b < n$ or $0 < b < a < n$.

For the Brusselator $k = 2$, $x = (u, v)$, and f is defined as follows:

$$\begin{pmatrix} u_{l,m} \\ v_{l,m} \end{pmatrix} = \begin{pmatrix} 1 - (\alpha + 1)u_{l,m} + \beta u_{l,m}^2 v_{l,m} + D_u(\Delta u)_{l,m} \\ \alpha u_{l,m} - \beta u_{l,m}^2 v_{l,m} + D_v(\Delta v)_{l,m}) \end{pmatrix}, \qquad (5.6)$$

where β, α, D_u, and D_v are positive constants. Equation (5.6) has the trivial equilibrium

$$u_{l,m} = 1 \quad \text{and} \quad v_{l,m} = \frac{\alpha}{\beta}.$$

Rescale by setting $D_u = rD_v$ and $D_{a,b} = (4 - \ell_a - \ell_b)D_v$. Then

$$L_{a,b} = \begin{bmatrix} \alpha - 1 - rD_{a,b} & \beta \\ -\alpha & -\beta - D_{a,b} \end{bmatrix}$$

It follows that by varying parameters we can force each $L_{a,b}$ to have a zero eigenvalue, which then implies that each of the representations listed in Section 3 will occur for this example.

6. PROOFS

In this section we verify the results stated in Section 3. We begin by showing that the spaces V_k listed in Table 1 give all the $(\mathbf{Z}_{2n})^2$ irreducible representations in $\mathbf{V}$.

Proof of Lemma 3.1. (a) The translations act on $\mathbf{V_k}$ as

$$\begin{aligned}(r,s)\cdot\left(\mathrm{Re}\left(z\exp\left[\frac{\pi i}{n}(l,m)\cdot\mathbf{k}\right]\right)\right) &= \left(\mathrm{Re}\left(z\exp\left[\frac{\pi i}{n}(l+r,m+s)\cdot\mathbf{k}\right]\right)\right)\\ &= \left(\mathrm{Re}\left(z\exp\left[\frac{\pi i}{n}(r,s)\cdot\mathbf{k}\right]\exp\left[\frac{\pi i}{n}(l,m)\cdot\mathbf{k}\right]\right)\right)\in\mathbf{V_k}\end{aligned}$$

so that $\mathbf{V_k}$ is $(\mathbf{Z}_{2n})^2$-invariant and clearly irreducible.

(b) In coordinates, the translations (r,s) act on $\mathbf{V_k}$ as rotations:

$$(r,s)\cdot z = \exp\left[\frac{\pi i}{n}(r,s)\cdot\mathbf{k}\right]z. \tag{6.1}$$

Note that

$$\mathbf{V_k} = \mathbf{V_{-k}} \tag{6.2}$$

and

$$\mathbf{V_k} = \mathbf{V}_{\mathbf{k}+(2pn,2qn)} \tag{6.3}$$

for $p,q\in\mathbf{Z}$.

To enumerate the distinct $(\mathbf{Z}_{2n})^2$-irreducibles $\mathbf{V_k}$, begin by using (6.3) to see that we may assume

$$0\le a,b<2n.$$

Note that (6.2) and (6.3) combine to yield

$$\mathbf{V}_{(a,b)} = \mathbf{V}_{(2n-a,2n-b)}, \tag{6.4}$$

$$\mathbf{V}_{(a,0)} = \mathbf{V}_{(2n-a,0)}, \tag{6.5}$$

$$\mathbf{V}_{(0,b)} = \mathbf{V}_{(0,2n-b)}. \tag{6.6}$$

It follows from (6.5) that the distinct $\mathbf{V}_{(a,0)}$ are of types (1), (3), and (6) and from (6.6) that the distinct $\mathbf{V}_{(0,b)}$ are of types (1), (4), and (7). Similarly, it follows from (6.4) that the distinct $\mathbf{V}_{(a,a)}$ are of types (1), (2), and (5). Now we assume that

$$1\le a,b\le 2n-1 \quad\text{and}\quad a\ne b.$$

Finally, we use (6.4) to reduce to case (8). We claim that the $(\mathbf{Z}_{2n})^2$ representations $\mathbf{V_k}$ for $\mathbf{k}\in I$ are distinct irreducible representations. This claim is verified by noting that a direct calculation using (6.1) implies that for $\mathbf{k}, \mathbf{k}' \in I$, $\mathbf{V}_{\mathbf{k}} \cong \mathbf{V}_{\mathbf{k}'}$ if and only if

$$\exp\left(\frac{\pi i}{n}(r,s)\cdot\mathbf{k}\right) = \exp\left(\frac{\pi i}{n}(r,s)\cdot\mathbf{k}'\right)$$

for each $1 \le r, s \le 2n$, which then implies that $\mathbf{k} = \mathbf{k}'$.

(c) Note that $\dim \mathbf{V}_{\mathbf{k}} = 1$ for $\mathbf{k}$ of types (1)–(4) and $\dim \mathbf{V}_{\mathbf{k}} = 2$ for $\mathbf{k}$ of types (5)–(8). A calculation shows that

$$\sum_{\mathbf{k}\in I} \dim \mathbf{V}_{\mathbf{k}} = 4n^2. \tag{6.7}$$

Since the $\mathbf{V}_{\mathbf{k}}$ are all distinct irreducible representations, (3.2) follows from (6.7). ∎

Proof of Lemma 3.2. The lemma follows from (3.3) by a calculation using (6.2)–(6.6). ∎

Next we calculate the kernels for each of the remaining representations.

LEMMA 6.1. *Let d, $\mathbf{K}_1$, and $\mathbf{K}_2$ be defined as in* (3.4), (3.5), *and* (3.6), *respectively. Then the kernel $\mathbf{K}$ is one of the two subgroups $\mathbf{K}_1, \mathbf{K}_2$.*

Proof. A calculation using Table 10 shows that no element of Γ having a nontrivial component in $\mathbf{D}_4$ is in the kernel. Thus $\mathbf{K} \subset (\mathbf{Z}_{2n})^2$. Let r be a

TABLE 10

Action of Γ: $\alpha = \exp(-a\pi i/n)$, $\beta = \exp(-b\pi i/n)$, $\hat{\gamma} = \gamma\cdot(r,s)$

$\hat{\gamma}$	I(b)	II(b)	III
$\hat{1}$	$(\alpha^{-r-s}z_1, \bar{a}^{r-s}z_2)$	$(\alpha^{-r}z_1, \alpha^{-s}z_2)$	$(\alpha^{-r}\beta^{-s}z_1, \alpha^{-s}\beta^{-r}z_2,$ $\alpha^{-r}\beta^{s}z_3, \alpha^{s}\beta^{-r}z_4)$
$\hat{\kappa}$	$(\alpha^{r-s+1}\bar{z}_2, \alpha^{r+s+1}rz_1)$	$(\alpha^{r+1}\bar{z}_1, \alpha^{-s}z_2)$	$(\alpha^{r+1}\beta^{-s}\bar{z}_3, \alpha^{-s}\beta^{r+1}\bar{z}_4,$ $\alpha^{r+1}\beta^{s}\bar{z}_1, \alpha^{s}\beta^{r+1}\bar{z}_2)$
$\hat{\delta}$	$(\alpha^{-r-s}z_1, \alpha^{r-s}\bar{z}_2)$	$(\alpha^{-s}z_2, \alpha^{-r}z_1)$	$(\alpha^{-s}\beta^{-r}z_2, \alpha^{-r}\beta^{-s}z_1,$ $\alpha^{-s}eta^{r}\bar{z}_4, \alpha^{r}\beta^{-s}\bar{z}_3)$
$\widehat{\kappa\delta}$	$(\alpha^{-r+s+1}z_2, \alpha^{r+s+1}\bar{z}_1)$	$(\alpha^{s+1}\bar{z}_2, \alpha^{-r}z_1)$	$(\alpha^{-s+1}\beta^{-r}z_4, \alpha^{-r}\beta^{s+1}z_3,$ $\alpha^{s+1}\beta^{r}\bar{z}_2, \alpha^{r}\beta^{s+1}\bar{z}_1)$
$\hat{\eta}$	$(\alpha^{-r+s+1}z_2, \alpha^{-r-s-1}z_1)$	$(\alpha^{-r}z_1, \alpha^{s+1}\bar{z}_2)$	$(\alpha^{-r}\beta^{s+1}z_3, \alpha^{s+1}\beta^{-r}z_4,$ $\alpha^{-r}\beta^{-s-1}z_1, \alpha^{s-1}\beta^{-r}z_2)$
$\widehat{\kappa\eta}$	$(\alpha^{r+s+2}\bar{z}_1, \alpha^{r-s}\bar{z}_2)$	$(\alpha^{r+1}\bar{z}_1, \alpha^{s+1}\bar{z}_2)$	$(\alpha^{r+1}\beta^{s+1}\bar{z}_1, \alpha^{s+1}\beta^{r+1}\bar{z}_2,$ $\alpha^{r+1}\beta^{-s-1}\bar{z}_3, \alpha^{-s-1}\beta^{r+1}\bar{z}_4)$
$\widehat{\delta\kappa}$	$(\alpha^{r-s+1}\bar{z}_2, \alpha^{r-s-1}z_1)$	$(\alpha^{-s}z_2, \alpha^{r+1}\bar{z}_1)$	$(\alpha^{-s}\beta^{r+1}\bar{z}_4, \alpha^{r+1}\beta^{-s}\bar{z}_3,$ $\alpha^{-s}\beta^{-r-1}z_2, \alpha^{-r-1}\beta^{s+1}z_1)$
$\widehat{\kappa\delta\kappa}$	$(\alpha^{r+s+2}\bar{z}_1, \alpha^{-r+s}z_2)$	$(\alpha^{r+1}\bar{z}_2, \alpha^{s+1}\bar{z}_1)$	$(\alpha^{s+1}\beta^{r+1}\bar{z}_2, \alpha^{r+1}\beta^{s+1}\bar{z}_1,$ $\alpha^{s+1}\beta^{-r-1}z_4, \alpha^{-r-1}\beta^{s+1}z_3)$

real number. A calculation shows that

(a) if ar/n is even, then r must be an integer multiple of $2n/d$, and
(b) if a/d is even, then $d \nmid n$.

It follows from Table 10 that the kernel contains the translations $(2n/d, 0)$ and $(0, 2n/d)$; the kernel may also contain $(n/d, n/d)$, depending on the particular representation. Indeed, using (a) and (b), we derive the kernels listed in (3.5)–(3.6). ∎

Proof of Theorem 3.3. We begin by using (6.1) to compute Fix(Δ) for each of the representations listed in Table 2; we list our results in Table 11. Note that the subgroup Δ fixes only the point 0 for representations of types I(a) and II(a); hence representations of this type cannot have any axial subgroups which contain Δ.

Next we explicitly compute the action of Γ in coordinates. If we let δ be the diagonal flip in $\mathbf{D}_4$, defined on $\mathbf{R}^{4n^2}$ by

$$\delta \cdot (x_{l,m}) = (x_{m,l}),$$

then δ, κ, and the translations (r, s) generate Γ. We list the results of the calculations in Table 10.

Our last step is to verify that the subgroups presented in Table 3 are axial and that, up to conjugacy, each axial subgroup is in the table. It is a straightforward exercise to check that each of the subgroups listed in Table 3 has a one-dimensional fixed-point subspace, and to verify that these subgroups are isotropy subgroups. This verification is made easier by noting that the subgroup $\Delta \dotplus \mathbf{K}$ acts trivially on all points in Fix(Δ) and that $\Delta \dotplus \mathbf{K}$ is of index 2 in each of the proposed isotropy subgroups.

TABLE 11
Computation of Fix(Δ): $\alpha = \exp(-a\pi i/n)$, $\beta = \exp(-b\pi i/n)$

Type	Γ action	Fix(Δ)	dim
I(a)	$\kappa := -x$ $\eta := -x$	$\{0\}$	0
I(b)	$\kappa := (\alpha\bar{z}_2, \alpha\bar{z}_1)$ $\eta := (\alpha z_2, \bar{\alpha} z_1)$	$\{(\alpha x_2, x_2)\}$	1
II(a)	$\kappa := (-x_1, x_2)$ $\eta := (x_1, -x_2)$	$\{0\}$	0
II(b)	$\kappa := (\alpha\bar{z}_1, z_2)$ $\eta := (z_1, \alpha\bar{z}_2)$	$\{(\alpha^{1/2}x_1, \alpha^{1/2}x_2)\}$	2
III	$\kappa := (\alpha\bar{z}_3, \beta\bar{z}_4, \alpha\bar{z}_1, \beta\bar{z}_2)$ $\eta := (\beta z_3, \alpha z_4, \bar{B} z_1, \bar{\alpha} z_2)$	$\{((\alpha\beta)^{1/2}x_1, (\alpha\beta)^{1/2}x_2, (\alpha\bar{B})^{1/2}x_1, (\bar{\alpha}\beta)^{1/2}x_2)\}$	2

Hence the verification that Σ, Ω_1, Ω_2, and R have the appropriate fixed-point subspaces relies on the verification that one element in each of these subgroups fixes the listed points. These distinguished elements are δ, $\delta(n/d, n/d)$, $\delta(n/d, 0)$, and $(0,1)$, respectively. We list the fixed-point subspaces for each of the axial subgroups in Table 12.

All that remains is to verify that up to conjugacy these are the only axial subgroups. For type I(b) representations, $\mathrm{Fix}(\Delta)$ is one-dimensional, and no other choices are possible.

For the remaining representations, let $\Sigma_v \supset \Delta \dot{+} \mathbf{K}$ be the isotropy subgroup of a point $v \in \mathrm{Fix}(\Delta)$, and assume $\gamma \in \Sigma_v/\Delta \dot{+} \mathbf{K}$. We claim that γ is in one of Σ, Ω, or R. To see this, recall that we may write γ uniquely as $\gamma = \sigma(r,s)$, where $\sigma \in \mathbf{D}_4$ and $(r,s) \in (\mathbf{Z}_{2n})^2$. Note that, for $\tau \in \mathbf{D}_4$, we have that $\tau\gamma\tau^{-1} = \tau\sigma\tau^{-1}(r', s')$ for some $(r', s') \in (\mathbf{Z}_{2n})^2$. Recalling that δ, κ generate $\mathbf{D}_4$, it follows that, up to conjugacy, we may assume γ equals (r,s), $\delta(r,s)$, or $\kappa\delta(r,s)$. Since $\kappa \in \Delta$ and $\kappa^2 = 1$, it follows that $\kappa\delta(r,s) \in \Sigma_v$ if and only if $\delta(r,s) \in \Sigma_v$. We may therefore assume either $\gamma = (r,s)$ or $\gamma = \delta(r,s)$.

Assume $\gamma = (r,s)$. By assumption $(r,s) \notin \mathbf{K}$. For type II(b) representations, a quick calculation shows that a nonkernel translation (r,s) is in the isotropy subgroup of a point $(\exp(-a\pi i/2n)x, \exp(-a\pi i/2n)y)$ if and if $x = 0$ or $y = 0$. Points of this type lie on the same Γ orbit, and hence have conjugate isotropy subgroups. Using Table 3, it follows that $\gamma \in R$. For type III representations, we may use coordinates to see that all nonkernel translations act fixed-point freely on $\mathrm{Fix}(\Delta)$. Next assume that $\gamma = \delta(r,s)$. If $(r,s) \in \mathbf{K}$ then $\gamma \in \Sigma$, so suppose $(r,s) \notin \mathbf{K}$. A calculation in coordinates shows, for both type II(b) and type III representations, that a point $v \in \mathrm{Fix}(\Delta)$ is fixed by $\delta(r,s)$ if and only if (r,s) acts as minus the identity on that representation. A straightforward calculation shows that this will

TABLE 12
Axial subgroups: $\alpha = \exp(-a\pi i/n)$, $\beta = \exp(-b\pi i/n)$

Type	Axials	Fix(Axial)
I(b)	Σ_1	$\mathbf{R}\{(\alpha, 1)\}$
	Σ_2	$\mathbf{R}\{(\alpha, 1)\}$
II(b)	Σ_1	$\mathbf{R}\{(\alpha^{1/2}, \alpha^{1/2})\}$
	Ω_1	$\mathbf{R}\{(\alpha^{1/2}, -\alpha^{1/2})\}$
	R	$\mathbf{R}\{\alpha^{1/2}, 0\}$
III	Σ_1	$\mathbf{R}\{((\alpha\beta)^{1/2}, (\alpha\beta)^{1/2}, (\alpha\bar{\beta})^{1/2}, (\bar{\alpha}\beta)^{1/2})\}$
	Σ_2	$\mathbf{R}\{((\alpha\beta)^{1/2}, (\alpha\beta)^{1/2}, (\alpha\bar{\beta})^{1/2}, (\bar{\alpha}\beta)^{1/2})\}$
	Ω_2	$\mathbf{R}\{((\alpha\beta)^{1/2}, -(\alpha\beta)^{1/2}, (\alpha\bar{\beta})^{1/2}, -(\bar{\alpha}\beta)^{1/2})\}$
	Ω_1	$\mathbf{R}\{((\alpha\beta)^{1/2}, -(\alpha\beta)^{1/2}, (\alpha\bar{\beta})^{1/2}, -(\bar{\alpha}\beta)^{1/2})\}$

occur if and only if $d|n$. It follows that $v \in \mathbf{R}\{\alpha^{1/2}, -\alpha^{1/2}\}$ (for type II(b) representations) or $v \in \mathbf{R}\{((\alpha\beta)^{1/2}, -(\alpha\beta)^{1/2}, (\alpha\bar{\beta})^{1/2}, -(\bar{\alpha}\beta)^{1/2})\}$ (for type III representations) and hence $\gamma \in \Omega$. Note for type III representations care must be taken when deciding which translations act as -1; the results will depend on the parity of the numbers a/d and b/d. ∎

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