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Network periodic solutions: patterns of phase-shift synchrony

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Abstract

We prove the *rigid phase conjecture* of Stewart and Parker. It then follows from previous results (of Stewart and Parker and our own) that rigid phase-shifts in periodic solutions on a transitive network are produced by a cyclic symmetry on a quotient network. More precisely, let $X(t) = (x_1(t), \ldots, x_n(t))$ be a hyperbolic T-periodic solution of an admissible system on an n-node network. Two nodes c and d are *phase-related* if there exists a phase-shift $\theta_{cd} \in [0, 1)$ such that $x_d(t) = x_c(t + \theta_{cd}T)$. The conjecture states that if phase relations persist under all small admissible perturbations (that is, the phase relations are *rigid*), then for each pair of phase-related cells, their input signals are also phase-related to the same phase-shift. For a transitive network, rigid phase relations can also be described abstractly as a $\boldsymbol{Z}_m$ permutation symmetry of a quotient network. We discuss how patterns of phase-shift synchrony lead to rigid synchrony, rigid phase synchrony, and rigid multirhythms, and we show that for each phase pattern there exists an admissible system with a periodic solution with that phase pattern. Finally, we generalize the results to nontransitive networks where we show that the symmetry that generates rigid phase-shifts occurs on an extension of a quotient network.

Mathematics Subject Classification: 34C15

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we discuss properties of periodic solutions of networks of differential equations that are rigid; that is, every small perturbation of the network equations yields a periodic solution with the same property as the solution to the unperturbed equations. It has been known for many years [15] that networks with symmetries can support periodic solutions that have rigid synchrony, rigid phase-shift synchrony, or rigid multirhythms. In this paper we complete a program begun by Stewart and Parker [25] that shows that if these properties of periodic solutions are rigid on a transitive (or path connected) network, then rigid phase-shifts and rigid multirhythms are indeed generated by symmetry, where the symmetry does not necessarily act on the original network but rather on some quotient network. The fact that periodic solutions with these rigid properties can occur in a network with no symmetry, but be generated by symmetry on a quotient network, was first noted by Pivato in Stewart *et al* [23]. We also show that the assumption of transitivity is not needed; however, then the symmetry results need to be posed on an extended network of the quotient network.

Let $\mathcal{G}$ be an n-node network and let

$$X(t) = (x_1(t), \ldots, x_n(t))$$

be a T-periodic solution to a *$\mathcal{G}$-admissible system*. Admissibility will be discussed later in this section. We describe the rigid properties more precisely.

- Two nodes c and d are *phase-related* if
$$x_d(t) = x_c(t + \theta_{cd}T), \tag{1.1}$$
where $\theta_{cd} \in S^1 = [0, 1)$ is a phase-shift expressed as a fraction of the period. Note that two nodes are *synchronous* if they are phase-related with $\theta_{cd} = 0$.
- Two nodes c and d have *rationally related periods* or *multirhythms* if the quotient of the minimal periods of $x_c(t)$ and $x_d(t)$ is a rational number unequal to 1.

Note that the existence of multirhythms is equivalent to at least one node being phase-related to itself. If two nodes c, d have different periods, then at least one of them, say node c, will have a period $\theta_{cc}T$ that is less than T and $x_c(t) = x_c(t + \theta_{cc}T)$ where $0 < \theta_{cc} < 1$. Conversely, if node c is phase-related to itself and $0 < \theta_{cc} < 1$, then multirhythms follow since the period of $x_c(t)$ is less than the full period of $X(t)$.

Definition 1.1. Let $X(t)$ be a hyperbolic periodic solution. Suppose two nodes c and d are phase-related as in (1.1). The phase relation is *rigid* if periodic solutions associated with perturbed network admissible systems always have the same phase-relation. A periodic solution $X(t)$ has *rigid phase-shifts* if all phase relations among the nodes in $X(t)$ are rigid.

Review of coupled cell networks. We now review part of the theory of coupled cell systems developed in [16, 23]. A *coupled cell network* is a graph that consists of a finite set of cells (or nodes) partitioned into cell types and a finite set of directed arrows or edges partitioned into edge types. Arrows indicate which cells are coupled to which. The *input set* of a cell c is the set of arrows that terminate at cell c. Two cells are *input equivalent* if there exists a bijection between the input sets of the cells that preserves coupling type.

Suppose that cell j receives signals from the m_j cells $\sigma_j(1), \ldots, \sigma_j(m_j)$. Then an *admissible system* of ODEs associated with this network has the form

$$\dot{x}_j = f_j(x_j, x_{\sigma_j(1)}, \ldots, x_{\sigma_j(m_j)}) \tag{1.2}$$

for $j = 1, \ldots, n$. Moreover, if the arrows from cells $\sigma_j(p)$ and $\sigma_j(q)$ to cell j are arrow equivalent, then f_j is assumed to be invariant under the transposition of coordinates $x_{\sigma_j(p)}$ and $x_{\sigma_j(q)}$. If cells i and j are input equivalent, then $f_i = f_j$.

Polydiagonals are subspaces of phase space consisting of points $x = (x_1, \ldots, x_n)$ that satisfy a set of equalities $x_i = x_j$. Note that every solution $X_0(t) = (x_1^0(t), \ldots, x_n^0(t))$ leads to a polydiagonal $\Delta(X_0)$ which is defined as

$$\Delta(X_0) = \{(x_1, \ldots, x_n) : x_i = x_j \text{ when } x_i^0(t) = x_j^0(t) \text{ for } t \in \boldsymbol{R}\}. \tag{1.3}$$

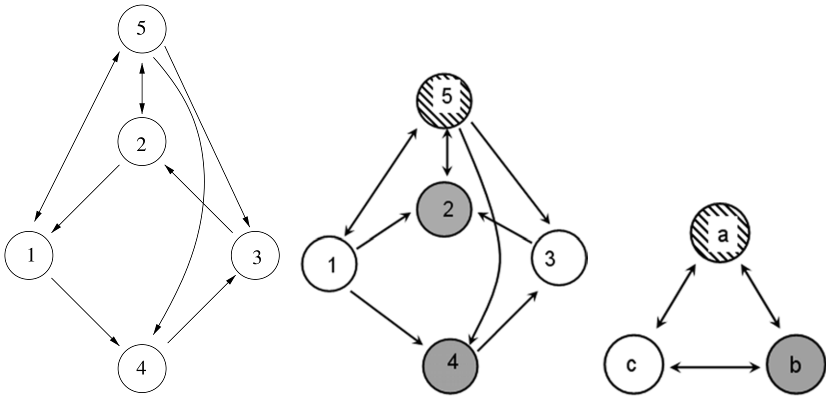

Figure 1. A regular five-cell network with a balanced colouring leading to a quotient three-cell bidirectional ring.

Every polydiagonal can be associated with a colouring of network nodes by colouring two cells i, j with the same colour if cells i and j are cell equivalent and $x_i = x_j$. We can also colour network arrows so that two arrows have the same arrow-colour if and only if their coupling types are the same, their head cells have the same node-colour, and their tail cells have the same node-colour.

The node-colouring is called *balanced* if there exists an arrow-colour preserving bijection between the input sets for each pair of nodes with the same node-colour. It is proved in [16] (see also [23]) that polydiagonals are flow-invariant with respect to all admissible vector fields if and only if the colouring associated with the polydiagonal is balanced.

We associate a balanced colouring with a *quotient network* by identifying all cells with the same colour to a cell and mapping input sets of those cells to the input set of the cell in the quotient network. The vector field obtained by restricting the original vector field to the polydiagonal is an admissible vector field of the quotient network. It is shown in [16] that each admissible vector field of a quotient network can be lifted to an admissible vector field of the original network.

An example of rigid properties. A five-node network that illustrates the three rigid properties of synchrony, phase-shift synchrony, and multirhythms is given in example 7.1 of [23]. We reproduce that example in figure 1(left). This example has the three-colour balanced colouring shown in figure 1(centre); the associated quotient network is the three-cell $\boldsymbol{D}_3$ symmetric bidirectional ring shown in figure 1(right).

Symmetry-breaking Hopf bifurcation with $\boldsymbol{D}_3$ symmetry yields three families of solutions one of which consists of discrete rotating waves where two nodes are a half-period out of phase while the third cell has twice the frequency of the other two [13, 15]. These solutions are also hyperbolic in the five-cell system. Typical simulations are shown in figure 2; the two simulations are obtained just by changing initial conditions. Observe that the pairs of cells $1, 3$ and $2, 4$ are synchronous. On the left the synchronous pair $2, 4$ oscillates at twice the frequency of the other three nodes (multirhythms) and the synchronous pair $1, 3$ oscillates a half-period out of phase from node 5. These properties are rigid.

Pattern of phase-shift synchrony. Recall that a *transitive* (or *path connected* or *strongly connected*) network is a network such that every cell i can be connected to a every other

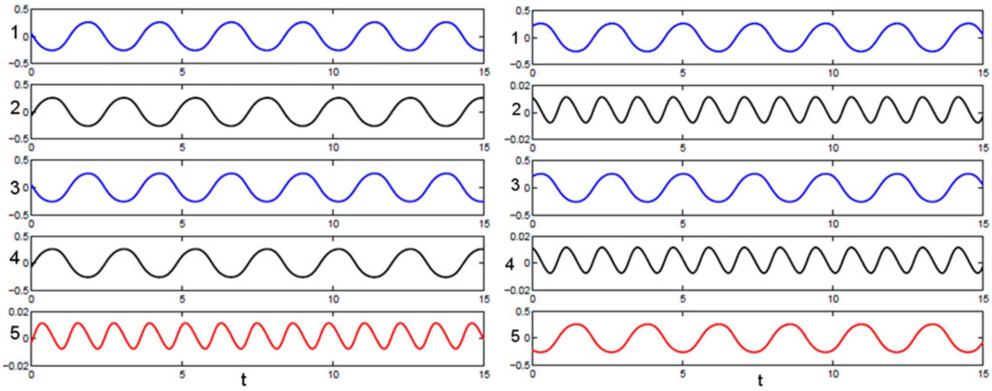

Figure 2. Simulations in the five-cell network in figure 1. (Left) double frequency in cells 2 and 4; (right) double frequency in cell 5.

cell j by a sequence of arrows. We now explain how the above example is representative of a general theorem for transitive networks.

Definition 1.2. A *pattern of phase-shift synchrony* on a transitive network $\mathcal{G}$ is a pair consisting of a quotient network $\mathcal{Q}$ of $\mathcal{G}$ and a cyclic symmetry group $\boldsymbol{Z}_m$ of $\mathcal{Q}$.

The definition of a pattern of phase-shift synchrony depends only on the graph associated with $\mathcal{G}$ and not specifically on the set of admissible differential equations associated with $\mathcal{G}$. Let $\Delta_{\mathcal{Q}}$ denote the synchrony subspace corresponding to the quotient network $\mathcal{Q}$. In the five-node example $\mathcal{Q}$ is the bidirectional three-node ring (indexed by a, b, c) and

$$\Delta_{\mathcal{Q}} = \{x : x_1 = x_3, x_2 = x_4\} = \{(x_1, x_2, x_1, x_2, x_5)\}.$$

Note that $\sigma_1 = (a)(b\ c)$ and $\sigma_2 = (a\ c)(b)$ are order 2 symmetries of the quotient network $\mathcal{Q}$. Hence the pairs $(\mathcal{Q}, \boldsymbol{Z}_2(\sigma_1))$ and $(\mathcal{Q}, \boldsymbol{Z}_2(\sigma_2))$ are patterns of phase-shift synchrony.

Definition 1.3. A T-periodic solution $X(t)$ to a $\mathcal{G}$-admissible system has the *pattern of phase-shift synchrony* $(\mathcal{Q}, \boldsymbol{Z}_m)$ if two conditions are satisfied. First, $\{X(t)\} \subset \Delta_{\mathcal{Q}}$. Hence we can view $X(t)$ as a solution $Y(t)$ of the $\mathcal{G}$-admissible system restricted $\Delta_{\mathcal{Q}}$. Second, there is a generator $\sigma \in \boldsymbol{Z}_m$ such that

$$\sigma Y(t) = Y\left(t + \frac{T}{m}\right). \tag{1.4}$$

The phase relations of solutions in figure 2 are forced by σ_1 amd σ_2. Indeed, (1.4) for σ_1 implies

$$\begin{aligned} &\sigma_1 Y(t) = \sigma_1(Y_a(t), Y_b(t), Y_c(t)) = (Y_a(t), Y_c(t), Y_b(t)) \\ &Y\left(t + \frac{T}{2}\right) = \left(Y_a\left(t + \frac{T}{2}\right), Y_b\left(t + \frac{T}{2}\right), Y_c\left(t + \frac{T}{2}\right)\right). \end{aligned}$$

Thus

$$Y_c(t) = Y_b\left(t + \frac{T}{2}\right) \qquad \text{and} \qquad Y_a(t) = Y_a\left(t + \frac{T}{2}\right).$$

Hence, cells b and c oscillate with a half-period phase-shift and cell a oscillates at twice the frequency.

The results in this paper and [11] when combined with Stewart and Parker's theorem in [25] prove the following.

Theorem 1.4. *Let $\mathcal{G}$ be a transitive network and $X(t)$ be a hyperbolic periodic solution. Suppose the phase-shifts of $X(t)$ are rigid, then there exists a pattern of phase-shift synchrony that generates the rigid phase-shifts.*

In other words, suppose the phase-shifts of $X(t)$ are rigid and the network is transitive, then there exists a symmetry on a quotient network that generates the rigid phase-shifts. We prove two theorems about patterns of phase-shift synchrony in section 2. First, in theorem 2.1, which is a converse of theorem 1.4, we explain why periodic solutions having a pattern of phase-shift synchrony can exhibit the three synchrony properties. We also show why the synchrony properties implied by a pattern of phase-shift synchrony are rigid. Second, in theorem 2.2, we show that there are periodic solutions of some admissible system of differential equations associated with each abstractly defined pattern of phase-shift synchrony. We cannot yet prove that the associated periodic solutions are hyperbolic; hence we cannot yet conclude rigidity of the synchrony properties in this periodic solution.

A discussion of the Stewart–Parker results. Stewart and Parker [25] assume the validity of three conjectures in order to construct the quotient network and the symmetry on that quotient network from the existence of rigid phase-shifts. The conjectures are the following:

(1) *Fully oscillatory* is a generic property of hyperbolic periodic solutions of admissible systems on a transitive network.
(2) The *rigid synchrony property* is valid.
(3) The *rigid phase property* is valid.

We now explain these three conjectures while also sketching the proof of the Stewart and Parker theorem. In [11] we verified that the first two conjectures are valid and in this paper we validate the third conjecture. See theorem 3.1.

The quotient network $\mathcal{Q}$ is obtained as follows. Suppose that all phase-shifts (including the 0 phase-shift) in $X(t)$ are rigid. Colour two nodes i, j with the same colour if and only if $x_j(t) = x_i(t)$ for all t. The *rigid synchrony property* (proved in [11]) states that this colouring is balanced and hence the associated polydiagonal $\Delta(X(t))$ is flow-invariant with respect to all admissible vector fields of the network. Let $\mathcal{Q}$ be the quotient network associated with the polydiagonal $\Delta(X(t))$ and suppose that $\mathcal{Q}$ has q nodes. Note that by definition the trajectory $\{X(t)\} \subset \Delta(X(t))$. Let $Y(t) = (y_1(t), \ldots, y_q(t))$ be the periodic solution $X(t)$ viewed on the q nodes of the quotient network $\mathcal{Q}$. By construction the solution $Y(t)$ has no zero phase-shifts between the nodes of $\mathcal{Q}$.

The Stewart and Parker [25] construction of the permutation symmetry σ on $\mathcal{Q}$ assumes that the network is transitive and proceeds as follows. Assume $X(t)$ is fully oscillatory (for each node j the cell coordinate $x_j(t)$ is nonconstant) and hence $Y(t)$ is also fully oscillatory. (We proved in [11] that *fully oscillatory* is a generic property of periodic solutions in transitive networks; hence rigidity implies that this assumption can be made without loss of generality.) It follows that there is a minimal nonzero period for each node projection $y_c(t)$. Let c be a node and let d be the node such that $y_d(t)$ is phase-shifted from $y_c(t)$ by the smallest phase-shift $\theta > 0$. This smallest θ might come from node c itself. Note that d is uniquely defined for if d_1 and d_2 were two such nodes then $x_{d_1}(t) = x_{d_2}(t)$, which contradicts the fact that there are no 0 phase-shifts on $\mathcal{Q}$. Therefore, we can define $\sigma c = d$. Similarly $\sigma : \mathcal{Q} \to \mathcal{Q}$ is 1 : 1. For if $\sigma c_1 = \sigma c_2$, then $y_{c_1}(t) = y_{c_2}(t)$ and the lack of 0 phase-shifts implies that $c_1 = c_2$. Hence σ permutes the nodes of $\mathcal{Q}$.

In order to conclude that the permutation σ is actually a symmetry of the network $\mathcal{Q}$, Stewart and Parker [25] assume the *rigid phase property*.

Definition 1.5. *A network $\mathcal{Q}$ satisfies the rigid phase property if whenever two cells in a hyperbolic periodic solution has rigid phase-shifts, then the tail cells of their inputs are also rigid phase-related to the same phase-shift.*

In theorem 3.1 we prove that the *rigid phase property* is valid for all networks that can support a hyperbolic periodic solution. Hence we can conclude that σ is a permutation symmetry of the network $\mathcal{Q}$.

Note that the *rigid synchrony property* is a special case of the *rigid phase property*. As we show, the proof of the *rigid synchrony property* in [11, theorem 6.1] can be adapted with the addition of a new idea to a proof of the *rigid phase property*. We prove theorem 3.1 by first lifting the original system to a doubled system that consists of two exact copies of the original system, and then by employing the strategy we developed for proving the *rigid synchrony property*. We note that Aldis [2] also discusses a method for proving the *rigid phase property* based on the double network idea, but he needs to make additional assumptions and his analysis proceeds using different methods.

Transitive components in nontransitive networks. In section 7 we generalize the results on phase-shift synchrony to nontransitive networks. To begin we recall that all directed networks can be decomposed into transitive components.

We say that $d \to c$ if there exists an arrow in $\mathcal{G}$ whose head is c and tail is d. A *directed path* from node d to node c is a sequence of nodes $d = a_0, a_1, \ldots, a_k, a_{k+1} = c$ such that $a_j \to a_{j+1}$. Next, we define relations $\rightharpoonup$ and $\rightleftharpoons$ on the nodes of the network $\mathcal{G}$.

- $c \rightharpoonup d$ if there exists a directed path in $\mathcal{G}$ that connects c to d.
- $c \rightleftharpoons d$ if $c \rightharpoonup d$ and $d \rightharpoonup c$.

The relation $\rightleftharpoons$ defines an equivalence relation on the nodes of $\mathcal{G}$ and equivalence classes under $\rightleftharpoons$ are called *transitive components*. Note that the relation $\rightharpoonup$ induces a partial ordering on the set of transitive components so that we can speak of one component being *higher* or *lower* in the network than another.

Given a transitive component $\mathcal{J} \subset \mathcal{G}$, we define $L(\mathcal{J})$ to be the union of $\mathcal{J}$ and all the transitive components lower than $\mathcal{J}$. We define $H(\mathcal{J})$ to be the union of $\mathcal{J}$ and all the transitive components higher than $\mathcal{J}$ and we also define $\overline{H}(\mathcal{J})$ to be the union of $\mathcal{J}$ and all transitive components not in $L(\mathcal{J})$. Note that $H(\mathcal{J}) \subset \overline{H}(\mathcal{J})$. For any node $c \in \mathcal{G}$ we define $H(c) = H(\mathcal{J})$ and $L(c) = L(\mathcal{J})$ where $\mathcal{J}$ is the transitive component containing c.

Hyperbolicity implies a maximal oscillating transitive component. Let X_0 be a hyperbolic periodic solution of an admissible system of network $\mathcal{G}$. It follows from the proof of *fully oscillatory* on transitive networks (see [11, theorem 2.1]) that the admissible system can be perturbed so that on each transitive component the perturbed periodic solution is either fully oscillatory or constant. It also follows that if the perturbed periodic solution oscillates on a transitive component $\mathcal{J}$, then generically it oscillates on every transitive component lower than $\mathcal{J}$.

Definition 1.6. A periodic solution is *hub-like* on a transitive component $\mathcal{J}$ if the periodic solution is oscillating on every node in $L(\mathcal{J})$ and constant on all other nodes.

For a given X_0, $\mathcal{J}$ is unique if it exists. Josić and Török [18] prove the following.

Theorem 1.7 ([18]). *Let X_0 be a hyperbolic periodic solution of an admissible system of $\mathcal{G}$. Let cells c and d be two oscillating cells. Then $H(c) \cap H(d)$ is nonempty and generically X_0 oscillates on $H(c) \cap H(d)$. Moreover, generically there exists a unique transitive component $\mathcal{J}_{\max}(X_0)$ on which X_0 is hub-like.*

Network extensions. On nontransitive networks we prove that if rigid phase-shift synchrony exists for a hyperbolic periodic solution of an admissible vector field, then that synchrony is caused by symmetry, but in a way that is more complicated than in the transitive networks case. As in transitive networks we can use the *rigid synchrony property* to restrict the periodic solution to a quotient network $\mathcal{Q}$ on which the solution has no rigid 0 phase-shifts. However, the symmetry that drives the rigid phase-shifts in feed-forward networks occurs on a network extension $\hat{\mathcal{Q}}$ of $\mathcal{Q}$.

A network $\hat{\mathcal{Q}}$ is an extension of $\mathcal{Q}$ if transitive components are added to the bottom of $\mathcal{Q}$ in such a way that every new transitive component in $\hat{\mathcal{Q}}$ is network isomorphic to a transitive component in $\mathcal{Q}$. See definition 7.11. We show that every admissible system F in $\mathcal{Q}$ extends uniquely to an admissible system $\hat{F}$ in $\hat{\mathcal{Q}}$ (see lemma 7.12).

Suppose that $Y_0(t)$ is a T-periodic solution to an admissible vector field on $\mathcal{Q}$ such that Y_0 has no synchronous cells. Suppose nodes c and d are phase-related by phase-shift θ; that is,

$$y_d(t) = y_c(t + \theta T). \tag{1.5}$$

Then d is the only node in $\mathcal{Q}$ that is phase-related to c by phase-shift θ because Y_0 has no synchronous cells.

Definition 1.8. Suppose c and d satisfy (1.5). Then we define $\sigma_\theta(c) = d$. If θ is the minimum positive phase-shift between two cells of $\mathcal{Q}$, we write $\theta_{\min}$ for θ and $\sigma_{\min}$ for $\sigma_{\theta_{\min}}$.

Note that σ_θ need not be defined on all nodes of $\mathcal{Q}$, but lemma 7.15 proves that $\sigma_{\min}$ is defined on every node in $\mathcal{J}_{\max}(Y_0)$ and in fact permutes these nodes. Note that if $y_c(t)$ is constant, then $\sigma_\theta(c) = c$. Our main result is the following.

Theorem 1.9. *Let Y_0 be a hyperbolic hub-like periodic solution of an admissible system $\dot{Y} = F(Y)$ of the network $\mathcal{Q}$. Suppose Y_0 has rigid phase-shifts and has no synchronous cells. Then there exists a unique extension $(\hat{F}, \hat{\mathcal{Q}})$ of $(F, \mathcal{Q})$ such that*

(a) $\hat{F}$ has a unique hyperbolic periodic solution $\hat{Y}_0$ whose projection to the cells on $\mathcal{Q}$ is Y_0.
(b) $\hat{Y}_0$ has no synchronous cells.
(c) $\sigma_{\min}$ is defined on every node in $\hat{\mathcal{Q}}$ and $\hat{\mathcal{Q}}$ is the union of $\sigma_{\min}$ orbits of nodes in $\mathcal{Q}$.
(d) Every rigid phase-shift in Y_0 is generated by $\sigma_{\min}$ acting on $\hat{\mathcal{Q}}$.

Patterns of phase-shift synchrony in nontransitive networks. We end this introduction with a definition of a pattern of phase-shift synchrony for feed-forward networks.

Definition 1.10. A *pattern of phase-shift synchrony* for a network $\mathcal{G}$ is a quotient network $\mathcal{Q}$ with a hub $\mathcal{J}$, an extension $\hat{\mathcal{Q}}$ of $\mathcal{Q}$, whose new transitive components are not higher than any transitive component in $\mathcal{Q}$, and a permutation symmetry $\sigma : \hat{\mathcal{Q}} \to \hat{\mathcal{Q}}$ such that

(a) σ acts as the identity on transitive components not in $L(\mathcal{J})$,
(b) σ maps $\mathcal{J}$ to $\mathcal{J}$, and
(c) $\hat{\mathcal{Q}}$ is the union of σ orbits of nodes in $\mathcal{Q}$.

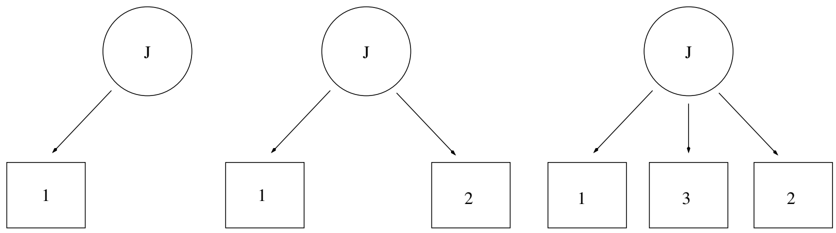

Figure 3. (Left) a two-node feed-forward network; (centre) an extension with one new node and $\mathbf{Z}_2$ symmetry; (right) an extension with two new nodes and $\mathbf{Z}_3$ symmetry.

It follows from theorem 1.9 that every hyperbolic periodic solution on $\mathcal{Q}$ leads to a pattern of phase-shift synchrony on an extended network $\hat{\mathcal{Q}}$ where the permutation symmetry is just $\sigma_{\min}$.

Definition 1.11. A T-periodic solution $X_0(t)$ to a $\mathcal{G}$-admissible system *has the pattern of phase-shift synchrony* $(\mathcal{Q}, \hat{\mathcal{Q}}, \sigma)$ if two conditions are satisfied. First, $\{X_0(t)\} \subset \Delta_{\mathcal{Q}}$. Hence, we can view $X_0(t)$ as a solution $Y_0(t)$ of the equations $\dot{Y} = F(Y)$ restricted to the quotient network $\mathcal{Q}$. Second, let $\hat{Y}_0(t)$ be the unique periodic solution to the extended equations $\dot{\hat{Y}} = \hat{F}(Y)$ that projects onto Y_0. Then $\hat{Y}_0$ satisfies

$$\sigma \hat{Y}_0(t) = \hat{Y}_0\left(t + \frac{T}{m}\right) \tag{1.6}$$

where m is the order of σ.

We note that the extension $\hat{\mathcal{Q}}$ need not be unique. For example, consider the two-node network in figure 3(left) where $\mathcal{J}$ is the top node and 1 is the other node. Figures 3(centre) and (right) are two extensions. In both cases the symmetry σ fixes $\mathcal{J}$. In the first the symmetry σ is the transposition (1 2) and in the second σ is the cyclic permutation (1 2 3). In the first extension the frequency of the solution in $\mathcal{J}$ is twice the frequency of the solution in 1, whereas in the second extension it is three times the frequency.

Structure of the paper. In section 2 we discuss two theorems (theorems 2.1 and 2.2) concerning patterns of phase-shift synchrony and also give a proof for a theorem in [12] that states that a hyperbolic equilibrium of a quotient network system can be lifted to a hyperbolic equilibrium of an admissible vector field of the original network. This proof illustrates the difficulties in proving the corresponding result for periodic solution. See theorem 2.3. In section 3, we state theorem 3.1, which asserts the validity of the *rigid phase property*. The rest of the paper is devoted to the proof of theorem 1.4. As we have discussed, this theorem follows from the rigid phase conjecture (that is, theorem 3.1). Sections 4–6 give a proof of theorem 3.1. In section 4, we show that the proof of theorem 3.1 can be reduced to that of proposition 4.4. In section 5, we prove several lemmas that will be needed in the proof of proposition 4.4. Then in section 6 we prove the proposition. Section 7 generalizes the results to the case when $\mathcal{Q}$ is nontransitive (or *feed-forward*). Note that this last section can be read directly after the statement of theorem 3.1.

2. Patterns of phase-shift synchrony

We begin this section by reviewing spatiotemporal symmetries of periodic solutions. This review will motivate the definition of a pattern of phase-shift synchrony. Then we discuss the relationship between patterns of phase-shift synchrony and rigid synchrony properties.

Review of spatiotemporal symmetries. The symmetry properties of periodic solutions of systems of differential equations have been studied actively [15]. Assume that Γ is a finite symmetry group acting on $\boldsymbol{R}^N$ and the differential equations are Γ-equivariant. Then the symmetries of a periodic solution $X(t) \in \boldsymbol{R}^N$ are of two types (see Fiedler [7])

$$K = \{\gamma \in \Gamma : \gamma X(t) = X(t) \text{ for all } t\}$$
$$H = \{\gamma \in \Gamma : \gamma\{X(t)\} = \{X(t)\}\}.$$

The space symmetries K are those symmetries that fix the periodic solution pointwise and the spatiotemporal symmetries H are those symmetries that fix the periodic trajectory setwise. It follows from uniqueness of solutions that when $h \in H$ there exists a phase-shift θ (normalized by the period T) such that $hX(t) = X(t + \theta T)$. It is well-known that H/K is cyclic [15]. It is also known that the symmetry groups H and K of a hyperbolic periodic solution are *rigid* in the sense that if the equivariant vector field is perturbed by an equivariant perturbation, thus leading to a unique perturbed period solution $\hat{X}(t)$ near $X(t)$, then the symmetry groups of $\hat{X}(t)$ will also be H and K.

Buono and Golubitsky [6] proved the H/K theorem that gives necessary and sufficient conditions for the existence of a hyperbolic periodic solution with space symmetries K and spatiotemporal symmetries H. See also [13]. The theorem contains three conditions on H and K in addition to the fact that H/K is cyclic.

Relationship with equivariant bifurcation theory and selected previous work. There is a huge literature on equivariant bifurcation from a symmetric equilibrium, much of which has been discussed in [13, 15]. Restriction to the case where the symmetry group is finite and the bifurcation is a symmetry-breaking Hopf bifurcation [7, 15] is the case that most parallels the coupled cell theory in this paper. In coupled cell theory these bifurcations are sometimes called synchrony-breaking bifurcations. Indeed, the study of periodic solutions obtained by synchrony-breaking Hopf bifurcation in coupled systems is one area where the manifestation of H symmetries as phase-shift synchrony has proved especially helpful. Early work focused on rings of oscillators [3, 15], where discrete travelling waves are observed, and on models for locomotor central pattern generators [19], where phase-shift synchrony is crucially important in identifying animal gaits. There are many other examples and applications.

Ashwin and Stork [5] note that there is a natural network associated with every finite group and hence for finite groups the Hopf bifurcation theory of spatiotemporal symmetries of periodic solutions is a proper subset of the coupled systems theory. Josić and Török [18] prove an analogue of the H/K theorem for networks with symmetry. Here $K \subset H$ are subgroups of the network symmetry group. As noted, this paper shows that symmetry is the source of all rigid phase-shift synchrony in coupled systems, though that symmetry may act on a quotient network, rather than on the original network.

There are other kinds of structurally stable dynamics associated with equivariant bifurcations and these dynamics have coupled cell analogues. For example, sequences of saddle-sink connections in fixed-point subspaces of equivariant systems can lead through symmetry-breaking bifurcations to structurally stable heteroclinic cycles [17, 20, 22]. Field and co-workers [1, 4, 8] have studied analogues and extensions of these results for coupled cell systems.

In other directions, bifurcation from periodic solutions in finite symmetry equivariant systems has been studied (see Lamb and Melbourne [21], in particular), as has the global symmetry of chaotic attractors (see Field *et al* [9]). The network analogue of attractor symmetry in equivariant systems has not been much studied.

Patterns of phase-shift synchrony for transitive networks. Theorem 1.4 together with theorem 2.1 provides an analogue of the H/K-Theorem for periodic solutions of networks of differential equations. In this analogy the quotient network $\mathcal{Q}$ plays the role of the group K and the cyclic group $\boldsymbol{Z}_m$ plays the role of the group H/K.

Theorem 2.1. *Suppose that a period T solution $X(t)$ that has a pattern of phase-shift synchrony $(\mathcal{Q}, \boldsymbol{Z}_m)$ where $\mathcal{Q} \neq \mathcal{G}$, $m > 1$, and $\mathcal{G}$ is transitive. Then $X(t)$ has synchronous nodes and phase-related nodes. If a generator of $\boldsymbol{Z}_m$ has a decomposition into disjoint cycles of different lengths, then the solution will also have multirhythms. Moreover, if $X(t)$ is hyperbolic, then all of the synchrony properties are rigid.*

Proof. Let $\hat{\sigma}$ be a generator of $\boldsymbol{Z}_m$ and a spatiotemporal symmetry of $Y(t)$, the periodic solution on the quotient network corresponding to $X(t)$. It follows from definition 1.3 that $\hat{\sigma}Y(t) = Y(t + \frac{k}{m}T)$ for some integer k. Since there is no synchrony between nodes in the quotient network, there is an integer ℓ such that $k\ell \equiv 1 \bmod m$. Then $\sigma = \hat{\sigma}^\ell$ satisfies

$$\sigma Y(t) = Y\left(t + \frac{T}{m}\right). \tag{2.1}$$

Next we show that the three rigid properties for a periodic solution mentioned in section 1 follow from a pattern of phase-shift synchrony. Suppose that $X(t) = (x_1(t), \ldots, x_n(t))$ is a periodic solution that has the pattern of phase-shift synchrony $(\mathcal{Q}, \boldsymbol{Z}_m)$. Then

$$x_c(t) = x_d(t)$$

whenever nodes c and d have the same colour in the balanced colouring corresponding to the quotient network $\mathcal{Q}$; that is, nodes c and d are synchronous.

Since $\sigma \in \boldsymbol{Z}_m$ defined by (2.1) is a permutation on the q nodes of $\mathcal{Q}$, we can decompose $\sigma = \sigma_1 \cdots \sigma_s$ as a product of disjoint cycles of orders $m_1, \ldots, m_s$. Suppose we renumber the nodes so that σ_j is the cyclic permutation $(1 \cdots m_j)$. Then (2.1) implies

$$\begin{aligned} y_2(t) &= y_1\left(t + \frac{T}{m}\right) \\ y_3(t) &= y_2\left(t + \frac{T}{m}\right) \\ &\vdots \\ y_{m_j}(t) &= y_{m_{j-1}}\left(t + \frac{T}{m}\right) \\ y_1(t) &= y_{m_j}\left(t + \frac{T}{m}\right). \end{aligned} \tag{2.2}$$

Thus, a nontrivial symmetry on a quotient network can imply phase-related nodes. It also follows from (2.2) that $y_1(t) = y_1(t + \frac{m_j}{m}T)$ and hence the nodes corresponding to the jth cycle have period

$$T_j = \frac{m_j}{m}T. \tag{2.3}$$

Hence, if there are cycles of different lengths in σ, then the solution $X(t)$ will exhibit multirhythms.

Finally, suppose that the solution $X(t)$ is hyperbolic. Then we show that the three properties discussed above are rigid. Note that hyperbolicity implies that if the admissible system that produced $X(t)$ is perturbed by a small admissible perturbation, then the perturbed system will have a unique $\hat{T}$-periodic solution $\hat{X}(t)$. Since $\Delta_{\mathcal{Q}}$ is also flow-invariant for the perturbed system, it follows from uniqueness that $\{\hat{X}(t)\} \subset \Delta_{\mathcal{Q}}$. Let $\hat{Y}(t)$ be the solution $\hat{X}(t)$ viewed on the quotient network $\mathcal{Q}$. Note that $\sigma\hat{Y}(t - \frac{\hat{T}}{m})$ is also a perturbed solution of $Y(t)$. Hence uniqueness of the perturbed periodic solution also implies that

$$\sigma\hat{Y}(t) = \hat{Y}\left(t + \frac{\hat{T}}{m}\right)$$

and hence the perturbed solution $\hat{X}(t)$ has the same pattern of phase-shift synchrony $(\mathcal{Q}, \mathbf{Z}_m)$ as does $X(t)$ and the associated properties are rigid. □

Solutions with a given pattern of phase-shift synchrony always exist. Using a result of Josić and Török [18], we prove the following.

Theorem 2.2. *Suppose that the transitive network $\mathcal{G}$ has a pattern of phase-shift synchrony $(\mathcal{Q}, \mathbf{Z}_m)$. Then there exists a periodic solution $X(t)$ that is associated with the pattern of phase-shift synchrony that is hyperbolic in $\Delta_{\mathcal{Q}}$.*

Proof. Josić and Török [18] show that there exists an admissible system with a hyperbolic periodic solution on the quotient network $\mathcal{Q}$ that satisfies $(H, K) = (\mathbf{Z}_m, 1)$. This result is somewhat subtle since if the permutation σ consists of more than one cycle (say two cycles whose lengths are unequal, greater than 1, and prime), then such solutions cannot be obtained using (generic) Hopf bifurcation [10, 13]. We can view the solution in $\Delta_{\mathcal{Q}}$ and then extend the admissible system on $\Delta_{\mathcal{Q}}$ to an admissible system on the whole network. □

A discussion of hyperbolicity. Note that although the hyperbolic periodic solution in $\Delta_{\mathcal{Q}}$ is a periodic solution for the extended system on the whole of phase space, it is not necessarily hyperbolic in the directions transverse to $\Delta_{\mathcal{Q}}$. Indeed, we have not been able to prove that $X(t)$ is hyperbolic in the whole phase space. We believe that for transitive networks this extension is always possible so that the extension is also hyperbolic, though one might have to perturb the $\mathcal{Q}$ admissible system, and hence the periodic solution $Y(t)$, first before doing the extension. Then rigidity would follow from hyperbolicity. Nevertheless, there is still a limited form of rigidity that follows from hyperbolicity on $\Delta_{\mathcal{Q}}$. In any perturbed admissible system, there exists a perturbed periodic solution in $\Delta_{\mathcal{Q}}$ with the desired symmetry and synchrony properties. However, there might exist other periodic solutions near $X(t)$ that are not in $\Delta_{\mathcal{Q}}$.

It also follows from theorem 1.7 that the extension will not be hyperbolic if X_0 is not hub-like and Y_0 is hub-like. A simple example occurs when $\mathcal{G}$ consists of two identical cells 1 and 2 that are identically coupled to a third cell 3 and $\mathcal{Q}$ is the quotient network given by the flow-invariant subspace $x_1 = x_2$.

The reason that we might need to perturb the $\mathcal{Q}$ admissible system first before extending it to a $\mathcal{G}$ admissible system can already be seen from the corresponding result about hyperbolic equilibria in $\Delta_{\mathcal{Q}}$. This result was presented in [12], but there was an error in the published proof. So for completeness, we state and prove theorem 2.3 here.

Theorem 2.3. *Suppose that $X_0 \in \Delta_{\mathcal{Q}}$ is a generic point that is a hyperbolic equilibrium for an admissible system f on the quotient network $\mathcal{Q}$. Then there exists an admissible perturbation* *of any lift F of f such that the perturbation has an equilibrium at X_0 that is hyperbolic in the phase space of $\mathcal{G}$.*

Proof. Let

$$\dot{x} = f(x) \tag{2.4}$$

be the admissible system on $\mathcal{Q}$. Let X_0 be a hyperbolic equilibrium to (2.4). Let

$$\dot{X} = F(X) \tag{2.5}$$

be a lift of (2.4) to the whole phase space.

Note that $F(X_0) = 0$ since $f = F|\Delta_{\mathcal{Q}}$, and X_0 is an equilibrium of the lift. If X_0 is a hyperbolic equilibrium for F, we are done. Suppose X_0 is not a hyperbolic equilibrium. We show that there exists an $\mathcal{G}$-admissible P such that $P(X_0) = 0$ and such that X_0 is a hyperbolic equilibrium for $F + \varepsilon P$ for all small $\varepsilon \neq 0$. We claim that an admissible P exists such that $P(X_0) = 0$ and $(DP)_{X_0} = -I$. It follows that

$$D(F + \varepsilon P)_{X_0} = (DF)_{X_0} - \varepsilon I$$

and for small enough ε the perturbed admissible vector field has a hyperbolic equilibrium at X_0, since the real parts of the eigenvalues are just shifted by ε.

Without loss of generality, we assume all cells are input equivalent (otherwise, we can divide the cells into input equivalent classes and then discuss class by class). Let P be strongly admissible; that is, let $P(X) = (p(x_1), \ldots, p(x_n))$. Let $X_0 = (x_1, \ldots, x_n)$ and let $x_{c_1}, \ldots, x_{c_m}$ be the distinct cell coordinates that appear in X_0. Let $U_{c_i} \subset \boldsymbol{R}^k$ be an open neighbourhood of x_{c_i} and $\overline{U}_{c_i}$ be the closure of U_{c_i}. We choose the open sets small enough such that $\overline{U}_{c_i} \cap \overline{U}_{c_j} = \emptyset$ if $i \neq j$. Define

$$p(x) = x_{c_i} - x$$

for $x \in U_{c_i}$. It follows from the fact that the $\overline{U}_{c_i}$ are disjoint that there exists a smooth extension of p to all of $\boldsymbol{R}^k$. Then $P(X_0) = 0$ and $(DP)_{X_0} = -I$. Alternatively, one can construct a polynomial P by defining by interpolation a polynomial p that satisfies $p(x_{c_i}) = 0$ and $(Dp)_{x_{c_i}} = -I$ for all i. $\square$

Note that the restriction of the admissible perturbation constructed in this proof (namely, $P|\Delta_{\mathcal{Q}}$) need not be zero, so that we cannot guarantee the existence of a hyperbolic equilibrium from an extension of the original f.

In the nontransitive case we can again apply the results of Josić and Török [18] to obtain hyperbolic periodic solutions Y_0 on $\Delta_{\mathcal{Q}}$ having a given pattern of phase-shift synchrony. As in the transitive case we have not proved that the corresponding periodic solution X_0 on $\mathcal{G}$ is hyperbolic.

3. The *rigid phase property*

In this section, we state the *rigid phase property* and give an overview of its proof. Suppose that $X_0(t)$ is a hyperbolic periodic solution of the admissible system

$$\dot{X} = F(X) \tag{3.1}$$

of $\mathcal{G}$ with minimum period T. Perturb this system by a small admissible vector field. Hyperbolicity guarantees that there is a unique periodic solution $\hat{X}_0(t)$ to this perturbed system near X_0.

Let $\mathcal{I}(c)$ denote an *input set* $\{e_1, \ldots, e_m\}$ of cell c. Let $\mathcal{T}(e)$ denote the tail cell of the arrow e, and define $\mathcal{T}(\mathcal{I}(c)) = \{c_1, \ldots, c_m\}$, where $c_i = \mathcal{T}(e_i)$. For $X = (x_1, \cdots, x_n) \in \mathcal{P}_{\mathcal{G}}$, we define $x_{\mathcal{T}(\mathcal{I}(c))} = (x_{c_1}, \ldots, x_{c_m})$.

Suppose c and d are input equivalent and let $\beta : \mathcal{I}(c) \to \mathcal{I}(d)$ be a bijection. If $\mathcal{T}(\mathcal{I}(d)) = (d_1, \ldots, d_m)$, we define the pull-back map $\beta^* : \mathcal{P}_{d_1} \times \cdots \times \mathcal{P}_{d_m} \to \mathcal{P}_{c_1} \times \cdots \times \mathcal{P}_{c_m}$ by

$$(\beta^* x)_{\mathcal{T}(i)} = x_{\mathcal{T}(\beta(i))}$$

for each $x \in \mathcal{P}_{d_1} \times \cdots \times \mathcal{P}_{d_m}$ and each $i \in \mathcal{I}(c)$.

Theorem 3.1 (Rigid phase property). *Let $X_0(t)$ be the hyperbolic T-periodic solution to (3.1). Suppose that X_0 has rigid phase-shifts. Then for each pair of phase-related cells c, d there exists an arrow-type preserving bijection $\beta : \mathcal{I}(c) \to \mathcal{I}(d)$ such that*

$$x^0_{\mathcal{T}(\mathcal{I}(c))}(t) = \beta^* x^0_{\mathcal{T}(\mathcal{I}(d))}(t + \theta T) \tag{3.2}$$

for all $t \in \boldsymbol{R}$.

Overview of the proof of theorem 3.1. As discussed in the introduction we prove theorem 3.1 by first lifting the original system to a doubled system that consists of two exact copies of the original system, and then by employing the strategy we developed for proving the *rigid synchrony property* [11, theorem 6.1]. In the following we recall the proof of the *rigid synchrony property*, explain why a doubled system is needed, and then outline the proof.

The *rigid synchrony property* is a special case of theorem 3.1, where only the rigid synchrony relation (or 0 phase-shift) is considered. Indeed, the *rigid synchrony property* is proved by showing that if the colouring associated with $\Delta(X_0)$ is rigid, then the polydiagonal $\Delta(X_0)$ is flow-invariant with respect to all admissible vector fields and hence that the colouring associated with $\Delta(X_0)$ is balanced. Therefore, for two synchronous cells c and d on X_0, there exists an arrow-type preserving bijection $\beta : \mathcal{I}(c) \to \mathcal{I}(d)$ such that (3.2) is valid with $\theta = 0$.

In the general rigid phase case, there is no simple relation between the colouring of the network and the phase pattern of the periodic solution. Now if we lift the original system to an admissible system of the network $2\mathcal{G}$ that consists of two identical copies of $\mathcal{G}$, then each periodic solution $X(t)$ of the original system lifts to a torus of periodic solutions $(X(t + \zeta T), X(t + \theta T))$ (for any $\theta, \zeta \in S^1$) of the $2\mathcal{G}$ network system. Thus the individual lifted periodic solutions are not hyperbolic, so that the results of [11] cannot be applied directly. Nevertheless, for a given phase-shift θ of the periodic solution X_0 with minimal period T, we can lift X_0 to the periodic solution $(X_0(t), X_0(t + \theta T))$, to which we can further associate a polydiagonal in $\mathcal{P}^2_{\mathcal{G}}$. Suppressing the dependence on X_0 and θ, let

$$Z_0(t) = \left(z^0_1(t), \cdots, z^0_{2n}(t)\right) = (X_0(t), X_0(t + \theta T)) \in \mathcal{P}^2_{\mathcal{G}}, \tag{3.3}$$

where $z^0_i(t) = x^0_i(t)$ and $z^0_{n+i}(t) = x^0_i(t + \theta T)$ for $1 \leqslant i \leqslant n$. Let

$$\Delta(Z_0) = \left\{(z_1, \ldots, z_{2n}) \in \mathcal{P}^2_{\mathcal{G}} : z_i = z_j \text{ if } z^0_i(t) = z^0_j(t) \text{ for } t \in \boldsymbol{R} \text{ and } 1 \leqslant i, j \leqslant 2n\right\}.$$

Thus we can associate $\Delta(Z_0)$ with the phase pattern of X_0 restricted to the given phase-shift θ. We also can define a polydiagonal by the values of the periodic solution on an open interval.

$$\Delta(Z_0, J) = \left\{(z_1, \ldots, z_{2n}) \in \mathcal{P}^2_{\mathcal{G}} : z_i = z_j \text{ if } z^0_i(t) = z^0_j(t) \text{ for } t \in J \text{ and } 1 \leqslant i, j \leqslant 2n\right\}$$

Our aim is to prove that if X_0 has a rigid phase-shift θ, then $\Delta(Z_0)$ is flow-invariant with respect to all admissible systems of $2\mathcal{G}$. In section 4, we reduce the proof to that of proposition 4.4, which states that if X_0 is 'θ-nondegenerately rigid' on an open interval J, then $\Delta(Z_0, J)$ is flow-invariant with respect to all admissible systems. As in the proof for the *rigid synchrony property*, we prove proposition 4.4 by contradiction. Suppose the polydiagonal is not flow-invariant. Then we show that there exists an admissible perturbation of the original single system, such that the phase pattern of the perturbed periodic solution is not the same as X_0.

4. The proof of theorem 3.1 from proposition 4.4

In this section, we set notation, give the definition for a hyperbolic periodic solution being θ-*nondegenerately rigid*, state proposition 4.4, and then give the proof for theorem 3.1 assuming that proposition 4.4 is valid.

Let $J \in \boldsymbol{R}$ be an open interval and let

$$\mathcal{C}(X_0, J) = \{i : \dot{x}_i^0(t) = 0 \text{ for all } t \in J\}$$
$$\mathcal{O}(X_0, J) = \{i : \dot{x}_i^0(t) \neq 0 \text{ for all } t \in J\}.$$

Let $\Theta(X_0)$ consist of all of the possible phase-shifts.

Definition 4.1. A property is *rigid* if and only if that property remains unchanged under all sufficiently small admissible perturbations.

For example, we can consider the set $\mathcal{C}(X_0, J)$ to be a *property* of the periodic solution X_0. That property is rigid if the set does not change on perturbation of the periodic solution by an admissible perturbation of the vector field. Note in particular if θ is a rigid phase-shift, then $\mathcal{C}(Z_0, J)$, $\mathcal{O}(Z_0, J)$ and $\Delta(Z_0, J)$ are properties of X_0 since the lifted solution (3.3) $Z_0(t) = (X_0(t), X_0(t + \theta T))$ is a function of X_0.

Definition 4.2. Let X_0 be a hyperbolic T-periodic solution and $J \subset \boldsymbol{R}$ be an open interval. Let $\theta \in \Theta(X_0)$ be a phase-shift and let Z_0 be the corresponding lifted solution (3.3). We say X_0 is θ-*nondegenerately rigid* on J if:

(a) $\mathcal{C}(Z_0, J)$, $\mathcal{O}(Z_0, J)$, and $\Delta(Z_0, J)$ are rigid, and
(b) for each pair i, j, either $z_i^0(t) = z_j^0(t)$ for all $t \in J$ or $z_i^0(J) \cap z_j^0(J) = \emptyset$. $\diamondsuit$

Before stating proposition 4.4, we discuss the network $2\mathcal{G}$. Formally, define $2\mathcal{G}$ to be the network consisting of two identical copies of the network $\mathcal{G}$. Assign the phase space of each copy $\mathcal{P}_\mathcal{G}$. So the whole phase space of $2\mathcal{G}$ is $\mathcal{P}_\mathcal{G} \times \mathcal{P}_\mathcal{G}$, which we denote by $\mathcal{P}_\mathcal{G}^2$. Let $X = (x_1, \ldots, x_n) \in \mathcal{P}_\mathcal{G}$ be the state variables of the cells in the first copy and let $Y = (y_1, \ldots, y_n) \in \mathcal{P}_\mathcal{G}$ be the state variables of the cells in the second copy, where x_i and y_i correspond to the same (doubled) cell for $i = 1, \ldots, n$. Then each admissible vector field of $2\mathcal{G}$ must be in the form of $(F(X), F(Y))$. It also follows that:

Lemma 4.3. *$\dot{X} = F(X)$ is an admissible system on $\mathcal{G}$ if and only if*

$$\begin{aligned} \dot{X} &= F(X) \\ \dot{Y} &= F(Y) \end{aligned} \tag{4.4}$$

is an admissible system on $2\mathcal{G}$.

In what follows, let $Z = (z_1, \ldots, z_{2n})$ denote the vector $(X, Y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ of state variables of $\mathcal{P}_\mathcal{G}^2$, and let $\mathcal{F} = (F, F)$ so that (4.4) becomes

$$\dot{Z} = \mathcal{F}(Z). \tag{4.5}$$

Now we state proposition 4.4.

Proposition 4.4. *Let X_0 be a hyperbolic T-periodic solution of (3.1) and let $J \subset \boldsymbol{R}$ be an open interval. Let θ be a rigid phase-shift and let $Z_0(t)$ be defined as in (3.3). Suppose X_0 is θ-nondegenerately rigid on J. Then the polydiagonal $\Delta(Z_0, J) \subset \mathcal{P}_\mathcal{G}^2$ is flow-invariant for all admissible vector fields of $2\mathcal{G}$ on $\mathcal{P}_\mathcal{G}^2$.*

The proof of proposition 4.4 is given in section 6. Next we show by lemma 4.5 that, generically, every hyperbolic periodic solution is θ-nondegenerately rigid; that is, for every hyperbolic periodic solution X_0, there exists an open interval J such that X_0 is θ-nondegenerately rigid on J.

Lemma 4.5. *Let X_0 be a hyperbolic periodic solution. Then for every $\theta \in [0, 1)$ there exists an open interval $J \subset \boldsymbol{R}$, such that X_0 can be perturbed by an arbitrarily small admissible perturbation to a perturbed periodic solution $\hat{X}$ that is θ-nondegenerately rigid on J.*

Proof. We must prove that there exists an open interval J and an arbitrarily small admissible perturbation such that on the perturbed periodic solution $\hat{X}$, the sets and properties in definition 4.2(a)–(b) are rigid. The rigidity of the sets in (a) follows from the proof of lemma 2.6 in [11]. By continuity we can shrink J if necessary such that condition (b) is valid on $\hat{X}$. Also note that small perturbations can only decrease the number of the pairs of equal cells. Since there are a finite number of cells, we need to only make a finite number of small perturbations to reach a state where the property (b) is rigid. Note that the sum of a finite number of small perturbations is again a small perturbation. It follows that after shrinking J, if necessary, there exists an admissible perturbation such that the perturbed periodic solution $\hat{X}$ is θ-nondegenerately rigid on J. □

Proof of theorem 3.1 By lemma 4.5, for a given phase-shift, we can assume that there exists an open interval J such that X_0 is θ-nondegenerately rigid on J. Note that because the number of nodes in the network is finite, there is only a finite number of phase-shifts. It follows that we can shrink J if necessary, such that X_0 is θ-nondegenerately rigid on J for every phase-shift θ.

Let $Z_0(t)$ be defined as in (3.3). By proposition 4.4, $\Delta(Z_0, J)$ is flow-invariant with respect to all admissible vector fields of $2\mathcal{G}$. That means $\Delta(Z_0) = \Delta(Z_0, \boldsymbol{R}) \subset \Delta(Z_0, J)$. On the other hand, note that $z_i(t)$ and $z_j(t)$ equal on $\boldsymbol{R}$ implies that they are equal on J. By the definition for the coordinates in $Z_0(t)$, more cells can be equal on J than are equal on $\boldsymbol{R}$. Therefore $\Delta(Z_0, J) \subset \Delta(Z_0, \boldsymbol{R})$. Hence $\Delta(Z_0) = \Delta(Z_0, J)$. This implies that $\Delta(Z_0)$ is flow-invariant for all admissible vector fields of $2\mathcal{G}$. It follows that its associated colouring is balanced. That is, for each pair of cells c and d synchronous on X_0, there exists an arrow-type preserving bijection β such that (3.2) is valid. □

5. Results needed for the proof of proposition 4.4

In this section, we prove lemmas 5.1–5.3 that are needed to prove proposition 4.4. Lemma 5.1 is the rigid input theorem [24, theorem 5.1] on a local interval.

Lemma 5.1. *Let X_0 be a hyperbolic T-period solution of (3.1) that is X_0 is θ-nondegenerately rigid on J. If c and d are cells that are synchronous for Z_0, then c and d are input equivalent.*

Proof. We argue by contradiction. Suppose cells c and d are not input equivalent. Let $\Phi = (\varphi_1, \ldots, \varphi_{2n})$ be a strongly admissible change of coordinates. Since cells c and d are not input equivalent, φ_c and φ_d are independent maps. For example, we can choose φ_d to be the identity and φ_c to be any diffeomorphism. Hence, we can choose a strongly admissible, near identity, change of coordinates Φ, such that $\varphi_c(z_c^0(t)) \neq z_d^0(t)$ for some $t \in J$. This contradicts the assumption that X_0 is θ-nondegenerately rigid on J. □

Next we define some terms and prove lemma 5.2, which is an analogue of lemma 5.3 in [11]. Let X_0 be a hyperbolic periodic solution of (3.1) that is θ-nondegenerately rigid on J.

Let $Z_0(t)$ defined as in (3.3). We can associate a colouring with $\Delta(Z_0, J)$ by assigning the same colour to the cells i and j in $2\mathcal{G}$ if $z_i^0(t) = z_j^0(t)$ for $t \in J$. We may thus identify a colour with the set L of all cells of that colour. We will call a colour L an $\mathcal{O}$-colour if the cells in L oscillate for Z_0 on J, and call L a $\mathcal{C}$-colour otherwise. An *$\mathcal{O}$-coloured sum* associated with f is a function on J of the form

$$\sum_{i\in L} D_{z_i} f(Z_0(t)) \tag{5.1}$$

where L is an $\mathcal{O}$-colour and D_{z_i} is the partial derivative about z_i.

If $P = (B, B)$ be an admissible vector field on $2\mathcal{G}$, let X_ε be the periodic solution to the perturbed system

$$\dot{X} = F(X) + \varepsilon B(X) \tag{5.2}$$

let Z_ε denote the lifted solution $(X_\varepsilon(t), X_\varepsilon(t+\theta T_\varepsilon))$ to the corresponding perturbed system

$$\dot{Z} = \mathcal{F}(Z) + \varepsilon P(Z), \tag{5.3}$$

and define

$$\beta(P) = \left.\frac{\mathrm{d}Z_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}.$$

Let

$$\mathsf{Q} = \{\Phi : \mathcal{P}_{\mathcal{G}}^2 \to \mathcal{P}_{\mathcal{G}}^2 \mid \Phi \text{ is a strongly admissible map of } 2\mathcal{G}\}.$$

Lemma 5.2. *Assume that the hyperbolic periodic solution X_0 to (3.1) is θ-nondegenerately rigid on J. Let $g : \mathcal{P}_{\mathcal{G}}^2 \to \boldsymbol{R}^k$ be a smooth function into the phase space of one of the cells of $2\mathcal{G}$. Suppose there exists an $\mathcal{O}$-colour L such that the corresponding $\mathcal{O}$-coloured sum associated with g is nonzero. Then the set*

$$\left\{D_Z g(Z_0(t))\beta^t(\Phi)(t) : \Phi \in \mathsf{Q}, t \in J\right\} \tag{5.4}$$

spans an infinite-dimensional function subspace, where β^t is the transpose of β.

Proof. Let $\beta(\Phi)$ be as defined above. For convenience, we sometimes drop Φ in the expression of β. In order to characterize the set (5.4), we calculate β explicitly.

Note that the perturbed periodic solution Z_ε satisfies

$$\dot{Z}_\varepsilon = \mathcal{F}(Z_\varepsilon) + \varepsilon\Phi(Z_\varepsilon). \tag{5.5}$$

On differentiating both sides of (5.5) about ε and evaluating at $\varepsilon = 0$, we have

$$\dot{\beta} = D_Z\mathcal{F}(Z_0)\beta + \Phi(Z_0). \tag{5.6}$$

Let $V(t)$ be a fundamental solution to the homogeneous system

$$\dot{\beta} = D_Z\mathcal{F}(Z_0)\beta \tag{5.7}$$

with $V(t_0) = I$, where $t_0 \in J$ and I is the identity matrix. Then the general solution to (5.6) has the form of

$$\begin{aligned}\beta(t) &= V(t)\left(\int_{t_0}^{t} V^{-1}(s)\Phi(Z_0(s))\,\mathrm{d}s + K\right)\\ &= V(t)\int_{t_0}^{t} V^{-1}(s)\Phi\big(Z_0(s)\big)\,\mathrm{d}s + V(t)K \end{aligned}\tag{5.8}$$

where $K = \beta(t_0)$.

When t is sufficiently near t_0, $V(t)$ can be approximated by I, and written as

$$V(t) = I + O(\tau),$$

where $\tau \ll 1$. Hence, when t is sufficiently near t_0, β can be written as

$$\beta(\Phi, t) = \int_{t_0}^{t} \Phi\big(X_0(s)\big)\,\mathrm{d}s + K + O(\tau).$$

Note that K is a constant vector. We will have proved the lemma if

$$D_Z g(Z_0(t)) \int_{t_0}^{t} \Phi\big(Z_0(s)\big)\,\mathrm{d}s$$

generates an infinite-dimensional function space when Φ varies in Q.

By the condition in this lemma, there exists an $\mathcal{O}$-colour L such that the corresponding $\mathcal{O}$-coloured sum associated with g is nonzero. That is,

$$\sum_{i\in L} D_{z_i} g(Z_0(t)) \neq 0. \tag{5.9}$$

Let

$$\mathsf{Q}_L = \{\Phi \in \mathsf{Q} : \varphi_i(z_i^0(t)) = 0 \qquad \text{for } i \notin L \text{ and } t \in J\}.$$

Since X_0 is θ-nondegenerately rigid on J, choose $\Phi \in \mathsf{Q}_L$, then for $t \in J$

$$D_Z g(Z_0(t)) \int_{t_0}^{t} \Phi\big(Z_0(s)\big)\,\mathrm{d}s = \sum_{i\in L} D_{z_i} g(Z_0(t)) \int_{t_0}^{t} \varphi^L(z^L(s))\,\mathrm{d}s \tag{5.10}$$

where $\varphi^L = \varphi_i$ for $i \in L$ and $z^L(s)$ is the common value of $z_i^0(s)$ for $i \in L$ and $s \in J$. Note that $z^L(t)$ is time-varying on J since L is an $\mathcal{O}$-colour. Then it follows from (5.9) that the set

$$\left\{ \sum_{l\in L} D_{z_l} g(Z_0(t)) \int_{t_0}^{t} \varphi^L(z^L(s))\,\mathrm{d}s : \Phi \in \mathsf{Q}_L \right\}$$

spans an infinite-dimensional function space since Q_L spans an infinite-dimensional function space on J. □

Lemma 5.3. *Let X_0 be a hyperbolic T-period solution to (3.1) that is θ-nondegenerately rigid on J, and let Z_0 be defined as (3.3). Suppose cells i and j of $2\mathcal{G}$ are in the same colour class of the colouring of $\Delta(Z_0, J)$. Let f_i and f_j be the ith and jth of $\mathcal{F}$ components, respectively. Let*

$$g(Z) = f_i(Z) - f_j(Z).$$

Then for each $\mathcal{O}$-colour L

$$\sum_{l\in L} D_{z_l} g(Z_0) = 0. \tag{5.11}$$

Proof. First we show that $g(Z) = 0$ on the periodic solution to

$$\dot{Z} = \mathcal{F}(Z) + \varepsilon\Phi(Z) \tag{5.12}$$

where $\Phi = (\varphi_1, \ldots, \varphi_{2n}) \in \mathsf{Q}$ and $\mathcal{F} = (f_1, \ldots, f_{2n})$.

Since cells i and j in $2\mathcal{G}$ are in the same colour class, $z_i^0(t) = z_j^0(t)$ for $t \in J$. By lemma 5.1, cells i and j in $\mathcal{G}$ are input equivalent. Hence, $\varphi_i = \varphi_j$ and $f_i = f_j$. Since X_0 is θ-nondegenerately rigid on J, we have

$$z_i^{\varepsilon}(t) = z_j^{\varepsilon}(t) \qquad \text{for } t \in J,$$

for all small ε. Therefore,

$$0 = \dot{z}_i^\varepsilon(t) - \dot{z}_j^\varepsilon(t) = f_i(Z_\varepsilon(t)) - f_j(Z_\varepsilon(t)) = g(Z_\varepsilon(t)). \tag{5.13}$$

On differentiating both sides of (5.13) about ε and evaluating at $\varepsilon = 0$, we have

$$D_Z g(Z_0(t))\beta^t(\Phi)(t) = 0. \tag{5.14}$$

By lemma 5.2 and (5.14), all $\mathcal{O}$-colour sum of g must be zero. Hence (5.11) is valid. □

6. The proof of proposition 4.4

Proposition 4.4 asserts the flow-invariance of $\Delta(Z_0, J) \subset \mathcal{P}_\mathcal{G}^2$ with respect to all admissible vector fields on $2\mathcal{G}$. We will prove this by contradiction, but first we establish some results that limit the way in which $\Delta(Z_0, J)$ can fail to be flow-invariant. In particular, in corollary 6.2 we show that a necessary condition for this to happen is the existence of a pair of cells c and d of $2\mathcal{G}$, an admissible vector field P on $2\mathcal{G}$, and a point $Q_0 \in \Delta(Z_0, J)$ such that

(a) c and d are of the same colour for Z_0 on J,
(b) c and d are oscillating cells for Z_0 on J and
(c) $P(Q_0)_c \neq P(Q_0)_d$.

We then prove proposition 4.4 by showing that these conditions imply that there exist arbitrarily small perturbations of X_0 such the cells c and d are no longer of the same colour for the corresponding lifted solution. This contradicts the hypothesis that X_0 is θ-nondegenerately rigid on J, and thus completes the proof of proposition 4.4.

Lemma 6.1. *Suppose that cells i and j of $2\mathcal{G}$ are of the same colour for $Z_0(t)$ on J, and that there exists an arrow-type preserving bijection $\beta : \mathcal{I}(i) \to \mathcal{I}(j)$ such that for each edge $e \in \mathcal{I}(c)$ the cells $\mathcal{T}(e)$ and $\mathcal{T}(\beta(e))$ are of the same colour. Then for each $Q \in \Delta(Z_0, J) \subset \mathcal{P}_\mathcal{G}^2$ and each admissible vector field P on $2\mathcal{G}$, we have $P(Q)_i = P(Q)_j$.*

Proof. The argument is identical to the argument in the proof of theorem 4.1 of [16] that the polydiagonal corresponding to a balanced equivalence relation is flow-invariant, but we reproduce it here for the convenience of the reader.

Consider $Q = (q_1, \ldots, q_{2n}) \in \Delta(Z_0, J)$ and let $P = (p_1, \ldots, p_{2n})$ be an admissible vector field on $2\mathcal{G}$. Since i and j are of the same colour and $Q \in \Delta(Z_0, J)$, we have that $q_i = q_j$. Now let $\beta : \mathcal{I}(i) \to \mathcal{I}(j)$ be an arrow-type preserving bijection such that for each edge $e \in \mathcal{I}(c)$ the cells $\mathcal{T}(e)$ and $\mathcal{T}(\beta(e))$ are of the same colour. Then for each edge $e \in \mathcal{I}(c)$ we have $q_{\mathcal{T}(e)} = q_{\mathcal{T}(\beta(e))}$, so that $\beta^*(q_{\mathcal{T}(\mathcal{I}(j))}) = q_{\mathcal{T}(\mathcal{I}(i))}$. Finally, the existence of β implies that c and d are input equivalent, so that

$$p_i\left(q_j, \beta^*\left(q_{\mathcal{T}(\mathcal{I}(j))}\right)\right) = p_j(q_j, q_{\mathcal{T}(\mathcal{I}(j))}),$$

and thus

$$P(Q)_i = p_i(q_i, q_{\mathcal{T}(\mathcal{I}(i))}) = p_i\left(q_j, \beta^*\left(q_{\mathcal{T}(\mathcal{I}(j))}\right)\right) = p_j(q_j, q_{\mathcal{T}(\mathcal{I}(j))}) = P(Q)_j. \qquad \square$$

Corollary 6.2. *Suppose that cells i and j are of the same colour on Z_0, and let Q be an element of $\Delta(Z_0, J) \subset \mathcal{P}_\mathcal{G}^2$. If i and j are both constant cells of Z_0, then $P(Q)_i = P(Q)_j$ for every admissible vector field P on $2\mathcal{G}$.*

Proof. Consider $Q \in \Delta(Z_0, J)$ and let P be an admissible vector field on $2\mathcal{G}$.

(a) Suppose i and j are both from the same copy of $\mathcal{G}$. Then the existence of an arrow-type preserving bijection as required by lemma 6.1 is a consequence of the fact that X_0 is θ-nondegenerately rigid on J. This implies that $\Delta(X_0, J)$ and $\Delta(Y_0, J)$ are both rigid, so that by theorem 6.1 of [11], the associated colourings are balanced.

(b) Now suppose that i and j are constant cells of Z_0. From case (a), we may assume that i is from the first copy of $\mathcal{G}$ and j is from the second copy; let us further assume that j is actually the copy of i and treat the remaining case below. Since i is constant and X_0 is nondegenerately rigid on J, it follows from theorem 2.1 of [11] that each cell in $\mathcal{T}(\mathcal{I}(i))$ must be constant as well, so that

$$y_k^0(t) = x_k^0(t + \theta T) = x_k^0(t)$$

for each cell $k \in \mathcal{T}(\mathcal{I}(i))$. Thus if $\beta : \mathcal{I}(i) \to \mathcal{I}(j)$ is the map that assigns to each edge $e \in \mathcal{I}(i)$ its copy in $\mathcal{I}(j)$, then β naturally satisfies the conditions of lemma 6.1.

Finally, suppose that i and j are constant cells of Z_0, where i is from the first copy of $\mathcal{G}$ and j is from the second, and let i' be the copy of cell i. Since i is constant, we have

$$y_i^0(t) = x_i^0(t + \theta T) = x_i^0(t),$$

so that i and i' are of the same colour, and thus $P(Q)_i = P(Q)_{i'}$ by the argument of the previous paragraph. On the other hand, i' and j are cells of the same colour from the same copy of $\mathcal{G}$, so that $P(Q)_{i'} = P(Q)_j$ by case (a). □

We can now begin our proof of proposition 4.4.

Proof of proposition 4.4. Our proof is almost identical to that of [11, theorem 6.1].

Suppose $\Delta(Z_0, J)$ is not flow-invariant. Then there exist a point $Q_0 \in \Delta(Z_0, J)$ and an admissible map P such that $P(Q_0) \notin \Delta(Z_0, J)$. In particular, there exist cells c and d of the same colour such that

$$P(Q_0)_c \neq P(Q_0)_d, \tag{6.1}$$

and, from corollary 6.2, it follows that c and d must oscillate on Z_0.

Note that since X_0 is θ-nondegenerately rigid, cells c and d are rigidly of the same colour. We will show, however, that (6.1) and the oscillation of c and d imply that there exist arbitrarily small admissible perturbations of X_0 such that cells c and d are not of the same colour for the corresponding lifted periodic solutions. We now construct such a family of admissible perturbations.

Since X_0 is θ-nondegenerately rigid on J, each point $Z_0(t)$ for $t \in J$ is a generic point of $\Delta(Z_0, J)$. Thus, if we fix $s \in J$, by lemma 7.5 in [16] there exists a strongly admissible map Φ of $2\mathcal{G}$ such that

$$Q_0 = \Phi\left(Z_0(s)\right). \tag{6.2}$$

From (6.1) and (6.2), we have

$$\left(P\Phi\left(Z_0(s)\right)\right)_c \neq \left(P\Phi\left(Z_0(s)\right)\right)_d, \tag{6.3}$$

and thus by lemma 6.4 in [11], we can pick a strongly admissible Ψ such that either

$$\frac{\mathrm{d}}{\mathrm{d}t}(P\Phi Z_0(t))_c\bigg|_{t=s} \neq 0 \tag{6.4}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(P\Phi Z_0(t) + \Phi Z_0(t))_c\bigg|_{t=s} \neq 0. \tag{6.5}$$

Suppose Φ satisfies (6.4), and consider the system

$$\dot{Z} = \mathcal{F}(Z) + \varepsilon \Phi P \Psi(Z) \tag{6.6}$$

obtained by perturbing (4.5), where Ψ is an arbitrary strongly admissible map. Let $Z_\varepsilon = (z_1^\varepsilon, \ldots, z_{2n}^\varepsilon)$ be the perturbed periodic solution, and let f_c and f_d be the components of F corresponding to cells c and d, respectively. Then z_c^ε and z_d^ε satisfy

$$\begin{aligned} \dot{z}_c^\varepsilon &= f_c(Z_\varepsilon) + \varepsilon\big((\Psi P \Phi)_c(Z_\varepsilon)\big) \\ \dot{z}_d^\varepsilon &= f_d(Z_\varepsilon) + \varepsilon\big((\Psi P \Phi)_d(Z_\varepsilon)\big). \end{aligned} \tag{6.7}$$

Letting $g = f_c - f_d$ and $u = (\Psi P \Phi)_c - (\Psi P \Phi)_d$, it follows that

$$0 = \dot{z}_c^\varepsilon - \dot{z}_d^\varepsilon = f(Z_\varepsilon) + \varepsilon u(Z_\varepsilon). \tag{6.8}$$

Now, if we define

$$\beta_i(t) = \left.\frac{\partial z_i^\varepsilon(t)}{\partial \varepsilon}\right|_{\varepsilon=0},$$

then on differentiating (6.8) with respect to ε and evaluating at $\varepsilon = 0$, we obtain

$$0 = \sum_{\text{colours } L} \sum_{i \in L} f_{z_i}(Z_0(t))\beta_i(t) + u(Z_0(t)) \tag{6.9}$$

$$= \sum_{\text{colours } L} \Big(\sum_{i \in L} f_{z_i}\big(Z_0(t)\big)\Big)\beta^L(t) + u(Z_0(t)), \tag{6.10}$$

where β^L denotes the common value of β_i for $i \in L$. By lemma 5.3, all the $\mathcal{O}$-coloured sums associated with f must be zero, so that (6.10) becomes

$$0 = \sum_{\mathcal{C}\text{-colours } L} \Big(\sum_{i \in L} f_{z_i}\big(Z_0(t)\big)\Big)\beta^L(t) + u(Z_0(t)). \tag{6.11}$$

Note that for any $\mathcal{C}$-colour L, the function $\beta^L(t)$ is constant, so that as Ψ varies, the function

$$\sum_{\mathcal{C}\text{-colours } L} \Big(\sum_{i \in L} f_{z_i}\big(Z_0(t)\big)\Big)\beta^L(t)$$

is constrained to lie in a finite-dimensional function space. However, recalling that $u = (\Psi P \Phi)_c - (\Psi P \Phi)_d$, we claim that having fixed P and Φ,

$$\mathcal{B} = \{(\Psi P \Phi)_c(Z_0(t)) - (\Psi P \Phi)_d(Z_0(t)) : \Psi \text{ is strongly admissible}, t \in J\}$$

contains an infinite-dimensional function space on J. Recall that

$$P\Phi(Z_0(s))_c \neq P\Phi(Z_0(s))_d.$$

By continuity, there exists an open neighbourhood $J_s \subset J$ of s, such that

$$P\Phi(Z_0(J_s))_c \cap P\Phi(Z_0(J_s))_d = \emptyset.$$

Note that (6.4) implies $P\Phi(Z_0(t))_c$ is time-varying on J_s and Ψ can be any strongly admissible map. It follows that

$$\mathcal{B}_c = \{(\Psi P \Phi)_c(Z_0(t)) : \psi_c((P\Phi(Z_0(J_s)))_d) = 0, t \in J.\}$$

contains an infinite-dimensional function space on J. Since $\mathcal{B}_c \subset \mathcal{B}$, we can always find strongly admissible maps Ψ such that (6.11) is invalid.

Suppose Φ satisfies (6.5). Then we consider the perturbed system

$$\dot{Z} = F(Z) + \varepsilon \Psi(P\Phi(Z) + \Phi(Z)). \tag{6.12}$$

The rest of the argument follows exactly as the previous case. □

7. Nontransitive networks

Theorem 1.4 and its proof require that the network $\mathcal{G}$ is transitive. In this section we discuss the extension of this theorem to nontransitive or feed-forward networks. In particular, we prove theorem 1.9. As before, suppose X_0 has rigid phase-shifts. By the *rigid synchrony property*, $\Delta(X_0)$ defined in (1.3) is flow-invariant. Let $\mathcal{Q}$ be the quotient network corresponding to $\Delta(X_0)$. We can therefore view $X_0(t)$ as a hyperbolic periodic solution $Y_0(t)$ in the quotient network $\mathcal{Q}$. By construction, no two nodes of Y_0 are synchronous. Throughout this section we assume that no two nodes of Y_0 are synchronous.

Rigid phase-shifts form a cyclic group

We begin by proving that the set of rigid phase-shifts of Y_0 forms a cyclic subgroup of S^1 and these phase-shifts occur in the hub $\mathcal{J}_{\max}(Y_0)$. Specifically we prove the following theorem.

Theorem 7.1. *Let Y_0 be a hyperbolic hub-like periodic solution in the quotient network $\mathcal{Q}$ and suppose two nodes have rigid phase-shift synchrony. Then the nodes are either in the same transitive component or in isomorphic transitive components. Moreover, all rigid phase-shift synchronies between nodes of Y_0 are generated by a single cyclic symmetry that acts on the transitive component $\mathcal{J}_{\max}(Y_0)$.*

Theorem 7.1 follows from lemma 7.3, lemma 7.4, and proposition 7.9.

Lemma 7.2. *Let Y_0 be a hyperbolic hub-like T-periodic solution. Suppose cells c, d in $\mathcal{Q}$ are rigidly phase-related in Y_0 with phase-shift θ. Suppose that there is a directed path from node a to node c. Then there is a node b and a directed path from b to d such that a, b are also phase-shfted by θ.*

Proof. Let $a = a_0 \to \cdots \to a_k = c$ be a directed path from a to c. By the *rigid phase property*, there exists a node b_{k-1} that goes to $b_k = d$ that is phase-shifted by θ to a_k. Inductively we create a directed path from $b = b_0$ to d such that $\sigma_\theta(a_j) = b_j$ for each j. So $\sigma_\theta(a) = b$. $\square$

Lemma 7.3. *Let Y_0 be a hyperbolic hub-like T-periodic solution. Suppose cells c, d in $\mathcal{Q}$ are rigidly phase-related in Y_0. Then there exists a pair of rigidly phase-related nodes q, p in $\mathcal{J}_{\max}(Y_0)$ with the same phase-shift.*

Proof. Let cells c, d be rigidly phase-related by phase-shift θ; that is $y_d(t) = y_c(t + \theta T)$. Theorem 1.7 implies that there is a directed path from a cell $a \in \mathcal{J}_{\max}(Y_0)$ to c. Hence, by lemma 7.2, there is a directed path from a node $b \in \mathcal{J}(d)$ to d such that at every step along the two paths the nodes are rigidly phase-shifted by θ. Next, there is also a directed path from node $p \in \mathcal{J}_{\max}(Y_0)$ to b and a corresponding path from node q to a such that at each step on the two paths the cells are oscillating and rigidly phase-shifted by θ. It also follows from theorem 1.7 that each node in the directed path from q to a is in $\mathcal{J}_{\max}(Y_0)$, because all transitive components above $\mathcal{J}_{\max}(Y_0)$ are constant. $\square$

Lemma 7.4. *Let Y_0 be a hyperbolic T-periodic solution of an admissible system of network $\mathcal{Q}$. Suppose that two nodes c, d are rigidly phase-related to the phase-shift $\theta \neq 0$. Then the input networks of c and d are isomorphic; specifically, σ_θ restricts to an isomorphism of $H(c) \to H(d)$ where $\sigma_\theta(c) = d$. If $H(c) = H(d)$, then σ_θ restricts to a symmetry of the network $H(c)$.*

The proof of lemma 7.4 is a slight adaptation of the proof in Stewart and Parker [25, lemma 6.1], who assumed 'fully oscillatory.'

Proof. Network symmetries permute nodes and arrows in a consistent way. We use the *rigid phase property* to define the symmetry associated with θ first on nodes and then on arrows.

Definition of σ_θ on nodes in $H(c)$. We need to prove that for every node $a \in H(c)$ there exists a node $b \in H(d)$ such that $\sigma_\theta(b) = a$, that is,

$$y_b(t) = y_a(t + \theta T). \tag{7.1}$$

Since $a \in H(c)$ there is a directed path that connects a to c. If y_a is constant then by definition 1.8 $\sigma_\theta(a) = a$ and $b = a$. Suppose that $y_a(t)$ is nonconstant, then by lemma 7.2 there exists a node $b \in H(d)$ that connects to d such that (7.1) is valid.

Next we show that σ_θ gives a bijection between $H(c)$ and $H(d)$. It follows from (7.1) (applied to $-\theta$) that

$$y_c(t) = y_d(t - \theta T).$$

Hence the argument above shows that the map $\sigma_{-\theta}$ is defined on $H(d)$ and that $\sigma_\theta^{-1} = \sigma_{-\theta}$ on $H(d)$. Hence $\sigma_\theta : H(c) \to H(d)$ is a bijection.

Definition of σ_θ on arrows. Suppose cells a, b are rigidly phase-related to phase-shift θ. By the *rigid phase property* (theorem 3.1), there exists an isomorphism between the input sets of a and b, such that the corresponding tail cells are rigidly phase-related to phase-shift θ. Define σ_θ on the arrows of the input set of a to be this isormophism between the input sets. Since a is arbitrary in $H(c)$, σ_θ is defined for all arrows on $H(c)$ and is a well defined permutation of arrows.

Hence σ_θ restricts to a bijection between $H(c)$ and $H(d)$ that preserves the network structure, and if $H(c) = H(d)$, this bijection is a permutation symmetry of the subnetwork $H(c)$. □

Remark 7.5. Let Y_0 be a hyperbolic periodic solution of the network $\mathcal{Q}$ that has rigid phase-shifts. Suppose two nodes c, d are rigidly phase-related to θ on Y_0 and they are in disjoint transitive components. Let $\mathcal{J}_c$ and $\mathcal{J}_d$ be the transitive components that contain c and d, respectively. Then it follows from lemma 7.4 that $\mathcal{J}_c$ and $\mathcal{J}_d$ have the same network structure.

Lemma 7.6. *Let $\mathcal{J}$ be a transitive component. Suppose that Y_0 is a hyperbolic T-periodic solution of an admissible system of the network $\mathcal{Q}$ that has rigid phase-shifts on $\mathcal{J}$. Let $\Theta \subset S^1$ be the set of all rigid phase-shifts in Y_0 on $\mathcal{H}$. Then Θ is a cyclic subgroup of S^1 and there is a cyclic symmetry group Γ acting on $H(\mathcal{J})$ that generates these phase-shifts.*

Proof. We claim that Θ is a finite subgroup of S^1 and hence cyclic. Suppose $\theta \in \Theta$. Lemma 7.4 states that there exists a permutation symmetry σ_θ acting on $\mathcal{J}(\mathcal{H})$ such that $\sigma_\theta(c) = d$ if and only if node d is phase-shifted from node c by θ. Suppose $\theta_1, \theta_2 \in \Theta$. If $\sigma_{\theta_1}(c) = d$ and $\sigma_{\theta_2}(d) = e$ that node e is phase-shifted by $\theta_1 + \theta_2$ from c. Moreover, if θ is a phase-shift, then so is $-\theta$. Therefore, Θ is a subgroup. Because the number of nodes is finite, the number of phase-shifts must also be finite. Hence Θ is finite and cyclic. Our discussion also shows that the map $Z(\theta) = \sigma_\theta$ is a group homomorphism from rigid phase-shifts to permutations symmetries on $\mathcal{H}(c)$. Hence, $\Gamma = Z(\Theta)$ is a cyclic group. □

We note that the groups Γ and Θ need not be isomorphic. Consider the feed-forward network in figure 4 consisting of a unidirectional ring of three cells $\mathcal{J}_{\max}$ connected all-to-all

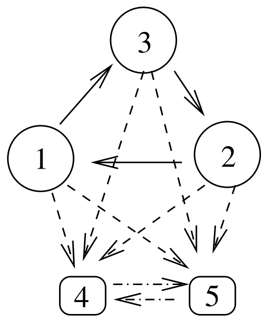

Figure 4. A feed-forward network consisting of a unidirectional three-node network forcing a two-node bidirectional ring.

to a bidirectional ring of two cells. This network can exhibit a 1-periodic solution Y_0 of the form $(u(t), u(t+\frac{1}{6}), u(t+\frac{2}{6}), v(t), v(t+\frac{1}{6}))$, where $u(t)$ is $\frac{1}{2}$-periodic and $v(t)$ is $\frac{1}{3}$-periodic. The group Γ acting on $\mathcal{J}_{\max}$ is $\boldsymbol{Z}_3$ since $\sigma_{\frac{1}{2}}$ acts as the identity on the three-cell ring $\mathcal{J}_{\max}(Y_0)$. Of course, the group of permutation symmetries on the whole network is $\boldsymbol{Z}_3 \times \boldsymbol{Z}_2 = \boldsymbol{Z}_6$.

Corollary 7.7. *If the hyperbolic periodic solution Y_0 has a rigid phase-shift, then that phase-shift must be a rational number.*

Remark 7.8. There is a subtlety in lemma 7.6 due to the fact that the minimal period of different nodes may be different. Let Y_0 be a hyperbolic periodic solution of minimal period T and let $T_{\mathcal{J}}$ be the minimal period of Y_0 restricted to the nodes in $\mathcal{J}_{\max}$. Note that T can be larger than $T_{\mathcal{J}}$; indeed, T must be a multiple of $T_{\mathcal{J}}$. So set $T_{\mathcal{J}} = T/n$. We claim that every rigid phase-shift down the network can have a corresponding phase-shift in $\mathcal{J}_{\max}(Y_0)$. Let c, d be phase-related to phase-shift θ. By corollary 7.7, θ must be a rational number. Let $\theta = l/m$, where $0 \leqslant l \leqslant m$ and $m > 0$. We have

$$y_d(t) = y_c\left(t + \frac{l}{m}T\right).$$

By lemma 7.3, there exist cells c' and d' in $\mathcal{J}_{\max}(Y_0)$ that are phase-related by θ. So we have

$$y_{d'}(t) = y_{c'}\left(t + \frac{l}{m}T\right).$$

Rewrite the above equality as

$$y_{d'}(t) = y_{c'}\left(t + \frac{ln}{m}\frac{T}{n}\right) = y_{c'}\left(t + \frac{k}{m}T_{\mathcal{J}}\right)$$

where $k = ln \bmod m$. So phase relations down the network correspond to the same phase-shifts in $\mathcal{J}_{\max}(Y_0)$, but because of differing periods of the different nodes may look different.

Proposition 7.9. *Let Y_0 be a hyperbolic T-periodic hub-like solution of the network $\mathcal{Q}$. If two nodes c and d are rigidly phase-related on Y_0 and there is a directed path from c to d, then c and d must belong to the same transitive component.*

Proof. Without loss of generality, assume that there exists an arrow from cell c to d; that is, $c \to d$. We must show that there exists a directed path from $d \to c$. Let the phase-shift be θ so that $y_c(t) = y_d(t + \theta T)$. By the *rigid phase property* applied to $c \to d$, there exists a cell $a_1 \to c$ such that

$$y_{a_1}(t) = y_c(t + \theta T) = y_d(t + 2\theta T).$$

Next use the *rigid phase property* applied to $a_1 \to c$ to find a node $a_2 \to a_1$ such that

$$y_{a_2}(t) = y_{a_1}(t + \theta T) = y_d(t + 3\theta T).$$

By induction we can find nodes $a_m \to a_{m-1}$ such that

$$y_{a_m}(t) = y_d(t + (m+1)\theta T).$$

Since θ is a rational number (corollary 7.7), there must exist a cell a_m such that $y_d(t) = y_{a_m}(t)$. It follows that $d = a_m$ since there are no synchronous cells in $\mathcal{Q}$. Hence, there is a directed path from $d = a_m$ to c, and c and d belong to the same transitive component. □

Remark 7.10. We can be more explicit about the generator of the cyclic symmetries of $\mathcal{J}_{\max}(Y_0)$ whose existence is asserted in theorem 7.1. Applying lemma 7.6 to $\mathcal{J}_{\max}(Y_0)$, let $\Theta_{\max}$ be the cyclic group of all phase-shifts of Y_0 on nodes in $\mathcal{J}_{\max}(Y_0)$. Lemma 7.3 shows that $\Theta_{\max}$ contains every rigid phase-shift between any pair of nodes of Y_0. Let $\theta_{\min}$ be the minimum positive phase-shift in $\Theta_{\max}$. Then $\theta_{\min}$ is a generator of $\Theta_{\max}$. Moreover, let $\Gamma_{\max}$ be the cyclic permutation group acting on $\mathcal{J}_{\max}(Y_0)$ corresponding to $\Theta_{\max}$ whose existence is asserted in lemma 7.3. Then the permutation $\sigma_{\min} = \sigma_{\theta_{\min}}$ is a generator of $\Gamma_{\max}$ and is the symmetry defined in (1.4) for transitive networks.

It follows from lemma 7.6 that $\sigma_{\min}$ acts on the transitive component $\mathcal{J}_{\max}(Y_0)$. Indeed, since Y_0 is constant on all transitive components of $\mathcal{Q}$ not in $L(\mathcal{J}_{\max}(Y_0))$, it follows that $\sigma_{\min}$ acts as the identity on these other transitive components.

In general, however, $\sigma_{\min}$ is not necessarily defined on components lower than $\mathcal{J}_{\max}(Y_0)$ in $\mathcal{Q}$. Consider the network in figure 5. This five-node network has T-periodic solutions of the form

$$Y_0(t) = \left(u(t), u\left(t + \tfrac{1}{3}T\right), u\left(t + \tfrac{2}{3}T\right), v(t), v\left(t + \tfrac{1}{3}T\right)\right)$$

where $u(t)$ and $v(t)$ are T-periodic. Moreover, the phase-shift from node 4 to node 5 is rigid. Clearly, the network $\mathcal{Q}$ has no symmetry (although the three-cell unidirectional ring $\mathcal{J}_{\max}$ does have $\Gamma_{\max} = \mathbf{Z}_3$ symmetry). Moreover, technically there is no symmetry that generates the phase-shift between nodes 4 and 5. Clearly, if we add a phantom node, then we can make the network $\mathbf{Z}_3$ symmetric and generate the 'surprising' phase-shift. Indeed, we show in the next subsection that $\mathcal{Q}$ can always be extended to a network $\hat{\mathcal{Q}}$ that has a cyclic symmetry $\sigma_{\min}$ that generates all rigid phase-shifts in $\mathcal{Q}$.

Finally, note that the subnetwork $H(\mathcal{J}_{\max}(Y_0))$ can have more symmetry than $\Gamma_{\max}$. For example, just consider any transitive network (such as a three-cell bidirectional ring) that has noncyclic symmetries. The symmetries in $\Gamma_{\max}$ are just those symmetries of $H(\mathcal{J}_{\max}(Y_0))$ that correspond to the rigid phase-shifts in Y_0.

Network completions

We begin with the definition of a network extension that was discussed in the introduction.

Definition 7.11. A network $\hat{\mathcal{Q}}$ is an *extension* of the network $\mathcal{Q}$ if it satisfies the following.

(a) Each transitive component of $\mathcal{Q}$ is a transitive component of $\hat{\mathcal{Q}}$.
(b) $\hat{\mathcal{Q}}$ is connected and there are no arrows from $\hat{\mathcal{Q}} \backslash \mathcal{Q}$ to $\mathcal{Q}$.
(c) Every node in $\hat{\mathcal{Q}}$ is input equivalent to a node in $\mathcal{Q}$.

Lemma 7.12. *Let $\hat{\mathcal{Q}}$ be an extension of $\mathcal{Q}$. Then every admissible vector field of $\mathcal{Q}$ can be extended uniquely to an admissible vector field of $\hat{\mathcal{Q}}$.*

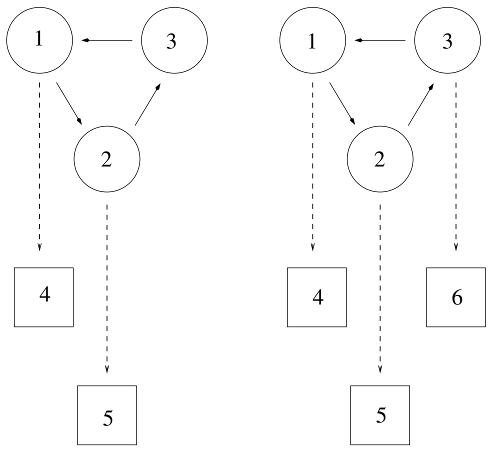

Figure 5. (Left) A three-cell unidirectional ring forcing two hanging nodes; (right) the completion with $\boldsymbol{Z}_3$ symmetry.

Proof. Since every node in $\hat{\mathcal{Q}}$ is input equivalent to a cell in $\mathcal{Q}$, the equation of each cell in $\hat{\mathcal{Q}} \setminus \mathcal{Q}$ must be the same as that of a cell in $\mathcal{Q}$ (up to substitution of coordinates specified by the input isomorphism). It follows that every admissible vector field of $\mathcal{Q}$ can be extended uniquely to an admissible vector field of $\hat{\mathcal{Q}}$. $\square$

Definition 7.13. Let $\dot{Y} = F(Y)$ be an admissible system on the network $\mathcal{Q}$. The pair $(\hat{F}, \hat{\mathcal{Q}})$ is an *extension* of $(F, \mathcal{Q})$ if $\hat{\mathcal{Q}}$ is an extension of $\mathcal{Q}$ and $\hat{F}$ is the unique extension of F guaranteed by lemma 7.12.

Let Y_0 be a hyperbolic T-periodic solution of an admissible system on the network $\mathcal{Q}$. We continue to assume that Y_0 has rigid phase-shifts and has no synchronous cells. Let $\sigma_{\min}$ be the permutation symmetry of $\mathcal{J}_{\max}(Y_0)$ defined in remark 7.10. Recall from definition 1.8 that $\sigma_{\min} c = d$ if

$$y_d(t) = y_c(t + \theta_{\min} T).$$

Also recall that $\sigma_{\min}$ is defined on the nodes in $\overline{H}(\mathcal{J}_{\max}(Y_0))$, but $\sigma_{\min}$ need not be defined on all of $\mathcal{Q}$.

Definition 7.14. A subnetwork $\mathcal{U} \subset \mathcal{Q}$ is Y_0*-complete* if $\sigma_{\min}$ is defined on every cell in $\mathcal{U}$.

Note that $\overline{H}(\mathcal{J}_{\max}(Y_0))$ is always a nonempty subnetwork that is Y_0-complete.

Lemma 7.15. *Let Y_0 be a hyperbolic hub-like periodic solution of network $\mathcal{Q}$. Suppose Y_0 has rigid phase-shifts but no synchronous cells. Suppose $\sigma_{\min}$ is defined for cell c. Then $\sigma_{\min}$ is defined for each cell in $H(c)$.*

Proof. By assumption there exists a cell d such that cells c, d are rigid phase-related by $\theta_{\min}$. The *rigid phase property* implies that the tail cells of their inputs are also phase-related by $\theta_{\min}$. Hence, $\sigma_{\min}$ is defined on each cell in $H(c)$. $\square$

Proposition 7.16. *Let Y_0 be a hyperbolic hub-like periodic solution of network $\mathcal{Q}$. Suppose Y_0 has rigid phase-shifts and has no synchronous cells. Then the largest Y_0-complete subnetwork is connected and contains $\overline{H}(\mathcal{J}_{\max}(Y_0))$.*

Proof. Proposition 7.16 follows from theorem 1.7 and lemma 7.15. □

Lemma 7.17. *Let Y_0 be a hyperbolic hub-like periodic solution of network $\mathcal{Q}$. Suppose Y_0 has rigid phase-shifts and has no synchronous cells. Suppose $\mathcal{Q}$ is not Y_0-complete and let $\mathcal{U}$ be the largest Y_0-complete subnetwork. Then there exists a transitive component that is not in $\mathcal{U}$ that receives external inputs only from cells in the largest Y_0-complete subnetwork $\mathcal{U}$.*

Proof. Because $\mathcal{Q}$ is not Y_0-complete, there exists a transitive component $\mathcal{M}_1$ that is not in $\mathcal{U}$. If all external inputs to cells in $\mathcal{M}_1$ are from cells in $\mathcal{U}$, then the lemma is proved.

If not, there exists a transitive component $\mathcal{M}_2 \not\subset \mathcal{U}$ such that $\mathcal{M}_2 \to \mathcal{M}_1$. Note that $\mathcal{M}_1 \to \mathcal{M}_2$ is not possible since $\mathcal{M}_1$ and $\mathcal{M}_2$ are distinct transitive components. If $\mathcal{M}_2$ receives all its external inputs from $\mathcal{U} \cup \mathcal{M}_1$, then the lemma is proved since $\mathcal{M}_2$ cannot receive external inputs from $\mathcal{M}_1$. Inductively, we can choose

$$\mathcal{M}_k \to \cdots \to \mathcal{M}_2 \to \mathcal{M}_1$$

such that $\mathcal{M}_j \not\subset \mathcal{U}$ receives external inputs from cells not in $\mathcal{U} \cup \mathcal{M}_{j-1} \cup \cdots \cup \mathcal{M}_1$. At each stage either $\mathcal{M}_k$ receives all external inputs from $\mathcal{U} \cup \mathcal{M}_{k-1} \cup \cdots \cup \mathcal{M}_1$ and the lemma is proved (since by construction $\mathcal{M}_k$ cannot receive inputs from $\mathcal{M}_{k-1} \cup \cdots \cup \mathcal{M}_1$) or the sequence can be extended. Since the number of transitive components is finite the sequence must terminate and $\mathcal{M}_k$ must receive all of its external inputs from $\mathcal{U}$. □

The strategy of the proof of theorem 1.9 is based on the following observation. We call a periodic solution $Y(t)$ of an admissible vector field on a network $\mathcal{Q}$ *good* if $Y(t)$ is hyperbolic, has no synchronous nodes and has only rigid nonzero phase-shifts. The following lemma follows directly from the definitions.

Lemma 7.18. *Suppose that network $\mathcal{Q}_2$ is an extension of network $\mathcal{Q}_1$ which itself is an extension of network $\mathcal{Q}_0$. Suppose that $Y_0(t)$ is a good periodic solution to an admissible vector field F_0 on $\mathcal{Q}_0$. Suppose that F_1 is the unique extension of F_0 on $\mathcal{Q}_1$ and F_2 is the unique extension of F_1 on $\mathcal{Q}_2$. Suppose that $Y_1(t)$ is a good periodic solution of F_1 that projects onto $Y_0(t)$ and that $Y_2(t)$ is a good periodic solution of F_2 that projects onto $Y_1(t)$. Then $(\mathcal{Q}_2, F_2)$ is an extension of $(\mathcal{Q}_0, F_0)$ and $Y_2(t)$ is a good periodic solution that projects onto $Y_0(t)$.*

Proposition 7.19. *Let Y_0 be a good periodic solution of an admissible system $\dot{Y} = F(Y)$ of the network $\mathcal{Q}$. Let $\mathcal{U} \subset \mathcal{Q}$ be the largest Y_0-complete subnetwork of $\mathcal{Q}$ and assume that $\mathcal{U} \neq \mathcal{Q}$. Then there exists an extension $(\hat{F}, \hat{\mathcal{Q}})$ of $(F, \mathcal{Q})$ such that*

(a) $\hat{F}$ has a unique good periodic solution $\hat{Y}_0$ whose projection to $\mathcal{Q}$ is Y_0.

(b) The nodes in $\hat{\mathcal{Q}} \setminus \mathcal{Q}$ are in $\sigma_{\min}$ orbits of nodes in $\mathcal{Q}$.

(c) The number of transitive components in $\mathcal{Q} \setminus \mathcal{U}$ is greater than the number of transitive components in $\hat{\mathcal{Q}} \setminus \hat{\mathcal{U}}$, where $\hat{\mathcal{U}}$ is the largest $\hat{Y}_0$-complete subnetwork of $\hat{\mathcal{Q}}$.

Lemma 7.18 and proposition 7.19 together provide a recursive proof of theorem 1.9. We just continue extending the network until the number of transitive components in $\hat{\mathcal{Q}} \setminus \hat{\mathcal{U}}$ is zero; that is, until $\hat{\mathcal{U}} = \hat{\mathcal{Q}}$.

Proof. By lemma 7.17, there exists a transitive component $\mathcal{N}_0 \subset \mathcal{Q} \setminus \mathcal{U}$ that receives external signals only from $\mathcal{U}$. By definition $\sigma_{\min}$ is not defined on $\mathcal{N}_0$. Let $\mathcal{N}_1$ be a copy of $\mathcal{N}_0$. Also, let $\tau : \mathcal{N}_0 \to \mathcal{N}_1$ be the isomorphism that identifies the nodes of $\mathcal{N}_0$ with the nodes of $\mathcal{N}_1$ and the internal arrows of $\mathcal{N}_0$ with the internal arrows of $\mathcal{N}_1$. We can adjoin the component $\mathcal{N}_1$ to $\mathcal{Q}$ so that τ extends to the external arrows of $\mathcal{N}_0$, as follows. Let e be an arrow from $\mathcal{U}$ to $\mathcal{N}_0$. Since $\mathcal{U}$ is Y_0-complete, $\sigma_{\min}(\mathcal{T}(e))$ is well defined. So we can add an arrow from $\sigma_{\min}(\mathcal{T}(e))$ to $\tau(\mathcal{H}(e))$ with the same arrow type as e. Let $\mathcal{Q}_1 = \mathcal{Q} \cup \mathcal{N}_1$. From definition 7.11 we see that $\mathcal{Q}_1$ is an extension of $\mathcal{Q}$.

Existence of a unique good $Y_1(t)$ *on* $\mathcal{Q}_1$. Recall that F is the vector field on $\mathcal{Q}$ that has a good periodic solution Y_0. Lemma 7.12 implies that F has a unique extension to F_1 on $\mathcal{Q}_1$. We claim that F_1 has a unique good periodic solution Y_1 such that the projection of $Y_1(t)$ to $\mathcal{Q}$ is Y_0. Divide the variables associated with nodes in $\mathcal{Q}_1$ into groups. Let

- u denote the variables corresponding to nodes in $\mathcal{U}$,
- x_j denote the variables corresponding to nodes in $\mathcal{N}_j$,
- w denote the variables corresponding to the remaining nodes in $\mathcal{Q} \setminus (\mathcal{U} \cup \mathcal{N}_0)$.

We also let u_j denote the variables corresponding to nodes in $\mathcal{U}$ that have arrows going to $\mathcal{N}_j$. Note that the u_j variables are a subset of the variables u and need not be distinct.

Observe that the original vector field F has the form

$$\begin{aligned} \dot{u} &= f(u) \\ \dot{x}_0 &= g(x_0, u_0) \\ \dot{w} &= h(w, u, x_0) \end{aligned} \tag{7.2}$$

and in these coordinates

$$Y_0(t) = (u(t), x_0(t), w(t)).$$

Using these coordinates we can explicitly write the form of the extended vector field F_1 as

$$\begin{aligned} \dot{u} &= f(u) \\ \dot{x}_0 &= g(x_0, u_0) \\ \dot{x}_1 &= g(x_1, u_1) \\ \dot{w} &= h(w, u, x_0). \end{aligned} \tag{7.3}$$

Moreover, we have constructed the extension $\mathcal{Q}_1$ so that the extended periodic solution is

$$Y_1(t) = (u(t), x_0(t), x_0(t + \theta_{\min} T), w(t));$$

that is, the adjoined coordinates of Y_1, those that are in $\mathcal{N}_1$, are $v(t + \theta_{\min} T)$. By construction $Y_1|\mathcal{Q} = Y_0$. The T-periodic state Y_1 is a solution to the equation F_1 because we have adjoined the cells using $\sigma_{\min}$ so that Y_1 would be a solution to the extended equation. So Y_1 is a periodic solution of (7.3) and the projection of $Y_1(t)$ onto $\mathcal{Q}$ is $Y_0(t)$.

We need to show that $Y_1(t)$ is good; that is, we must show that Y_1 has no synchronous nodes, is hub-like, is hyperbolic, and has only rigid phase-shifts.

First, we note that Y_1 has no synchronous nodes. If it did there would be a node in $Y_0(t)$ that is synchronous with one of the new nodes $x_0(t + \theta_{\min} T)$. Then the new node would be synchronous to a node in $\mathcal{Q}$, which would imply that $\sigma_{\min}$ would have been defined on a node in $\mathcal{N}_0$, contradicting the choice of the transitive component $\mathcal{N}_0$.

Second, $Y_0(t)$ is hub-like. The new coordinates of $Y_1(t)$ in $\mathcal{N}_1$ are derived from those in $\mathcal{N}_0$ by phase-shift; hence $Y_1(t)$ is fully oscillatory on $\mathcal{N}_1$. Moreover, by construction $\mathcal{N}_1$ is lower than $\mathcal{J}_{\max}(Y_0)$ and since Y_0 is hub-like $\mathcal{J}_{\max}(Y_1) = \mathcal{J}_{\max}(Y_0)$.

Third, we prove that $Y_1(t)$ is hyperbolic. Since we have assumed that $Y_0(t)$ is hyperbolic, the calculation of the Floquet matrix will give us information about the Jacobian matrix J_0 along $Y_0(t)$. Specifically,

$$J_0(t) = \begin{bmatrix} Df_{u(t)} & 0 & 0 \\ * & D_x g_{(x_0(t), u_0(t))} & 0 \\ * & * & D_w h_{(w(t), u_0(t))} \end{bmatrix}. \tag{7.4}$$

Hyperbolicity of $Y_0(t)$ implies the following. The matrix $\int_0^T Df_{u(t)}\, \mathrm{d}t$ has one 0 eigenvalue and all other eigenvalues off of the imaginary axis, because $u(t)$ is the hyperbolic periodic solution of the equation $\dot{u} = f(u)$ on the $\mathcal{U}$ nodes. Similarly, the eigenvalues of $\int_0^T D_w h_{(w(t),u(t),y(t))}\,\mathrm{d}t$ correspond to eigenvalues of Y_0 on the $\mathcal{W}$ variables. Here we use the fact that the inputs to $\mathcal{W}$ coming from $\mathcal{N}$ are only from those nodes that were originally in $\mathcal{Q}$. Moreover, $\int_0^T D_v g(x_0(t), u_0(t))\,\mathrm{d}t$ must have eigenvalues off the imaginary axis because of the hyperbolicity of $Y_0(t)$ in the $\mathcal{N}_0$ directions.

Next we compute the Floquet equations that show that Y_1 is hyperbolic. The Jacobian matrix J_1 along $Y_1(t)$ has the form

$$J_1(t) = \begin{bmatrix} Df_{u(t)} & 0 & 0 & 0 \\ * & D_x g_{(x_0(t),u_0(t))} & 0 & 0 \\ * & 0 & D_x g_{(x_0(t+\theta_{\min}T),u_1(t))} & 0 \\ * & * & * & D_w h_{(w(t),u_0(t))} \end{bmatrix}. \tag{7.5}$$

Comparing the matrices $J_0(t)$ in (7.4) and $J_1(t)$ in (7.5) we see that $Y_1(t)$ is hyperbolic if the eigenvalues of $\int_0^T D_x g(x_0(t+\theta_{\min}T), u_1(t))\,\mathrm{d}t$ have no eigenvalues on the imaginary axis. However, the construction of the $\mathcal{N}_1$ nodes implies

$$\begin{aligned} \int_0^T D_x g(x_0(t+\theta_{\min}T), u_1(t))\,\mathrm{d}t &= \int_0^T D_x g(x_0(t+\theta_{\min}T), u_0(t+\theta_{\min}T))\,\mathrm{d}t \\ &= \int_0^T D_x g(x_0(t), u_0(t))\,\mathrm{d}t \end{aligned}$$

and $\int_0^T D_v g(x_0(t), u_0(t))\,\mathrm{d}t$ has no eigenvalues on the imaginary axis since $Y_0(t)$ is hyperbolic. So the periodic solution Y_1 is hyperbolic.

Finally, we claim that Y_1 has only rigid phase-shifts. Observe that every admissible perturbation on $\mathcal{Q}_1$ projects to an admissible perturbation on $\mathcal{Q}$ and the associated perturbed periodic solution of Y_1 projects to a periodic solution that is a perturbation of Y_0. So the perturbed phase-shifts of Y_1 that involve pairs of nodes in $\mathcal{Q}$ are rigid. The only other phase-shifts in Y_1 that occur are ones between pairs of nodes in $\mathcal{N}_0$ and $\mathcal{N}_1$ and they are rigid by construction because of the *rigid phase property*.

Convergence of the extensions. In this construction, we adjoined a copy $\mathcal{N}_1$ of a transitive component $\mathcal{N}_0$ in $\mathcal{Q}$ that is not in the largest complete subnetwork $\mathcal{U}$. We did so in such a way that $\sigma_{\min}$ is defined on $\mathcal{N}_0$ and the new $\mathcal{U}_1 \supset \mathcal{U} \cup \mathcal{N}_0$. In fact, since $\mathcal{Q}_1 = \mathcal{Q} \cup \mathcal{N}_1$, there are only two possible outcomes: either

(a) $\mathcal{N}_1 \subset \mathcal{U}_1$, in which case $\mathcal{U}_1 = \mathcal{U} \cup \mathcal{N}_0 \cup \mathcal{N}_1$ and the number of transitive components in $\mathcal{Q}_1 \setminus \mathcal{U}_1$ is less than the number of transitive components in $\mathcal{Q} \setminus \mathcal{U}$ and we are done, or
(b) $\mathcal{N}_1 \not\subset \mathcal{U}_1$, in which case $\mathcal{U}_1 = \mathcal{U} \cup \mathcal{N}_0$ and the number of transitive components in $\mathcal{Q}_1 \setminus \mathcal{U}_1$ is the same as the number of transitive components in $\mathcal{Q} \setminus \mathcal{U}$.

Since by construction $\mathcal{N}_1$ receives external input only from $\mathcal{U}_1$, if (b) holds, we may repeat this construction and adjoin a copy $\mathcal{N}_2$ of $\mathcal{N}_1$ to $\mathcal{Q}_1$ so that $\mathcal{Q}_2 = \mathcal{Q}_1 \cup \mathcal{N}_2$ extends $\mathcal{Q}_1$ and has a hyperbolic periodic solution $Y_2(t)$ that extends $Y_1(t)$. It follows that we are done as soon repeated applications of this construction yields case (a).

Let $\theta_{\min} = \frac{1}{m}$. Either the repeated application of this construction yields case (a) in fewer than $m-1$ applications, in which case we are done; or it does not, in which case it produces a sequence of transitive components $\mathcal{N}_0, \ldots, \mathcal{N}_{m-1}$ and a corresponding sequence networks $\mathcal{Q} = \mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_{m-1}$, such that for each j we have $\mathcal{Q}_{j+1} = \mathcal{Q}_j \cup \mathcal{N}_{j+1}$ and $\mathcal{N}_{j+1} = \sigma_{\min}(\mathcal{N}_j)$ for $j = 0, \ldots, m-1$.

We claim that $\sigma_{\min}$ in defined on $\mathcal{N}_{m-1}$; for if $c \in \mathcal{N}_{m-1}$, then by construction there exist $d \in \mathcal{N}_0$ such that $(\sigma_{\min})^{m-1}(d) = c$. Thus

$$y_c^{m-1}(t) = y_d^{m-1}\left(t + \frac{m-1}{m}T\right),$$

where $y_c^{m-1}(t)$ denotes the node c coordinate of $Y_{m-1}(t)$, so that

$$y_c^{m-1}\left(t + \frac{1}{m}T\right) = y_d^{m-1}(t+T) = y_d^{m-1}(t)$$

and hence $\sigma_{\min}(c) = d$. Thus $\mathcal{N}_{m-1} \subset \mathcal{U}_{m-1}$, and case (a) holds. □

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