PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A | MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1995 **352**, 125-168
doi: 10.1098/rsta.1995.0061

Time-periodic spatially periodic planforms in Euclidean equivariant partial differential equations

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In Rayleigh–Bénard convection, the spatially uniform motionless state of a fluid loses stability as the Rayleigh number is increased beyond a critical value. In the simplest case of convection in a pure Boussinesq fluid, the instability is a symmetry-breaking steady-state bifurcation that leads to the formation of spatially periodic patterns. However, in many double-diffusive convection systems, the heat-conduction solution actually loses stability via Hopf bifurcation. These hydrodynamic systems provide motivation for the present study of spatio-temporally periodic pattern formation in Euclidean equivariant systems. We call such patterns *planforms*.

We classify, according to spatio-temporal symmetries and spatial periodicity, many of the time-periodic solutions that may be obtained through equivariant Hopf bifurcation from a group-invariant equilibrium. Instead of focusing on planforms periodic with respect to a specified planar lattice, as has been done in previous investigations, we consider all planforms that are spatially periodic with

Phil. Trans. R. Soc. Lond. A (1995) **352**, 125–168

Printed in Great Britain

TEX Paper

respect to some planar lattice. Our classification results rely only on the existence of Hopf bifurcation and planar Euclidean symmetry and not on the particular differential equation.

1. Introduction

In Rayleigh–Bénard convection, the spatially uniform motionless state of the fluid loses stability as the Rayleigh number is increased beyond a critical value. In the simplest case of convection in a pure Boussinesq fluid, the instability is a symmetry-breaking steady-state bifurcation that leads to the formation of spatially periodic patterns (Chandrasekhar 1961). However, in many doubly diffusive convection systems, the heat-conduction solution actually loses stability via Hopf bifurcation. This is the case, for example, in magnetoconvection (Chandrasekhar 1961), thermosolutal convection (Veronis 1968), and binary fluid convection (Hurle & Jakeman 1971). These hydrodynamic systems provide motivation for the present study of spatio-temporally periodic pattern formation in Euclidean equivariant systems (we call such patterns *planforms*).

We consider systems of partial differential equations (PDEs), which we write in evolutionary form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}, \lambda) = \mathbf{0}, \tag{1.1}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is a vector-valued function of $\boldsymbol{x} \in \boldsymbol{R}^2$ and time t. Here we have suppressed any possible dependence of $\boldsymbol{u}$ on a third (bounded) spatial coordinate since this does not enter into our characterization of the symmetry of the problem. We assume that (1.1) commutes with the action of the Euclidean group $E(2)$. Recall that the Euclidean group $E(2)$ is the group of all motions in the plane that preserve distances (translations, rotations, reflections). The action of $E(2)$ on functions $\boldsymbol{u}$ that we consider is defined by

$$g \cdot \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(g^{-1}\boldsymbol{x}, t) \tag{1.2}$$

for all $g \in E(2)$. As pointed out to us by Ian Melbourne, there are other actions of $E(2)$ that we could consider, but in the Hopf bifurcation analysis of the hydrodynamic examples that we mention, (1.2) is the relevant action.

We are interested in the situation where (1.1) undergoes a Hopf bifurcation from a time-independent group-invariant equilibrium $\boldsymbol{u}_0$ as the bifurcation parameter λ crosses a critical value λ_c. Our goal is to find all the time-periodic spatially periodic solutions to (1.1) that can be obtained using the equivariant Hopf theorem (Golubitsky *et al.* 1988). Thus, we restrict attention to solutions of (1.1) that are spatially periodic with respect to some planar lattice $\mathcal{L}$. This guarantees that the real generalized eigenspace of

$$\mathrm{D}_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}_0, \lambda), \tag{1.3}$$

associated with the imaginary eigenvalues at $\lambda = \lambda_c$, is finite dimensional.

When (1.1) is posed in the space $\mathcal{X}_{\mathcal{L}}$ of $\mathcal{L}$-periodic functions, then the symmetry group of the problem is reduced from $E(2)$ to $\Gamma = H \dot{+} \boldsymbol{T}^2$, where H is the holohedry of the lattice $\mathcal{L}$ and $\boldsymbol{T}^2 = \boldsymbol{R}^2/\mathcal{L}$ (Dionne & Golubitsky 1992). We scale time so that the only purely imaginary eigenvalues of (1.3) are $\pm\mathrm{i}$. We further assume that the real generalized eigenspace associated with $\pm i$ is of the form $V \oplus V$, where V is a Γ-absolutely irreducible subspace of $\mathcal{X}_{\mathcal{L}}$ and Γ acts diagonally on $V \oplus V$. Finally, we assume that the eigenvalues of (1.3) that cross the imaginary axis as λ crosses λ_c do so with non-zero speed.

The assumptions stated above guarantee that the hypotheses of the equivariant Hopf theorem are satisfied (Golubitsky *et al.* 1988). This theorem states that for each isotropy subgroup $\Sigma \subset \Gamma \times \boldsymbol{S}^1$ that fixes a two-dimensional subspace of $V \oplus V$, there exists a unique branch of (group orbits of) small-amplitude time-periodic solutions to (1.1) with period near 2π. These solution branches all bifurcate from the invariant equilibrium as λ crosses λ_c and have Σ as their group of symmetries. The additional $\boldsymbol{S}^1$ symmetry in the equivariant Hopf theorem is induced by the Liapunov–Schmidt reduction and corresponds to a phase-shift symmetry of the 2π time-periodic functions in $V \oplus V$ (Golubitsky *et al.* 1988).

Previous studies of Hopf bifurcation leading to spatially periodic solutions in Euclidean equivariant systems have focused on specific representations of $\Gamma = H \dot{+} \boldsymbol{T}^2$. Roberts *et al.* (1986) considered the six-dimensional irreducible representation of $\boldsymbol{D}_6 \dot{+} \boldsymbol{T}^2$, which is associated with a particular hexagonal lattice. They determined the symmetries Σ of those solutions guaranteed by the equivariant Hopf theorem, together with their stability properties for perturbations that lie on the same lattice. Their results were used to investigate pattern selection via symmetry-breaking Hopf bifurcation in the two-layer Bénard problem (Renardy & Renardy 1988), in the Bénard problem for a viscoelastic liquid (Renardy & Renardy 1992), and in thermosolutal convection (Renardy 1993). Silber & (1991) Knobloch carried out a similar analysis for the Hopf bifurcation problem associated with the four-dimensional irreducible representation of $\boldsymbol{D}_4 \dot{+} \boldsymbol{T}^2$. They classified the possible bifurcation diagrams, and also showed existence of more complicated dynamical states that can bifurcate from the trivial solution at the Hopf bifurcation. This classification proved useful in interpreting results of a numerical study of three-dimensional compressible magnetoconvection in a square box with periodic boundary conditions in the horizontal directions (Matthews *et al.* 1995). Finally, Hopf bifurcation for the rhombic lattice, with holohedry $H = \boldsymbol{D}_2$, arose in work on the oscillatory instability of spatially anisotropic two-dimensional hydrodynamic systems (Silber *et al.* 1992). This paper focused on the formation of a structurally stable heteroclinic cycle between three of the periodic solutions that are guaranteed to exist by the equivariant Hopf theorem.

The present paper differs fundamentally from the previous investigations in that it does not focus on a particular representation of the group $\Gamma = H \dot{+} \boldsymbol{T}^2$. Instead, following Dionne (1990, 1993) and Dionne & Golubitsky (1992), we consider those solutions that are spatially periodic with respect to *some* lattice $\mathcal{L}$. We classify, according to their spatio-temporal symmetries and spatial periodicity, the time-periodic solutions that may be obtained through equivariant Hopf bifurcation from a group-invariant equilibrium. A similar (partial) classification of the steady-state, spatially periodic solutions of Euclidean-equivariant PDEs, in two and three space dimensions, is presented in Dionne (1990, 1993) and Dionne & Golubitsky (1992). These papers classify the solutions that are guaranteed to exist by the equivariant branching lemma (Vanderbauwhede 1982; Golubitsky *et al.* 1988).

Our classification of the time-periodic solutions includes the maximal isotropy periodic solutions that were found in the previous Hopf bifurcation studies; these are associated with the rhombic lattice and the short spatial period hexagonal and square lattices. However, we also find a countable infinity of new solutions that are periodic on larger hexagonal and square lattices. Specifically, we find five new solution branches for each of the countable infinity of twelve-dimensional irreducible representations of $\boldsymbol{D}_6 \dot{+} \boldsymbol{T}^2$, and four new solution branches for each of the countable infinity of eight-dimensional irreducible representations of $\boldsymbol{D}_4 \dot{+} \boldsymbol{T}^2$. In our classification, each doubly periodic solution is associated with a unique finest lattice $\mathcal{L}$; we guarantee that a solution is not periodic with respect to a *finer* lattice by requiring that the solution has no (non-trivial) pure translation symmetries (Dionne 1990). This innocuous seeming observation is the algebraic basis for the classification of steady-state planforms in Dionne (1990) and for the classification of time-periodic planforms given here.

The approach taken in this paper, which is entirely group theoretic, is described in the next section. The analysis breaks up into three separate cases, depending on whether the holohedry is $\boldsymbol{D}_6$, $\boldsymbol{D}_4$ or $\boldsymbol{D}_2$; these are treated consecutively in each of the sections. The solutions are characterized further by whether or not their symmetries are continuous or discrete. The rotating waves have symmetry Σ, where Σ is one dimensional; these solutions correspond to travelling waves in $\boldsymbol{R}^2$ and rotating waves in $\boldsymbol{T}^2$. The discrete waves do not have any continuous symmetries; these include not only the standing-wave patterns, but also solutions with slightly more complicated spatio-temporal symmetry. The main theorems for the discrete waves appear in § 9, and the results for the rotating waves are in § 10. Finally, in § 11 we interpret the results by presenting pictures of representative solutions as they might appear in shadowgraph images of thermal convection patterns in hydrodynamic systems.

2. Theory

Our goal is to find all time-periodic spatially doubly periodic solutions that typically bifurcate from a group-invariant equilibrium of a Euclidean invariant planar system of PDEs by Hopf bifurcation. This goal is, of course, too ambitious; there is, however, a restricted problem that gives much information about bifurcating solutions and which is tractable. We describe this restricted problem in four steps.

First, observe that any spatially doubly periodic solution lies on a planar lattice $\mathcal{L}$. Let $\mathcal{X}_{\mathcal{L}}$ denote the space of $\mathcal{L}$-periodic functions on $\boldsymbol{R}^2$. We think of $\mathcal{X}_{\mathcal{L}}$ as a model for the phase space of the differential equation. (In the actual PDE, the phase space will consist of several copies of $\mathcal{X}_{\mathcal{L}}$ – the number of copies depending on the number of equations.) The symmetries of $\mathcal{X}_{\mathcal{L}}$ have the form $\Gamma = H \dot{+} \boldsymbol{T}^2$, where H is the holohedry of the lattice and $\boldsymbol{T}^2 = \boldsymbol{R}^2/\mathcal{L}$ is the torus of translations modulo the lattice. The symmetries of a time-periodic solution in $\mathcal{X}_{\mathcal{L}}$ are described by a pair of subgroups $K \subset G$ of Γ, where the elements of G map the periodic trajectory in phase space onto itself and the elements of K fix the periodic trajectory pointwise. It follows from standard theory that K is a normal subgroup of G and that G/K is a Lie subgroup of $\boldsymbol{S}^1$; that is, either G/K is cyclic or G/K is isomorphic to $\boldsymbol{S}^1$.

Second, observe that if there is a pure translation in K (that is, $K \cap \boldsymbol{T}^2 \neq \{0\}$), then the periodic solution is supported on a smaller lattice. Thus, when we search for solutions using equivariant Hopf-bifurcation-theory techniques, we need only search for solutions that are translation-free, that is, $K \cap \boldsymbol{T}^2 = \{0\}$. We call such subgroups K shifted subgroups. Let $\pi_{\mathrm{H}} : \Gamma \to H$ be the projection of Γ

onto the holohedry H. The shifted subgroups K are mapped isomorphically by π_H onto $\pi_H(K)$. Thus, we can classify shifted subgroups by the subgroups of the holohedry.

Once a lattice $\mathcal{L}$ is fixed, the PDE is defined on a compact domain (with periodic boundary conditions) and the standard hypotheses for Hopf bifurcation will typically be satisfied. That is, the eigenspace corresponding to the critical imaginary eigenvalues will have the form $V \oplus V$, where Γ acts orthogonally on V, and by the diagonal action on $V \oplus V$. Moreover, the only irreducible representations V that can occur (in the action of Γ on $L^2(\mathcal{L})$) are absolutely irreducible representations. (The domains of linear PDEs of the kind we consider are, typically, subspaces of L^2.) We shall consider only Hopf bifurcations where Γ acts (absolutely) irreducibly on V. (What we have excluded are the cases where the critical eigenspace is not irreducible.) From the previous discussion, we may assume that no pure translation in Γ acts trivially on V. If a translation were to act trivially, then all solutions obtained by that Hopf bifurcation would have a proper translation symmetry and be supported on a smaller lattice. Thus, we may assume that the representation of Γ on V is translation-free as well as supposing that it is absolutely irreducible.

Third, we observe that the equivariant Hopf theorem guarantees the existence of branches of solutions under the assumption that a certain fixed-point subspace is two-dimensional. To see this, identify $V \cong \boldsymbol{R}^n$ and $V \oplus V \cong \boldsymbol{C}^n$. Then there is an action of $\Gamma \times \boldsymbol{S}^1$ on $\boldsymbol{C}^n$, where the action of Γ is as described previously and the action of $\boldsymbol{S}^1$ is given by scalar multiplication (by scalars of unit modulus). Let $\Sigma \subset \Gamma \times \boldsymbol{S}^1$ be an isotropy subgroup of this action and assume that $\dim \operatorname{Fix}_{\boldsymbol{C}^n}(\Sigma) = 2$. Then the equivariant Hopf bifurcation theorem guarantees the existence of a unique branch of time-periodic solutions with symmetry exactly Σ.

The standard theory of equivariant Hopf bifurcation provides an identification of Σ with the pair of subgroups $K \subset G$ of Γ described previously. In this identification, $K = \Sigma \cap \Gamma$ and $G = \pi_\Gamma(\Sigma)$, where $\pi_\Gamma : \Gamma \times \boldsymbol{S}^1 \to \Gamma$ is projection. Moreover, Σ has the form of a twisted subgroup G^Θ in the following sense. Since $\boldsymbol{S}^1$ acts fixed-point freely on $\boldsymbol{C}^n$, there is a unique homomorphism $\Theta : G \to \boldsymbol{S}^1$, such that

$$\Sigma = G^\Theta \equiv \{(g, \Theta(g)) \in \Gamma \times \boldsymbol{S}^1 : g \in G\}.$$

In these terms, $K = \ker(\Theta)$ and $\Theta(G) \cong G/K$ (which provides one method for proving that G/K is a Lie subgroup of $\boldsymbol{S}^1$).

Fourth, we observe from corollary 2.2 of Golubitsky & Stewart (1993) that there is a further group-theoretic restriction on the pair (G, K) that follows from the two-dimensional fixed-point subspace condition.

Theorem 2.1. *If* $\dim \operatorname{Fix}(G^\Theta) = 2$, *then* G/K *is a maximal Abelian subgroup of* $N_\Gamma(K)/K$.

We can summarize the previous discussion as follows.

Definition 2.2. *The pair of subgroups* (G, K) *of* Γ *forms a* wave pair *if*
(1) K *is a shifted subgroup; i.e.* $K \cap \boldsymbol{T}^2 = \{0\}$,
(2) $K \lhd G$ *and* G/K *is a Lie subgroup of* $\boldsymbol{S}^1$, *and*
(3) G/K *is a maximal Abelian subgroup of* $N_\Gamma(K)/K$.

We observe that being a wave pair is a property of subgroups of Γ and has absolutely nothing to do with the representation of Γ on $V \cong \boldsymbol{R}^n$. For this reason,

Table 1. *Trace formulae for twisted subgroups*

G/K	$\dim \mathrm{Fix}\,(G^{\Theta})$
$\mathbf{1}$	$2 \dim \mathrm{Fix}\,(G)$
$\boldsymbol{Z}_2$	$2(\dim \mathrm{Fix}\,(K) - \dim \mathrm{Fix}\,(G))$
$\boldsymbol{Z}_3$	$\dim \mathrm{Fix}\,(K) - \dim \mathrm{Fix}\,(G)$
$\boldsymbol{Z}_4$	$\dim \mathrm{Fix}\,(K) - \dim \mathrm{Fix}\,(M)$, where $K \subset M \subset G$ and $\lvert G/M\rvert = 2$
$\boldsymbol{Z}_6$	$\dim \mathrm{Fix}\,(K) - \dim \mathrm{Fix}\,(M) - \dim \mathrm{Fix}\,(L) + \dim \mathrm{Fix}\,(G)$, where $K \subset M \subset G$, $\lvert G/M\rvert = 2$, $K \subset L \subset G$ and $\lvert G/L\rvert = 3$.

wave pairs give us a strategy for finding translation-free spatially doubly periodic time-periodic solutions.

(1) Classify shifted subgroups $K \subset \Gamma$ up to conjugacy in Γ.

(2) Find all subgroups $G \subset \Gamma$ such that:

(i) $K \subset G \subset N_{\Gamma}(K)$;

(ii) G/K is a Lie subgroup of $\boldsymbol{S}^1$; and

(iii) G/K is a maximal Abelian subgroup of $N_{\Gamma}(K)/K$.

(3) For each translation-free irreducible representation of Γ, determine those wave pairs that correspond to twisted subgroups G^{Θ} such that $\dim \mathrm{Fix}\,(G^{\Theta}) = 2$.

We now comment on part 3 of this strategy. Recall that if G/K is cyclic, then the corresponding solution is called a discrete wave; if $G/K \cong \boldsymbol{S}^1$, then the solution is a rotating wave. Note that since K is shifted, it is isomorphic to the subgroup of the holohedry $\pi_{\mathrm{H}}(K)$ and is therefore finite. Hence, the wave is discrete if G is finite and the wave is rotating if $\dim G = 1$.

For the symmetry groups Γ of the planar lattices, the quotient groups of discrete waves $G/K \cong \boldsymbol{Z}_k$ occur only for $k = 1, 2, 3, 4, 6$. (This is related to the crystallographic restriction on lattice-point groups.) For precisely these cyclic quotients, the dimensions of $\mathrm{Fix}_{\boldsymbol{C}^n}(G^{\Theta})$ can be determined from the dimensions of fixed-point subspaces of $\boldsymbol{R}^n$ through the use of trace formulae (see, for example, Golubitsky *et al.* 1988). In particular, the form of Θ in the construction of G^{Θ} is not needed, which is a substantial simplification. The formulae for $\dim \mathrm{Fix}\,(G^{\Theta})$ are given in table 1 for each twist type.

Finally, we note that for rotating waves it is necessary to compute $\dim \mathrm{Fix}\,(G^{\Theta})$ directly. Fortunately, in these cases, the computations are tractable.

We can now describe in detail the structure of the paper. As noted previously, we need consider only translation-free irreducible representations of the planar lattice groups Γ. It was shown in Dionne (1990) and Dionne & Golubitsky (1992) that translation-free representations occur only for the hexagonal, square and rhombic lattices. Thus, in each succeeding section we break the discussion into three parts – one for each lattice.

The shifted subgroups K are indexed by subgroups of the holohedry. In § 3, we enumerate up to conjugacy in H all subgroups of the holohedry. The shifted subgroups K are then classified up to conjugacy in Γ in § 5 after some preliminary discussion of pertinent lattice information in § 4. The normalizers $N_{\Gamma}(K)$ of the shifted subgroups K are computed in § 6, followed by a complete classification of wave pairs (G, K) in § 7.

To this point, only group-theoretic information in Γ has been tabulated. In order to find time-periodic solutions using the equivariant Hopf theorem, it is necessary to discuss the irreducible representations of Γ and this is performed in § 8. Discrete waves are found in § 9 using the trace formulae listed in table 1 and rotating waves are found in § 10 using explicit calculations. Finally, pictures of all of the time-periodic spatially doubly periodic planforms are presented in § 11.

3. Subgroups of the holohedries

In propositions 3.1–3.3, we classify up to conjugacy the subgroups of the holohedry on the hexagonal, square and rhombic lattices. We use the notation $\boldsymbol{Z}_k$ to indicate a cyclic group of order k consisting only of rotations. The notation $\boldsymbol{D}_k$ indicates a dihedral group of order $2k$. Note that $\boldsymbol{D}_1$ has order two and is generated by a reflection. The group $\boldsymbol{D}_2$ has four elements, is group isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ and has three non-trivial order-two elements, two of which are reflections and one is a rotation. In each of the holohedries, there are, up to conjugacy, three order-two groups – one generated by rotation through π and two generated by reflections. We have tried to develop a notation that will distinguish these subgroups and allow for an easy enumeration of the shifted subgroups in § 5.

(*a*) *Hexagonal lattice*

The holohedry of a hexagonal lattice is $H = \boldsymbol{D}_6$. Without loss of generality, we may assume that the hexagonal lattice is generated by $\ell_1 = (1/\sqrt{3}, 1)$ and $\ell_2 = (2/\sqrt{3}, 0)$. We enumerate the subgroups of $\boldsymbol{D}_6$ as follows. Let R denote rotation counterclockwise by 60° and let h denote the reflection across the x-axis. Then the twelve elements of $\boldsymbol{D}_6$ are enumerated by $e, R, \ldots, R^5, h, hR, \ldots, hR^5$.

Proposition 3.1. *Up to conjugacy in* $\boldsymbol{D}_6$*, the subgroups of* $\boldsymbol{D}_6$ *are*
(1) $\boldsymbol{1} = \{e\}$,
(2) $\boldsymbol{Z}_2[R^3] = \{e, R^3\}$,
(3) $\boldsymbol{Z}_3[R^2] = \{e, R^2, R^4\}$,
(4) $\boldsymbol{Z}_6[R] = \{e, R, R^2, R^3, R^4, R^5\}$,
(5) $\boldsymbol{D}_1[h] = \{e, h\}$,
(6) $\boldsymbol{D}_1[hR^3] = \{e, hR^3\}$,
(7) $\boldsymbol{D}_2[h, R^3] = \{e, h, hR^3, R^3\}$,
(8) $\boldsymbol{D}_3[h, R^2] = \{e, R^2, R^4, h, hR^2, hR^4\}$,
(9) $\boldsymbol{D}_3[hR, R^2] = \{e, R^2, R^4, hR, hR^3, hR^5\}$, *and*
(10) $\boldsymbol{D}_6[h, R] = \boldsymbol{D}_6$.

The proof uses routine group-theory arguments. As noted previously, there are three two-element subgroups in this list. One contains a rotation ($\boldsymbol{Z}_2[R^3]$), one contains a reflection across a line connecting opposite vertices of a regular hexagon ($\boldsymbol{D}_1[h]$), and one contains a reflection across a line connecting midpoints of opposite sides in the hexagon ($\boldsymbol{D}_1[hR^3]$). Up to conjugacy, there is only one $\boldsymbol{D}_2$, since each $\boldsymbol{D}_2$ contains one reflection of each type. There are two non-conjugate $\boldsymbol{D}_3$s; each containing the reflections of a given type.

(*b*) *Square lattice*

Without loss of generality, we assume that the square lattice $\mathcal{L}$ is generated by $\ell_1 = (1, 0)$ and $\ell_2 = (0, 1)$. The holohedry of a square lattice is the group $H = \boldsymbol{D}_4$, generated by R, the rotation counterclockwise by 90°, and h, the reflection across the x-axis. Note that Rh is the reflection across the line $x = y$.

The following proposition is proved using elementary group-theory arguments.

Proposition 3.2. *Up to conjugacy, the subgroups of* $\boldsymbol{D}_4$ *are*
(1) $\boldsymbol{1} = \{e\}$,
(2) $\boldsymbol{Z}_2[R^2] = \{e, R^2\}$,
(3) $\boldsymbol{Z}_4[R] = \{e, R, R^2, R^3\}$,
(4) $\boldsymbol{D}_1[h] = \{e, h\}$,
(5) $\boldsymbol{D}_1[Rh] = \{e, Rh\}$,
(6) $\boldsymbol{D}_2[h, R^2] = \{e, h, R^2, R^2h\}$,
(7) $\boldsymbol{D}_2[Rh, R^2] = \{e, Rh, R^2, R^3h\}$, *and*
(8) $\boldsymbol{D}_4[h, R] = \{e, R, R^2, R^3, h, Rh, R^2h, R^3h\}$.

In the square lattice, as in the hexagonal lattice, there are two types of reflection yielding, up to conjugacy, the two element subgroups $\boldsymbol{D}_1[h]$ (generated by a reflection across an axis) and $\boldsymbol{D}_1[Rh]$ (generated by a reflection across a diagonal). On this lattice, however, there are two non-conjugate subgroups isomorphic to $\boldsymbol{D}_2$ each containing the reflections of a given type.

(c) *Rhombic lattice*

Without loss of generality, we may assume that the rhombic lattice is generated by $\ell_1 = (1, -\cot\theta)$ and $\ell_2 = (0, \operatorname{cosec}\theta)$ where $0 < \theta < \frac{1}{2}\pi$ and $\theta \neq \frac{1}{3}\pi$. The holohedry of the rhombic lattice is $\boldsymbol{D}_2$ generated by P, a rotation by π, and h, a reflection that interchanges ℓ_1 and ℓ_2.

Proposition 3.3. *Up to conjugacy, the subgroups of* $\boldsymbol{D}_2$ *are*
(1) $\boldsymbol{1} = \{e\}$,
(2) $\boldsymbol{Z}_2[P] = \{e, P\}$,
(3) $\boldsymbol{D}_1[h] = \{e, h\}$,
(4) $\boldsymbol{D}_1[Ph] = \{e, Ph\}$, *and*
(5) $\boldsymbol{D}_2[h, P] = \{e, h, P, Ph\}$.

4. Lattice geometry

We introduce the following notation. Suppose that g is a reflection in the holohedry H. Then the eigenvalues of g viewed as a matrix on $\boldsymbol{R}^2$ are $+1$ and -1. We define two circles in $\boldsymbol{T}^2$ as follows:

$$E^+(g) = \text{ the projection of the } +1 \text{ eigenspace into } \boldsymbol{T}^2$$
$$E^-(g) = \text{ the projection of the } -1 \text{ eigenspace into } \boldsymbol{T}^2.$$

Note that for each vector $w \in \boldsymbol{R}^2$ we can write $w = w^+ + w^-$, where $w^+ \in E^+(g)$ and $w^- \in E^-(g)$. Moreover, this decomposition is also valid for points $w \in \boldsymbol{T}^2$.

The half lattice $\frac{1}{2}\mathcal{L}$ generated by $v_1 = \frac{1}{2}\ell_1$ and $v_2 = \frac{1}{2}\ell_2$, where ℓ_1 and ℓ_2 are generators for the lattice $\mathcal{L}$, plays an important role in our analysis. By abuse of notation, we denote the group obtained by projecting the half-lattice $\frac{1}{2}\mathcal{L}$ into $\boldsymbol{T}^2$ by $\frac{1}{2}\mathcal{L}$. This subgroup of $\boldsymbol{T}^2$ has four elements and is generated by v_1 and v_2. The other non-trivial element in this subgroup is $v_d = v_1 + v_2$.

Next we define subsets of $\boldsymbol{T}^2$ by

$$F^+(g) = \{v \in \boldsymbol{T}^2 : gv = v\},$$
$$F^-(g) = \{v \in \boldsymbol{T}^2 : gv = -v\}.$$

Clearly $E^+(g) \subset F^+(g)$ and $E^-(g) \subset F^-(g)$. We will see in the subsequent sections that whether or not $F^+(g)$ equals $E^+(g)$ plays an important role in the computation.

Similarly, the intersection of $F^+(g)$ with $F^-(g)$ also plays a significant role. Note that

$$F^+(g) \cap F^-(g) \subset \tfrac{1}{2}\mathcal{L},$$

since if $gv = v$ modulo $\mathcal{L}$ and $gv = -v$ modulo $\mathcal{L}$, then $v = -v$ modulo $\mathcal{L}$ and $v \in \frac{1}{2}\mathcal{L}$. Below, we compute these intersections for each type of reflection on each lattice.

Finally, let $g^\perp$ be the reflection across the line perpendicular to the line of reflection of g. Note that when viewed as a linear mapping on $\boldsymbol{R}^2$, $g^\perp = -g$; hence $E^+(g^\perp) = E^-(g)$ and $E^-(g^\perp) = E^+(g)$.

We first consider the hexagonal lattice with $g = h$, where h is the reflection with respect to the x-axis. In this case,

$$E^+(h) = \{\alpha\ell_2 : \alpha \in \boldsymbol{R}\} = F^+(h)$$
$$E^-(h) = \{-2\alpha\ell_1 + \alpha\ell_2 : \alpha \in \boldsymbol{R}\} = F^-(h).$$

For this reflection,

$$F^+(h) \cap F^-(h) = \boldsymbol{Z}_2[v_2].$$

As noted previously, the intersection $F^+(h) \cap F^-(h)$ in $\boldsymbol{T}^2$ consists of half lattice vectors. A calculation on the hexagonal lattice shows that this intersection consists precisely of the elements of $\frac{1}{2}\mathcal{L}$ that are also in the circle $F^+(h)$.

The reflection $h^\perp$ is a reflection of the other type on the hexagonal lattice. Hence, it is a straightforward exercise to compute the relevant data for $h^\perp$ from the data for h.

We now consider the square lattice with $g = h$, where h is again the reflection with respect to the x-axis. We find that

$$E^+(h) = \{\alpha\ell_1 : \alpha \in \boldsymbol{R}\},$$
$$E^-(h) = \{\alpha\ell_2 : \alpha \in \boldsymbol{R}\},$$

and

$$F^+(h) = \{(x,y) \in \boldsymbol{T}^2 : 0 \leqslant x < 1, y = 0 \text{ or } y = \tfrac{1}{2}\} = E^+(h) \oplus \boldsymbol{Z}_2[v_2],$$
$$F^-(h) = \{(x,y) \in \boldsymbol{T}^2 : 0 \leqslant y < 1, x = 0 \text{ or } x = \tfrac{1}{2}\} = E^-(h) \oplus \boldsymbol{Z}_2[v_2].$$

These groups consists of two disjoint circles on the torus. Moreover, in this case

$$F^+(h) \cap F^-(h) = \tfrac{1}{2}\mathcal{L}.$$

When $g = Rh$, the reflection across the diagonal line $x = y$, we have that

$$E^+(Rh) = \{\alpha(\ell_1 + \ell_2) : \alpha \in \boldsymbol{R}\} = F^+(Rh),$$
$$E^-(Rh) = \{\alpha(\ell_1 - \ell_2) : \alpha \in \boldsymbol{R}\} = F^-(Rh).$$

Moreover,

$$F^+(Rh) \cap F^-(Rh) = \boldsymbol{Z}_2[v_d].$$

Finally, we consider the rhombic lattice with $g = h$, the reflection that interchanges ℓ_1 and ℓ_2. It is easy to see that

$$E^+(h) = \{a(\ell_1 + \ell_2) : a \in \boldsymbol{R}\} = F^+(h),$$
$$E^-(h) = \{a(\ell_1 - \ell_2) : a \in \boldsymbol{R}\} = F^-(h).$$

Moreover,

$$F^+(h) \cap F^-(h) = \boldsymbol{Z}_2[v_d].$$

As with the hexagonal lattice, $h^\perp$ is the second type of reflection on the rhombic lattice.

When $E^+(g)$ is equal to $F^+(g)$, we sometimes denote $F^+(g)$ by $E^+(g)$. Similarly, when $E^-(g)$ is equal to $F^-(g)$, we sometimes denote $F^-(g)$ by $E^-(g)$.

5. Shifted subgroups

A *shifted subgroup* of Γ is a subgroup K whose intersection with $\boldsymbol{T}^2$ is trivial. Such subgroups project isomorphically into the holohedry H, i.e. K is isomorphic to $\pi_{\mathrm{H}}(K)$. We can classify such subgroups by their image in H – a shifted $\boldsymbol{D}_1$ is just a shifted subgroup isomorphic to $\boldsymbol{D}_1 \subset H$, and so on.

(a) Hexagonal lattice

We first classify the shifted groups in $\Gamma = \boldsymbol{D}_6 \dot{+} \boldsymbol{T}^2$.

Theorem 5.1. *Every shifted group $\Sigma \subset \Gamma$ is conjugate to a subgroup of the holohedry $\boldsymbol{D}_6$.*

Proof. Let Σ be a shifted subgroup and let $K = \pi_{\mathrm{H}}(\Sigma) \subset H$. Since Σ is shifted, K and Σ are isomorphic. The proof proceeds by considering in turn each of the possible Ks enumerated in proposition 3.1.

Cases 1–4. $K = \boldsymbol{Z}_k$, where $k = 1, 2, 3, 6$. Suppose that Σ contains an element (r, v), where r is a rotation. Then we may conjugate (r, v) by (e, w) to obtain

$$(e, w)(r, v)(e, -w) = (r, v + w - rw).$$

Since $I - r$ is invertible, we can solve for w so that $v + w - rw = 0$. It follows that any shifted subgroup Σ of Γ that is π_{H}-isomorphic to $\mathbf{1}, \boldsymbol{Z}_2[R^3], \boldsymbol{Z}_3[R^2], \boldsymbol{Z}_6[R]$ is actually conjugate to that group.

Cases 5, 6. $K = \boldsymbol{D}_1[h]$ or $K = \boldsymbol{D}_1[hR^3]$. We now consider the reflections. The two-element group Σ is generated by an element of the form (g, v), where $g \in H$ is a reflection. Recall that $E^+(g)$ and $E^-(g)$ equal the circle subgroups $F^+(g)$ and $F^-(g)$, respectively, in $\boldsymbol{T}^2$, which we also denote by $E^+(g)$ and $E^-(g)$. Moreover, the intersection $E^+(g) \cap E^-(g)$ in $\boldsymbol{T}^2$ is $\boldsymbol{Z}_2[v_2]$.

For (g, v) to have order two it is necessary that $gv + v = 0$ modulo the lattice $\mathcal{L}$. Suppose we write $v = v^+ + v^-$, where $v^+ \in E^+(g)$ and $v^- \in E^-(g)$. It follows that $2v^+ = 0$ modulo $\mathcal{L}$ or $v^+ \in \frac{1}{2}\mathcal{L}$. However, such v^+ are also in $E^-(g)$; so we may assume that (g, v) has the form (g, v^-). Next, we conjugate this group element by (e, w^-) obtaining

$$(e, w^-)(g, v^-)(e, -w^-) = (g, v^- + 2w^-).$$

It follows that we can choose $w^- = -\frac{1}{2}v^-$ so that the result of the conjugation is $(g, 0)$. Hence, a shifted $\boldsymbol{D}_1$ is conjugate to $\boldsymbol{D}_1$, as desired.

Case 7. $K = \boldsymbol{D}_2[h, R^3]$. The shifted group Σ is generated by two reflections across perpendicular axes of reflection (since the reflections must commute). Thus, $\Sigma = \langle (h, v^-), (h^\perp, v^+) \rangle$ since, as above, each reflection must be of order two. Using the result for shifted $\boldsymbol{D}_1$s we can conjugate Σ to have the form $\Sigma = \langle (h, 0), (h^\perp, v^+) \rangle$. If we now conjugate Σ again by an element of the form (e, w^+), then $(h, 0)$ remains fixed and $(h^\perp, v^+)$ is transformed to $(h^\perp, v^+ + 2w^+)$. Thus, we can choose w^+ so that the shifted $\boldsymbol{D}_2$ becomes the standard $\boldsymbol{D}_2[h, R^3]$.

Cases 8, 9. $K = \boldsymbol{D}_3[h, R^2]$ *or* $K = \boldsymbol{D}_3[hR, R^2]$. Let Σ be a shifted $\boldsymbol{D}_3$. To verify that Σ can be conjugated to a standard $\boldsymbol{D}_3$, we first conjugate Σ so that the rotations are of the form $(r, 0)$. Next, let (g, v) be a reflection in Σ. Then the group laws in $\boldsymbol{D}_3$ demand that

$$(r, 0)(g, v) = (g, v)(r^2, 0).$$

This identity implies that $rv = v$. Since r is a rotation, this means that $v = 0$. Hence, $\Sigma = \boldsymbol{D}_3$, as desired. Note that this proof is valid for either subgroup $\boldsymbol{D}_3[h, R^2]$ or $\boldsymbol{D}_3[hR, R^2]$.

Case 10. $K = \boldsymbol{D}_6[h, R]$. The proof for the shifted $\boldsymbol{D}_3$s works identically for the shifted $\boldsymbol{D}_6$s. First, we conjugate rotations so that the shifted $\boldsymbol{Z}_6$ is just the standard $\boldsymbol{Z}_6[R]$. The group operations then imply that the reflections must already be in the holohedry. ■

(*b*) *Square lattice*

We now classify the shifted subgroups up to conjugacy in $\Gamma = \boldsymbol{D}_4 \dot{+} \boldsymbol{T}^2$.

Theorem 5.2. *Every shifted subgroup K in Γ is conjugate in Γ to one of the following subgroups*

(1) $\boldsymbol{1}$,
(2) $\boldsymbol{Z}_2[R^2]$,
(3) $\boldsymbol{Z}_4[R]$,
(4) $\boldsymbol{D}_1[h]$ *and* $\boldsymbol{D}_1[(h, v_1)]$,
(5) $\boldsymbol{D}_1[Rh]$,
(6) $\boldsymbol{D}_2[h, R^2]$, $\boldsymbol{D}_2[(h, v_1), R^2]$ *and* $\boldsymbol{D}_2[(h, v_d), R^2]$,
(7) $\boldsymbol{D}_2[Rh, R^2]$, *or*
(8) $\boldsymbol{D}_4[h, R]$ *and* $\boldsymbol{D}_4[(h, v_d), R]$.

Proof. Let $\pi_{\mathrm{H}} : \Gamma \to \boldsymbol{D}_4$ be the projection of Γ onto the holohedry $\boldsymbol{D}_4$. We proceed by classifying up to conjugacy those subgroups Σ that project by π_{H} onto one of the eight subgroups of $\boldsymbol{D}_4$ listed in proposition 3.2. Note that since Σ is a shifted subgroup of $\Gamma = \boldsymbol{D}_4 \dot{+} \boldsymbol{T}^2$, it is isomorphic to $\pi_{\mathrm{H}}(\Sigma)$.

Cases 1–3. $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{1}, \boldsymbol{Z}_2[R^2], \boldsymbol{Z}_4[R]$. In these cases, Σ is conjugate to $\mathbf{1}$, $\boldsymbol{Z}_2[R^2]$, or $\boldsymbol{Z}_4[R]$. The proof is identical to the proof of cases 1–4 in theorem 5.1.

Cases 4, 5. $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{D}_1[h], \boldsymbol{D}_1[Rh]$. In these cases, $|\Sigma| = 2$ and is generated by (g, v), where g is a reflection and $v \in \boldsymbol{T}^2$. We begin by considering the conjugacy

$$(e, -w)(g, v)(e, w) = (g, v + gw - w).$$

We can choose $w = \frac{1}{2}v^-$ to conjugate (g, v) to (g, v^+). It follows from the fact that (g, v^+) is of order two that $v^+ \in \frac{1}{2}\mathcal{L}$; hence either $v^+ = 0$ or $v^+ = v_1$ in the case $g = h$, and either $v^+ = 0$ or $v^+ = v_d$ in the case $g = Rh$.

It remains to decide whether or not the two groups in each case are conjugate. If they are, then the conjugacy has to have the form

$$(k, w)(g, v^+)(k^{-1}, -k^{-1}w) = (kgk^{-1}, kv^+ + w - kgk^{-1}w).$$

It follows that $kgk^{-1} = g$ and that the result of the conjugacy is $(g, kv^+ + w - gw) = (g, kv^+ + 2w^-)$. Now the only group elements in $\boldsymbol{D}_4$ that commute with g are e, g, R^2 and R^2g, and each of these group elements preserves the circle $E^+(g)$. Thus, when v^+ is non-zero, the only way for $kv^+ + 2w^-$ to equal zero mod $\mathcal{L}$ is for v^+ to be in $E^+(g) \cap E^-(g)$. This cannot happen for the case $g = h$ but does happen for the case $g = Rh$. In particular, the intersection of $E^+(g)$ and $E^-(g)$ in $\boldsymbol{T}^2$ is $\{0\}$ when $g = h$, and $\{0, v_d\}$ when $g = Rh$. Thus, in the case $g = Rh$ it is possible to choose w^- so that the two groups $\boldsymbol{D}_1[Rh]$ and $\boldsymbol{D}_1[(Rh, v_d)]$ are conjugate.

Cases 6, 7. $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{D}_2[h, R^2], \boldsymbol{D}_2[Rh, R^2]$. In these cases, $|\Sigma| = 4$ and Σ is generated by (g, v) and (R^2, w), where v and w are in $\boldsymbol{T}^2$. We begin by conjugating, as we did previously, (R^2, w) to $(R^2, 0)$. The third non-trivial element in the shifted $\boldsymbol{D}_2$ must have the form $(g^\perp, v')$. Group multiplication implies that

$$(g, v)(R^2, 0) = (g^\perp, v').$$

Hence $v' = v$. Next, note that the commutativity relation

$$(g, v)(R^2, 0) = (R^2, 0)(g, v) \tag{5.1}$$

implies that $v \in \frac{1}{2}\mathcal{L}$. Hence, the shifted $\boldsymbol{D}_2$ equals $\boldsymbol{D}_2[R^2, (g, v)]$ for some $v \in \frac{1}{2}\mathcal{L}$. There are four possibilities: $v = 0$, $v = v_1$, $v = v_d$ and $v = v_2$. For the case $g = Rh$, only $\boldsymbol{D}_2[Rh, R^2]$ and $\boldsymbol{D}_2[(Rh, v_d), R^2]$ are translation-free subgroups and we show that these groups are conjugate. We also show that that $\boldsymbol{D}_2[(h, v_1), R^2]$ is conjugate to $\boldsymbol{D}_2[(h, v_2), R^2]$.

If two of these groups are conjugate, then the conjugacy must fix the element $(R^2, 0)$. Note that any group element (k, w) commutes with $(R^2, 0)$ as long as $w \in \frac{1}{2}\mathcal{L}$. Next, compute the conjugacy

$$(k, w)(g, v)(k^{-1}, -k^{-1}w) = (kgk^{-1}, kv + w - kgk^{-1}w).$$

In order to be a conjugacy between two of the $\boldsymbol{D}_2[R^2, (g, v)]$s it is necessary that either kgk^{-1} equal g or $g^\perp$. So if we write $w = w^+ + w^-$, then either

$$(k, w)(g, v)(k^{-1}, -k^{-1}w) = (g, kv + 2w^-)$$

or

$$(k, w)(g, v)(k^{-1}, -k^{-1}w) = (g^\perp, kv + 2w^+).$$

Here there is a difference between the cases $g = h$ and $g = Rh$. In the case $g = h$, we can check that $w \in \frac{1}{2}\mathcal{L}$ implies that $w^+, w^- \in \frac{1}{2}\mathcal{L}$ and, hence, that $2w^+, 2w^- \in \mathcal{L}$. Therefore, the second coordinate in the conjugacy is kv. In the case of the $\boldsymbol{D}_2[R^2, (g, v)]$, where $g = h$, we can set $k = R$ to show that $\boldsymbol{D}_2[(h, v_1), R^2]$ is conjugate to $\boldsymbol{D}_2[(h, v_2), R^2]$. Moreover, no further conjugacies of the $\boldsymbol{D}_2[R^2, (g, v)]$ are possible.

For the case $g = Rh$, we choose $k = e$, $w = w^- = \frac{1}{2}(\frac{1}{2}, -\frac{1}{2})$ and $v = 0$ to see that

$$v + 2w^- \equiv v_d,$$

which proves that $\boldsymbol{D}_2[Rh, R^2]$ and $\boldsymbol{D}_2[(Rh, v_d), R^2]$ are conjugate.

Case 9. $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{D}_4[h, R]$. In this case, Σ is generated by (h, v) and (R, w), where $v, w \in \boldsymbol{T}^2$. Since R is a rotation, we may conjugate Σ so that Σ is generated by (h, v) and $(R, 0)$ for some $v \in \boldsymbol{T}^2$, and Σ equals $\boldsymbol{D}_4[(h, v), R]$. Since (h, v) and (hR^2, v) are of order two, it follows that $v + hv \in \mathcal{L}$ and that $v - hv \in \mathcal{L}$; hence, $v \in \frac{1}{2}\mathcal{L}$. There are now four possibilities for v; namely, $v = 0$, $v = v_d$, $v = v_1$ and $v = v_2$. Note, however, that $(Rh, v) = (h, v)(R^3, 0)$ is also in $\boldsymbol{D}_4[(h, v), R]$ and is of order two. It follows that $v + Rhv$ must be in $\mathcal{L}$, which is false when $v = v_1$ or $v = v_2$. ∎

(c) *Rhombic lattice*

We now classify the shifted groups in $\Gamma = \boldsymbol{D}_2 \dot{+} \boldsymbol{T}^2$.

Theorem 5.3. *Every shifted subgroup of Γ is conjugate to a subgroup of the holohedry $\boldsymbol{D}_2$.*

Proof. Let Σ be a shifted subgroup of $\Gamma = \boldsymbol{D}_2 \dot{+} \boldsymbol{T}^2$; hence, Σ is isomorphic to $\pi_{\mathrm{H}}(\Sigma)$.

Case 1–2. $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{1}, \boldsymbol{Z}_2[P]$. In these cases, Σ is conjugate to $\mathbf{1}$ or $\boldsymbol{Z}_2[P]$. The proof is identical to the proof of cases 1–4 in theorem 5.1.

Cases 3, 4. $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{D}_1[h]$ *or* $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{D}_1[Ph]$. Suppose that Σ is generated by (g, v), where $g = h$ or $g = Ph$. Recall that $E^+(g)$ and $E^-(g)$ equal the circle subgroups $F^+(g)$ and $F^-(g)$, respectively, in $\boldsymbol{T}^2$, which we also denote by $E^+(g)$ and $E^-(g)$. We can write v as $v = v^+ + v^-$, with $v^+ \in E^+(g)$ and $v^- \in E^-(g)$. Since

$$(e, w)(g, v)(e, -w) = (g, 2w^- + v) = (g, v^+)$$

for some w^-, we may assume that $v = v^+$. Since $v^+ \in \frac{1}{2}\mathcal{L} \cap E^+(g) = \{0, v_d\}$ (as (g, v^+) is of order two), it follows that $v = 0$ or $v = v_d$. The subgroup generated by (g, v_d) is conjugate to the subgroup generated by $(g, 0)$ for

$$(e, v_1)(g, v_d)(e, v_1) = (g, 0).$$

Case 5. $\pi_{\mathrm{H}}(\Sigma) = \boldsymbol{D}_2[h, P]$. After a preliminary conjugation we may assume that Σ is generated by $(P, 0)$ and (h, v), where $v = v^+ + v^-$ with $v^+ \in E^+(h)$ and $v^- \in E^-(h)$. From

$$(h, v)(P, 0) = (P, 0)(h, v),$$

we see that $v \in \frac{1}{2}\mathcal{L}$. However, Σ is not a shifted subgroup if $v = v_1$ or $v = v_2$ for $(h, v)(h, v) = (e, v_d) \in \Sigma$ for these values of v. Hence, $v = 0$ or $v = v_d$. The group generated by $(P, 0)$ and (h, v_d) is conjugate to $\boldsymbol{D}_2[h, P]$ for

$$(e, v_2)(h, v_d)(e, v_2) = (h, 0) \quad \text{and} \quad (e, v_2)(P, 0)(e, v_2) = (P, 0).$$

■

6. Normalizers

To determine the possible Gs that go with each of the shifted subgroups K of Γ, namely the Gs that satisfy theorem 2.1, we need to compute the normalizers of the shifted subgroups K. The following lemma will be useful for this purpose. Note that the conclusion of this lemma is obviously true if K is of order two and $\mathcal{L}$ is any planar lattice.

It is worth recalling the definitions of the commutator, normalizer and centralizer subgroups. The commutator of two subgroups G and K is

$$[G, K] = \{gkg^{-1}k^{-1} : g \in G \text{ and } k \in K\}.$$

The normalizer of K in Γ is

$$N_\Gamma(K) = \{\gamma \in \Gamma : \gamma K \gamma^{-1} = K\}.$$

Finally, the centralizer of G in Γ is

$$C_\Gamma(G) = \{\gamma \in \Gamma : \gamma g = g\gamma, \forall\, g \in G\}.$$

Lemma 6.1. *Let $\mathcal{L}$ be a rhombic or hexagonal lattice. Suppose that K is a shifted $\boldsymbol{D}_2$. Then $N_\Gamma(K) = C_\Gamma(K)$.*

Proof. Let $\boldsymbol{Z}_1 = \{e\}$. We begin by claiming that the commutator subgroup $[\Gamma,\Gamma] = \boldsymbol{Z}_k \dot{+} \boldsymbol{T}^2$, where $k = 3$ on the hexagonal lattice and $k = 1$ on the rhombic lattice. To verify the claim, first recall that if $\Delta \triangleleft \Gamma$ and if Γ/Δ is Abelian, then $[\Gamma,\Gamma] \subset \Delta$. Observe that $\Delta = \boldsymbol{Z}_k \dot{+} \boldsymbol{T}^2$ is normal in $\Gamma = \boldsymbol{D}_{2k} \dot{+} \boldsymbol{T}^2$ and that $\Gamma/\Delta = \boldsymbol{Z}_2^2$ is Abelian. Hence, $[\Gamma,\Gamma] \subset \boldsymbol{Z}_k \dot{+} \boldsymbol{T}^2$. On the other hand, an explicit calculation shows that each element of $\boldsymbol{Z}_k \dot{+} \boldsymbol{T}^2$ is a commutator.

Next, let $N = N_\Gamma(K)$. Since K is normal in N, it follows that $[N,K] \subset K$. In addition, $[N,K] \subset [\Gamma,\Gamma] = \boldsymbol{Z}_k \dot{+} \boldsymbol{T}^2$. Thus, $[N,K] \subset K \cap (\boldsymbol{Z}_k \dot{+} \boldsymbol{T}^2)$. However, this intersection is trivial since K is a shifted $\boldsymbol{D}_2$ and, therefore, N commutes with K. Hence, $N \subset C_\Gamma(K)$, but the centralizer is always contained in the normalizer and the lemma is proved. ∎

(a) *Hexagonal lattice*

The three-element subgroup of $\boldsymbol{T}^2$ generated by $v_t = \frac{1}{3}\ell_1 + \frac{1}{3}\ell_2$ plays an important role in the analysis that follows. We denote this group by $\boldsymbol{Z}_3[v_t]$.

Proposition 6.2. *Let $\mathcal{L}$ be a hexagonal lattice and let $q = h$ or $q = h^\perp$. Then*

$$C_\Gamma(\boldsymbol{D}_1) = \boldsymbol{D}_2[h, R^3] \dot{+} E^+(q),$$

where $\boldsymbol{D}_1$ is generated by $(q, 0)$, and

$$C_\Gamma(\boldsymbol{D}_2[h, R^3]) = \boldsymbol{D}_2[h, R^3] \times \boldsymbol{Z}_2[v_2].$$

Proof. Let (g, w) commute with $\boldsymbol{D}_1$, i.e. (g, w) commutes with $(q, 0)$. It follows that g commutes with q; hence, g is either also a reflection (and either equals h or $h^\perp$), $g = R^3$ or $g = e$.

If $g = e$, then (e, w) commutes with $(q, 0)$ when $w = w^+ \in E^+(q)$. If $g = R^3$, $g = h$ or $g = h^\perp$, then (g, w) also commutes with $(q, 0)$ if $w = w^+$. These statements verify that $C_\Gamma(\boldsymbol{D}_1) = \boldsymbol{D}_2 \dot{+} E^+(q)$.

To verify the second statement, observe that

$$C_\Gamma(\boldsymbol{D}_2[h, R^3]) = C_\Gamma(\boldsymbol{D}_1[h]) \cap C_\Gamma(\boldsymbol{D}_1[hR^3]) = \boldsymbol{D}_2[h, R^3] \dot{+} (E^+(h) \cap E^-(h)),$$

and $E^+(h) \cap E^-(h) = \boldsymbol{Z}_2[v_2]$. ∎

Next we compute the normalizers of the shifted subgroups K of $\boldsymbol{D}_6[h, R]$ given in theorem 5.1.

Theorem 6.3. *The normalizers in Γ of the subgroups K of theorem 5.1 are*
(1) $N_\Gamma(\boldsymbol{1}) = \Gamma$,
(2) $N_\Gamma(\boldsymbol{Z}_2[R^3]) = \boldsymbol{D}_6[h, R] \dot{+} \frac{1}{2}\mathcal{L}$,
(3) $N_\Gamma(\boldsymbol{Z}_3[R^2]) = \boldsymbol{D}_6[h, R] \dot{+} \boldsymbol{Z}_3[v_t]$,
(4) $N_\Gamma(\boldsymbol{Z}_6[R]) = \boldsymbol{D}_6[h, R]$,
(5) $N_\Gamma(\boldsymbol{D}_1[h]) = \boldsymbol{D}_2[h, R^3] \dot{+} E^+(h)$,
(6) $N_\Gamma(\boldsymbol{D}_1[hR^3]) = \boldsymbol{D}_2[h, R^3] \dot{+} E^-(h)$,
(7) $N_\Gamma(\boldsymbol{D}_2[h, R^3]) = \boldsymbol{D}_2[h, R^3] \times \boldsymbol{Z}_2[v_2]$,
(8) $N_\Gamma(\boldsymbol{D}_3[h, R^2]) = \boldsymbol{D}_6[h, R]$, *and*
(9) $N_\Gamma(\boldsymbol{D}_3[hR, R^2]) = \boldsymbol{D}_6[h, R] \dot{+} \boldsymbol{Z}_3[v_t]$ *where* $v_t = \frac{1}{3}(\ell_1 + \ell_2)$, *and*
(10) $N_\Gamma(\boldsymbol{D}_6[h, R]) = \boldsymbol{D}_6[h, R]$.

Proof. We proceed by computing the normalizers for each of the subgroups listed in proposition 3.1.

Case 1. $K = \mathbf{1}$. This case is trivial.

Case 2. $K = \mathbf{Z}_2[R^3]$. We conjugate $\mathbf{Z}_2[R^3]$ by (a, v) obtaining

$$(a, v)(R^3, 0)(a^{-1}, -a^{-1}v) = (R^3, 2v).$$

Thus, $(a, v) \in N_\Gamma(\mathbf{Z}_2[R^3])$ precisely when $v \in \frac{1}{2}\mathcal{L}$.

Cases 4, 10. $K = \mathbf{Z}_6[R]$ *or* $K = \mathbf{D}_6[h, R]$. Suppose $(q, 0) \in K$ and $(a, v) \in N_\Gamma(K)$. Then, conjugating by (a, v) produces

$$(a, v)(q, 0)(a^{-1}, -a^{-1}v) = (aqa^{-1}, v - aqa^{-1}v). \tag{6.1}$$

Since $K \subset H$, it follows that $v - aqa^{-1}v = 0$ modulo $\mathcal{L}$.

We first observe that if K contains the rotation R, then $N_\Gamma(K) = N_{\mathrm{H}}(K)$. If $q = R$ or R^5, then aqa^{-1} is either R or R^5 for all $a \in \mathbf{D}_6$. Moreover, $w - Sw = 0$ in $\mathbf{T}^2$, where $S = R$ or $S = R^5$ and $w = \alpha\ell_1 + \beta\ell_2$, implies that $\alpha \equiv \beta \equiv 0 \pmod 1$. It follows that $v - aqa^{-1}v = 0$ modulo $\mathcal{L}$ is satisfied only if $v = 0$ and then $N_\Gamma(K) \subset N_{\mathrm{H}}(K)$. The reverse inclusion is always valid. Standard calculations now prove cases 4 and 10.

Cases 3, 9. $K = \mathbf{Z}_3[R^2]$ *or* $K = \mathbf{D}_3[hR, R^2]$. We observe that if K contains the rotation R^2, then

$$N_\Gamma(K) \subset N_{\mathrm{H}}(K) \dot{+} \mathbf{Z}_3[v_t]. \tag{6.2}$$

Suppose that (a, v) is in $N_\Gamma(K)$. From (6.1) it is easy to see that we must have $a \in N_{\mathrm{H}}(K)$. Moreover, to satisfy the relation $v - aqa^{-1}v = 0 \in \mathbf{T}^2$, where $q = R^2$ or $q = R^4$, we must have that $v \in \mathbf{Z}_3[v_t]$. This follows from two facts: aqa^{-1} is either R^2 or R^4 for all $a \in \mathbf{D}_6$, and $w - Sw = 0$ in $\mathbf{T}^2$, where $S = R^2$ or $S = R^4$ and $w = \alpha\ell_1 + \beta\ell_2$, implies that $\alpha \equiv \beta \equiv 0, \frac{1}{3}$ or $\frac{2}{3} \pmod 1$.

The normalizers of $K = \mathbf{Z}_3[R^2]$ and $K = \mathbf{D}_3[hR, R^2]$ are easy to compute when we realize that $\mathbf{Z}_3[v_t]$ is a subset of $N_\Gamma(K)$ and $N_{\mathrm{H}}(\mathbf{Z}_3[R^2]) = N_{\mathrm{H}}(\mathbf{D}_3[hR, R^2]) = \mathbf{D}_6[h, R]$. Note that

$$(e, 2v_t)(hR, 0)(e, v_t) = (hR, 0).$$

Cases 5–7. $K = \mathbf{D}_1[h]$, $K = \mathbf{D}_1[hR^3]$ *or* $K = \mathbf{D}_2[h, R^3]$. These cases follow directly from proposition 6.2.

Case 8. $K = \mathbf{D}_3[h, R^2]$. Suppose that (a, v) is in the normalizer of $\mathbf{D}_3[h, R^2]$, where $v \in \mathbf{Z}_3[v_t]$ (note that (6.2) is still true here). It is always possible to find $q \in \mathbf{D}_3[h, R^2]$ such that $aqa^{-1} = h$. With this choice of q in (6.1), we obtain $v - hv = 0 \in \mathbf{T}^2$. However, $v - hv = 0 \in \mathbf{T}^2$, where $v \in \mathbf{Z}_3[v_t]$, is satisfied only if $v = 0$. This shows that $N_\Gamma(\mathbf{D}_3[h, R^2]) = N_{\mathrm{H}}(\mathbf{D}_3[h, R^2]) = \mathbf{D}_6[h, R]$. ■

(*b*) *Square lattice*

Theorem 6.4. *The normalizers in Γ of the shifted subgroups in theorem 5.2 are*

(1) $N_\Gamma(\mathbf{1}) = \Gamma$,
(2) $N_\Gamma(\mathbf{Z}_2[R^2]) = \mathbf{D}_4[h, R] \dot{+} \frac{1}{2}\mathcal{L}$,
(3) $N_\Gamma(\mathbf{Z}_4[R]) = \mathbf{D}_4[h, R] \dot{+} \mathbf{Z}_2[v_d]$,
(4) $N_\Gamma(\mathbf{D}_1[h]) = N_\Gamma(\mathbf{D}_1[(h, v_1)]) = \mathbf{D}_2[h, R^2] \dot{+} F^+(h)$,
(5) $N_\Gamma(\mathbf{D}_1[Rh]) = \mathbf{D}_2[Rh, R^2] \dot{+} E^+(Rh)$,
(6) $N_\Gamma(\mathbf{D}_2[h, R^2]) = N_\Gamma(\mathbf{D}_2[(h, v_d), R^2]) = \mathbf{D}_4[h, R] \dot{+} \frac{1}{2}\mathcal{L}$,
(7) $N_\Gamma(\mathbf{D}_2[(h, v_1), R^2]) = \mathbf{D}_2[h, R^2] \times \frac{1}{2}\mathcal{L}$,

(8) $N_\Gamma(\boldsymbol{D}_2[Rh, R^2]) = \boldsymbol{D}_4[h, R] \times \boldsymbol{Z}_2[v_d]$, *and*
(9) $N_\Gamma(\boldsymbol{D}_4[h, R]) = N_\Gamma(\boldsymbol{D}_4[(h, v_d), R]) = \boldsymbol{D}_4[h, R] \times \boldsymbol{Z}_2[v_d]$.

Proof.

Case 1. Trivial.

Case 2. We have shown in (5.1) that (g, v) commutes with $\boldsymbol{Z}_2[R^2]$ precisely when $v \in \frac{1}{2}\mathcal{L}$.

Case 3. We know that $(g, 0)$ normalizes $\boldsymbol{Z}_4[R]$ for any $g \in \boldsymbol{D}_4[h, R]$. Any (e, w) that normalizes $\boldsymbol{Z}_4[R]$ must commute with $(R^2, 0)$; hence, $w \in \frac{1}{2}\mathcal{L}$ by case 2. Now compute $(e, -w)(R, 0)(e, w) = (R, Rw - w)$. Note that when $w = 0$ or $w = v_d$, the element $Rw - w \equiv 0$ and $(R, Rw - w)$ is in $\boldsymbol{Z}_4[R]$; however, this conclusion is invalid when $w = v_1$ or $w = v_2$.

Case 4. Suppose that (g, w) commutes with either $\boldsymbol{D}_1[h]$ or $\boldsymbol{D}_1[(h, v_1)]$. Then,

$$(g, w)(h, v) = (h, v)(g, w),$$

where either $v = 0$ or $v = v_1$. This implies that $gh = hg$ and $w + gv = v + hw$. Hence, g may be e, h, R^2 or hR^2. For any such g, we have $gv = v$ in $\boldsymbol{T}^2$; hence, $hw = w$. By definition, $(e, w) \in F^+(h)$ and we have computed the normalizer.

Case 5. Suppose that (g, w) commutes with $\boldsymbol{D}_1[Rh]$. Then

$$(g, w)(Rh, 0) = (Rh, 0)(g, w)$$

implies that $gRh = Rhg$ and $w = Rhw$. Hence, g must be either e, Rh, R^2 or R^3h; and $w \in E^+(Rh)$.

Case 6. Observe that if $K = \boldsymbol{D}_2[R^2, (g, v)]$, where $g = h$, $v = 0$, $v = v_1$ or $v = v_d$, then

$$\tfrac{1}{2}\mathcal{L} = N_\Gamma(K) \cap \boldsymbol{T}^2.$$

To verify this point, note that from $(e, w)K(e, -w) \subset K$, it follows that

$$(e, w)(h, v)(e, -w) = (h, v) \quad \text{and} \quad (e, w)(R^2h, v)(e, -w) = (R^2h, v).$$

Hence, $w - hw = 0$ and $w - R^2hw = w + hw = 0$ in $\boldsymbol{T}^2$. Thus, $w \in \frac{1}{2}\mathcal{L}$.

In the cases where $v = 0$ or $v = v_d$, we may check that $\boldsymbol{D}_4[h, R] \subset N_\Gamma(K)$. (Note that 0 and v_d are fixed under the action of $\boldsymbol{D}_4[h, R]$.) Hence, we have that

$$N_\Gamma(K) \supset \boldsymbol{D}_4[h, R] \dot{+} \tfrac{1}{2}\mathcal{L}.$$

In fact we have equality. Suppose that $(g, w) \in N_\Gamma(K)$. Since $(g, 0) \in N_\Gamma(K)$, it follows that (e, w) is also in $N_\Gamma(K)$. Hence, $w \in \frac{1}{2}\mathcal{L}$ and $(g, w) \in \boldsymbol{D}_4[h, R] \dot{+} \frac{1}{2}\mathcal{L}$.

Case 7. In the case where $K = \boldsymbol{D}_2[(h, v_1), R^2]$, it is easy to check that $\boldsymbol{D}_2[h, R^2] \subset N_\Gamma(K)$. (Note that the elements of $\frac{1}{2}\mathcal{L}$ are fixed under the action of $\boldsymbol{D}_2[h, R^2]$.) Hence,

$$N_\Gamma(K) \supset \boldsymbol{D}_2[h, R^2] \dot{+} \tfrac{1}{2}\mathcal{L}.$$

Again, we have equality. Suppose that we have strict inclusion, then $(R, w) \in N_\Gamma(K)$ for some $w \in \boldsymbol{T}^2$. However, this is impossible since

$$(R, w)(h, v_1)(R^{-1}, -R^{-1}w) = (hR^2, w + v_2 + hw) \in K$$

implies that $w + hw + v_2 \equiv 0$. However, $w + hw \in E^+(h)$ and $0 \neq v_2 \in E^-(h)$ and $E^+(h) \cap E^-(h) = \{0\}$.

Case 8. We consider the case $\boldsymbol{D}_2[Rh, R^2]$ and note that

$$N_\Gamma(\boldsymbol{D}_2[Rh, R^2]) \cap \boldsymbol{T}^2 = \boldsymbol{Z}_2[v_d].$$

From $(e, w)\boldsymbol{D}_2[Rh, R^2](e, -w) \subset \boldsymbol{D}_2[Rh, R^2]$, it follows that

$$(e, w)(Rh, 0)(e, -w) = (Rh, 0)$$

and

$$(e, w)(R^3h, 0)(e, -w) = (R^3h, 0).$$

Hence, $w - Rhw = 0$ and $w - R^3hw = w + Rhw = 0$ in $\boldsymbol{T}^2$. Thus, $w \in \frac{1}{2}\mathcal{L}$. We now check that $w - Rhw = 0 \in \boldsymbol{T}^2$ only when $w = 0$ or $w = v_d$, as desired.

It is easy to see that $\boldsymbol{D}_4[h, R] \subset N_\Gamma(K)$. Hence,

$$N_\Gamma(K) \supset \boldsymbol{D}_4[h, R] \dot{+} \boldsymbol{Z}_2[v_d].$$

To prove that we have equality, we suppose that $(g, w) \in N_\Gamma(K)$. Since $(g, 0) \in N_\Gamma(K)$, it follows that $(e, w) \in N_\Gamma(K)$. Hence, $w \in \boldsymbol{Z}_2[v_d]$ and $(g, w) \in \boldsymbol{D}_4[h, R] \dot{+} \boldsymbol{Z}_2[v_d]$.

Case 9. Let $K = \boldsymbol{D}_4[h, R]$ or $K = \boldsymbol{D}_4[(h, v_d), R]$. Observe that any group element that normalizes K automatically normalizes $\boldsymbol{Z}_4$. It then follows that

$$K \dot{+} \boldsymbol{Z}_2[v_d] \subset N_\Gamma(K) \subset N_\Gamma(\boldsymbol{Z}_4[R]) = K \dot{+} \boldsymbol{Z}_2[v_d],$$

where the first inclusion is found by direct calculation. ■

(c) *Rhombic lattice*

Theorem 6.5. *The normalizers in Γ of the shifted subgroups in theorem 5.3 are*

(1) $N_\Gamma(\boldsymbol{1}) = \Gamma$,
(2) $N_\Gamma(\boldsymbol{Z}_2[P]) = \boldsymbol{D}_2[h, P] \dot{+} \frac{1}{2}\mathcal{L}$,
(3) $N_\Gamma(\boldsymbol{D}_1[h]) = \boldsymbol{D}_2[h, P] \dot{+} E^+(h)$,
(4) $N_\Gamma(\boldsymbol{D}_1[Ph]) = \boldsymbol{D}_2[h, P] \dot{+} E^-(h)$, *and*
(5) $N_\Gamma(\boldsymbol{D}_2[h, P]) = \boldsymbol{D}_2[h, P] \times \boldsymbol{Z}_2[v_d]$.

Proof. Since $C_\Gamma(K) = N_\Gamma(K)$ for any of the groups that we have to consider, we only have to compute the centralizers of these groups. See lemma 6.1.

Case 1. Trivial.

Case 2. It follows from $(g, v)(P, 0) = (P, 0)(g, v)$ that $gP = Pg$ and $-v = Pv = v$. Hence, g is any element of $\boldsymbol{D}_2[h, P]$ and $v \in \frac{1}{2}\mathcal{L}$.

Case 3. It follows from $(g, v)(h, 0) = (h, 0)(g, v)$ that $gh = hg$ and $hv = v$. Hence, g is any element of $\boldsymbol{D}_2[h, P]$ and $v \in E^+(h)$.

Case 4. Similar to case 3; just replace h by Ph.

Case 5. This result follows from cases 3 and 4

$$\begin{aligned} C_\Gamma(\boldsymbol{D}_2[h, P]) &= C_\Gamma(\boldsymbol{D}_1[h]) \cap C_\Gamma(\boldsymbol{D}_1[Ph]) \\ &= \boldsymbol{D}_2[h, P] \dot{+} E^+(h) \cap E^-(h). \end{aligned}$$

Note that $E^+(h) \cap E^-(h) = \boldsymbol{Z}_2[v_d]$ and that $\boldsymbol{Z}_2[v_d]$ commutes with $\boldsymbol{D}_2[h, P]$. ■

7. Wave pairs

We now list up to conjugacy the wave-pair subgroups $K \subset G$. See definition 2.2.

(*a*) *Hexagonal lattice*

We first begin with the hexagonal lattice.

Theorem 7.1. *Let $\Gamma = \boldsymbol{D}_6 \dot{+} \boldsymbol{T}^2$ be the group of symmetries of the hexagonal lattice. Up to conjugacy, the pairs of subgroups $K \subset G$, where K is a shifted subgroup, $G \subset N_\Gamma(K)$, G/K is cyclic or isomorphic to $\boldsymbol{S}^1$ and G/K is maximal Abelian in $N_\Gamma(K)/K$, are*

(1) $K = \boldsymbol{1}$ and $G = \boldsymbol{Z}_6$,
(2) $K = \boldsymbol{Z}_2[R^3]$ and $G = \boldsymbol{Z}_2[R^3] \times \boldsymbol{Z}_4[(h, v_1)]$,
(3) $K = \boldsymbol{Z}_2[R^3]$ and $G = \boldsymbol{Z}_6[R]$,
(4) $K = \boldsymbol{D}_2[h, R^3]$ and $G = \boldsymbol{D}_2[h, R^3] \times \boldsymbol{Z}_2[v_2]$,
(5) $K = \boldsymbol{Z}_6[R]$ and $G = \boldsymbol{D}_6[h, R]$,
(6) $K = \boldsymbol{D}_3[h, R^2]$ and $G = \boldsymbol{D}_6[h, R]$,
(7) $K = \boldsymbol{D}_3[hR, R^2]$ and $G = \boldsymbol{D}_6[h, R]$,
(8) $K = \boldsymbol{D}_3[hR, R^2]$ and $G = \boldsymbol{D}_3[hR, R^2] \dot{+} \boldsymbol{Z}_3[v_t]$ where $v_t = \frac{1}{3}(\ell_1 + \ell_2)$,
(9) $K = \boldsymbol{D}_6[h, R]$ and $G = \boldsymbol{D}_6[h, R]$,
(10) $K = \boldsymbol{D}_1[h]$ and $G = \boldsymbol{D}_1[h] \times E^+(h)$, and
(11) $K = \boldsymbol{D}_1[hR^3]$ and $G = \boldsymbol{D}_1[hR^3] \times E^-(h)$.

Proof. Again the proof proceeds by considering each of the subgroups listed in proposition 3.1.

Case 1. $K = \boldsymbol{1}$. From theorem 2.1, we see that G must be a maximal Abelian subgroup of Γ. If G is isomorphic to $\boldsymbol{S}^1$, then G is a strict subgroup of $\boldsymbol{T}^2$ and therefore is not maximal Abelian. Suppose that G is cyclic and let a generator for G be (S, v), where $S \in H$ and $v \in \boldsymbol{T}^2$. There are three possibilities: $S = r$, a rotation; $S = q$, a reflection; or $S = e$. In the last case, $G \subset \boldsymbol{T}^2$ and, hence, G is not maximal Abelian. In the second case $G \subset G \times E^+(q)$ and again G is not maximal Abelian. In the first case G is generated by (r, v) and, by proposition 3.1, G is conjugate to $\boldsymbol{Z}_k$ for $k = 2, 3, 6$. Only when $G = \boldsymbol{Z}_6[R]$ is G maximal Abelian.

Case 2. $K = \boldsymbol{Z}_2[R^3]$. From proposition 6.3 we see that

$$N_\Gamma(\boldsymbol{Z}_2[R^3]) = \boldsymbol{Z}_2[R^3] \times (\boldsymbol{D}_3[h, R^2] \dot{+} \tfrac{1}{2}\mathcal{L})$$

and hence that

$$N_\Gamma(\boldsymbol{Z}_2[R^3])/\boldsymbol{Z}_2[R^3] \cong \Delta \equiv \boldsymbol{D}_3[h, R^2] \dot{+} \tfrac{1}{2}\mathcal{L}.$$

We proceed by first determining the cyclic maximal Abelian subgroups G' of Δ. Let (S, v) be a generator of G'. We claim that up to conjugacy in Δ either (S, v) equals $(R^2, 0)$ or (h, v_1). To verify the claim, note that up to conjugacy in $\boldsymbol{D}_3[h, R^2]$ it is enough to consider $S = e$, $S = R^2$ or $S = h$. When $S = e$, G' is strictly included in $\frac{1}{2}\mathcal{L}$ and G' is not maximal Abelian. When $S = R^2$, then (R^2, v) is conjugate to $(R^2, 0)$. (Use (e, w), where $w \in \frac{1}{2}\mathcal{L}$ to perform the conjugacy.) In this case, $G' = \boldsymbol{Z}_3[R^2]$ is a maximal-Abelian cyclic subgroup of $\boldsymbol{D}_3[h, R^2] \dot{+} \frac{1}{2}\mathcal{L}$. Now consider the case $S = h$. There are four possibilities for (h, v) and they are given by $v = 0$, $v = v_2$, $v = v_1$ or $v = v_d$. For the first two possibilities, G' is strictly included in $\boldsymbol{D}_1[h] \times \boldsymbol{Z}_2[v_2]$, which is Abelian but not cyclic. For the last two possibilities, $G' = \boldsymbol{Z}_4[(h, v_1)]$ which is a maximal Abelian and cyclic.

In the previous paragraph we showed that, up to conjugacy, any subgroup $G \subset N_\Gamma(\boldsymbol{Z}_2[R^3])$ that contains $\boldsymbol{Z}_2[R^3]$ and whose quotient with $\boldsymbol{Z}_2[R^3]$ is cyclic and maximal Abelian in $N_\Gamma(\boldsymbol{Z}_2[R^3])/\boldsymbol{Z}_2[R^3]$ must project onto one of two subgroups: $\boldsymbol{Z}_3[R^2]$ or $\boldsymbol{Z}_4[(h, v_1)]$. Only $G = \boldsymbol{Z}_6[R]$ projects onto the first group and, up to conjugacy, only $G = \boldsymbol{Z}_2[R^3] \times \boldsymbol{Z}_4[(h, v_1)]$ projects onto the second group.

Case 3. $K = \boldsymbol{Z}_3[R^2]$. In this case there is no G associated with K. Since

$$N_\Gamma(K) = \boldsymbol{Z}_3[R^2]\dot{+}(\boldsymbol{D}_2[h, R^3]\dot{+}\boldsymbol{Z}_3[v_t]),$$

we need to show that each cyclic subgroup G' of

$$N_\Gamma(K)/K \cong \boldsymbol{D}_2[h, R^3]\dot{+}\boldsymbol{Z}_3[v_t]$$

is strictly contained in an Abelian subgroup Δ. Let $v \in \boldsymbol{Z}_3[v_t]$. If G' is generated by (e, v) or (hR^3, v), we take $\Delta = \boldsymbol{D}_1[hR^3] \times \boldsymbol{Z}_3[v_t]$. If G' is generated by (R^3, v), then we take $\Delta = \boldsymbol{D}_1[hR^3] \times \boldsymbol{Z}_2[(R^3, v)]$. Finally, if G' is generated by (h, v), then we take $\Delta = \boldsymbol{D}_1[(h, v)] \times \boldsymbol{Z}_2[(R^3, v)]$.

Cases 4, 7, 8, and 10. $K = \boldsymbol{Z}_6[R]$, $K = \boldsymbol{D}_2[h, R^3]$, $K = \boldsymbol{D}_3[h, R^2]$ *or* $K = \boldsymbol{D}_6[h, R]$. Observe that in each of these cases $N_\Gamma(K)/K$ is cyclic. The requirement that G/K be maximal Abelian implies that $G = N_\Gamma(K)$ is the only possibility.

Cases 5, 6. $K = \boldsymbol{D}_1[h]$ *or* $K = \boldsymbol{D}_1[hR^3]$. We first show that there are no finite Gs associated with these Ks. It follows from proposition 6.3 that

$$N_\Gamma(K) = K \times (\boldsymbol{D}_1[q^\perp]\dot{+}E^+(q)),$$

where $K = \boldsymbol{D}_1[q]$. To satisfy theorem 2.1, we need to find cyclic subgroups G' of $N_\Gamma(K)/K \cong \boldsymbol{D}_1[q^\perp]\dot{+}E^+(q)$ which are also maximal Abelian. Let (S, v) be the generator of G'. If $S = e$, then G' is strictly included in $E^+(q)$. Hence, G' is not maximal Abelian. Suppose that $S = q^\perp$, the other reflection in $\boldsymbol{D}_2$. We may choose $w \in E^+(q)$ such that $(e, w)(S, v)(e, -w) = (S, 2w + v) = (S, 0)$ and $(e, w)\boldsymbol{D}_1(e, -w) = \boldsymbol{D}_1$. Hence, we may assume that G' is generated by $(S, 0)$. However, then G' is strictly included in $\boldsymbol{D}_1[q^\perp] \times \boldsymbol{Z}_2[v_2]$, which is Abelian. Hence, G' is not maximal Abelian.

The group $G = K \times E^+(q)$ satisfies $G/K \cong \boldsymbol{S}^1$ and G/K is maximal Abelian in $N_\Gamma(K)/K$. The continuous group $E^+(q)$ is maximal Abelian in $\boldsymbol{D}_1[q^\perp]\dot{+}E^+(q)$ since $q^\perp$ does not commute with all the elements of $E^+(q)$.

Case 9. $K = \boldsymbol{D}_3[hR, R^2]$. We show that, up to conjugacy, we have $G = \boldsymbol{D}_6[h, R]$ or $G = \boldsymbol{D}_3[hR, R^2]\dot{+}\boldsymbol{Z}_3[v_t]$. Since

$$N_\Gamma(K) = \boldsymbol{D}_3[hR, R^2]\dot{+}(\boldsymbol{Z}_2[R^3]\dot{+}\boldsymbol{Z}_3[v_t]),$$

we need to find maximal-Abelian cyclic subgroups G' of

$$N_\Gamma(K)/K \cong \boldsymbol{Z}_2[R^3]\dot{+}\boldsymbol{Z}_3[v_t].$$

If G' is generated by (e, v), where $0 \neq v \in \boldsymbol{Z}_3[v_t]$, then $G' = \boldsymbol{Z}_3[v_t]$ is maximal Abelian. If $v = 0$, then $G' = \boldsymbol{1}$ is strictly included in the Abelian group $\boldsymbol{Z}_2[R^3]$, say. If G' is generated by (R^3, v), where $v \in \boldsymbol{Z}_3[v_t]$, then $G' = \boldsymbol{Z}_2[(R^3, v)]$ is maximal Abelian. Note that $\boldsymbol{D}_3[hR, R^2]\dot{+}\boldsymbol{Z}_2[(R^3, v_t)]$ and $\boldsymbol{D}_3[hR, R^2]\dot{+}\boldsymbol{Z}_2[(R^3, 2v_t)]$ are conjugate to $\boldsymbol{D}_6[h, R]$. The conjugacy for the first group is given by (e, v_t) and for the second group by $(e, 2v_t)$. ∎

(b) *Square lattice*

For the square lattice, we have the following result.

Theorem 7.2. *Up to conjugacy, the pairs of subgroups $K \subset G$, where K is a shifted subgroup, $G \subset N_\Gamma(K)$, G/K is cyclic or isomorphic to $\boldsymbol{S}^1$ and G/K is maximal Abelian in $N_\Gamma(K)/K$, are*

(1) $K = \boldsymbol{D}_2[h, R^2]$ and $G = \boldsymbol{D}_2[h, R^2]\dot{+}\boldsymbol{Z}_4[(Rh, v_1)]$,
(2) $K = \boldsymbol{D}_2[(h, v_d), R^2]$ and $G = \boldsymbol{D}_2[(h, v_d), R^2]\dot{+}\boldsymbol{Z}_4[(Rh, v_1)]$,
(3) $K = \boldsymbol{D}_4[h, R]$ and $G = \boldsymbol{D}_4[h, R]\dot{+}\boldsymbol{Z}_2[v_d]$,
(4) $K = \boldsymbol{D}_4[(h, v_d), R]$ and $G = \boldsymbol{D}_4[h, R]\dot{+}\boldsymbol{Z}_2[v_d]$, and
(5) $K = \boldsymbol{D}_1[Rh]$ and $G = \boldsymbol{D}_1[Rh] \times E^+(Rh)$.

Proof. The method of proof is to determine, for each K in theorem 6.4, the possible Gs for which G/K is cyclic or isomorphic to $\boldsymbol{S}^1$, and maximal Abelian in $N_\Gamma(K)/K$.

Case 1. $K = \boldsymbol{1}$. We show that there are no subgroups G that pair with $\mathbf{1}$; the proof is similar to the one in theorem 7.1. If G is isomorphic to $\boldsymbol{S}^1$, then G is a strict subgroup of $\boldsymbol{T}^2$ and therefore is not maximal Abelian. The subgroup G must be a cyclic maximal Abelian subgroup of Γ. Let $(S, v) \in \Gamma$ be the generator of G. If $S = e$, then G is strictly included in $\boldsymbol{T}^2$ and therefore G is not maximal Abelian. If S is a reflection, then G is strictly included in $G \times E^+(S)$ and again G is not maximal Abelian. Finally, if S is a rotation, then we may assume that G is generated by $(S, 0)$ after conjugacy. Hence, $G = \boldsymbol{Z}_2[R^2]$ or $\boldsymbol{Z}_4[R]$. In either case, G is strictly included in $G \times \boldsymbol{Z}_2[v_d]$ and G is not maximal Abelian.

Case 2. $K = \boldsymbol{Z}_2[R^2]$. We show that there are no subgroups G that pair with K. We know that $\boldsymbol{Z}_2[R^2] \subset G \subset N_\Gamma(\boldsymbol{Z}_2[R^2]) = \boldsymbol{D}_4[h, R]\dot{+}\frac{1}{2}\mathcal{L}$. In this case,

$$N_\Gamma(K)/K \cong (\boldsymbol{D}_4[h, R]/\boldsymbol{Z}_2[R^2])\dot{+}\tfrac{1}{2}\mathcal{L}$$

is a sixteen-element group. By direct calculation one can show that the elements of $N_\Gamma(K)/K$ have order one, two or four. The elements of order two can be embedded either in a subgroup isomorphic to $\boldsymbol{Z}_4[R]$ or $\boldsymbol{D}_2[h, R^2]$. In either case these cyclic groups are not maximal Abelian. There are four elements of order four; they are the cosets of (R, v_1), (R, v_2), (Rh, v_1) and (Rh, v_2). Each of these commutes with the coset of $(h, 0)$ in $(\boldsymbol{D}_4[h, R]/\boldsymbol{Z}_2[R^2])\dot{+}\frac{1}{2}\mathcal{L}$ and the element of order four together with $(h, 0)$, generates an eight-element non-cyclic Abelian group.

Case 3, 7, 8. $K = \boldsymbol{Z}_4[R]$, $K = \boldsymbol{D}_2[(h, v_1), R^2]$ or $K = \boldsymbol{D}_2[Rh, R^2]$. In these cases $N_\Gamma(K)/K$ is Abelian but not cyclic. Hence, there are no Gs associated with these Ks.

Case 4. $K = \boldsymbol{D}_1[(h, v)]$ where $v = 0$ or $v = v_1$. We first show that there are no finite subgroups G associated with these Ks. We note that

$$N_\Gamma(K) = \boldsymbol{D}_1[(h, v)] \times (\boldsymbol{Z}_2[R^2]\dot{+}F^+(h)).$$

We have to find cyclic maximal Abelian subgroups G' of $N_\Gamma(K)/K \cong \boldsymbol{Z}_2[R^2]\dot{+}F^+(h)$. Let $(S, w) \in N_\Gamma(K)$ be the generator of G'. As usual, if $S = e$ then G' is strictly included in $F^+(h)$. Thus, G' is not maximal Abelian. If $S = R^2$, then (S, w) is of order two. Moreover, (S, w) commutes with (e, v_1); and G' is not maximal Abelian. There is no group G such that $G/K \cong \boldsymbol{S}^1$ and G/K is maximal Abelian in $N_\Gamma(K)/K$. Note that there is no subgroup G' of $F^+(h)$ such that $G' \cong \boldsymbol{S}^1$ and G' is maximal Abelian in $F^+(h)$ for $F^+(h) \cong E^+(h) \times \boldsymbol{Z}_2[v_2] \ncong \boldsymbol{S}^1$.

Case 5. $K = \boldsymbol{D}_1[Rh]$. There are no finite Gs such that G/K is cyclic and maximal Abelian in $N_\Gamma(K)/K$; the proof proceeds exactly as in case 4 when $K = \boldsymbol{D}_1[h]$.

However, there is a group G such that $G/K \cong \boldsymbol{S}^1$ and G/K is maximal Abelian in $N_\Gamma(K)/K$. It is $G = \boldsymbol{D}_1[Rh] \times E^+(Rh)$. Note that

$$N_\Gamma(K) = \boldsymbol{D}_1[Rh] \times (\boldsymbol{Z}_2[R^2]\dot{+}E^+(Rh)).$$

and $E^+(Rh) \cong \boldsymbol{S}^1$ is maximal Abelian in $\boldsymbol{Z}_2[R^2]\dot{+}E^+(Rh)$.

Case 6. $K = \boldsymbol{D}_2[R^2,(h,v)]$ where $v = 0$ or $v = v_d$. We show that we can have $G = \boldsymbol{D}_2[R^2,(h,v)]\dot{+}\boldsymbol{Z}_4[(Rh, v_1)]$. Since

$$N_\Gamma(K) = K \times (\boldsymbol{D}_1[Rh]\dot{+}\tfrac{1}{2}\mathcal{L}),$$

we need to find cyclic maximal Abelian subgroups G' of $N_\Gamma(K)/K \cong \boldsymbol{D}_1[Rh]\dot{+}\frac{1}{2}\mathcal{L}$. Let (S, v) be the generator of G'.

If $S = e$, then G' is strictly included in $\frac{1}{2}\mathcal{L}$ and G' is not maximal Abelian.

If $S = Rh$ and $v = 0$ or $v = v_d$, then G' is strictly included in $\boldsymbol{D}_1[Rh] \times \boldsymbol{Z}_2[v_d]$ and again G' is not maximal Abelian.

If $S = Rh$ and $v = v_1$ or $v = v_2$, then $G' = \boldsymbol{Z}_4[(Rh, v_1)]$ is cyclic and maximal Abelian. Moreover, only the subgroup $K\dot{+}\boldsymbol{Z}_4[(Rh, v_1)]$ of $N_\Gamma(K)$ projects onto $\boldsymbol{Z}_4[(Rh, v_1)]$ in $N_\Gamma(K)/K$.

Case 9. $K = \boldsymbol{D}_4[h, R]$ or $K = \boldsymbol{D}_4[(h, v_d), R]$. In this case G must be $N_\Gamma(K)$ since $N_\Gamma(K)/K \cong \boldsymbol{Z}_2[v_d]$ is cyclic. ∎

(*c*) *Rhombic lattice*

Finally, we consider the rhombic lattice.

Theorem 7.3. *Up to conjugacy, the pairs of subgroups $K \subset G$ where K is a shifted subgroup, $G \subset N_\Gamma(K)$, G/K is cyclic or isomorphic to $\boldsymbol{S}^1$, and G/K is maximal Abelian in $N_\Gamma(K)/K$ are*

(1) $K = \boldsymbol{Z}_2[P]$ and $G = \boldsymbol{Z}_2[P] \times \boldsymbol{Z}_4[(h, v_1)]$,
(2) $K = \boldsymbol{D}_2[h, P]$ and $G = \boldsymbol{D}_2[h, P] \times \boldsymbol{Z}_2[v_d]$,
(3) $K = \boldsymbol{D}_1[h]$ and $G = \boldsymbol{D}_1[h] \times E^+(h)$, and
(4) $K = \boldsymbol{D}_1[Ph]$ and $G = \boldsymbol{D}_1[Ph] \times E^-(h)$.

Proof. We consider in order the five Ks listed in proposition 3.3.

Case 0. $K = \boldsymbol{1}$. We first show that there is no finite G associated with K. The subgroup G must be a cyclic maximal Abelian subgroup of Γ. Let $(S, v) \in \Gamma$ be the generator of G. If $S = e$, then G is strictly included in $\boldsymbol{T}^2$ and therefore G is not maximal Abelian. If S is a reflection, then G is strictly included inside $G \times E^+(S)$, where $E^+(S) = \{v \in \boldsymbol{T}^2 : Sv = v\}$, and again G is not maximal Abelian. Finally, if $S = P$, then we may assume that G is generated by S alone after conjugacy. Hence, $G = \boldsymbol{Z}_2[P]$ and G is strictly included in $G \times \frac{1}{2}\mathcal{L}$ and G is still not maximal Abelian. If $G \cong \boldsymbol{S}^1$, then G is a strict subgroup of $\boldsymbol{T}^2$ and therefore is not maximal Abelian.

Case 1. $K = \boldsymbol{Z}_2[P]$. We have that

$$N_\Gamma(K) = \boldsymbol{Z}_2[P] \times (\boldsymbol{D}_1[h]\dot{+}\tfrac{1}{2}\mathcal{L}).$$

We wish to find cyclic subgroups G' of $N_\Gamma(K)/K \cong \boldsymbol{D}_1[h]\dot{+}\frac{1}{2}\mathcal{L}$ which are also maximal Abelian. Let $(S, v) \in N_\Gamma(K)$ be the generator of G'. If $S = e$, then G' is not maximal Abelian since it is strictly included in $\frac{1}{2}\mathcal{L}$. Similarly, if $S = h$ and $v = 0$ or $v = v_d$, then G' is strictly included in $\boldsymbol{D}_1[h] \times \boldsymbol{Z}_2[v_d]$. However, if $S = h$ and $v = v_1$ or $v = v_2$, then $G' = \boldsymbol{Z}_4[(h, v_1)]$, which is maximal Abelian.

Case 2. $K = \boldsymbol{D}_2[h, P]$. The only choice for G is $N_\Gamma(K)$ since $N_\Gamma(K)/K \cong \boldsymbol{Z}_2[v_d]$ is cyclic.

Case 3. $K = \boldsymbol{D}_1[h]$. In this case,

$$N_\Gamma(K) = \boldsymbol{D}_1[h] \times (\boldsymbol{D}_1[Ph] \dot{+} E^+(h)).$$

First we show that there is no finite subgroup G associated with K by considering cyclic subgroups G' of $\boldsymbol{D}_1[Ph] \dot{+} E^+(h)$ and showing that they are not maximal Abelian. Let $(S, v) \in N_\Gamma(K)$ be the generator of G'. As usual, if $S = e$ then G' is strictly included in $E^+(h)$. Thus, G' is not maximal Abelian. If $S = Ph$, we may assume that $v = 0$ for $(e, w)(S, v)(e, -w) = (S, v + 2w) = (S, 0)$ and $(e, w)(h, 0)(e, -w) = (h, 0)$ for some $w \in E^+(h)$. Hence, G' is strictly included in $\boldsymbol{D}_1[Ph] \times \boldsymbol{Z}_2[v_d]$ and G' is again not maximal Abelian. However, there is a group G for which $G/K \cong \boldsymbol{S}^1$ and G/K is maximal Abelian in $N_\Gamma(K)/K$. It is $G = \boldsymbol{D}_1[h] \times E^+(h)$, since $E^+(h) \cong \boldsymbol{S}^1$ is maximal Abelian in $\boldsymbol{D}_1[Ph] \dot{+} E^+(h)$.

Case 4. $K = \boldsymbol{D}_1[Ph]$. The computations for this case are similar to those for $\boldsymbol{D}_1[h]$ above. There are no finite Gs such that G/K is cyclic and maximal Abelian in $N_\Gamma(K)/K$ and there is a group G such that $G/K \cong \boldsymbol{S}^1$ and G/K is maximal Abelian in $N_\Gamma(K)/K$. It is $G = \boldsymbol{D}_1[Ph] \times E^-(h)$. ∎

8. Irreducible representations

We wish to determine which of the wave pairs found in the previous section characterize symmetries of discrete or rotating waves that are guaranteed to exist by the equivariant Hopf theorem (see Golubitsky *et al.* 1988, theorem XVI, § 4.1). In particular, we need to determine those wave pairs that support twisted subgroups with two-dimensional fixed-point subspaces in $V \oplus V$, where V is a Γ-absolutely irreducible subspace of the space of $\mathcal{L}$-periodic functions. This is achieved in the following two sections. In this section we:

(i) describe the Γ-absolutely irreducible translation-free subspaces of the space of $\mathcal{L}$-periodic function; and

(ii) describe the action of Γ on these (finite) Γ absolutely irreducible, translation free subspaces.

First, in table 2, we give (without loss of generality) basis vectors for each planar lattice and its dual lattice.

It is shown in Dionne & Golubitsky (1992) that the Γ-absolutely irreducible translation-free subspaces of the space of $\mathcal{L}$-periodic functions are of the form

$$V = \left\{ \sum_{j=1}^{s} (z_j \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{K}_j \cdot \boldsymbol{x}} + \mathrm{c.c.}) : z_j \in \boldsymbol{C} \right\} \cong \boldsymbol{C}^s,$$

where the possible values of $\dim V = 2s$ and the $\boldsymbol{K}_j$s are given in table 3 for the pertinent lattices. Note that the rectangular and oblique lattices do not support translation-free absolutely irreducible subspaces (Dionne & Golubitsky 1992).

To compute the dimensions of the fixed-point subspaces of the finite twisted subgroups obtained in theorems 7.1–7.3 (see § 9), we need only define the action of some finite subgroups of Γ on V. This is performed in the next three subsections. To compute the dimensions of the fixed-point subspaces of the infinite twisted subgroups obtained in theorems 7.1–7.3, the action of $\Gamma \times \boldsymbol{S}^1$ on $V \oplus V$ will have to be considered (see § 10).

Table 2. *Lattices in two dimensions*

name	holohedry	basis of $\mathcal{L}$	basis of $\mathcal{L}^*$
hexagonal	$\boldsymbol{D}_6$	$\ell_1 = (1/\sqrt{3}, 1)$	$\boldsymbol{k}_1 = (0, 1)$
		$\ell_2 = (2/\sqrt{3}, 0)$	$\boldsymbol{k}_2 = (\sqrt{3}/2, -1/2)$
square	$\boldsymbol{D}_4$	$\ell_1 = (1, 0)$	$\boldsymbol{k}_1 = (1, 0)$
		$\ell_2 = (0, 1)$	$\boldsymbol{k}_2 = (0, 1)$
rhombic	$\boldsymbol{D}_2$	$\ell_1 = (1, -\cot\theta)$	$\boldsymbol{k}_1 = (1, 0)$
		$\ell_2 = (0, \operatorname{cosec}\theta)$	$\boldsymbol{k}_2 = (\cos\theta, \sin\theta)$
		$0 < \theta < \pi/2$, $\theta \neq \pi/3$	
rectangular	$\boldsymbol{D}_2$	$\ell_1 = (1, 0)$	$\boldsymbol{k}_1 = (1, 0)$
		$\ell_2 = (0, c)$	$\boldsymbol{k}_2 = (0, 1/c)$
		$0 < c < 1$	
oblique	$\boldsymbol{Z}_2$	$\lvert\ell_1\rvert \neq \lvert\ell_2\rvert$	
		$\ell_1 \cdot \ell_2 \neq 0$	

Remark. The generalized eigenspace of

$$\mathrm{D}_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}_0, \lambda_c) \tag{8.1}$$

associated with the eigenvalues $\pm\mathrm{i}$ is generated by finite sums of expressions of the form

$$z\mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{K}\cdot x+t)} + w\mathrm{e}^{2\pi\mathrm{i}(-\boldsymbol{K}\cdot x+t)} + \text{c.c.},$$

where $\boldsymbol{K}$ is in the dual lattice $\mathcal{L}^*$ and $|\boldsymbol{K}| = \kappa_c$ is the *critical wavenumber.* Hence, this eigenspace may not be of the form $V \oplus V$ with V being Γ-absolutely irreducible. Instead, V may be the direct sum of Γ-absolutely irreducible subspaces of the space of $\mathcal{L}$-periodic functions. For the square lattice this is the case if there exist two distinct integer pairs, (α_1, β_1) and (α_2, β_2), such that $\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = \kappa_c^2$, e.g., $(\alpha_1, \beta_1) = (4, 3)$ and $(\alpha_2, \beta_2) = (5, 0)$. Similarly, for the hexagonal lattice, the assumption of absolute irreducibility may not be satisfied for certain positive integer pairs (α_1, β_1) if their exists another non-negative integer pair (α_2, β_2) such that $\alpha_1^2 + \beta_1^2 - \alpha_1\beta_1 = \alpha_2^2 + \beta_2^2 - \alpha_2\beta_2 = \kappa_c^2$.

(*a*) *Hexagonal lattice*

The action of $\boldsymbol{D}_6[h, R]$ on $\boldsymbol{C}^3$ is generated by

$$R(z_1, z_2, z_3) = (\overline{z_2}, \overline{z_3}, \overline{z_1}),$$
$$h(z_1, z_2, z_3) = (\overline{z_2}, \overline{z_1}, \overline{z_3}).$$

To compute the dimensions of the fixed-point subspaces of the finite twisted groups G^Θ in theorem 7.1, we also need to know that:

$$(e, v_1)(z_1, z_2, z_3) = (-z_1, z_2, -z_3),$$
$$(e, v_2)(z_1, z_2, z_3) = (-z_1, -z_2, z_3),$$
$$(e, v_t)(z_1, z_2, z_3) = \mathrm{e}^{2\pi\mathrm{i}/3}(z_1, z_2, z_3).$$

The action of $\boldsymbol{D}_6[h, R]$ on $\boldsymbol{C}^6$ is generated by

$$R(z_1, z_2, z_3, z_4, z_5, z_6) = (\overline{z_2}, \overline{z_3}, \overline{z_1}, \overline{z_5}, \overline{z_6}, \overline{z_4}),$$
$$h(z_1, z_2, z_3, z_4, z_5, z_6) = (z_6, z_5, z_4, z_3, z_2, z_1).$$

Table 3. *Translation-free irreducible representations.* (α, β) *denotes the greatest common divisor of* α *and* β.

Name	dim V	$\boldsymbol{K}_s$
rhombic	4	$\boldsymbol{K}_1 = \boldsymbol{k}_1$
		$\boldsymbol{K}_2 = \boldsymbol{k}_2$
square	4	$\boldsymbol{K}_1 = \boldsymbol{k}_1$
		$\boldsymbol{K}_2 = \boldsymbol{k}_2$
	8	$\boldsymbol{K}_1 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$
		$\boldsymbol{K}_2 = (-\alpha+\beta)\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2$
		$\boldsymbol{K}_3 = \beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2$
		$\boldsymbol{K}_4 = -\alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$
		α and β are integers,
		$\alpha > \beta > 0$
		$\alpha+\beta$ is odd and $(\alpha, \beta) = 1$.
hexagonal	6	$\boldsymbol{K}_1 = \boldsymbol{k}_1 + \boldsymbol{k}_2$
		$\boldsymbol{K}_2 = -\boldsymbol{k}_2$
		$\boldsymbol{K}_3 = -\boldsymbol{k}_1$
	12	$\boldsymbol{K}_1 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$
		$\boldsymbol{K}_2 = (-\alpha+\beta)\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2$
		$\boldsymbol{K}_3 = -\beta\boldsymbol{k}_1 + (\alpha-\beta)\boldsymbol{k}_2$
		$\boldsymbol{K}_4 = \alpha\boldsymbol{k}_1 + (\alpha-\beta)\boldsymbol{k}_2$
		$\boldsymbol{K}_5 = -\beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2$
		$\boldsymbol{K}_6 = (-\alpha+\beta)\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$
		α and β are integers,
		$\alpha > \beta > \alpha/\beta > 0$,
		$(\alpha, \beta) = 1$ and $(3, \alpha+\beta) = 1$.

Moreover,

$$(e, v_1)(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{cases} (-z_1, -z_2, z_3, -z_4, z_5, -z_6) & \text{if only } \alpha \text{ is odd,} \\ (z_1, -z_2, -z_3, z_4, -z_5, -z_6) & \text{if only } \beta \text{ is odd,} \\ (-z_1, z_2, -z_3, -z_4, -z_5, z_6) & \text{if both are odd.} \end{cases}$$

$$(e, v_2)(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{cases} (z_1, -z_2, -z_3, -z_4, -z_5.z_6) & \text{if only } \alpha \text{ is odd,} \\ (-z_1, z_2, -z_3, -z_4, z_5, -z_6) & \text{if only } \beta \text{ is odd,} \\ (-z_1, -z_2, z_3, z_4, -z_5, -z_6) & \text{if both are odd.} \end{cases}$$

$$(e, v_t)(z_1, z_2, z_3, z_4, z_5, z_6) = (\mathrm{e}^{-2\pi\mathrm{i}(\alpha+\beta)/3} z_1, \mathrm{e}^{2\pi\mathrm{i}(2\alpha-\beta)/3} z_2, \mathrm{e}^{2\pi\mathrm{i}(-\alpha+2\beta)/3} z_3, \\ \mathrm{e}^{2\pi\mathrm{i}(-2\alpha+\beta)/3} z_4, \mathrm{e}^{2\pi\mathrm{i}(\alpha+\beta)/3} z_5, \mathrm{e}^{2\pi\mathrm{i}(\alpha-2\beta)/3} z_1).$$

(b) *Square lattice*

To compute the dimensions of the fixed-point subspaces of the finite twisted groups G^{Θ} in theorem 7.2, it is enough to define the action of $\boldsymbol{D}_4[h, R]\dot{+}\frac{1}{2}\mathcal{L}$ on $\boldsymbol{C}^2$ and $\boldsymbol{C}^4$.

The action of $\boldsymbol{D}_4[h, R]\dot{+}\frac{1}{2}\mathcal{L}$ on $\boldsymbol{C}^2$ is generated by

$$h(z_1, z_2) = (z_1, \overline{z_2}),$$

$$R(z_1, z_2) = (\overline{z_2}, z_1),$$
$$(e, v_1)(z_1, z_2) = (-z_1, z_2),$$
$$(e, v_d)(z_1, z_2) = (-z_1, -z_2).$$

The action of $\boldsymbol{D}_4[h, R]\dot{+}\frac{1}{2}\mathcal{L}$ on $\boldsymbol{C}^4$ is generated by

$$R(z_1, z_2, z_3, z_4) = (\overline{z_2}, z_1, \overline{z_4}, z_3),$$
$$h(z_1, z_2, z_3, z_4) = (\overline{z_4}, \overline{z_3}, \overline{z_2}, \overline{z_1}),$$
$$(e, v_1)(z_1, z_2, z_3, z_4) = \begin{cases} (-z_1, z_2, z_3, -z_4) & \text{if } \alpha \text{ is odd,} \\ (z_1, -z_2, -z_3, z_4) & \text{if } \beta \text{ is odd,} \end{cases}$$
$$(e, v_d)(z_1, z_2, z_3, z_4) = (-z_1, -z_2, -z_3, -z_4).$$

(*c*) *Rhombic lattice*

To compute the dimensions of the fixed-point subspaces of the finite twisted groups G^Θ in theorem 7.3, it is enough to define the action of $\boldsymbol{D}_2[h, P]\dot{+}\frac{1}{2}\mathcal{L}$ on $\boldsymbol{C}^2$. It is generated by

$$h(z_1, z_2) = (z_2, z_1),$$
$$P(z_1, z_2) = (\overline{z_1}, \overline{z_2}),$$
$$(e, v_1)(z_1, z_2) = (-z_1, z_2),$$
$$(e, v_d)(z_1, z_2) = (-z_1, -z_2).$$

9. Discrete waves

The following theorems determine the branches of time-periodic solutions with discrete spatio-temporal symmetries (i.e. the discrete waves) which can be obtained by using the equivariant Hopf theorem (see Golubitsky *et al.* 1988, theorem XVI, § 4.1). To guarantee that the solutions associated with the twisted subgroups G^Θ listed in the theorems of this section do not have more symmetries, it is necessary to check that the G^Θs are isotropy subgroups. Easy computations, using the action of Γ on $V \oplus V$ (see § 10 and 11), show that this is indeed the case.

Remark. To obtain a complete list of (branches of) time-periodic solutions bifurcating from a group-invariant equilibrium by Hopf bifurcation, we need also to consider the possibility that $\boldsymbol{u}(\boldsymbol{x}, t)$ in (1.1) depends only on one space variable; that is, the solutions are constant along lines perpendicular to a vector $\ell \in \boldsymbol{R}^2$ and periodic in the direction of ℓ. Without loss of generality, we may assume that the period is one. The action of the Euclidean group on the only translation-free absolutely irreducible subspace

$$V = \{z\mathrm{e}^{2\pi \mathrm{i}x} + \text{c.c.} : z \in \boldsymbol{C}\} \cong \boldsymbol{C},$$

reduces to the action of $O(2) \cong \boldsymbol{Z}_2[-e]\dot{+}\boldsymbol{T}^1$ on $\boldsymbol{C}$ generated by

$$(-e, 0)z = \overline{z},$$
$$(0, \theta)z = \mathrm{e}^{-2\pi \mathrm{i}\theta}z,$$

where e is the identity on $\boldsymbol{R}$ and $\boldsymbol{T}^1 \cong \{\theta\ell : 0 \leqslant \theta < 1\}$. The results for Hopf bifurcation with $O(2)$-symmetry are well known and can be found in Golubitsky

Table 4. *Discrete and rotating waves in one dimension*

K	G	generators of G^{Θ}	fixed-point subspace
$\boldsymbol{Z}_2[-e]$	$\boldsymbol{Z}_2[-e] \times \boldsymbol{Z}_2[\frac{1}{2}]$	$((-e,0),0)$ and $((e,\frac{1}{2}),\frac{1}{2})$	$z_1 = z_2$
$\mathbf{1}$	$\boldsymbol{T}^1$	$((e,\theta),\theta)$, $0 \leqslant \theta < 1$	$z_2 = 0$

Table 5. *Pairs $K \subset G$ for the hexagonal lattice*

$\dim V = 6$		$\dim V = 12$	
K	G	K	G
$\mathbf{1}$	$\boldsymbol{Z}_6[R]$	$\boldsymbol{Z}_6[R]$	$\boldsymbol{D}_6[h,R]$
$\boldsymbol{Z}_2[R^3]$	$\boldsymbol{Z}_2[R^3] \times \boldsymbol{Z}_4[(h,v_1)]$	$\boldsymbol{D}_3[h,R^2]$	$\boldsymbol{D}_6[h,R]$
$\boldsymbol{Z}_2[R^3]$	$\boldsymbol{Z}_6[R]$	$\boldsymbol{D}_3[hR,R^2]$	$\boldsymbol{D}_6[h,R]$
$\boldsymbol{D}_2[h,R^3]$	$\boldsymbol{D}_2[h,R^3] \times \boldsymbol{Z}_2[v_2]$	$\boldsymbol{D}_3[hR,R^2]$	$\boldsymbol{D}_3[hR,R^2] \dot{+} \boldsymbol{Z}_3[v_t]$
$\boldsymbol{D}_3[hR,R^2]$	$\boldsymbol{D}_6[h,R]$	$\boldsymbol{D}_6[h,R]$	$\boldsymbol{D}_6[h,R]$
$\boldsymbol{D}_3[hR,R^2]$	$\boldsymbol{D}_3[hR,R^2] \dot{+} \boldsymbol{Z}_3[v_t]$		
$\boldsymbol{D}_6[h,R]$	$\boldsymbol{D}_6[h,R]$		

et al. (1988), § XVII, 1(c). There are two branches of time-periodic solutions bifurcating from the group-invariant equilibrium: a discrete wave and a rotating wave. They are given in table 4. Note that the action of $O(2) \times \boldsymbol{S}^1$ on

$$V \oplus V = \{z_1 e^{2\pi i(x+t)} + z_2 e^{2\pi i(-x+t)} + \text{c.c.} : z_j \in \boldsymbol{C}\} \cong \boldsymbol{C}^2$$

is given by

$$\begin{aligned} ((-e,0),0)(z_1,z_2) &= (z_2,z_1), \\ ((e,\psi),0)(z_1,z_2) &= (e^{-2\pi i\psi} z_1, e^{2\pi i\psi} z_2), \\ ((e,0),\theta)(z_1,z_2) &= e^{2\pi i\theta}(z_1,z_2). \end{aligned}$$

(a) *Hexagonal lattice*

Theorem 9.1. *Up to conjugacy, the pairs of finite groups $K \subset G$ in table 5 produce twisted subgroups G^{Θ} with two-dimensional fixed-point subspaces where Γ acts on $V \oplus V$; V is a translation free, absolutely irreducible representation of Γ in the space of $\mathcal{L}$-periodic functions.*

Proof. This proof proceeds by an easy calculation. For each of the the first nine cases of theorem 7.1, we compute $\dim \operatorname{Fix}(G^{\Theta})$ using the formulae listed in table 1. In particular, the precise formula that is used is determined by the twist type G/K. Then, the computation of $\dim \operatorname{Fix}(G^{\Theta})$ proceeds using the dimensions listed in table 6. The results are listed in table 7.

We note that in order to use the formulae in table 1 we must compute the dimension of the fixed-point subspace of certain intermediate groups. In particular, in case 1 we must compute the dimension of the fixed-point subspaces of

Table 6. *Fixed-point subspaces of V*

K	$\dim V = 6$ Fix(K)	dim	$\dim V = 12$ Fix(K)	dim
$\mathbf{1}$	$\mathbf{C}^3$	6	$\mathbf{C}^6$	12
$\mathbf{Z}_2[R^3]$	$z = \bar{z}$	3	$z = \bar{z}$	6
$\mathbf{Z}_2[R^3] \times \mathbf{Z}_2[v_2]$	$z_1 = z_2 = 0$, $z_3 = \overline{z_3}$	1	$*$	2
$\mathbf{Z}_3[R^2]$	$z_1 = z_2 = z_3$	2	$z_1 = z_2 = z_3$, $z_4 = z_5 = z_6$	4
$\mathbf{Z}_6[R]$	$z_1 = z_2 = z_3$, $z = \bar{z}$	1	$z_1 = z_2 = z_3,\ z_4 = z_5 = z_6$, $z = \bar{z}$	2
$\mathbf{D}_2[h, R^3]$	$z_1 = z_2$, $z = \bar{z}$	2	$z_1 = z_6,\ z_2 = z_5,\ z_3 = z_4$, $z = \bar{z}$	3
$\mathbf{D}_2[h, R^3] \times \mathbf{Z}_2[v_2]$	$z_1 = z_2 = 0$, $z_3 = \overline{z_3}$	1	$*$	1
$\mathbf{D}_3[h, R^2]$	$z_1 = z_2 = z_3$, $z = \bar{z}$	1	$z_1 = z_2 = z_3 = z_4 = z_5 = z_6$	2
$\mathbf{D}_3[hR, R^2]$	$z_1 = z_2 = z_3$	2	$z_1 = z_2 = z_3 = \overline{z_4} = \overline{z_5} = \overline{z_6}$	2
$\mathbf{D}_3[hR, R^2] \dot{+} \mathbf{Z}_3[v_t]$	$z = 0$	0	$z = 0$	0
$\mathbf{D}_6[h, R]$	$z_1 = z_2 = z_3$, $z = \bar{z}$	1	$z_1 = z_2 = z_3 = z_4 = z_5 = z_6$, $z = \bar{z}$	1

$*$ denotes fixed-point subspaces that vary with the parity of α and β.

Table 7. *Dimensions of fixed-point subspaces in* $V \oplus V$

Case	G/K	formula $d(\Sigma) \equiv \dim \mathrm{Fix}\,(\Sigma)$	$\dim \mathrm{Fix}\,(G^{\Theta})$: $\dim V = 6$	$\dim V = 12$
1	$\mathbf{Z}_6$	$d(\mathbf{Z}_6[R]) + d(\mathbf{1}) - d(\mathbf{Z}_2[R^3]) - d(\mathbf{Z}_3[R^2])$	2	4
2	$\mathbf{Z}_4$	$d(\mathbf{Z}_2[R^3]) - d(\mathbf{Z}_2[R^3] \times \mathbf{Z}_2[v_2])$	2	4
3	$\mathbf{Z}_3$	$d(\mathbf{Z}_2[R^3]) - d(\mathbf{Z}_6[R])$	2	4
4	$\mathbf{Z}_2$	$2(d(\mathbf{D}_2[h, R^3]) - d(\mathbf{D}_2[h, R^3] \times \mathbf{Z}_2[v_2])$	2	4
5	$\mathbf{Z}_2$	$2(d(\mathbf{Z}_6[R]) - d(\mathbf{D}_6[h, R]))$	0	2
6	$\mathbf{Z}_2$	$2(d(\mathbf{D}_3[h, R^2]) - d(\mathbf{D}_6[h, R]))$	0	2
7	$\mathbf{Z}_2$	$2(d(\mathbf{D}_3[hR, R^2]) - d(\mathbf{D}_6[h, R]))$	2	2
8	$\mathbf{Z}_3$	$d(\mathbf{D}_3[hR, R^2]) - d(\mathbf{D}_3[hR, R^2] \dot{+} \mathbf{Z}_3[v_t])$	2	2
10	$\mathbf{1}$	$2d(\mathbf{D}_6[h, R])$	2	2

the intermediate groups $L = \mathbf{Z}_2[R^3]$ and $M = \mathbf{Z}_3[R^2]$, while in case 2 we must compute the dimension of the fixed-point subspace of the intermediate group $M = \mathbf{Z}_2[R^3] \times \mathbf{Z}_2[v_2]$.

We also point out that to compute the fixed-point subspace of

Table 8. *Pairs $K \subset G$ for the square lattice*

dim $V=4$		dim $V=8$	
K	G	K	G
$\boldsymbol{D}_2[h,R^2]$	$\boldsymbol{D}_2[h,R^2]\times\boldsymbol{Z}_4[(Rh,v_1)]$	$\boldsymbol{D}_2[h,R^2]$	$\boldsymbol{D}_2[h,R^2]\times\boldsymbol{Z}_4[(Rh,v_1)]$
$\boldsymbol{D}_4[h,R]$	$\boldsymbol{D}_4[h,R]\times\boldsymbol{Z}_2[v_d]$	$\boldsymbol{D}_2[(h,v_d),R^2]$	$\boldsymbol{D}_2[(h,v_d),R^2]\times\boldsymbol{Z}_4[(Rh,v_1)]$
		$\boldsymbol{D}_4[h,R]$	$\boldsymbol{D}_4[h,R]\times\boldsymbol{Z}_2[v_d]$
		$\boldsymbol{D}_4[(h,v_d),R]$	$\boldsymbol{D}_4[h,R]\times\boldsymbol{Z}_2[v_d]$

Table 9. *Fixed-point subspaces of V*

	dim $V=4$		dim $V=8$	
K	Fix (K)	dim	Fix (K)	dim
$\boldsymbol{D}_2[h,R^2]$	$z=\bar{z}$	2	$z_1=z_4,\ z_2=z_3$ $z=\bar{z}$	2
$\boldsymbol{D}_2[(h,v_d),R^2]$	$z=0$	0	$z_1=-z_4,\ z_2=-z_3$ $z=\bar{z}$	2
$\boldsymbol{D}_2[h,R^2]\times\boldsymbol{Z}_2[v_d]$	$z=0$	0	$z=0$	0
$\boldsymbol{D}_2[(h,v_d),R^2]\times\boldsymbol{Z}_2[v_d]$	$z=0$	0	$z=0$	0
$\boldsymbol{D}_4[h,R]$	$z_1=z_2$ $z=\bar{z}$	1	$z_1=z_2=z_3=z_4$ $z=\bar{z}$	1
$\boldsymbol{D}_4[(h,v_d),R]$	$z=0$	0	$z_1=z_2=-z_3=-z_4$ $z=\bar{z}$	1
$\boldsymbol{D}_4[h,R]\dot{+}\boldsymbol{Z}_2[v_d]$	$z=0$	0	$z=0$	0

$\boldsymbol{D}_3[hR,R^2]\dot{+}\boldsymbol{Z}_3[v_t]$, when $\dim V=12$, we need to use the inequality $e^{2\pi i(\alpha+\beta)/3}\neq 1$, which comes from $(\alpha+\beta,3)=1$. ■

(b) *Square lattice*

Theorem 9.2. *Up to conjugacy, the pairs of finite groups $K\subset G$ in table 8 produce twisted subgroups G^{Θ} with two-dimensional fixed-point subspaces, where Γ acts on $V\oplus V$; V is a translation-free absolutely irreducible representation of Γ in the space of $\mathcal{L}$-periodic functions.*

Proof. We proceed as in the proof of theorem 9.1. We use the formulae of table 1 combined with the dimensions of the fixed-point subspaces given in table 9 to compute $\dim \mathrm{Fix}\,(G^{\Theta})$ for each of the first four cases of theorem 7.2. The results are listed in table 10. Note that in the first two cases, we need to compute the dimensions of the fixed-point subspaces of the intermediate groups $M=\boldsymbol{D}_2[h,R^2]\times\boldsymbol{Z}_2[v_d]$ and $M=\boldsymbol{D}_2[(h,v_d),R^2]\times\boldsymbol{Z}_2[v_d]$. ■

Table 10. *Dimensions of fixed-point subspaces in* $V \oplus V$

case	$G/\boldsymbol{K}$	formula $d(\Sigma) = \dim \mathrm{Fix}\,(\Sigma)$	$\dim \mathrm{Fix}\,(G^{\Theta})$	
			$\dim V = 4$	$\dim V = 8$
1	$\boldsymbol{Z}_4$	$d(\boldsymbol{D}_2[h, R^2]) - d(\boldsymbol{D}_2[h, R^2] \times \boldsymbol{Z}_2[v_d])$	2	2
2	$\boldsymbol{Z}_4$	$d(\boldsymbol{D}_2[(h, v_d), R^2]) - d(\boldsymbol{D}_2[(h, v_d), R^2] \times \boldsymbol{Z}_2[v_d])$	0	2
4	$\boldsymbol{Z}_2$	$2(d(\boldsymbol{D}_4[h, R]) - d(\boldsymbol{D}_4[h, R] \times \boldsymbol{Z}_2[v_d])$	2	2
5	$\boldsymbol{Z}_2$	$2(d(\boldsymbol{D}_4[(h, v_d), R]) - d(\boldsymbol{D}_4[h, R] \times \boldsymbol{Z}_2[v_d])$	0	2

Table 11. *Pairs* $K \subset G$ *for the rhombic lattice*

K	G
$\boldsymbol{Z}_2[P]$	$\boldsymbol{Z}_2[P] \times \boldsymbol{Z}_4[(h, v_1)]$
$\boldsymbol{D}_2[h, P]$	$\boldsymbol{D}_2[h, P] \times \boldsymbol{Z}_2[v_d]$

Table 12. *Fixed-point subspaces of* V

K	$\mathrm{Fix}\,(K)$	dim
$\boldsymbol{Z}_2[P]$	$z = \bar{z}$	2
$\boldsymbol{Z}_2[P] \times \boldsymbol{Z}_2[v_d]$	$z = 0$	0
$\boldsymbol{D}_2[h, P]$	$z_1 = z_2,\ z = \bar{z}$	1
$\boldsymbol{D}_2[h, P] \times \boldsymbol{Z}_2[v_d]$	$z = 0$	0

(*c*) *Rhombic lattice*

Theorem 9.3. *Up to conjugacy, the pairs of finite groups $K \subset G$ in table 11 produce twisted subgroups G^{Θ} with two-dimensional fixed-point subspaces, where Γ acts on $V \oplus V$; V is a translation-free absolutely irreducible representation of Γ in the space of $\mathcal{L}$-periodic functions.*

Proof. Using the formulae in table 1 and the dimensions of the fixed-point subspaces given in table 12, we compute the dimension of the fixed-point subspace for each discrete twisted group G^{Θ} given by theorem 7.3. The results are in table 13. In the first case, the dimension of the fixed-point subspace of the intermediate group $\boldsymbol{Z}_2[P] \times \boldsymbol{Z}_2[v_d]$ needs to be computed. ■

10. Rotating waves

The following theorem determines the branches of time-periodic solutions with continuous spatio-temporal symmetries (i.e. the rotating waves) which are obtained by using the equivariant Hopf theorem (see Golubitsky *et al.* 1988, theorem XVI, § 4.1).

Table 13. *Dimensions of fixed-point subspaces in* $V \oplus V$

case	G/K	formula	$\dim \mathrm{Fix}(G^{\Theta})$
1	$\boldsymbol{Z}_4[(h, v_1)]$	$\dim \mathrm{Fix}(\boldsymbol{Z}_2[P]) - \dim \mathrm{Fix}(\boldsymbol{Z}_2[P] \times \boldsymbol{Z}_2[v_d])$	2
2	$\boldsymbol{Z}_2[P]$	$2(\dim \mathrm{Fix}(\boldsymbol{D}_2[h, P]) - \dim \mathrm{Fix}(\boldsymbol{D}_2[h, P] \times \boldsymbol{Z}_2[v_d]))$	2

Table 14. *Pairs* $K \subset G$ *for all three lattices in the continuous case*

	Hexagonal Lattice		
	$\dim V$	K	G
TH1	6	$\boldsymbol{D}_1[h]$	$\boldsymbol{D}_1[h] \times E^{+}(h)$
TH2		$\boldsymbol{D}_1[hR^3]$	$\boldsymbol{D}_1[hR^3] \times E^{-}(h)$
	12		None
	Square Lattice		
	$\dim V$	K	G
TS	4	$\boldsymbol{D}_1[Rh]$	$\boldsymbol{D}_1[Rh] \times E^{+}(Rh)$
	8		None
	Rhombic Lattice		
	$\dim V$	K	G
TR1	4	$\boldsymbol{D}_1[h]$	$\boldsymbol{D}_1[h] \times E^{+}(h)$
TR2		$\boldsymbol{D}_1[Ph]$	$\boldsymbol{D}_1[Ph] \times E^{-}(h)$

Theorem 10.1. *Up to conjugacy, the pairs of finite groups* $K \subset G$ *such that* $G/K \cong \boldsymbol{S}^1$ *listed in table 14 produce isotropy subgroups* G^{Θ} *with two-dimensional fixed-point subspaces, where* Γ *acts on* $V \oplus V$*;* V *is a translation-free absolutely irreducible representation of* Γ *in the space of* $\mathcal{L}$*-periodic functions.*

The rest of this section is devoted to the proof of this theorem. We can not use, as we did in the previous section, the dimensions of the fixed-point subspaces of some subgroups of Γ acting on the absolutely irreducible subspace V to compute the dimensions of the fixed-point subspaces in $V \oplus V$ of the twisted subgroups associated with the wave pairs $K \subset G$, where $G/K \cong \boldsymbol{S}^1$. We have to directly compute the fixed-point subspaces of these twisted subgroups. To do this we will: (i) list the generators of the twisted subgroups associated with the wave pairs $K \subset G$; and (ii) define the actions of these generators on the translation-free subspaces of the form $V \oplus V$, where V is Γ-absolutely irreducible.

It is generally easy to give generators for the twisted groups G^{Θ} associated with the wave pairs of subgroups $K \subset G$. It is enough to note that $\Theta : G \to \boldsymbol{S}^1$ is a group homomorphism with kernel K – recall that $\boldsymbol{S}^1 \cong \boldsymbol{R}/\boldsymbol{Z}$.

If we introduce time, $V \oplus V$ is of the form

$$\left\{ \sum_{j=1}^{s} (z_j \mathrm{e}^{2\pi \mathrm{i}(t+\boldsymbol{K}_j\cdot\boldsymbol{x})} + w_j \mathrm{e}^{2\pi \mathrm{i}(t-\boldsymbol{K}_j\cdot\boldsymbol{x})} + \text{c.c.}) : z_j, w_j \in \boldsymbol{C} \right\} \cong \boldsymbol{C}^{2s}, \tag{10.1}$$

where $s(=\frac{1}{2}\dim V)$ and the $\boldsymbol{K}_j$s are defined in table 3.

(*a*) *Hexagonal lattice*

Theorem 7.1 shows that there are only two pairs $K \subset G$, where $G/K \cong \boldsymbol{S}^1$, that need to be considered. The twisted group G^{Θ} associated with the wave pair $K = \boldsymbol{D}_1[h]$ and $G = \boldsymbol{D}_1[h] \times E^+(h)$ is generated by h and $((e, \theta\ell_2), \theta)$ for $0 \leqslant \theta < 1$, and the twisted group G^{Θ} associated with the wave pair $K = \boldsymbol{D}_1[hR^3]$ and $G = \boldsymbol{D}_1[hR^3] \times E^-(h)$ is generated by $R^3 h$ and $((e, \theta(-2\ell_1 + \ell_2)), \theta)$ for $0 \leqslant \theta < 1$.

The actions of these elements on $V \oplus V$ can be deduced from the actions of the following elements.

When $\dim V = 6$: let $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3)$:

$$\left.\begin{aligned} R(z,w) &= (w_2, w_3, w_1, z_2, z_3, z_1), \\ h(z,w) &= (w_2, w_1, w_3, z_2, z_1, z_3), \\ ((e, \theta\ell_2), \theta)(z,w) &= (z_1, \mathrm{e}^{4\pi\theta i} z_2, \mathrm{e}^{2\pi\theta \mathrm{i}} z_3, \mathrm{e}^{4\pi\theta \mathrm{i}} w_1, w_2, \mathrm{e}^{2\pi\theta \mathrm{i}} w_3), \\ ((e, -2\theta\ell_1 + \theta\ell_2), \theta)(z,w) &= (\mathrm{e}^{4\pi\theta \mathrm{i}} z_1, \mathrm{e}^{4\pi\theta \mathrm{i}} z_2, \mathrm{e}^{-2\pi\theta \mathrm{i}} z_3, w_1, w_2, \mathrm{e}^{6\pi\theta \mathrm{i}} w_3). \end{aligned}\right\} \tag{10.2}$$

When $\dim V = 12$: let $z = (z_1, z_2, z_3, z_4, z_5, z_6)$, $w = (w_1, w_2, w_3, w_4, w_5, w_6)$ and $\chi = \mathrm{e}^{2\pi\theta \mathrm{i}}$,

$$\left.\begin{aligned} R(z,w) &= (w_2, w_3, w_1, w_5, w_6, w_4, z_2, z_3, z_1, z_5, z_6, z_4), \\ h(z,w) &= (z_6, z_5, z_4, z_3, z_2, z_1, w_6, w_5, w_4, w_3, w_2, w_1), \\ \mathrm{e}, \theta\ell_2), \theta)(z,w) &= \chi(\chi^{-\beta} z_1, \chi^{\alpha} z_2, \chi^{(\beta-\alpha)} z_3, \chi^{(\beta-\alpha)} z_4, \chi^{\alpha} z_5, \chi^{-\beta} z_6, \\ &\qquad \chi^{\beta} w_1, \chi^{-\alpha} w_2, \chi^{(\alpha-\beta)} w_3, \chi^{(\alpha-\beta)} w_4, \chi^{-\alpha} z_5, \chi^{\beta} z_6), \\ ((e, \theta(\ell_2 - 2\ell_1)), \theta)(z,w) &= \chi(\chi^{(2\alpha-\beta)} z_1, \chi^{(2\beta-\alpha)} z_2, \chi^{(-\alpha-\beta)} z_3, \chi^{(\alpha+\beta)} z_4, \\ &\qquad \chi^{(\alpha-2\beta)} z_5, \chi^{(\beta-2\alpha)} z_6, \chi^{(\beta-2\alpha)} w_1, \chi^{(\alpha-2\beta)} w_2, \\ &\qquad \chi^{(\alpha+\beta)} w_3, \chi^{(-\alpha-\beta)} w_4, \chi^{(2\beta-\alpha)} w_5, \chi^{(2\alpha-\beta)} w_6). \end{aligned}\right\} \tag{10.3}$$

In table 15, we list the fixed-point subspaces in $V \oplus V$ of the twisted subgroups associated with the wave pairs $K \subset G$, where $G/K \cong \boldsymbol{S}^1$, mentioned above. Simple computations show that the G^{Θ}s are isotropy subgroups when $\dim V \oplus V$ $=12$. However, they are not isotropy subgroups when $\dim V \oplus V = 24$. When $K = \boldsymbol{D}_1[h]$ and $G = \boldsymbol{D}_1[h] \times E^+(h)$, $v = (1/(\alpha+\beta))(\ell_1 + \beta\ell_2) \notin G^{\Theta}$ acts trivially on $\mathrm{Fix}_{V\oplus V}\, G^{\Theta}$. When $K = \boldsymbol{D}_1[hR^3]$ and $G = \boldsymbol{D}_1[hR^3] \times E^-(h)$, $v = (1/\alpha)\ell_2 \notin G^{\Theta}$ acts trivially on $\mathrm{Fix}_{V\oplus V}\, G^{\Theta}$.

The computations of the fixed-point subspaces when $\dim V \oplus V = 24$ are as follows.

First, we consider $\boldsymbol{D}_1[h] \times \{((e, \theta\ell_2), \theta) : 0 \leqslant \theta < 1\}$. The result follows easily from the following two remarks. From $h(z,w) = (z,w)$, we obtain that $z_j = z_{7-j}$ and $w_j = w_{7-j}$ for $j = 1, 2$ and 3. Moreover, if $\Delta \equiv \{((e, \theta\ell_2), \theta) : 0 \leqslant \theta < 1\}$ acts non-trivially on z_j (respectively w_i), then $z_j = 0$ (respectively $w_i = 0$). Since $\alpha > \beta > \alpha/\beta > 0$, we can see from the definition of the action of Δ on (z,w) given

Table 15. *Fixed-point subspaces for subgroups G^Θ of theorem 7.1, where $G/K \cong \boldsymbol{S}^1$*

G^Θ	Fix(G^Θ) dim $V = 6$	Fix(G^Θ) dim $V = 12$
$\boldsymbol{D}_1[h]\times$ $\{((e,\theta\ell_2),\theta) : 0 \leqslant \theta < 1\}$	$z_1 = w_2$ and $z_2 = z_3 = w_1 = w_3 = 0$	$z_3 = z_4$ and $z_j = w_i = 0$ otherwise if $\alpha - \beta = 1$ $z = w = 0$ if $\alpha - \beta \neq 1$
$\boldsymbol{D}_1[hR^3]\times$ $\{((e,-2\theta\ell_1+\theta\ell_2),\theta) : 0 \leqslant \theta < 1\}$	$w_1 = w_2$ and $z_1 = z_2 = z_3 = w_3 = 0$	$z_2 = w_5$ and $z_i = w_j = 0$ otherwise if $\alpha - 2\beta = 1$ $z_5 = w_2$ and $z_i = w_j = 0$ otherwise if $\alpha - 2\beta = -1$ $z = w = 0$ if $\alpha - 2\beta \neq \pm 1$

above that Δ acts trivially on some components of (z,w) only when $\alpha - \beta = 1$. In this case, it acts trivially on z_3 and z_4.

Second, we consider $\boldsymbol{D}_1[hR^3] \times \{((e,-2\theta\ell_1+\theta\ell_2),\theta) : 0 \leqslant \theta < 1\}$. From $hR^3(z,w) = (z,w)$, we obtain that $z_j = w_{7-j}$ for $j = 1,2,\ldots,6$. As before, if $\Lambda \equiv \{((e,-2\theta\ell_1+\theta\ell_2),\theta) : 0 \leqslant \theta < 1\}$ acts non-trivially on z_j (respectively w_i), then $z_j = 0$ (respectively $w_i = 0$). Since $\alpha > \beta > \alpha/\beta > 0$, Λ acts trivially on some components of (z,w) only when $\alpha - 2\beta = \pm 1$. It acts trivially on z_2 and w_5 if $\alpha - 2\beta = 1$, and on z_5 and w_2 if $\alpha - 2\beta = -1$. It is now easy to complete the computation of the fixed-point subspace.

(*b*) *Square lattice*

Theorem 7.2 determines that there is only one pair $K \subset G$, where $G/K \cong \boldsymbol{S}^1$, that needs to be considered. The twisted group G^Θ associated with the wave pair $K = \boldsymbol{D}_1[Rh]$ and $G = \boldsymbol{D}_1[Rh] \times E^+(Rh)$ is generated by R^3h and $((e,\theta(\ell_1+\ell_2)),\theta)$ for $0 \leqslant \theta < 1$.

The actions of these elements on $V \oplus V$ can be deduced from the actions of the following elements.

When dim $V = 4$: let $z = (z_1,z_2)$ and $w = (w_1,w_2)$.

$$\left.\begin{aligned} R(z,w) &= (w_2,z_1,z_2,w_1), \\ h(z,w) &= (z_1,w_2,w_1,z_2), \\ ((e,\theta(\ell_1+\ell_2)),\theta)(z,w) &= (z_1,z_2,\mathrm{e}^{4\pi\theta\mathrm{i}}w_1,\mathrm{e}^{4\pi\theta\mathrm{i}}w_2). \end{aligned}\right\} \qquad (10.4)$$

Table 16. *Fixed-point subspaces for subgroups* G^Θ *of theorem 7.2, where* $G/K \cong \boldsymbol{S}^1$

G^Θ	$\mathrm{Fix}(G^\Theta)$ $\dim V = 4$	$\mathrm{Fix}(G^\Theta)$ $\dim V = 8$
$\boldsymbol{D}_1[Rh] \times \{((e, \theta(\ell_1 + \ell_2)), \theta) : 0 \leqslant \theta < 1\}$	$z_1 = z_2$ $w_1 = w_2 = 0$	$z_2 = w_4$ and $z_j = w_i = 0$ otherwise if $\alpha - \beta = 1$ $z = w = 0$ if $\alpha - \beta \neq 1$

Table 17. *Fixed-point subspaces for subgroups* G^Θ *of theorem 7.3, where* $G/K \cong \boldsymbol{S}^1$

G^Θ	$\mathrm{Fix}\,(G^\Theta)$ $\dim V = 4$
$\boldsymbol{D}_1[h] \times \{((e, \theta(\ell_1 + \ell_2)), \theta) : 0 \leqslant \theta < 1\}$	$z_1 = z_2$ and $w_1 = w_2 = 0$
$\boldsymbol{D}_1[Ph] \times \{((e, \theta(\ell_1 - \ell_2)), \theta) : 0 \leqslant \theta < 1\}$	$z_1 = w_2$ and $z_2 = w_1 = 0$

Table 18. *Generators of twisted subgroups – hexagonal lattice*

K	G	generators of G^Θ
$\mathbf{1}$	$\boldsymbol{Z}_6[R]$	$((R, 0), \frac{1}{6})$
$\boldsymbol{Z}_2[R^3]$	$\boldsymbol{Z}_2[R^3] \times \boldsymbol{Z}_4[(h, v_1)]$	$(R^3, 0)$ and $((h, v_1), \frac{1}{4})$
$\boldsymbol{Z}_2[R^3]$	$\boldsymbol{Z}_6[R]$	$((R, 0), \frac{1}{3})$
$\boldsymbol{D}_2[h, R^3]$	$\boldsymbol{D}_2[h, R^3] \times \boldsymbol{Z}_2[v_2]$	$((R^3, 0), 0), ((h, 0), 0)$ and $((e, v_2), \frac{1}{2})$
$\boldsymbol{Z}_6[R]$	$\boldsymbol{D}_6[h, R]$	$((R, 0), 0)$ and $((h, 0), \frac{1}{2})$
$\boldsymbol{D}_3[h, R^2]$	$\boldsymbol{D}_6[h, R]$	$((R, 0), \frac{1}{2})$ and $((h, 0), \frac{1}{2})$
$\boldsymbol{D}_3[hR, R^2]$	$\boldsymbol{D}_6[h, R]$	$((R, 0), \frac{1}{2})$ and $((h, 0), 0)$
$\boldsymbol{D}_3[hR, R^2]$	$\boldsymbol{D}_3[hR, R^2] \dot{+} \boldsymbol{Z}_3[v_t]$	$((R^2, 0), 0)$, $((hR, 0), 0)$ and $((e, v_t), \frac{1}{3})$
$\boldsymbol{D}_6[h, R]$	$\boldsymbol{D}_6[h, R]$	$((R, 0), 0)$ and $((h, 0), 0)$

When $\dim V = 8$: let $z = (z_1, z_2, z_3, z_4)$, $w = (w_1, w_2, w_3, w_4)$ and $\chi = \mathrm{e}^{2\pi\theta\mathrm{i}}$.

$$\left.\begin{aligned}
R(z, w) &= (w_2, z_1, w_4, z_3, z_2, w_1, z_4, w_3),\\
h(z, w) &= (w_4, w_3, w_2, w_1, z_4, z_3, z_2, z_1),\\
((e, \theta\ell_1), \theta)(z, w) &= \chi(\chi^{-\alpha} z_1, \chi^{\beta} z_2, \chi^{-\beta} z_3, \chi^{\alpha} z_4, \chi^{\alpha} w_1, \chi^{-\beta} w_2, \chi^{\beta} w_3, \chi^{-\alpha} w_4),\\
((e, \theta(\ell_1 + \ell_2)), \theta)(z, w) &= \chi(\chi^{-(\beta+\alpha)} z_1, \chi^{(-\alpha+\beta)} z_2, \chi^{-(\alpha+\beta)} z_3, \chi^{(\alpha-\beta)} z_4,\\
&\qquad \chi^{(\alpha+\beta)} w_1, \chi^{(\alpha-\beta)} w_2, \chi^{(\beta+\alpha)} w_3, \chi^{(-\alpha+\beta)} w_4).
\end{aligned}\right\} \tag{10.5}$$

In table 16, we list the fixed-point subspace in $V \oplus V$ of the twisted subgroup G^Θ above. To compute this fixed-point subspace, we proceed as for the hexagonal lattices. Simple computations show that G^{Θ} is an isotropy subgroup when $\dim V \oplus V = 8$. However, it is not an isotropy subgroup when $\dim V \oplus V = 16$. The non-trivial translation $v = (1/(\alpha+\beta))(\ell_1+\ell_2) \notin G^{\Theta}$ acts trivially on $\mathrm{Fix}_{V\oplus V}\, G^{\Theta}$.

(c) *Rhombic lattice*

Theorem 7.3 shows that there are two pairs $K \subset G$, where $G/K \cong \boldsymbol{S}^1$, that need to be considered. The twisted group G^{Θ} associated with the wave pair $K = \boldsymbol{D}_1[h]$ and $G = \boldsymbol{D}_1[h] \times E^+(h)$ is generated by h and $((e, \theta(\ell_1+\ell_2)), \theta)$ for $0 \leqslant \theta < 1$, and the twisted group G^{Θ} associated with the wave pair $K = \boldsymbol{D}_1[Ph]$ and $G = \boldsymbol{D}_1[Ph] \times E^-(h)$ is generated by Ph and $((e, \theta(\ell_1-\ell_2)), \theta)$ for $0 \leqslant \theta < 1$.

The actions of these elements on $V \oplus V$, where $\dim V = 4$, can be deduced from the actions of the following elements.

$$\left.\begin{aligned} P(z_1,z_2,w_1,w_2) &= (w_1,w_2,z_1,z_2),\\ h(z_1,z_2,w_1,w_2) &= (z_2,z_1,w_2,w_1),\\ ((e,\theta(\ell_1+\ell_2)),\theta)(z_1,z_2,w_1,w_2) &= (z_1,z_2,\mathrm{e}^{4\pi\theta\mathrm{i}}w_1,\mathrm{e}^{4\pi\theta\mathrm{i}}w_2),\\ ((e,\theta(\ell_1-\ell_2)),\theta)(z_1,z_2,w_1,w_2) &= (z_1,\mathrm{e}^{4\pi\theta\mathrm{i}}z_2,\mathrm{e}^{4\pi\theta\mathrm{i}}w_1,w_2). \end{aligned}\right\} \tag{10.6}$$

In table 17, we list the fixed-point subspaces in $V \oplus V$ of the twisted subgroups G^{Θ} mentioned above. The computations of these fixed-point subspaces proceed as for the hexagonal and square lattices. One readily verifies that the G^{Θ}s are isotropy subgroups.

11. Pictures

In this section we present pictures of the planforms obtained in § 9 and 10 using the equivariant Hopf theorem. This is accomplished by introducing time explicitly into the problem and computing the fixed-point subspaces in $V \oplus V$. In particular, $V \oplus V$ has the form (10.1). The corresponding superposition of Fourier modes in (10.1) can then be represented by a two-dimensional density plot for discrete values of the time t during one period of the oscillation. For the application of oscillatory hydrodynamic convection problems, these plots indicate the possible appearance of shadowgraph images of (small-amplitude) convection patterns. Such images trace the (vertically averaged) fluid density in the convecting fluid layer; the dark regions correspond to hot (buoyant) rising fluid and the light regions correspond to cold (heavy) descending fluid.

In § 10 we determined the fixed-point subspaces for the rotating waves guaranteed by the equivariant Hopf theorem (see theorem 10.1). In figure 1 we present some 'snapshots' of the rotating-wave planforms. The remainder of this section is devoted to computing the fixed-point subspaces in $V \oplus V$ of the discrete waves obtained in § 9 (see tables 19, 21 and 23).

To achieve this we will (as we did for the rotating waves): (i) list the generators of the twisted subgroups G^{Θ} associated with the wave pairs $K \subset G$; and (ii) define the actions of these generators on the translation-free subspaces of the form $V \oplus V$, where V is Γ-absolutely irreducible.

(a) *Hexagonal lattice*

We list in table 18 generators for each of the twisted groups of theorem 9.1.

In table 19, we describe the fixed-point subspaces in $V \oplus V$ of the twisted

Table 19. *Fixed-point subspaces of twisted subgroups – hexagonal lattice*

	K	G	$\mathrm{Fix}\,(G^{\Theta})$
			$\dim V = 6$ $z = (z_1, z_2, z_3)$, $w = (w_1, w_2, w_3)$, $\chi = \mathrm{e}^{2\pi\mathrm{i}/3}$
H1	$\mathbf{1}$	$\boldsymbol{Z}_6[R]$	$z_1 = \chi z_2 = \chi^2 z_3$, $w = -z$
H2	$\boldsymbol{Z}_2[R^3]$	$\boldsymbol{Z}_2[R^3] \times \boldsymbol{Z}_4[(h, v_1)]$	$z_1 = -\mathrm{i}z_2$, $z_3 = 0$, $w = z$
H3	$\boldsymbol{Z}_2[R^3]$	$\boldsymbol{Z}_6[R]$	$z_1 = \chi^2 z_2 = \chi z_3$, $w = z$
H4	$\boldsymbol{D}_2[h, R^3]$	$\boldsymbol{D}_2[h, R^3] \times \boldsymbol{Z}_2[v_2]$	$z_1 = z_2$, $z_3 = 0$, $w = z$
H5	$\boldsymbol{D}_3[hR, R^2]$	$\boldsymbol{D}_6[h, R]$	$z_1 = z_2 = z_3$, $w = -z$
H6	$\boldsymbol{D}_3[hR, R^2]$	$\boldsymbol{D}_3[hR, R^2]\dot{+}\boldsymbol{Z}_3[v_t]$	$z = 0$, $w_1 = w_2 = w_3$
H7	$\boldsymbol{D}_6[h, R]$	$\boldsymbol{D}_6[h, R]$	$z_1 = z_2 = z_3$, $w = z$
			$\dim V = 12$ $z = (z_1, z_2, z_3, z_4, z_5, z_6)$, $w = (w_1, w_2, w_3, w_4, w_5, w_6)$, $\sigma w = (w_4, w_5, w_6, w_1, w_2, w_3)$
$\mathrm{H8}_{\alpha\beta}$	$\boldsymbol{Z}_6[R]$	$\boldsymbol{D}_6[h, R]$	$z_1 = z_2 = z_3 = -z_4 = -z_5 = -z_6$, $w = z$
$\mathrm{H9}_{\alpha\beta}$	$\boldsymbol{D}_3[h, R^2]$	$\boldsymbol{D}_6[h, R]$	$z_1 = z_2 = z_3 = -z_4 = -z_5 = -z_6$, $w = -z$
$\mathrm{H5}_{\alpha,\beta}$	$\boldsymbol{D}_3[hR, R^2]$	$\boldsymbol{D}_6[h, R]$	$z_1 = z_2 = z_3 = z_4 = z_5 = z_6$, $w = -z$
$\mathrm{H6}_{\alpha,\beta}$	$\boldsymbol{D}_3[hR, R^2]$	$\boldsymbol{D}_3[hR, R^2]\dot{+}\boldsymbol{Z}_3[v_t]$	$z_1 = z_2 = z_3$, $z_4 = z_5 = z_6 = 0$, $\sigma w = z$, if $\alpha + \beta \equiv 1 \pmod 3$ $z_1 = z_2 = z_3 = 0$, $z_4 = z_5 = z_6$, $\sigma w = z$, if $\alpha + \beta \equiv 2 \pmod 3$
$\mathrm{H7}_{\alpha,\beta}$	$\boldsymbol{D}_6[h, R]$	$\boldsymbol{D}_6[h, R]$	$z_1 = z_2 = z_3 = z_4 = z_5 = z_6$, $w = z$

subgroups in table 18. To compute the fixed-point subspaces of the twisted subgroups in table 18, we need to define the actions of their generators on $V \oplus V$. For $\dim V = 6$, the actions of h and R are given in (10.2). Let $z = (z_1, z_2, z_3)$, $w = (w_1, w_2, w_3)$ and $\chi = \mathrm{e}^{2\pi\mathrm{i}/3}$. Using

$$\begin{aligned}(e, v_1)(z, w) &= (-z_1, z_2, -z_3, -w_1, w_2, -w_3),\\ (e, v_2)(z, w) &= (-z_1, -z_2, z_3, -w_1, -w_2, w_3),\\ (e, v_t)(z, w) &= (\chi z, \chi^2 w),\\ ((e, 0), \phi)(z, w) &= \mathrm{e}^{2\pi\phi\mathrm{i}}(z, w),\end{aligned}$$

one obtains the action of the generators when $\dim V = 6$.

For $\dim V = 12$, the actions of h and R are given in (10.3). Let $\chi = \mathrm{e}^{2\pi\mathrm{i}/3}$, $z = (z_1, z_2, z_3, z_4, z_5, z_6)$ and $w = (w_1, w_2, w_3, w_4, w_5, w_6)$. Using

$$\begin{aligned}(e, v_t)(z, w) = (&\chi^{-\alpha-\beta} z_1, \chi^{2\alpha-\beta} z_2, \chi^{2\beta-\alpha} z_3, \chi^{-2\alpha+\beta} z_4, \chi^{\alpha+\beta} z_5, \chi^{\alpha-2\beta} z_6,\\ &\chi^{\alpha+\beta} w_1, \chi^{-2\alpha+\beta} w_2, \chi^{\alpha-2\beta} w_3, \chi^{2\alpha-\beta} w_4, \chi^{-\alpha-\beta} w_5, \chi^{2\beta-\alpha} w_6),\\ ((e, 0), \phi)(z, w) = \mathrm{e}&^{2\pi\phi\mathrm{i}}(z, w),\end{aligned}$$

one readily determines the action of the generators when $\dim V = 12$.

Usually, the computations of the fixed-point subspaces are simple. We mention that when $((e, v_t), \frac{1}{3})$ acts non-trivially on a coordinate of (z, w), then this

Table 20. *Generators of twisted subgroups – square lattice*

K	G	generators of G^{Θ}
$\boldsymbol{D}_2[h, R^2]$	$\boldsymbol{D}_2[h, R^2] \times \boldsymbol{Z}_4[(Rh, v_1)]$	$((R^2, 0), 0)$, $((h, 0), 0)$ and $((Rh, v_1), \frac{1}{4})$
$\boldsymbol{D}_2[(h, v_d), R^2]$	$\boldsymbol{D}_2[(h, v_d), R^2] \times \boldsymbol{Z}_4[(Rh, v_1)]$	$((R^2, 0), 0)$, $((h, v_d), 0)$ and $((Rh, v_1), \frac{1}{4})$
$\boldsymbol{D}_4[h, R]$	$\boldsymbol{D}_4[h, R]\dot{+}\boldsymbol{Z}_2[v_d]$	$((R, 0), 0), ((h, 0), 0)$ and $((e, v_d), \frac{1}{2})$
$\boldsymbol{D}_4[(h, v_d), R]$	$\boldsymbol{D}_4[h, R]\dot{+}\boldsymbol{Z}_2[v_d]$	$((R, 0), 0), ((h, v_d), 0)$ and $((e, v_d), \frac{1}{2})$

Table 21. *Fixed-point subspaces of twisted subgroups – square lattice*

	K	G	Fix (G^{Θ})
		$\dim V = 4$	
S1	$\boldsymbol{D}_2[h, R^2]$	$\boldsymbol{D}_2[h, R^2] \times \boldsymbol{Z}_4[(Rh, v_1)]$	$z_1 = -\mathrm{i}z_2 = w_1 = -\mathrm{i}w_2$
S2	$\boldsymbol{D}_4[h, R]$	$\boldsymbol{D}_4[h, R]\dot{+}\boldsymbol{Z}_2[v_d]$	$z_1 = z_2 = w_1 = w_2$
		$\dim V = 8$, $z = (z_1, z_2, z_3, z_4)$, $w = (w_1, w_2, w_3, w_4)$	
$\mathrm{S1}_{\alpha,\beta}$	$\boldsymbol{D}_2[h, R^2]$	$\boldsymbol{D}_2[h, R^2] \times \boldsymbol{Z}_4[(Rh, v_1)])$	$z_1 = -\mathrm{i}z_2 = -\mathrm{i}z_3 = z_4$, $w = z$ if α is odd; $z_1 = \mathrm{i}z_2 = \mathrm{i}z_3 = z_4$, $w = z$ if β is odd
$\mathrm{S3}_{\alpha,\beta}$	$\boldsymbol{D}_2[(h, v_d), R^2]$	$\boldsymbol{D}_2[(h, v_d), R^2] \times \boldsymbol{Z}_4[(Rh, v_1)]$	$z_1 = \mathrm{i}z_2 = -\mathrm{i}z_3 = -z_4$, $w = z$ if α is odd; $z_1 = -\mathrm{i}z_2 = \mathrm{i}z_3 = -z_4$, $w = z$ if β is odd
$\mathrm{S2}_{\alpha,\beta}$	$\boldsymbol{D}_4[h, R]$	$\boldsymbol{D}_4[h, R]\dot{+}\boldsymbol{Z}_2[v_d]$	$z_1 = z_2 = z_3 = z_4$, $w = z$
$\mathrm{S4}_{\alpha,\beta}$	$\boldsymbol{D}_4[(h, v_d), R]$	$\boldsymbol{D}_4[h, R]\dot{+}\boldsymbol{Z}_2[v_d]$	$z_1 = z_2 = -z_3 = -z_4$, $w = z$

coordinate must be zero. In particular, when $\dim V \oplus V = 24$ and $\alpha + \beta \equiv 1 \pmod 3$ then $((e, v_t), \frac{1}{3})$ acts trivially on the six coordinates $z_1, z_2, z_3, w_4, w_5, w_6$. When $\alpha + \beta \equiv -1 \pmod 3$ then $((e, v_t), \frac{1}{3})$ acts trivially on the six coordinates $z_4, z_5, z_6, w_1, w_2, w_3$. This observation follows from the definition of the action of $((e, v_t), \frac{1}{3})$ on (z, w) and the fact that $-\frac{1}{3}(\alpha + \beta) \equiv \frac{1}{3}(2\alpha - \beta) \equiv \frac{1}{3}(-\alpha + 2\beta) \pmod 1$. Note that we always have $\alpha + \beta \equiv \pm 1 \pmod 3$ since $\alpha + \beta$ and 3 are coprime.

Examples of the pictures associated with elements of these fixed-point subspaces are presented in figure 2.

(*b*) *Square lattice*

We list in table 20 generators for each of the twisted groups of theorem 9.2.

In table 21, we describe the fixed-point subspaces in $V \oplus V$ of the twisted subgroups in table 20. To compute the fixed-point subspaces of the twisted sub-

Table 22. *Generators for twisted subgroups – rhombic lattice*

K	G	generators of G^{Θ}
$\boldsymbol{Z}_2[P]$	$\boldsymbol{Z}_2[P]\times\boldsymbol{Z}_4[(h,v_1)]$	$((P,0),0)$ and $((h,v_1),\frac{1}{4})$
$\boldsymbol{D}_2[h,P]$	$\boldsymbol{D}_2[h,P]\times\boldsymbol{Z}_2[v_d]$	$((P,0),0),((h,0),0)$ and $((e,v_d),\frac{1}{2})$

Table 23. *Fixed-point subspaces of twisted subgroups – rhombic lattice*

	K	G	$\mathrm{Fix}\,(G^{\Theta})$
R1	$\boldsymbol{Z}_2[P]$	$\boldsymbol{Z}_2[P]\times\boldsymbol{Z}_4[(h,v_1)]$	$z_1=-\mathrm{i}z_2=w_1=-\mathrm{i}w_2$
R2	$\boldsymbol{D}_2[h,P]$	$\boldsymbol{D}_2[h,P]\times\boldsymbol{Z}_2[v_d]$	$z_1=z_2=w_1=w_2$

groups in table 20, we need to define the action of the generators on $V\oplus V$. For $\dim V=4$, the actions of h and R are given in (10.4). Using

$$\begin{aligned}(e,v_1)(z_1,z_2,w_1,w_2)&=(-z_1,z_2,-w_1,w_2),\\(e,v_d)(z_1,z_2,w_1,w_2)&=(-z_1,-z_2,-w_1,-w_2),\\((e,0),\phi)(z_1,z_2,w_1,w_2)&=\mathrm{e}^{2\pi\phi\mathrm{i}}(z_1,z_2,w_1,w_2),\end{aligned}$$

it is easy to determine the action of the generators when $\dim V=4$.

For $\dim V=8$, the actions of h and R are given in (10.5). Using

$$(e,v_1)(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4)$$
$$=\begin{cases}(-z_1,z_2,z_3,-z_4,-w_1,w_2,w_3,-w_4) & \text{if } \alpha \text{ odd},\\(z_1,-z_2,-z_3,z_4,w_1,-w_2,-w_3,w_4) & \text{if } \beta \text{ odd},\end{cases}$$

$$\begin{aligned}(e,v_d)(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4)&=-(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4),\\((e,0),\phi)(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4)&=\mathrm{e}^{2\pi\phi\mathrm{i}}(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4),\end{aligned}$$

one obtains the action of the generators when $\dim V=8$.

Representative elements of these fixed-point subspaces are presented in figure 3 as they might appear in shadowgraph images of a convecting fluid layer.

(*c*) *Rhombic lattice*

We list in table 22 generators for each of the twisted groups of theorem 9.3.

In table 23, we describe the fixed-point subspaces in $V\oplus V$ of the twisted subgroups in table 22. To compute the fixed-point subspaces of the twisted subgroups in table 22, we need to define the actions of their generators on $V\oplus V$, where $\dim V=4$. The actions of h and P are given in (10.6). Using

$$\begin{aligned}(e,v_1)(z_1,z_2,w_1,w_2)&=(-z_1,z_2,-w_1,w_2),\\(e,v_d)(z_1,z_2,w_1,w_2)&=(-z_1,-z_2,-w_1,-w_2),\\((e,0),\phi)(z_1,z_2,w_1,w_2)&=\mathrm{e}^{2\pi\phi\mathrm{i}}(z_1,z_2,w_1,w_2),\end{aligned}$$

it is easy to determine the action of the generators.

Representative elements of these fixed-point subspaces are presented in figure 4.

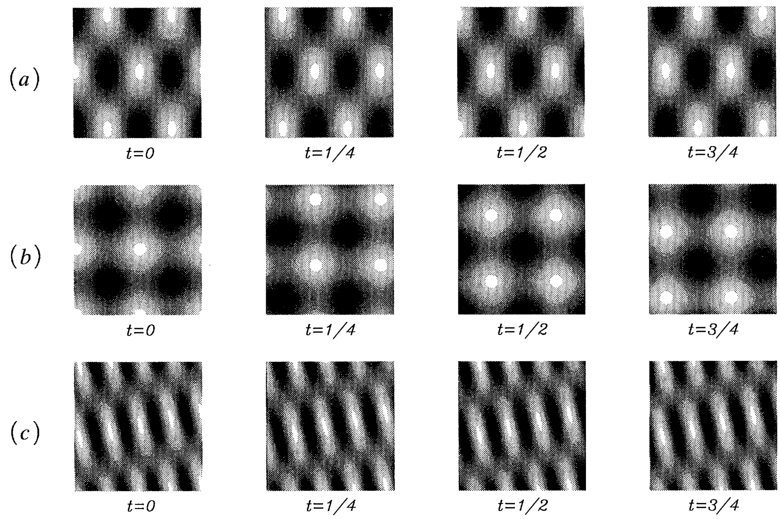

Figure 1. (*a*) Rotating wave TH1 for the six-dimensional representation V of the hexagonal lattice (see theorem 10.1). The Fourier sum in (10.1) is plotted at discrete times with $(z,w) = (1,0,0,0,1,0)$ and $-2/\sqrt{3} \leqslant x, y \leqslant 2/\sqrt{3}$. The pattern travels horizontally to the left. The pattern TH2 (not shown here) moves vertically. (*b*) Rotating wave TS for the eight-dimensional representation V of the square lattice. The Fourier sum in (10.1) is plotted at discrete times with $(z,w) = (1,1,0,0)$; the pattern travels diagonally to the left and down. (*c*) Rotating wave TR1 for the four-dimensional representation V of the rhombic lattice. The Fourier sum in (10.1) is plotted at discrete times with $(z,w) = (1,1,0,0)$ and $-\cot 25^\circ \leqslant x, y \leqslant \cot 25^\circ$. The pattern travels diagonally to the left and down. The pattern TR2 (not shown here) moves in the perpendicular direction.

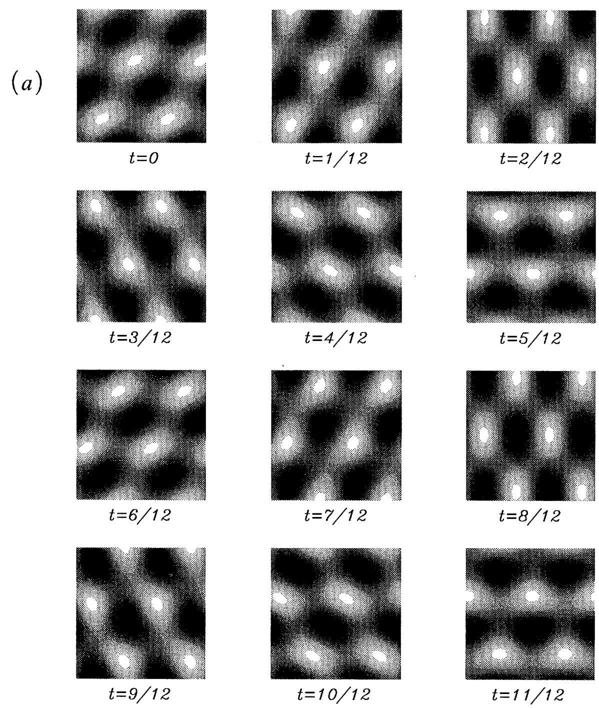

Figure 2. (*a*) For description see p. 165

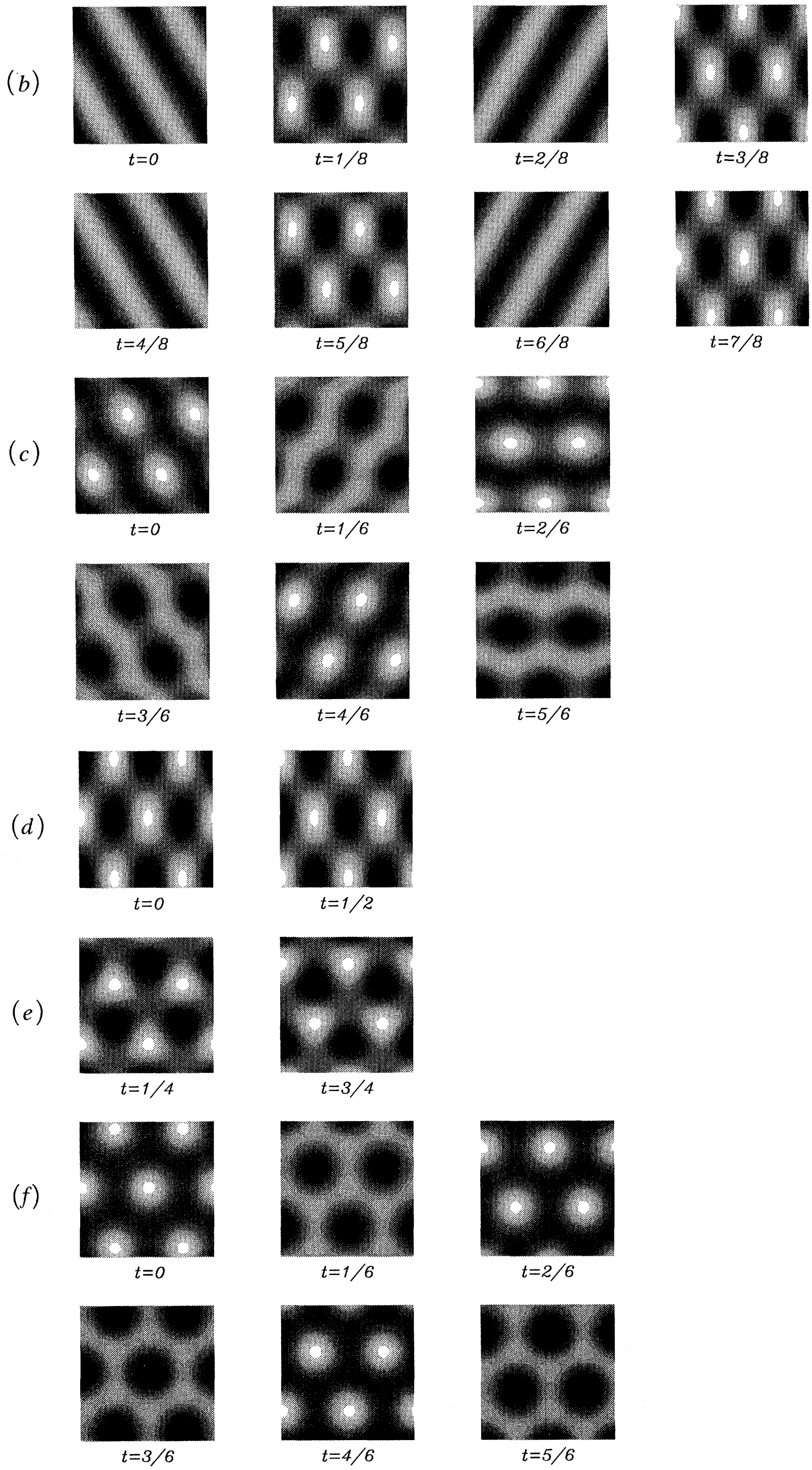

Figure 2. (*b*)–(*f*) For description see p. 165.

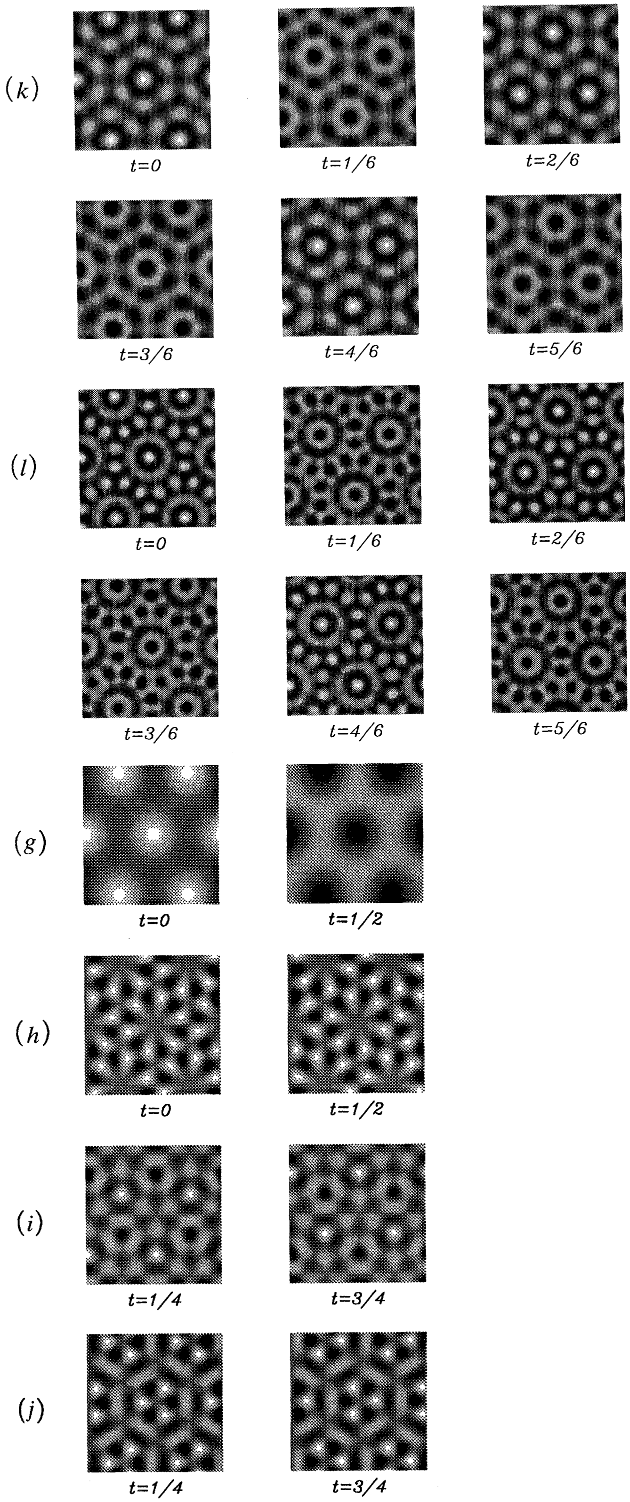

Figure 2. (*g*)–(*l*) For description see p. 165.

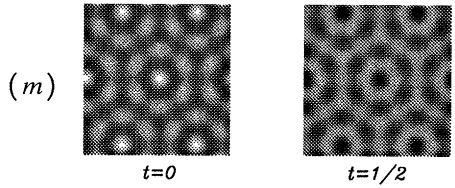

Figure 2. Discrete waves for the hexagonal lattice: (*a*) H1 [wavy rolls (2)]; (*b*) H2 [wavy rolls (1)]; (*c*) H3 [twisted patchwork quilt]; (*d*) H4 [standing patchwork quilt]; (e) *H*5 [standing regular triangle]; (*f*) H6 [oscillating triangle]; (*g*) H7 [standing hexagon]. The Fourier sum in (10.1) is plotted at discrete times for representative elements of the fixed-point subspaces listed in table 19. If the time step is divided by two in (*d*), (*e*), (*g*), (*h*), (*i*), (*j*) and (*m*), the extra pictures that we get show constant functions in the space variables. The first seven pictures illustrate discrete waves on the six-dimensional representation V. The names between square brackets are those given in Roberts *et al.* (1986). In each case the spatial domain is larger than the unit hexagonal cell (namely, $-2/\sqrt{3} \leqslant x, y \leqslant 2/\sqrt{3}$) and (z, w) is determined by $z_1 = 1$, except for (*f*) and (*k*) where (z, w) is determined by $w_1 = 1$. The last six pictures illustrate discrete waves on 12-dimensional representations on V. The subscripts α and β are the values used in table 3 to describe the 12-dimensional representations V: (*h*) $\mathrm{H8}_{3,2}$; (*i*) $\mathrm{H9}_{3,2}$; (*j*) $\mathrm{H5}_{3,2}$; (*k*) $\mathrm{H6}_{3,2}$; (*l*) $\mathrm{H6}_{4,3}$; (*m*) $\mathrm{H7}_{3,2}$.

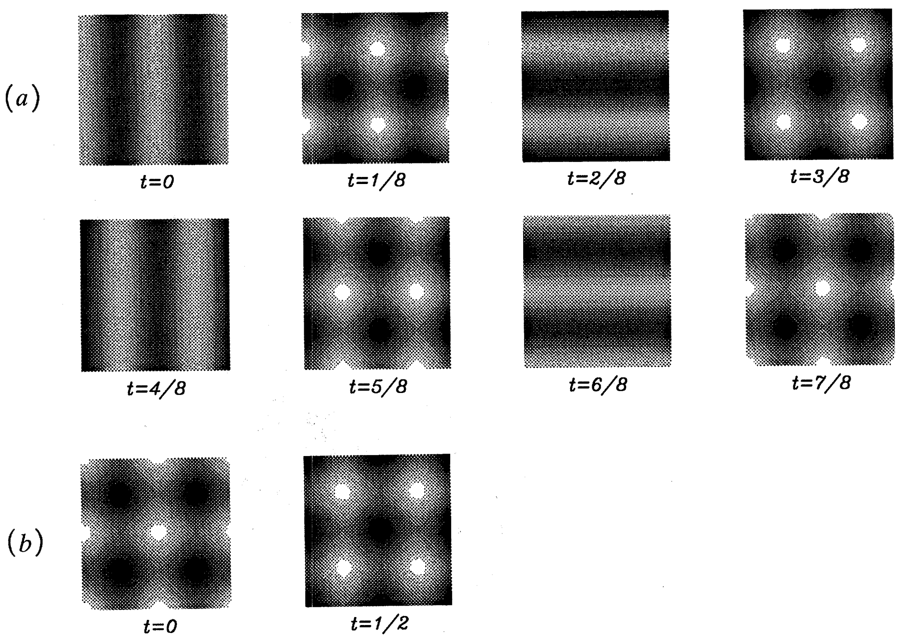

Figure 3. Discrete waves for the square lattice: (*a*) S1 (alternating rolls); (*b*) S2 (standing square). The Fourier sum in (10.1) is plotted at discrete times for representative elements of the fixed-point subspaces listed in table 21. If the time step is divided by two in (*b*), (*g*) and (*h*) the extra pictures that we get show constant functions in the space variables. The first two pictures illustrate discrete waves on the four-dimensional representation V. The names between square brackets are those given in Silber & Knobloch (1991). In each case the spatial domain contains four unit square cells (namely, $-1 \leqslant x, y \leqslant 1$) and (z,w) is determined by $z_1 = 1$. The last six pictures illustrate discrete waves on eight-dimensional representations V. The subscripts α and β are the values used in table 3 to describe the eight-dimensional representations V: (*c*) $\mathrm{S1}_{2,1}$; (*d*) $\mathrm{S1}_{3,2}$; (*e*) $\mathrm{S3}_{2,1}$; (*f*) $\mathrm{S3}_{3,2}$; (*g*) $\mathrm{S2}_{2,1}$; (*h*) $\mathrm{S4}_{2,1}$.

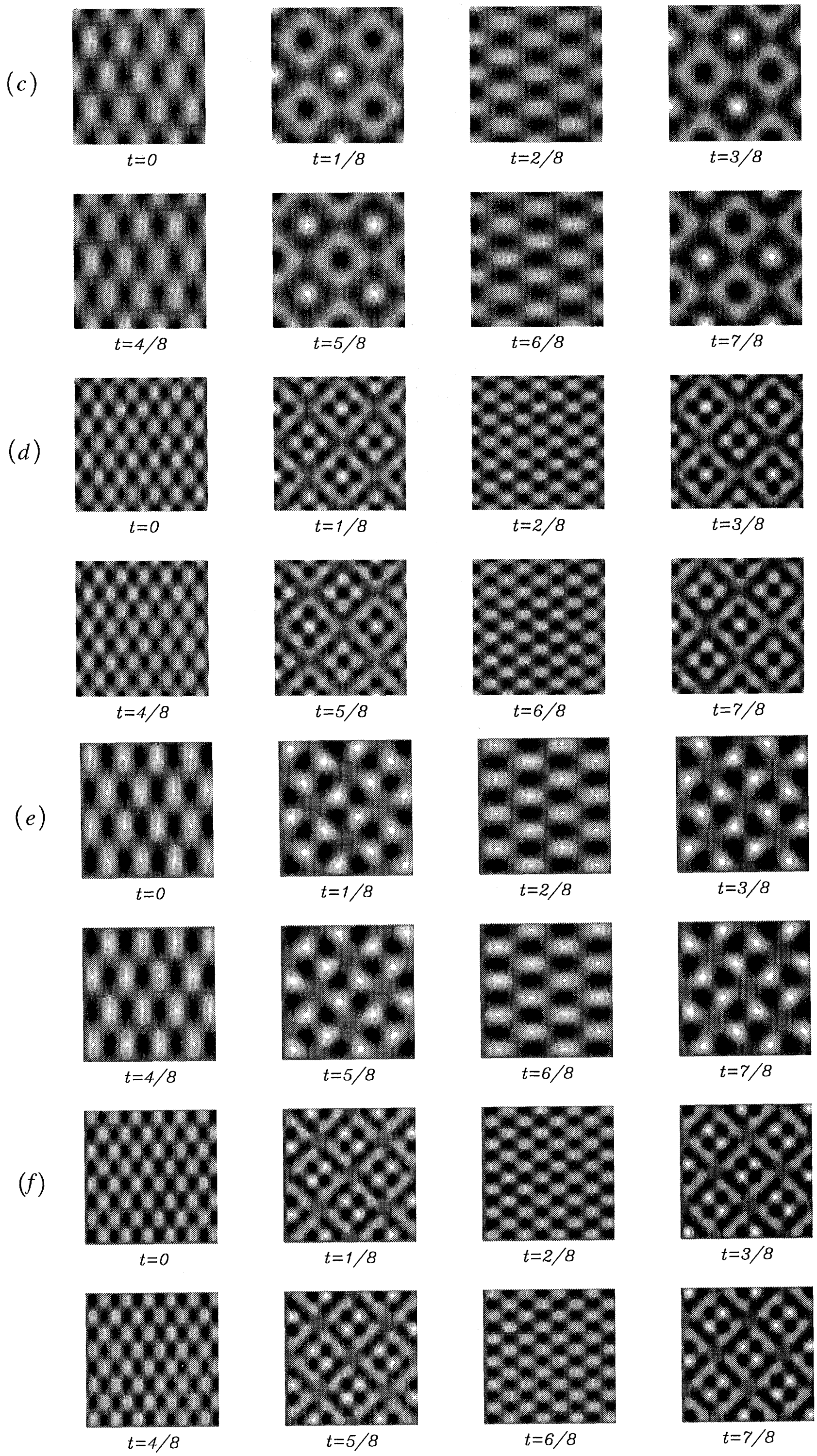

Figure 3. (*c*)–(*f*) For description see p. 165.

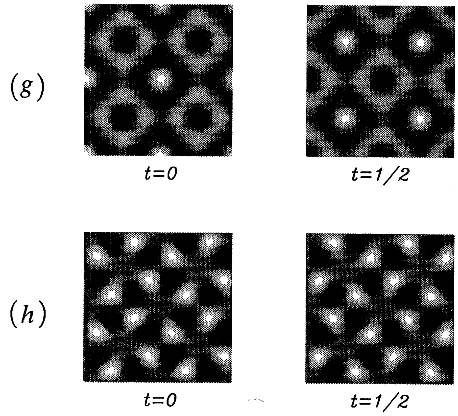

Figure 3. $(g), (h)$ For description see p. 165.

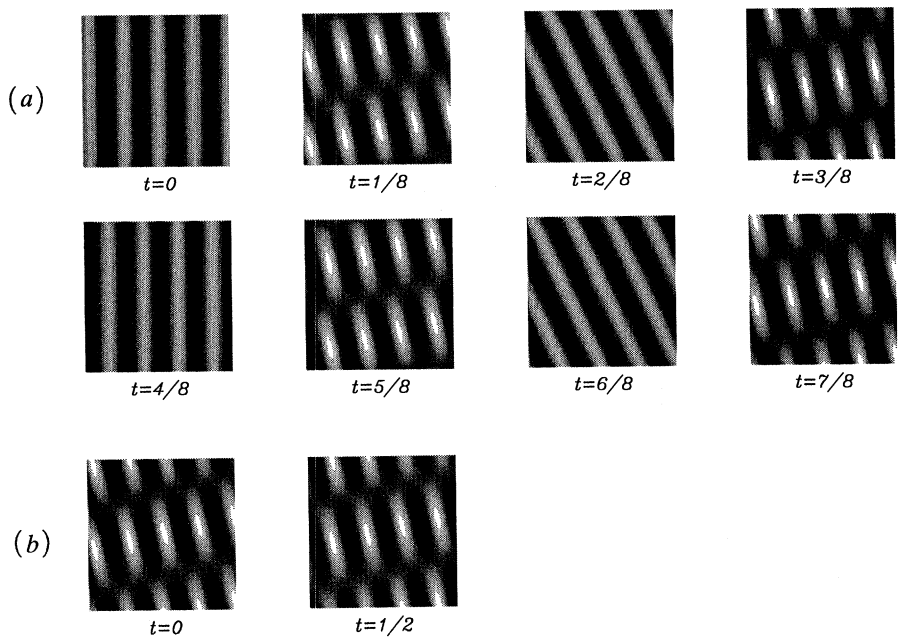

Figure 4. Discrete waves for the rhombic lattice: (a) R1 (alternating rolls); (b) R2 (standing rectangle). The two basic vectors ℓ_1 and ℓ_2 form a 25° angle. The Fourier sum in (10.1) is plotted at discrete times for representative elements of the fixed-point subspaces listed in table 23. If the time step is divided by two in (b) the extra pictures that we get show constant functions in the space variables. In each case the spatial domain is $-\cot 25^\circ \leqslant x, y \leqslant \cot 25^\circ$ and (z, w) is determined by $z_1 = 1$. The names between square brackets are those given in Silber *et al.* (1992).

The research of B.D. was supported by the Ministry of Colleges and Universities of Ontario and by NSERC through the Fields Institute for Research in the Mathematical Sciences. The research of MG was supported in part by NSF Grant DMS-9101836 and by the Texas Advanced Research Program (003652037). The research of I.N.S. was supported in part by a grant from the Science and Engineering Research Council of the UK and a European Community Laboratory Twinning Grant, and was carried out under the auspices of the European Bifurcation Theory Group. The authors also thank the Fields Institute for Research in the Mathematical Sciences for its hospitality and for its support of this research.

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Received 10 February 1993; accepted 26 August 1993

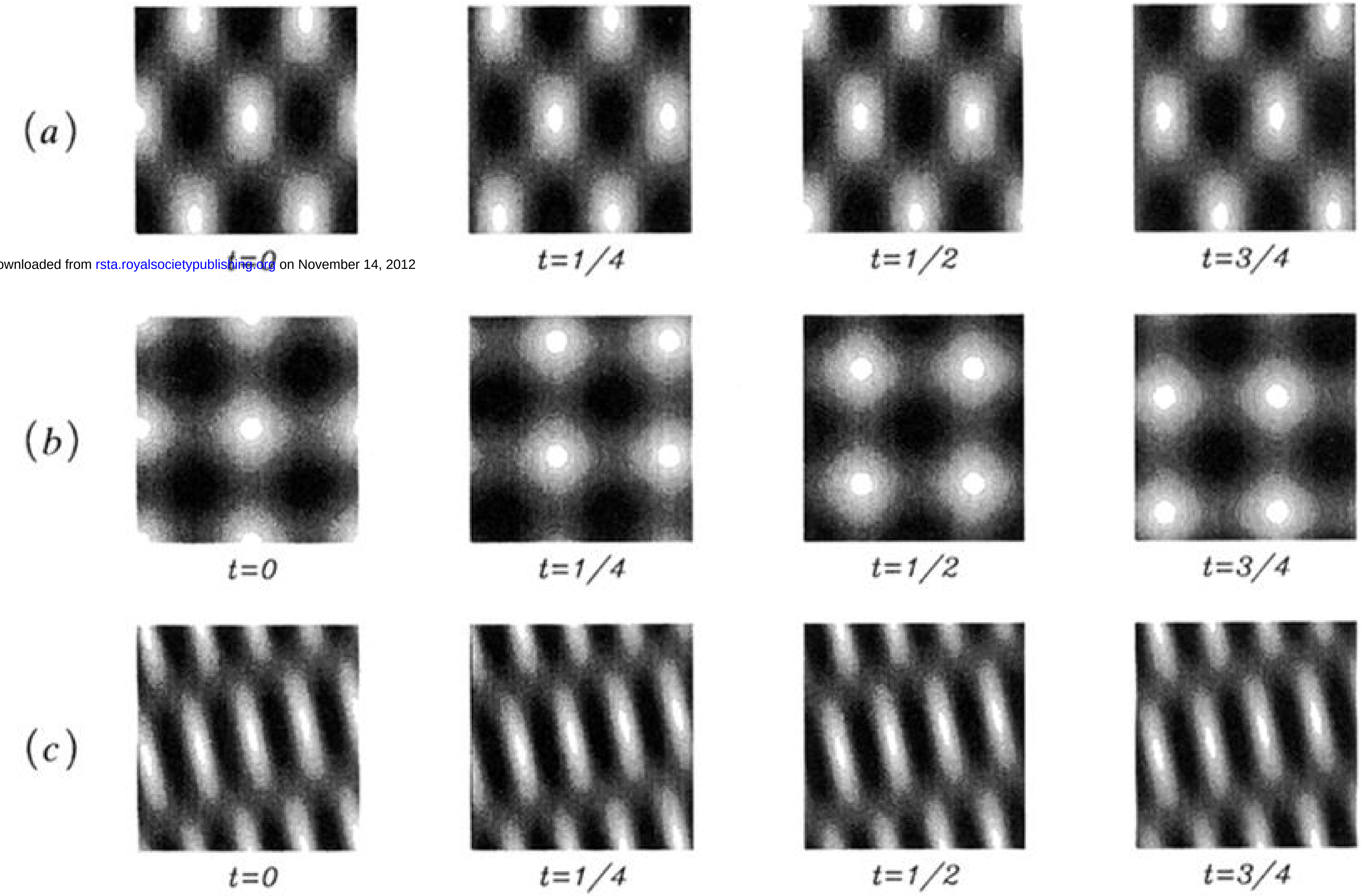


Figure 1. (a) Rotating wave TH1 for the six-dimensional representation V of the hexagonal lattice (see theorem 10.1). The Fourier sum in (10.1) is plotted at discrete times with $(z, w) = (1, 0, 0, 0, 1, 0)$ and $-2/\sqrt{3} \leqslant x, y \leqslant 2/\sqrt{3}$. The pattern travels horizontally to the left. The pattern TH2 (not shown here) moves vertically. (b) Rotating wave TS for the eight-dimensional representation V of the square lattice. The Fourier sum in (10.1) is plotted at discrete times with $(z, w) = (1, 1, 0, 0)$; the pattern travels diagonally to the left and down. (c) Rotating wave TR1 for the four-dimensional representation V of the rhombic lattice. The Fourier sum in (10.1) is plotted at discrete times with $(z, w) = (1, 1, 0, 0)$ and $-\cot 25^\circ \leqslant x, y \leqslant \cot 25^\circ$. The pattern travels diagonally to the left and down. The pattern TR2 (not shown here) moves in the perpendicular direction.

(a)

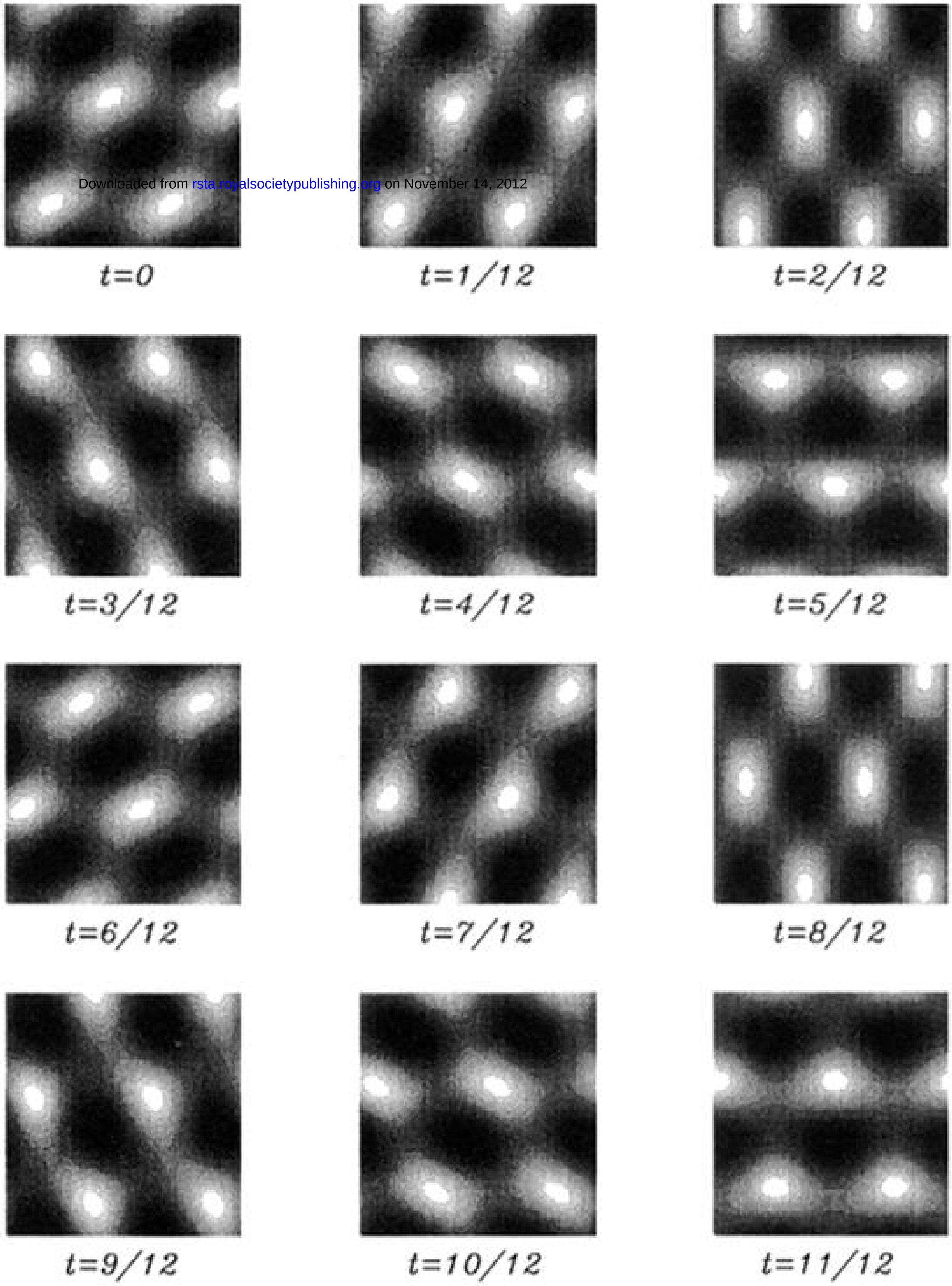


Figure 2. (a) For description see p. 165

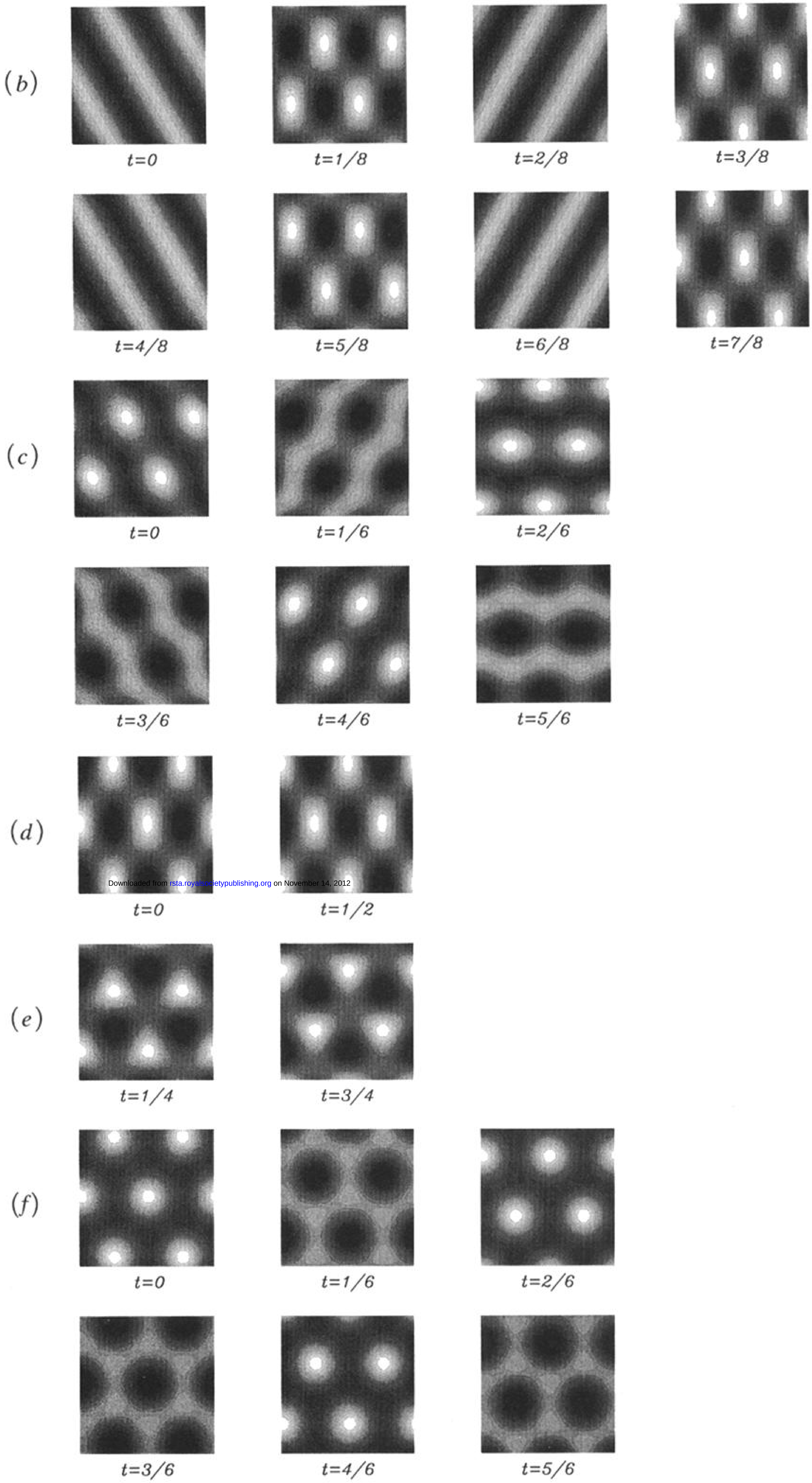


Figure 2. (b)–(f) For description see p. 165.

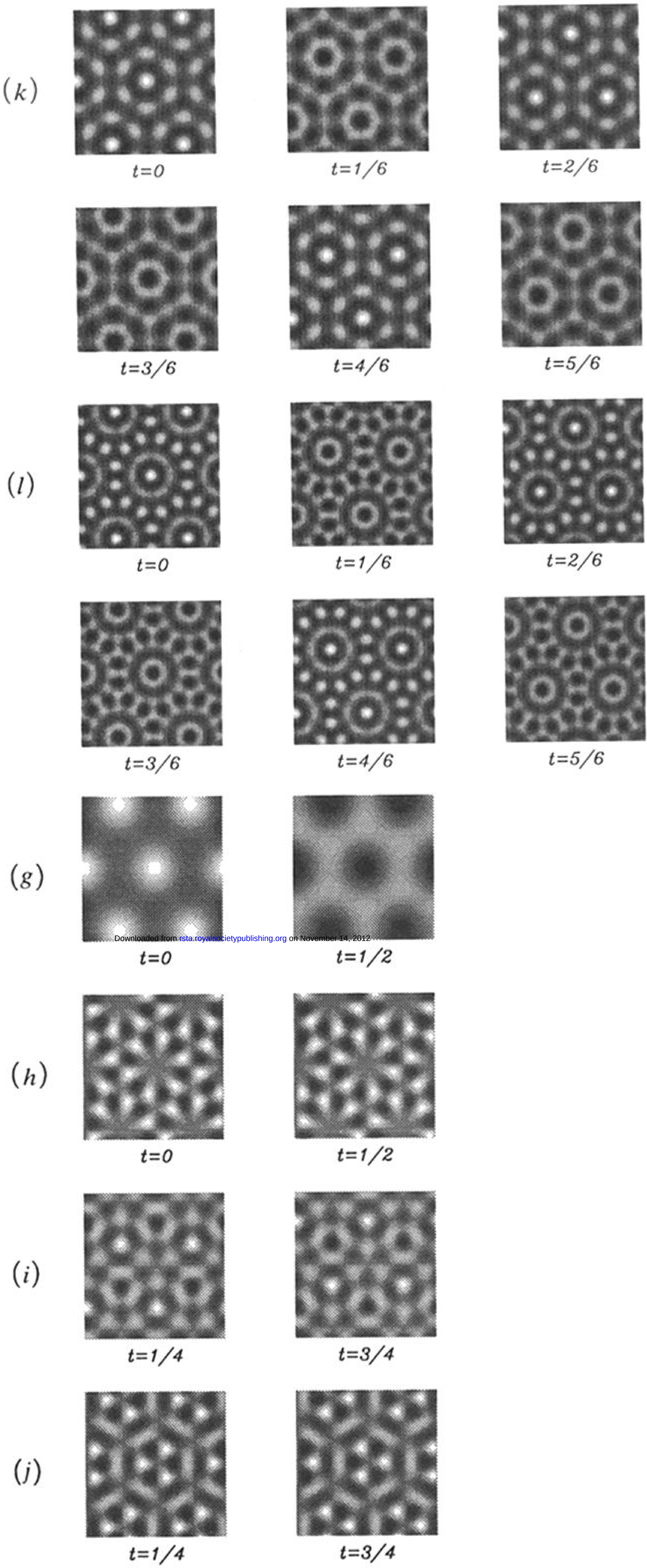
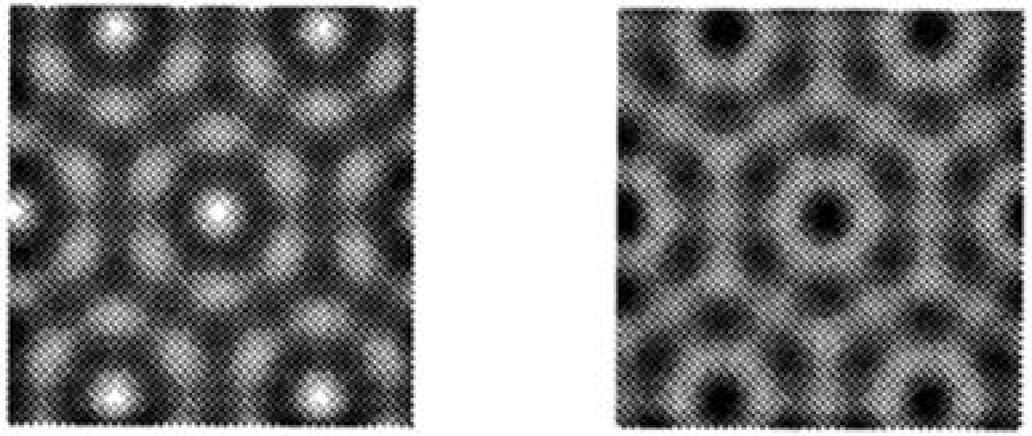


Figure 2. (g)–(l) For description see p. 165.

(m) 

t=0 t=1/2

Figure 2. Discrete waves for the hexagonal lattice: (*a*) H1 [wavy rolls (2)]; (*b*) H2 [wavy rolls (1)]; (*c*) H3 [twisted patchwork quilt]; (*d*) H4 [standing patchwork quilt]; (e) $H5$ [standing regular triangle]; (*f*) H6 [oscillating triangle]; (*g*) H7 [standing hexagon]. The Fourier sum in (10.1) is plotted at discrete times for representative elements of the fixed-point subspaces listed in table 19. If the time step is divided by two in (*d*), (*e*), (*g*), (*h*), (*i*), (*j*) and (*m*), the extra pictures that we get show constant functions in the space variables. The first seven pictures illustrate discrete waves on the six-dimensional representation V. The names between square brackets are those given in Roberts *et al.* (1986). In each case the spatial domain is larger than the unit hexagonal cell (namely, $-2/\sqrt{3} \leqslant x, y \leqslant 2/\sqrt{3}$) and (z, w) is determined by $z_1 = 1$, except for (*f*) and (*k*) where (z, w) is determined by $w_1 = 1$. The last six pictures illustrate discrete waves on 12-dimensional representations on V. The subscripts α and β are the values used in table 3 to describe the 12-dimensional representations V: (*h*) $\mathrm{H8}_{3,2}$; (*i*) $\mathrm{H9}_{3,2}$; (*j*) $\mathrm{H5}_{3,2}$; (*k*) $\mathrm{H6}_{3,2}$; (*l*) $\mathrm{H6}_{4,3}$; (*m*) $\mathrm{H7}_{3,2}$.

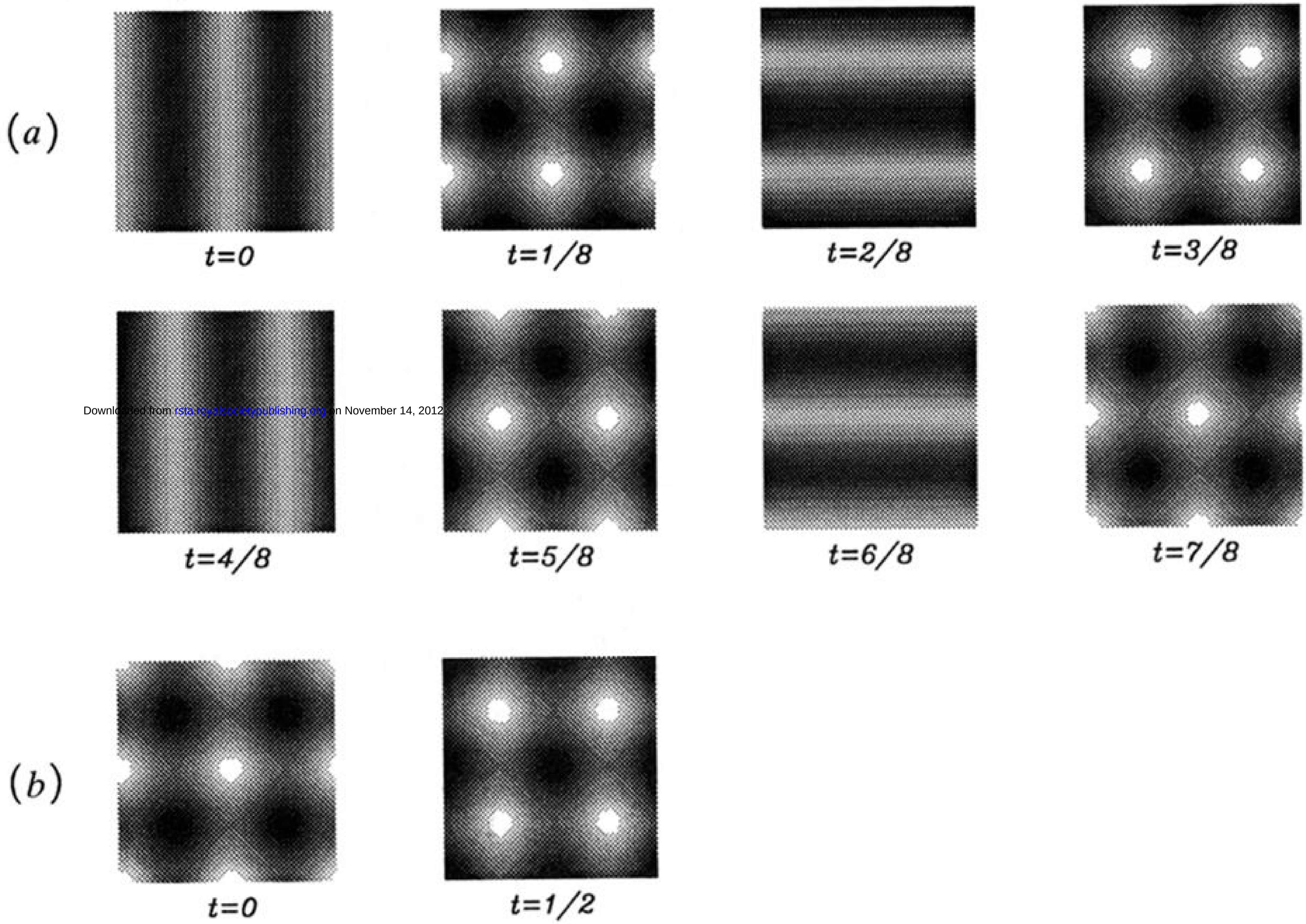


Figure 3. Discrete waves for the square lattice: (*a*) S1 (alternating rolls); (*b*) S2 (standing square). The Fourier sum in (10.1) is plotted at discrete times for representative elements of the fixed-point subspaces listed in table 21. If the time step is divided by two in (*b*), (*g*) and (*h*) the extra pictures that we get show constant functions in the space variables. The first two pictures illustrate discrete waves on the four-dimensional representation V. The names between square brackets are those given in Silber & Knobloch (1991). In each case the spatial domain contains four unit square cells (namely, $-1 \leqslant x, y \leqslant 1$) and (z,w) is determined by $z_1 = 1$. The last six pictures illustrate discrete waves on eight-dimensional representations V. The subscripts α and β are the values used in table 3 to describe the eight-dimensional representations V: (*c*) $\mathrm{S1}_{2,1}$; (*d*) $\mathrm{S1}_{3,2}$; (*e*) $\mathrm{S3}_{2,1}$; (*f*) $\mathrm{S3}_{3,2}$; (*g*) $\mathrm{S2}_{2,1}$; (*h*) $\mathrm{S4}_{2,1}$.

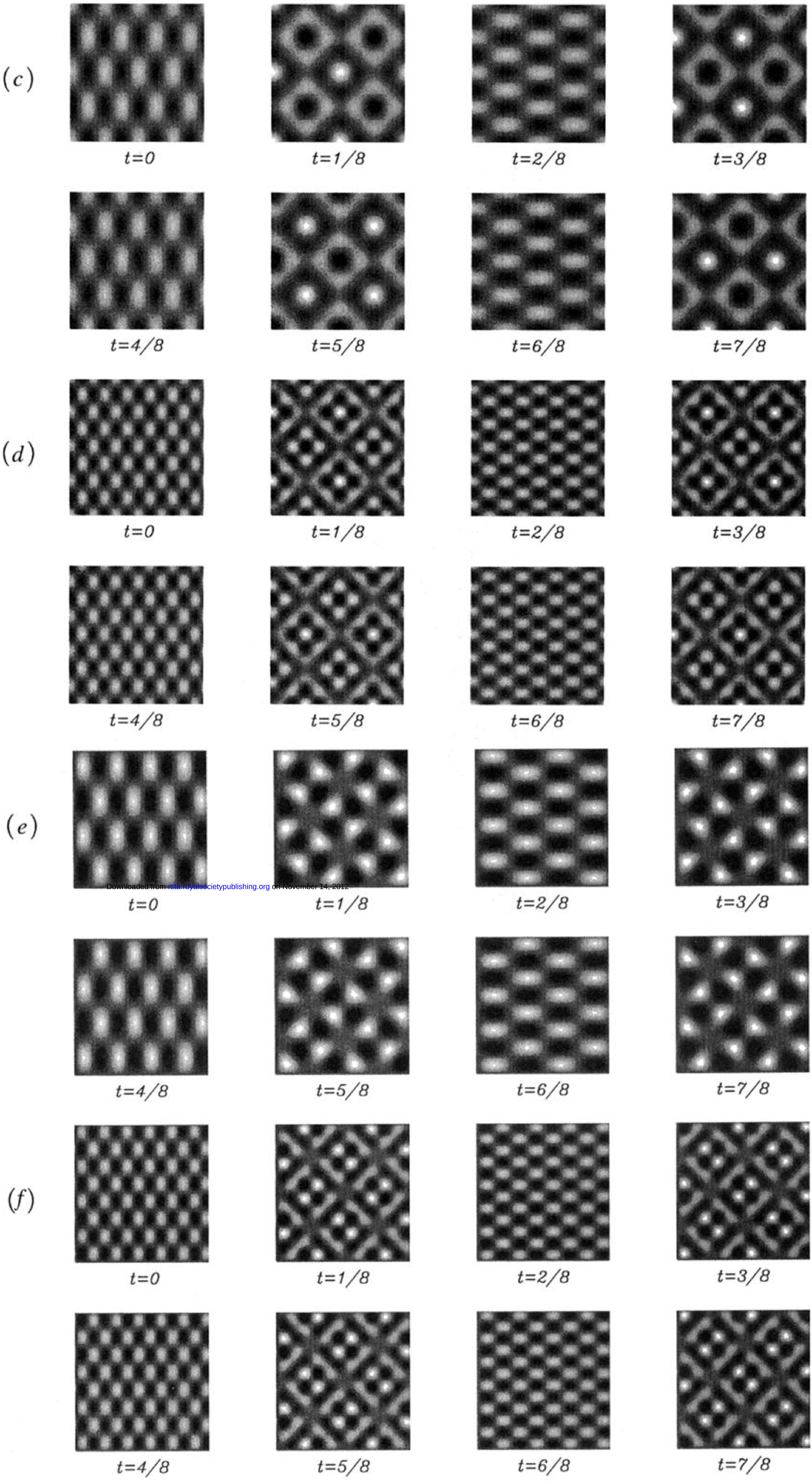


Figure 3. (c)–(f) For description see p. 165.

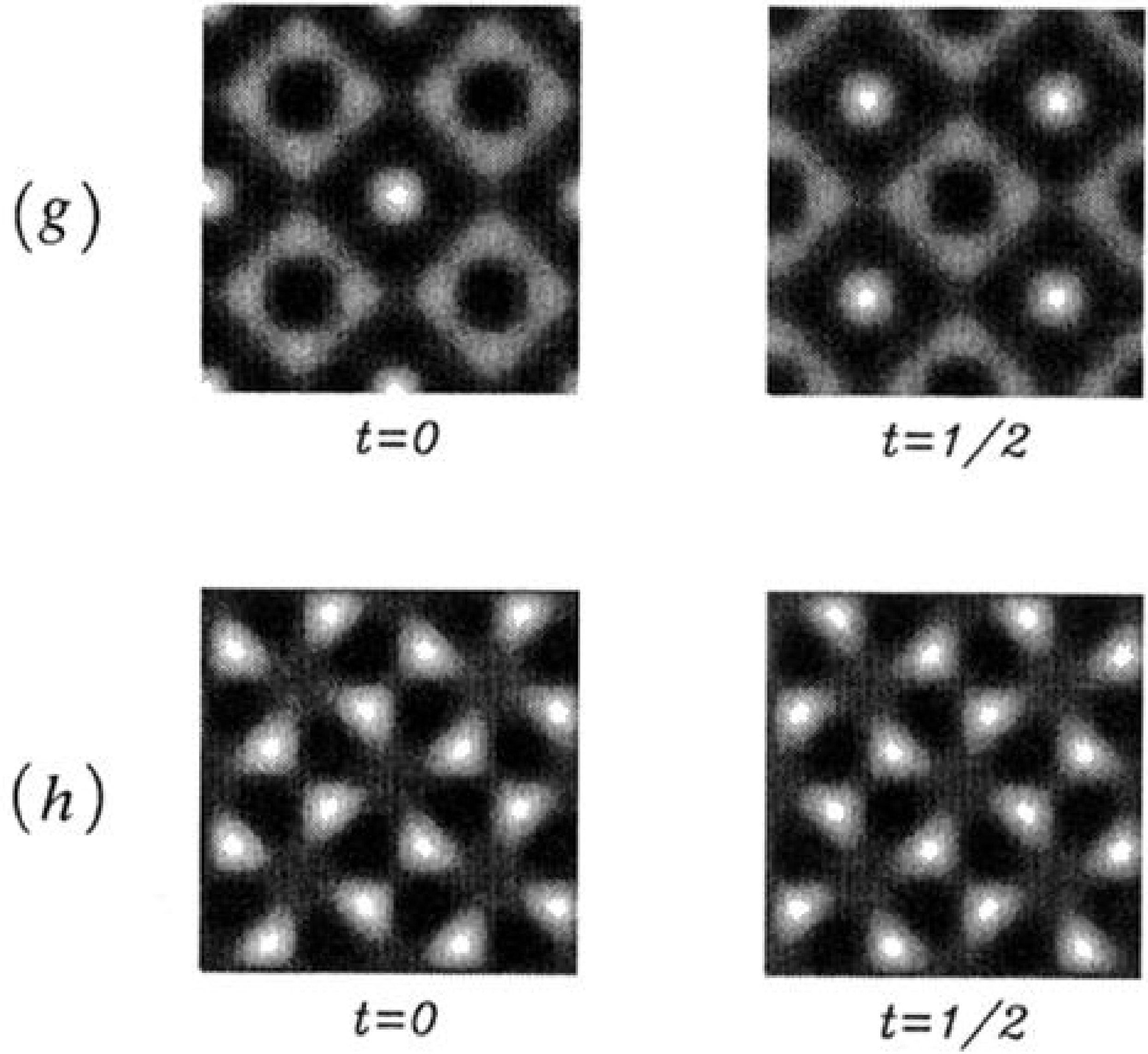


Figure 3. (g), (h) For description see p. 165.

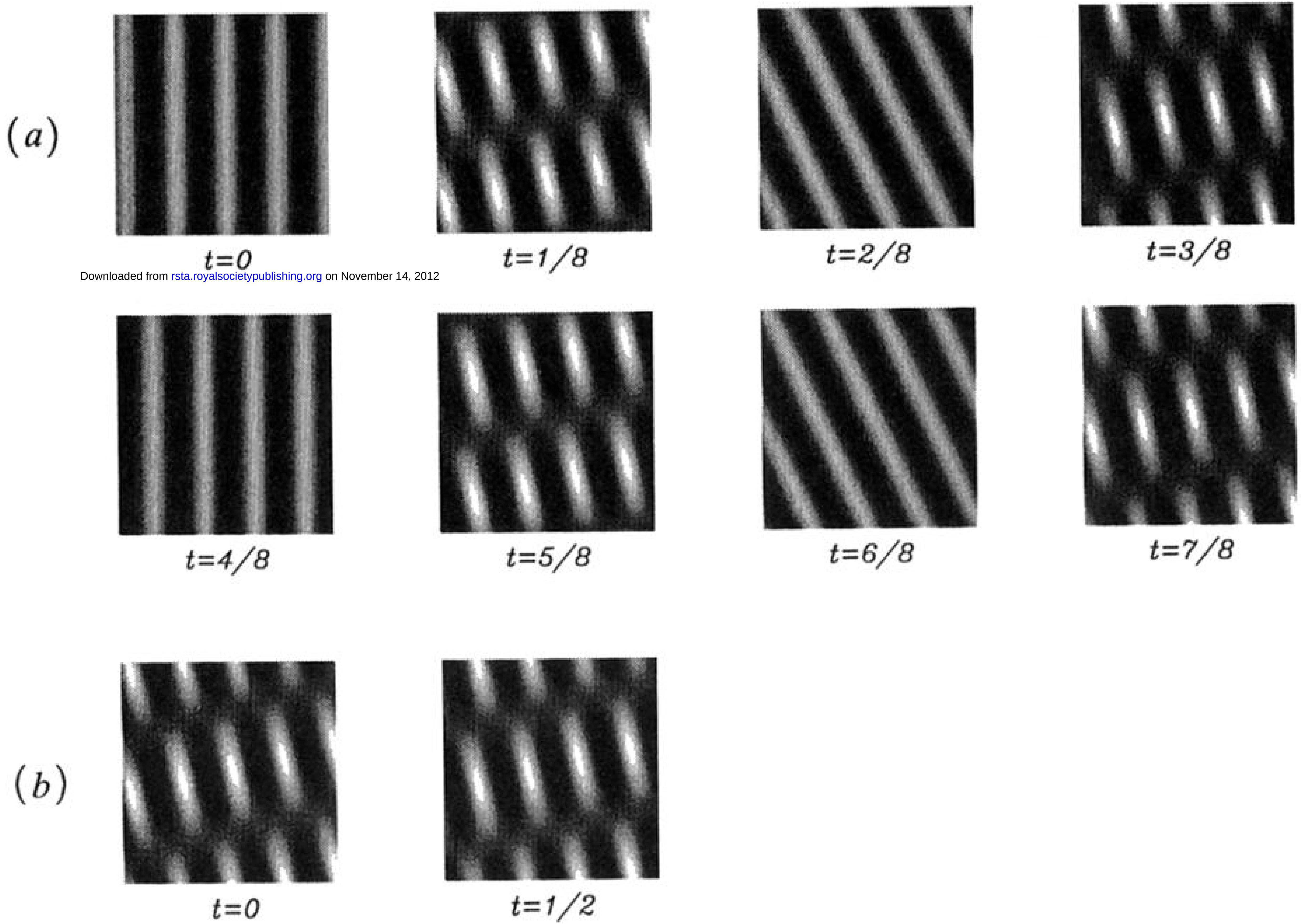


Figure 4. Discrete waves for the rhombic lattice: (*a*) R1 (alternating rolls); (*b*) R2 (standing rectangle). The two basic vectors ℓ_1 and ℓ_2 form a 25° angle. The Fourier sum in (10.1) is plotted at discrete times for representative elements of the fixed-point subspaces listed in table 23. If the time step is divided by two in (*b*) the extra pictures that we get show constant functions in the space variables. In each case the spatial domain is $-\cot 25^\circ \leqslant x, y \leqslant \cot 25^\circ$ and (z, w) is determined by $z_1 = 1$. The names between square brackets are those given in Silber *et al.* (1992).