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Patterns of synchrony in lattice dynamical systems

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Abstract

From the point of view of coupled systems developed by Stewart, Golubitsky and Pivato, lattice differential equations consist of choosing a phase space $\boldsymbol{R}^k$ for each point in a lattice, and a system of differential equations on each of these spaces $\boldsymbol{R}^k$ such that the whole system is translation invariant. The architecture of a lattice differential equation specifies the sites that are coupled to each other (nearest neighbour coupling (NN) is a standard example). A polydiagonal is a finite-dimensional subspace of phase space obtained by setting coordinates in different phase spaces as equal. There is a colouring of the network associated with each polydiagonal obtained by colouring any two cells that have equal coordinates with the same colour. A *pattern of synchrony* is a colouring associated with a polydiagonal that is flow-invariant for every lattice differential equation with a given architecture. We prove that every pattern of synchrony for a fixed architecture in planar lattice differential equations is spatially doubly-periodic, assuming that the couplings are sufficiently extensive. For example, nearest and next nearest neighbour couplings are needed for square and hexagonal couplings, but a third level of coupling is needed for the corresponding result to hold in rhombic and primitive cubic lattices. On planar lattices this result is known to fail if the network architecture consists only of NN. The techniques we develop to prove spatial periodicity and finiteness can be applied to other lattices as well.

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1. Introduction

Many physical and biological systems can be modelled by networks of systems of differential equations. Networks of differential equations possess additional structure, namely, canonical observables—the dynamical behaviour of the individual network nodes [18]. These observables can be compared, revealing such features as synchrony, or in periodic solutions, specified phase-relations. These features are important in many applications and any theoretical treatment of network dynamics must take this additional structure into account.

Stewart and co-workers [19, 23] formalized the concept of a *coupled cell network*, where a *cell* is a system of ordinary differential equations (ODEs) and a *coupled cell system* consists of cells whose equations are coupled. Stewart *et al* defined the architecture of coupled cell networks and developed a theory that shows how network architecture leads to synchrony. The architecture of a coupled cell network is a graph which indicates which cells have the same phase space, which cells are coupled to which and which couplings are the same (see also the development by Field [15]).

Synchrony is one of the most interesting features of coupled cell systems, and in order to study this, the concept needs to be formalized. We use a strong form of network synchrony, namely, robust synchrony, which we now define. A *polydiagonal* Δ is a subspace of the phase space of a coupled cell system that is defined by the equality of cell coordinates. The polydiagonal Δ is *robustly polysynchronous* if Δ is flow-invariant for every coupled cell system with the given network architecture. Solutions in a flow-invariant Δ have a collection of coordinates equal for all time. If we colour two cells the same when the coordinates are equal, then we can associate robustly polysynchronous polydiagonals with *patterns of synchrony*. A k-colour pattern of synchrony is that which is defined by exactly k colours.

Stewart *et al* [23, theorem 6.1] prove that a polydiagonal is robustly polysynchronous if, and only if, the colouring (given by colouring cells that have the same coordinates with the same colour) is balanced. (The definition of balanced is given in definition 2.11.) Thus, classifying robustly polysynchronous polydiagonals is equivalent to the combinatorial question of classifying balanced colourings.

Earlier works [14, 16], study periodic patterns of synchrony on one- and two-dimensional lattices using techniques from the equivariant bifurcation theory. These results depend on the fact that fixed-point subspaces of equivariant systems are always flow-invariant, and are therefore special cases of robustly polysynchronous polydiagonals. We emphasize that [14, 16] assume spatial periodicity, whereas in this study we show that periodicity is a consequence of flow invariance.

A lattice dynamical system is an infinite system of ODEs, indexed by points in a lattice (such as the n-dimensional integer lattice $\boldsymbol{Z}^n$). Since lattices have spatial structure, lattice differential equations resemble partial differential equations, although the former may also exhibit phenomena not found in the latter.

Studies done on lattice differential equations often focus on equilibria or travelling waves. In the case of equilibria, these studies discuss the spatial features associated with stable equilibria. Such features can take the form of regularly ordered patterns (pattern formation) on the one hand, and spatially disordered displays (spatial chaos) on the other. See Chow [6] for a review of the theory of lattice differential equations and also the survey papers of Chow *et al* [7, 8] and Mallet-Paret and co-workers [21, 22].

Lattice differential equations arise in many applications. For example, in the field of electrical circuit theory, much work has been done by Chua and his collaborators, particularly in their studies of cellular neural networks (CNN). The CNN is a lattice dynamical system, in which each cell connects only to neighbouring cells that are within a finite radius (see Chua and Roska [10] and Chua and Yang [11]). CNN can be realized as VLSI chips and can operate at very high speeds and complexity [5]; they have been used to solve image processing and pattern recognition problems (see [3, 12]). Lattice differential equations have also been used in metallurgy and specifically, to model solidification of alloys (see Cahn [4], Cook *et al* [13] and Hillert [20]).

Many lattice differential equations arise from the discretization of a system of partial differential equations. For example, the discretization of a reaction–diffusion system in one space variable leads to

$$\dot{u}_i = -\beta \Delta u_i - f(u_i) \qquad i \in \mathbf{Z}, \tag{1.1}$$

where $\beta > 0$ and

$$\Delta u_i = u_{i+1} + u_{i-1} - 2u_i$$

is the *discrete Laplace operator*. Discretization of a planar system leads to

$$\dot{u}_{i,j} = -\beta^+ \Delta^+ u_{i,j} - \beta^\times \Delta^\times u_{i,j} - f(u_{i,j}) \qquad (i, j) \in \mathbf{Z}^2, \tag{1.2}$$

where $\beta^+ > \beta^\times > 0$ and Δ^+ and $\Delta^\times$ are the *discrete two-dimensional Laplace operators* on $\mathbf{Z}^2$ based on the nearest neighbours and next nearest neighbours, respectively, and are given by

$$\begin{aligned} \Delta^+ u_{i,j} &= u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}, \\ \Delta^\times u_{i,j} &= u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} - 4u_{i,j}. \end{aligned}$$

In these equations the parameters β, β^+, $\beta^\times$ are coupling parameters and f is a given nonlinearity representing the 'internal dynamics.' These equations are special examples of lattice differential equations since the coupling is linear and the lattices are the simplest possible. Note that (1.2) is equivariant with respect to rotations, reflections and translations that preserve the lattice.

We shall define lattice dynamical systems in a general sense to include nonlinear coupling and all lattices. Properties of these general lattice dynamical systems include the following:

(i) the coupling between pairs of lattice points is a function of the distance between the pair of points,
(ii) the range of coupling is finite.

It follows that lattice differential equations are equivariant with respect to rotations, reflections and translations that preserve the lattice.

Using the formalism of coupled cell systems, it is possible to give such a general definition. Given a lattice $\mathcal{L}$, we consider a coupled cell network $G_{\mathcal{L}}$, indexed by the points of $\mathcal{L}$. Each cell has an ordered finite set of cells $I(c)$, that are coupled to c. (For convenience we assume that the cells in $I(c)$ are ordered so that cells of the same coupling type are contiguous, and lattice translation symmetries preserve the orderings in input sets.) A standard example of network architecture is given by *nearest neighbour coupling* (NN) in which case $I(c)$ consists of those cells in the lattice that are nearest to c. We can associate systems of differential equations with $G_{\mathcal{L}}$, as follows.

$$\dot{x}_c = g(x_c, x_{I(c)}) \qquad c \in \mathcal{L}, \tag{1.3}$$

where $x_c \in \mathbf{R}^n$, $I(c) = (c_1, \ldots, c_k)$, $x_{I(c)} = (x_{c_1}, \ldots, x_{c_k}) \in (\mathbf{R}^n)^k$ and $g : (\mathbf{R}^n)^{k+1} \to \mathbf{R}^n$ is a $G_{\mathcal{L}}$-admissible map, that is, a map that is invariant under any permutation of the cells in $I(c)$. One important consequence of this construction is that the system of differential equations so defined is equivariant with respect to the symmetries of the lattice consisting of translations, rotations and reflections.

Golubitsky *et al* [17] give an infinite class of two-colour patterns of synchrony on square lattice systems with NN. Wang and Golubitsky [25] classify all possible two-colour patterns of synchrony of square and hexagonal lattice differential equations with two different architectures—both NN and next nearest neighbour coupling (NNN). It follows from these results that in the NNN architecture, balanced 2-colourings are finite in number and spatially doubly-periodic. Thus, there is a profound difference between balanced 2-colourings in the

NN and NNN cases: one classification is finite, the other infinite; one set has spatially periodic and nonperiodic colourings, the other only periodic colourings.

In this paper, we show that each balanced k-colouring on a square and hexagonal lattice with NNN architecture is spatially periodic and that there are only a finite number of k-colourings for each k (see theorem 4.1). We developed general techniques to prove similar theorems for other lattices as well; the general principle seems to be that if there is enough coupling, then k-colourings are spatially periodic.

Chow *et al* [8,9] and Thiran *et al* [24] considered a related notion of a 'mosaic' solution to a square lattice dynamical system. *Mosaic* solutions are equilibria that take on a finite number of values ℓ and, as such, lie in a polydiagonal defined by ℓ colours. These authors considered only mosaic solutions where $\ell = 2, 3$. We note that the polydiagonal associated with a mosaic solution may or may not always be flow-invariant; that is, the ℓ-colouring may or may not be balanced. There is, however, a somewhat surprising relationship between robust mosaic solutions and balanced ℓ-colourings. A mosaic solution is *robust* if it is hyperbolic and all equilibria obtained by a small perturbation in the lattice differential equation are also mosaic solutions with the same pattern. In this case, the associated ℓ-colouring is balanced [19].

In section 2 we discuss the general structure of lattice differential equations. The techniques that we use to prove spatial periodicity and finiteness (namely, the notions of 'window' and 'determining boundaries') are discussed in section 3. The theorems on planar lattices are given in section 4 and a cubic lattice is discussed in section 5.

2. Lattice dynamical systems

In this section, we generalize the abstract definition of a *coupled cell network* given in [19] to include networks with a countable number of cells and define several terms associated with these networks. Each cell network corresponds to a class of differential equations called *admissible* coupled cell systems. A coupled cell system is itself a collection of dynamical systems (cells) coupled together. The *architecture* of that network can be represented schematically by a directed graph whose nodes correspond to cells, and whose edges represent coupling. Hence, one cell is *coupled* to another if the output of the first cell affects the time-evolution of the second one. This approach provides a framework within which statements about all coupled cell systems corresponding to a given coupled cell network can be proved. Our purpose here is to give a precise definition of a *lattice dynamical system* (see definition 2.5) and then, in later sections, to prove that under certain assumptions about the extent of coupling, all *balanced k-colourings* are spatially periodic.

Definition 2.1. *A* coupled cell network *G consists of:*

(a) a countable set $\mathcal{C}$ of cells,
(b) an equivalence relation $\sim_C$ on cells in $\mathcal{C}$,
(c) a countable set $\mathcal{E}$ of edges *or* arrows,
(d) an equivalence relation $\sim_E$ on edges in $\mathcal{E}$. The edge type *of edge e is the $\sim_E$-equivalence class of e,*
(e) (local finiteness) there is a head *map $\mathcal{H} : \mathcal{E} \to \mathcal{C}$ and a* tail *map $\mathcal{T} : \mathcal{E} \to \mathcal{C}$ such that for every $c \in \mathcal{C}$ the sets $\mathcal{H}^{-1}(c)$ and $\mathcal{T}^{-1}(c)$ are finite.*

We also require the consistency condition*:*

(f) equivalent arrows have equivalent tails and heads; that is, if $e_1 \sim_E e_2$ in $\mathcal{E}$, then $\mathcal{H}(e_1) \sim_C \mathcal{H}(e_2)$ and $\mathcal{T}(e_1) \sim_C \mathcal{T}(e_2)$.

We write $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. ◇

Remark 2.2. In the abstract setting of [19], multiple connections between cells are permitted since it is possible that $\mathcal{H}(e_1) = \mathcal{H}(e_2)$ and $\mathcal{T}(e_1) = \mathcal{T}(e_2)$ for $e_1 \neq e_2$. Also self-coupling is permitted since $\mathcal{H}(e) = \mathcal{T}(e)$ is permitted. Owing to this, it is most natural to think of inputs as arrows. This generality is not necessary in our discussion of lattice dynamical systems where multiple connections and self-coupling are not considered; therefore, we can identify input arrows by their tail and head cells, as was originally done in [23]. Indeed, we denote an arrow e by the pair $(\mathcal{T}(e), \mathcal{H}(e))$.

Definition 2.3. *Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network without self-coupling and multiarrows.*

(a) Let $c \in \mathcal{C}$. The input set *of c is*

$$I(c) = \mathcal{T}(\mathcal{H}^{-1}(c)), \tag{2.1}$$

with a fixed ordering. An element of the finite set $I(c)$ is called an input cell *of c.*

(b) Two input sets $I(c)$ and $I(d)$ are isomorphic *if there is a coupling type preserving bijection between the input sets; that is, there exists a bijection $\beta : I(c) \to I(d)$ such that for all $i \in I(c)$*

$$(i, c) \sim_E (\beta(i), d).$$

(c) A coupled cell network is homogeneous *if all input sets are isomorphic.* ◇

Note that if G is homogeneous, then it follows from definition 2.1(f) that all cells are cell-equivalent; that is, there is only one $\sim_C$-equivalence class. However, a homogeneous cell network can have many different $\sim_E$-equivalence classes.

An *n-dimensional lattice* $\mathcal{L}$ is a subset of $\boldsymbol{R}^n$ of the form

$$\mathcal{L} = \{\alpha_1 v_1 + \cdots + \alpha_n v_n : \alpha_i \in \boldsymbol{Z}\},$$

where $\{v_1, \ldots, v_n\}$ is a set of linearly independent vectors in $\boldsymbol{R}^n$ called the *generators of the lattice* $\mathcal{L}$. Note that $\mathcal{L}$ is a discrete subgroup of $\boldsymbol{R}^n$.

Definition 2.4. *We call a lattice $\mathcal{L}$ Euclidean if it satisfies the following:*

(a) all generators of $\mathcal{L}$ have the same length,

(b) the generators of $\mathcal{L}$ are exactly those lattice vectors that are nearest to the origin in Euclidean distance. ◇

Euclidean lattices are most relevant for applications of the bifurcation theory [18]. While planar square and hexagonal lattices are Euclidean, only certain rhombic lattices are so. The generators v_1, v_2 of a rhombic lattice can be assumed to be in the first quadrant. A rhombic lattice satisfies (b) only when the angle between v_1 and v_2 is greater than $\pi/3$. See, for example, Armstrong [2] (chapter 25), for details on planar lattices.

Let $r_0 < r_1 < \cdots$ be the possible lengths of vectors in a fixed lattice $\mathcal{L}$. We can partition the vectors in $\mathcal{L}$ by length as follows.

Let

$$J_i = \{v \in \mathcal{L} : |v| = r_i\},$$

where $|v|$ denotes the Euclidean norm of v. The vectors in $\mathcal{L}$ can be divided into classes of neighbours as follows: the nearest neighbours to a vector $c \in \mathcal{L}$ are the set of vectors $\{c + v : v \in J_1\}$. The next nearest neighbours to c are the set of vectors $\{c + v : v \in J_2\}$. The *pth nearest neighbours* to c are the set of vectors $\{c + v : v \in J_p\}$.

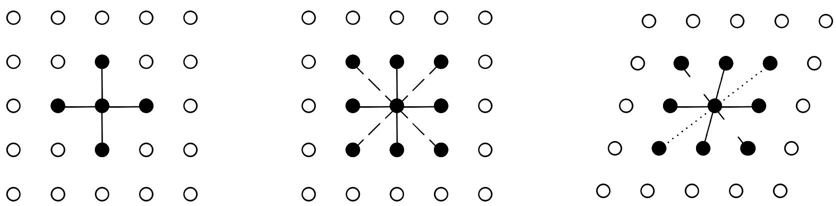

Figure 1. Left: square lattice network with NN (——). Centre: square lattice network with NN and NNN (- - - -). Right: rhombic lattice network with NN, NNN and next NNN (······).

Definition 2.5. *An* n-dimensional lattice network *consists of:*

(a) an n-dimensional lattice $\mathcal{L}$,
(b) a homogeneous coupled cell network $G_{\mathcal{L}}$ whose cells are indexed by $\mathcal{L}$,
(c) the set $I(0) = J_1 \cup \cdots \cup J_p$ for some p. This set is ordered so that the cells in J_i precede those in J_{i+1},
(d) the arrows connecting any two cells in $I(0)$ to 0 have the same edge type if, and only if, the cells are in the same set J_i for some i. ◇

We say that a lattice in which the cells are coupled to neighbours of order less than or equal to p is a *lattice with pth NN*. In particular, if $p = 1$ we have a *lattice with NN* and if $p = 2$ we have a *lattice with NN and NNN*. Figure 1 shows examples of two-dimensional lattice networks.

Remarks 2.6.

(a) Lattice networks are bidirectional, that is, for each arrow from c to d there is an arrow of the same type from d to c. This follows from definition 2.5.
(b) The symmetry group of the lattice is the symmetry group of the lattice network. In particular, translation by any vector in the lattice is a symmetry of the lattice network.

Example 2.7. Up to equivalence, there is exactly one lattice $\mathcal{L}$ in $\boldsymbol{R}$. If we normalize the length of the generator of the lattice to be 1, $\mathcal{L} \cong \boldsymbol{Z}$. In a network defined on $\boldsymbol{Z}$, each cell i has exactly two neighbours of order p, namely the left $(i - p)$ and the right $(i + p)$. ◇

Next, we make a precise connection between lattice dynamical systems and lattice networks. Again, we follow the treatment of Stewart and co-workers [19, 23] adapted to lattice networks. Given a lattice network $G_{\mathcal{L}}$, we wish to define a class of 'admissible' vector fields corresponding to $G_{\mathcal{L}}$. This class consists of all vector fields that are compatible with the labelled graph structure. With each cell c, we associate a *cell phase space* which we assume is $P_c = \boldsymbol{R}^k$. Moreover, since a lattice network is homogeneous, we have that all cells are cell-equivalent. Hence we choose the same phase space for all cells. A point x in the total phase space has coordinates x_c in cell c. Let $\mathcal{D} = (c_1, \ldots, c_\ell)$ be any finite ordered set of ℓ cells in $\mathcal{L}$ and define

$$x_{\mathcal{D}} = (x_{c_1}, \ldots, x_{c_\ell}),$$

where $x_{c_i} \in P_{c_i} = \boldsymbol{R}^k$.

Given $c, d \in \mathcal{L}$ there is a bijection $\beta : I(c) \to I(d)$ preserving coupling types such that $I(d) = \beta(I(c))$. Note that from definition 2.5(d), any such isomorphism has to map each class of p-neighbours of c to the class of p-neighbours of d. In particular, if $c = d$ then any such β is a permutation of $I(c)$ leaving invariant each class of neighbours of c. Recall from the paragraph preceding (1.3) that we order cells in $I(c)$ so that cells in $c + J_i$ are contiguous for each $1 \leqslant i \leqslant p$.

Definition 2.8. *An n-dimensional* lattice dynamical system *is a system of ODE associated with a n-dimensional lattice network $G_{\mathcal{L}}$, given by*

$$\dot{x}_c = f(x_c, x_{I(c)}) \qquad c \in \mathcal{L},$$

where $x_c \in \boldsymbol{R}^k$, $I(c) = (c_1, \ldots, c_\ell)$, $x_{I(c)} = (x_{c_1}, \ldots, x_{c_\ell}) \in \boldsymbol{R}^{k\,\ell}$ and the map $f : \boldsymbol{R}^{k(\ell+1)} \to \boldsymbol{R}^k$ is smooth and invariant under all permutations of the variables that map each class of neighbours of c into itself. The corresponding vector field is said to be $G_{\mathcal{L}}$-admissible. ◇

Example 2.9. Let $G_{\mathcal{L}}$ be the square lattice network with NN and NNN in figure 1 (centre). If we normalize the length of the generators of the lattice to be 1, $\mathcal{L} \cong \boldsymbol{Z}^2$. A square lattice dynamical system with NN and NNN corresponding to it has the form

$$\dot{x}_{i,j} = f(x_{i,j}, \overline{x_{i+1,j}, x_{i-1,j}, x_{i,j+1}, x_{i,j-1}}, \overline{x_{i+1,j+1}, x_{i-1,j+1}, x_{i+1,j-1}, x_{i-1,j-1}}),$$

where $(i, j) \in \boldsymbol{Z}^2$, $x_{i,j} \in \boldsymbol{R}^k$ and f is invariant under all permutations of the variables under the bars. The invariance of f under permutation of the four coordinates $x_{i+1,j}, x_{i-1,j}, x_{i,j+1}$ and $x_{i,j-1}$ reflects the fact that the edge types of the nearest neighbours of cell (i, j) are the same. Similarly, the invariance of f under permutation of the coordinates $x_{i+1,j+1}$, $x_{i-1,j+1}$, $x_{i+1,j-1}$ and $x_{i-1,j-1}$ derives from the same edge type of the next nearest neighbours of cell (i, j). ◇

Given an equivalence relation $\bowtie$ on the cells $\mathcal{L}$, the *polydiagonal* is defined as

$$\Delta_{\bowtie} = \{x \in \boldsymbol{R}^{k(\ell+1)} : x_c = x_d \text{ whenever } c, d \in \mathcal{L} \text{ and } c \bowtie d\}.$$

Note that if $\Delta_{\bowtie}$ is flow-invariant subspace for a given lattice dynamical system, then the solutions in $\Delta_{\bowtie}$ have a collection of coordinates equal for all time. A *robustly polysynchronous* equivalence relation is an equivalence relation $\bowtie$ on cells such that the associated polydiagonal $\Delta_{\bowtie}$ is flow-invariant under every $G_{\mathcal{L}}$-admissible vector field. We call the colouring associated with a robustly synchronous equivalence relation, a *pattern of synchrony*.

Stewart *et al* [19, 23] prove that classifying robustly polysynchronous polydiagonals is equivalent to the combinatorial issue of classifying certain equivalence relations on cells called 'balanced' and defined by the following definition.

Definition 2.10. *An equivalence relation $\bowtie$ on $\mathcal{L}$ is* balanced *if for all $c, d \in \mathcal{L}$ with $c \bowtie d$ and $c \neq d$, there exists an isomorphism $\gamma : I(c) \to I(d)$ preserving coupling types such that $i \bowtie \gamma(i)$ for all $i \in I(c)$.* ◇

It is proved in [19, 23] that an equivalence relation is robustly polysynchronous if, and only if, it is balanced.

Suppose we have a finite number ℓ of $\bowtie$-equivalence classes and we colour the cells in the lattice so that two cells have the same colour precisely when they are in the same $\bowtie$-equivalence class. That is, any equivalence relation can be represented by an ℓ*-colouring* of the cells. Now let $K_1, \ldots, K_\ell$ be the colours of an ℓ*-colouring* of a lattice network $G_{\mathcal{L}}$. Using the colourings, definition 2.10 can now be read as follows.

Definition 2.11. *The* ℓ-colouring *is* balanced *if and only if each cell of colour K_i receives the same number of inputs from cells of colour K_j $(j = 1, \ldots, \ell)$ of each edge type.* ◇

3. Techniques for proving spatial periodicity

Definition 3.1. *Let $G_{\mathcal{L}}$ be a lattice network and let $U \subset \mathcal{L}$ be a subset. The closure of U consists of all cells that are connected by some arrow to a cell in U, that is,*

$$\mathrm{cl}(U) = \{\mathcal{T}(e) : e \in \mathcal{E} \text{ and } \mathcal{H}(e) \in U\}.$$

The boundary of U is the set

$$\mathrm{bd}(U) = \mathrm{cl}(U) \backslash U. \qquad \diamond$$

Example 3.2. Let $\mathcal{L}$ be the square lattice of length 1 which we can identify with $\boldsymbol{Z}^2$. Let $G_{\mathcal{L}}$ be the associated lattice network such that each cell has four nearest neighbours at distance 1. See figure 1 (left). Then the boundary of the set W_2 in figure 2 is formed by the white cells with a cross. $\diamond$

For each Euclidean lattice network $G_{\mathcal{L}}$, there is a natural expanding sequence of finite subsets that covers the lattice. Let

$$W_0 = \{0\} \qquad \text{and} \qquad W_{i+1} = \mathrm{cl}(W_i), \tag{3.1}$$

for $i \geqslant 0$. Since the input set of each cell contains the generators of the lattice, we have

$$\mathcal{L} = W_0 \cup W_1 \cup \cdots .$$

It follows that for any colouring of a lattice $\mathcal{L}$ by k colours, there is some j such that all k colours are represented by cells in W_j. In fact, this is more true for balanced colourings.

Lemma 3.3. *Let $G_{\mathcal{L}}$ be a lattice network with a balanced k-colouring. Then W_{k-1} contains all k colours.*

Proof. We claim that if $\ell < k$, then W_ℓ contains at least $\ell + 1$ colours. The proof proceeds by induction on W_ℓ. $W_0 = \{0\}$ contains one cell and one colour.

Assume that the statement is true for $\ell < k - 1$; we prove that it is also true for $\ell + 1$. Suppose that the number m of colours contained in $W_{\ell+1} = \mathrm{cl}(W_\ell)$ is the same as the number of colours in W_ℓ. Then every cell $c \in W_{\ell+1}$ has a colour that is the same as the colour of a cell d in W_ℓ. Therefore, all cells connected to d lie in $W_{\ell+1}$ and are coloured by the m colours. Therefore, the term balanced implies that the cells connected to c must also be coloured by one of the m colours. It follows that the cells in $W_{\ell+2} = \mathrm{cl}(W_{\ell+1})$ are also coloured by these m colours. By induction, the entire lattice is coloured by m colours; hence $m = k$. So if $m < k$, the number of colours in $W_{\ell+1}$ must be greater than the number of colours in W_ℓ. That is, $W_{\ell+1}$ contains at least $\ell + 2$ colours. It follows that W_{k-1} contains all k colours. $\square$

Definition 3.4. *Let $G_{\mathcal{L}}$ be a lattice network and let $U \subset \mathcal{L}$ be a subset of cells. We say that U is* connected *if for every pair of cells $c, d \in U$ there is a sequence of cells $c = e_1, e_2, \ldots, e_j = d \in U$ such that $e_i \in I(e_{i+1})$ for all $i = 1, \ldots, j - 1$.* $\diamond$

In lemma 3.3 we show that if $G_{\mathcal{L}}$ is a lattice network with a balanced k-colouring, then the set W_{k-1} defined by (3.1) contains all k colours. It follows then, that if we know the pattern on W_k, given any colour of the pattern, there exists a cell e with that colour in W_{k-1}. Moreover, all the neighbours of e are in W_k so that all their colours are known. We may now raise a question about the extension of the pattern from W_k to the whole lattice, more specifically, about the extension of the pattern from W_k to W_{k+1}. This leads us to the concept of 'determinacy' that we define later. Recall that in a balanced k-colouring, for any two cells c and d of the same colour, there is a bijection between $I(c)$ and $I(d)$ that preserves arrow type and colour. In particular,

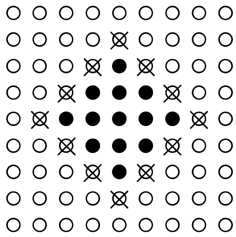

Figure 2. The set W_2 (black cells) and its boundary (white cells with a cross).

if we know the colour of all cells in $I(c)$ of a certain coupling type, and we also know the colour of all cells of the same coupling type except one in $I(d)$, then, since the colouring is balanced, the colour of the last cell with that coupling type in $I(d)$ can be determined.

Definition 3.5. *Let $G_{\mathcal{L}}$ be a lattice network and let $U \subset G_{\mathcal{L}}$ be a finite connected set.*

(a) Every cell $c \in U$ is called 0-determined.
*(b) A cell $c \in \mathrm{bd}(U)$ is p-*determined*, where $p \geqslant 1$ if there is a cell $d \in U$ such that c is in the input set of d and each cell in the input set of d that has the same coupling type as c, except c itself, is q-determined for some $q < p$.*
(c) A cell $c \in \mathrm{bd}(U)$ is determined if it is p-determined for some p.
(d) The set U determines its boundary if all cells in $\mathrm{bd}(U)$ are determined. ◇

Definition 3.6. *Let $G_{\mathcal{L}}$ be a lattice network. Then the set W_{i_0} is a* window *if W_i determines its boundary for all $i \geqslant i_0$.* ◇

Remark 3.7. Note that if there are no 1-determined cells then, by induction, there are no p-determined cells for any p. In particular, if there are no 1-determined cells, then windows do not exist.

Example 3.8. Let $\mathcal{L}$ be the square lattice of length 1 which we can identify with $\mathbf{Z}^2$. Let $G_{\mathcal{L}}$ be the associated lattice network such that each cell has four nearest neighbours at distance 1. See figure 1 (left). This network has no window, as we show. (Note that in this case, it is shown in [25] that there are infinitely many balanced 2-colourings.)

We claim that no set W_i is a window. Remark 3.7 suffices to show that there are no 1-determined cells. For example, consider W_2 and its boundary (figure 2). Since the cells on the boundary are in a diagonal line, it is not possible for them to be the only cell in the input set of a cell in W_2 that is not in W_2. Note that when $i > 2$ the set W_i has the same 'diamond shape' as W_2. Therefore, there are no 1-determined cells in $\mathrm{bd}(W_i)$. By remark 3.7, this network has no window. ◇

Example 3.9. Let $G_{\mathcal{L}}$ be the lattice network associated with the square lattice $\mathcal{L} = \mathbf{Z}^2$ such that each cell has four nearest neighbours at distance 1, and four next nearest neighbours at distance $\sqrt{2}$ (see figure 1 (centre)).

Let $W_0, W_1, \ldots$ be the sequence of sets generated by cell 0. It is clear that each set W_i is a square of size $2i + 1$. The size of a square is the number of cells in one (and hence all) of its sides.

We show that the sets W_i for $i \geqslant 2$ determine their boundaries. To show this, one of the corners of such a square should be analysed (by symmetry) because all the cells on each side, except the last two on both extremes, are 1-determined, since they are the only nearest neighbours outside the square (figure 3).

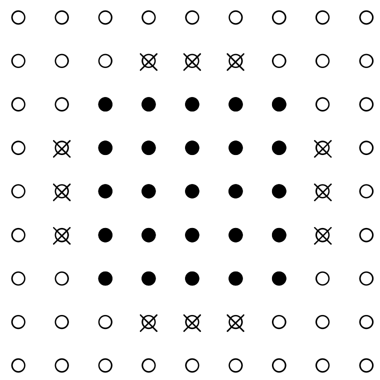

Figure 3. The set W_2 (black cells) and the 1-determined cells of its boundary (white cells with a cross).

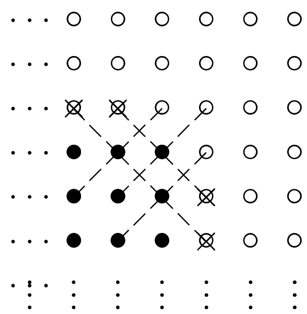

Figure 4. The corner of a set W_i (black cells), the 1-determined cells (white cells with a cross) and 2-determined cells (white cells connected to black cells by dashed lines).

The three cells in the corners of the square are 2-determined using the NNN as long as the square has size greater than 3 (see figure 4). ◇

Definition 3.10. *Let $G_{\mathcal{L}}$ be a lattice network and let $U \subset \mathcal{L}$ be a subset. The interior of U consists of all cells $c \in U$ such that any cell connected to c is also in U, that is,*

$$\mathrm{int}(U) = \{c \in U : I(c) \subseteq U\}. \qquad ◇$$

Lemma 3.11. *Let $G_{\mathcal{L}}$ be a lattice network where $\mathcal{L}$ is a Euclidean lattice and assume that W_{i_0} is a window. Suppose that a balanced k-colouring restricted to $\mathrm{int}(W_i)$ for some $i \geqslant i_0$ contains all k colours. Then the k-colouring is uniquely determined on the whole lattice by its restriction to W_i.*

Proof. Let K be a balanced k-colouring. Then for any two cells c and d of the same colour, there is a bijection $\beta : I(c) \to I(d)$ that preserves arrow type and colour. So if we know the colours of all cells in $I(c)$ and we know the colours of all cells except one in $I(d)$, then the fact that the colouring is balanced reveals the colour of the last cell in $I(d)$.

Since the colours of all cells in W_i are known, we can assume that the colours of all q-determined cells have been determined, where $0 \leqslant q < p$. Suppose $c \in \mathrm{bd}(W_i)$ is p-determined, then there exists $d \in W_i$ and $c \in I(d)$ such that all other input cells in $I(d)$ that have the same coupling type as c are q-determined for some $q < p$. Since $\mathrm{int}(W_i)$ contains all k colours, there exists a cell $e \in \mathrm{int}(W_i)$ with the same colour as d. Since all neighbours of

e are in W_i (by definition of interior), their colours are known. Now we apply the reasoning in the previous paragraph to deduce the colour of cell c.

Finally, we continue inductively to colour $W_{i+\ell}$ for $\ell \geqslant 1$. It follows that the balanced k-colouring restricted to W_i can be uniquely extended to the whole lattice. □

Theorem 3.12. *Let $\mathcal{L}$ be a Euclidean lattice and $G_{\mathcal{L}}$ a lattice network with a window. Fix $k \geqslant 1$. Then there is a finite number of balanced k-colourings on $\mathcal{L}$ and each balanced k-colouring is spatially multiply-periodic.*

Proof. Let W_j be a window for $G_{\mathcal{L}}$ where $j \geqslant k$. By lemma 3.3, the interior of W_j contains all k colours. Then by lemma 3.11, a balanced k-colouring is uniquely determined by its restriction to W_j. Since there is only a finite number of possible ways to distribute k colours on the cells in W_j, it follows that there are only a finite number of balanced k-colourings.

Let K be a balanced k-colouring on $G_{\mathcal{L}}$ and let $v \in \mathcal{L}$. Let $T_v(K)$ be the colouring obtained by shifting the colouring K by v, that is, the colour of cell c in $T_v(K)$ is the same as the colour of cell $c - v$ in K. Since translations are symmetries of the lattice network, $T_v(K)$ is also a balanced colouring.

Let v be a generator of the lattice and consider all translates of K in the direction of v. Since there is only a finite number of balanced k-colourings and an infinite number of translates of K, there must exist $N \in \mathbf{Z}^+$, such that K and $T_{Nv}(K)$ exhibit exactly the same colouring. It follows that K is invariant under the translation T_{Nv}. Hence K is periodic in the direction of v. The same argument can be applied to all the generators of the lattice, thus, all balanced k-colourings are spatially multiply-periodic. □

The fundamental property that we have identified in the course of the proof of the theorems in this section is determinacy, which is related to the architecture of the network defined by the choice of the structure of the input set.

Example 3.13. Consider the one-dimensional lattice $\mathcal{L} = \mathbf{Z}$. Let $G_{\mathcal{L}}$ be the lattice network with NN. The input set of a cell c consists of c plus its left and right neighbours. Let W_0, $W_1, \ldots$ be the sequence of sets defined in (3.1). Then $W_i = \{-i, \ldots, 0, \ldots, i\}$ is an *interval* of $2i + 1$ consecutive cells. Note that the boundary of any interval has two cells that are not in the interval and both of them are 1-determined. Therefore, the sets W_i for $i \geqslant 1$ are windows. Theorem 3.12 implies the finiteness of balanced k-colourings and spatial periodicity of all balanced k-colourings for the one-dimensional lattice network with NN. This special case is proved directly in [1]. ◇

4. Planar lattices

The main result about balanced colourings of planar lattice networks is the following.

Theorem 4.1. *Let*

$$\mathcal{L} = \{\alpha u + \beta v : \alpha,\ \beta \in \mathbf{Z}\},$$

be a planar lattice, where the generators u and v are norm 1 linearly independent vectors. Assume that the angle θ between u and v satisfies

$$\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{2}.$$

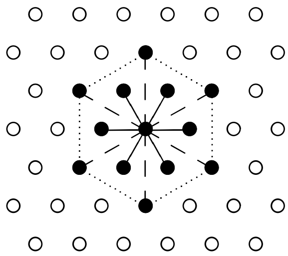

Figure 5. Hexagonal lattice network. NN (——) and NNN (- - - -). The dotted lines show the hexagonal region W_1.

Let $G_\mathcal{L}$ be the associated network such that the input set of each cell c contains cells whose distance from c is less than or equal to $|u + v|$. Then, for each $k > 0$ the network $G_\mathcal{L}$ admits only a finite number of balanced k-colourings each of which is spatially doubly-periodic.

Remark 4.2. Theorem 4.1 covers three types of lattices:

(a) square lattice: $u = (1, 0)$ and $v = (0, 1)$,
(b) hexagonal lattice: $u = (1, 0)$ and $v = (1, \sqrt{3})/2$,
(c) rhombic lattice: $u = (1, 0)$ and $v = (\cos\theta, \sin\theta)$ where $\pi/3 < \theta < \pi/2$.

For each of these lattices we define the *critical distance* as $|u + v|$. The couplings allowed by the critical distance are nearest and next nearest neighbour for all three lattices, and next next nearest neighbour for the rhombic lattices. Theorem 4.1 is the best possible in the sense that planar lattice networks with less coupling do admit an infinite number of aperiodic balanced 2-colourings. See [17, 25].

Proof. It is sufficient to show that the three types of lattices mentioned in the remark 4.2 have windows. More precisely, let $W_0, W_1, \ldots$ be the sets defined in (3.1) for one of the lattices satisfying the hypothesis of the theorem. We shall prove that W_i determines its boundary for all $i \geqslant 2$ and is a window. The conclusion follows from theorem 3.12.

First, let $\mathcal{L}$ be the square lattice. We have already shown in example 3.9 that for all $i \geqslant 2$ the set W_i determines its boundary.

Second, let $\mathcal{L}$ be a rhombic lattice with $\pi/3 < \theta < \pi/2$. Since this lattice is a deformation of the square lattice, the same argument as used in example 3.9 shows that W_i determines its boundary for all $i \geqslant 2$. The only new element is that the set of next nearest neighbours has four elements in the square lattice which breaks into two sets of two elements each in the rhombic lattice (see figure 1 (right)).

Third, let $\mathcal{L}$ be the hexagonal lattice. The input set of a cell c in the hexagonal lattice with NN and NNN has 12 cells: 6 nearest neighbours at a distance 1 from c, and 6 next nearest neighbours at a distance $\sqrt{3}$ from c (figure 5).

The set $W_{i+1} \setminus W_i$ is a hexagonal annulus surrounding W_i. Indeed, the cells in the input set of one cell c in W_i are within a distance less than or equal to $\sqrt{3}$ from c, so they must lie inside this region (see figure 6). Another observation is that the three lines through 0, and the next nearest neighbours of 0, divide each set W_i into six sectors. Since rotations by $\pi/3$ are symmetries of the lattice, we can restrict the analysis to any one of these sectors.

In the hexagonal lattice, the boundaries of the sets W_i in a given sector consist of three lines of cells (see figure 7). Note that cells on the first line of W_i are nearest neighbours of cells on the second line of W_{i-1}; cells on the second line of W_i are nearest neighbours of the cells

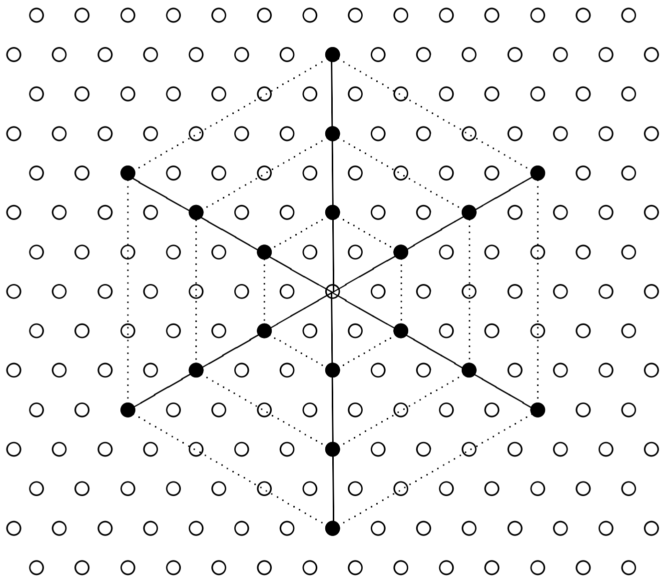

Figure 6. The next nearest neighbours of 0 and the sets W_1, W_2 and W_3 (hexagonal regions defined by dotted lines). The six sectors defined by the next nearest neighbours are separated by solid lines.

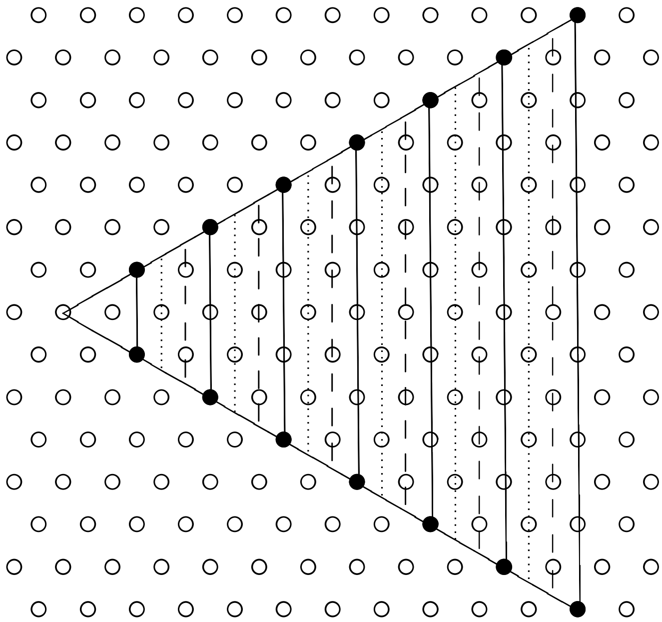

Figure 7. One sector of the sets $W_{i+1} \setminus W_i$ with the three lines of cells connected by dots, dashes, and solid.

on the third line of W_{i-1}; and cells on the third line of W_i are nearest neighbours of the cells on the first line of W_i.

The first line of the boundary of a set W_i is 1-determined. This follows from the fact that a cell c in the first line of $\mathrm{bd}(W_i)$ is a nearest neighbour of a cell d in the second line of W_i, and all other nearest neighbours of d are in W_i (see figure 8). The same argument illustrates

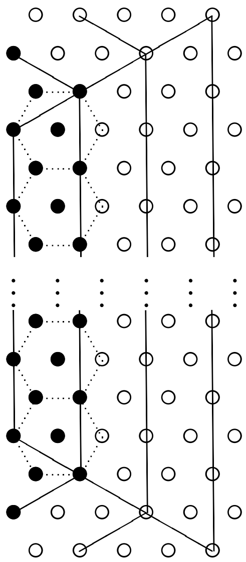

Figure 8. One sector of the set W_i (black cells) and a sector of its boundary $\mathrm{bd}(W_i)$. The first line of $\mathrm{bd}(W_i)$ is 1-determined.

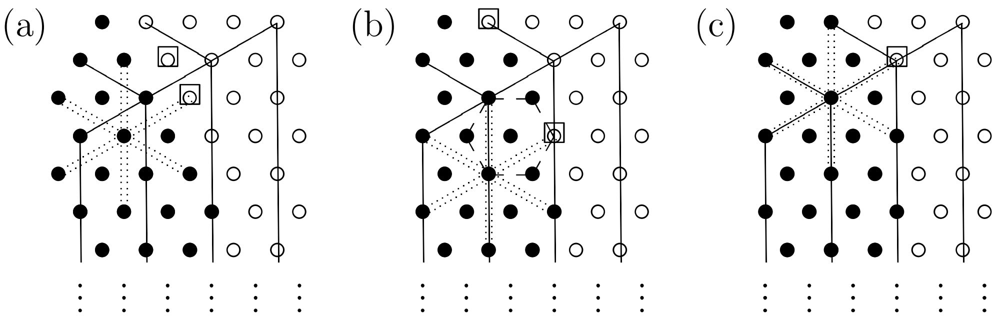

Figure 9. The corner of one sector of the set W_i. The three remaining cells are determined.

that cells in the second line, with the exception of the two cells nearest the sector boundary, are 2-determined; and cells in the third line of one sector, with the exception of the two cells on the sector boundary and the two cells nearest the sector boundary, are 3-determined. So far, we have shown that, except for six cells near or on the boundary of the sector, all cells are determined.

Reflections on the above allow us to restrict ourselves to cells near one of the corners of a sector. Thus, we show that the three remaining cells are determined. We now assume that $i \geqslant 2$.

To see that cell c_1 near the sector boundary on the second line of $\mathrm{bd}(W_i)$ is determined, consider the next nearest neighbours of the cell d_1 near the sector boundary on the second line of $\mathrm{bd}(W_{i-1})$. Since c_1 is the only next nearest neighbour of d_1 that has not yet been determined, c_1 is determined (see figure 9(*a*)).

To see that cell c_2 on the third line but not on the sector boundary is determined, consider the nearest neighbours of cell d_2 nearest the second line of $\mathrm{bd}(W_i)$. Since c_2 is the only nearest neighbour of d_2 that has not yet been determined, c_2 is determined (see figure 9(*b*)).

In order to determine cell c_3 on the third line of $\mathrm{bd}(W_i)$ and the sector boundary, consider the next nearest neighbours of cell d_3 on the sector boundary and the third line of $\mathrm{bd}(W_{i-1})$. Since c_3 is the only next nearest neighbour of d_3 that has not yet been determined, c_3 is also determined (see figure 9(c)). $\square$

5. The cubic lattice

In this section, we show that our techniques can work on three-dimensional lattices as well, by considering the standard (or primitive) cubic lattice $\mathcal{L} = \boldsymbol{Z}^3$. This lattice is the direct generalization to $\boldsymbol{R}^3$ of the linear lattice in $\boldsymbol{R}$, and the square lattice in $\boldsymbol{R}^2$.

Proposition 5.1. *Let $G_{\mathcal{L}}$ be the standard cubic lattice network with NN, second NN and third NN. Then W_s determines its boundary for all $s \geqslant 3$ and $G_{\mathcal{L}}$ admits a window.*

Proof. Note that if a cell with coordinates $y = (y_1, y_2, y_3)$ is in the input set of a cell with coordinates $x = (x_1, x_2, x_3)$, then the coordinates must satisfy

$$|y_i - x_i| \leqslant 1, \qquad \text{for } i = 1, 2, 3. \tag{5.1}$$

Therefore,

$$W_s = \{(x_1, x_2, x_3) : -s \leqslant x_i \leqslant s,\ x_i \in \mathbf{Z}\}$$

is the cube centred at the origin whose sides have $2s + 1$ cells. We prove that W_s ($s \geqslant 3$) determines its boundary. Observe that

$$\begin{aligned} \mathrm{bd}(W_s) &= \mathrm{cl}(W_s) \setminus W_s \\ &= W_{s+1} \setminus W_s \\ &= \{(x_1, x_2, x_3) \in W_{s+1} : \exists\, i \in \{1, 2, 3\} \text{ such that } |x_i| = s + 1\}. \end{aligned}$$

By symmetry it is sufficient to prove that all the cells in the set

$$Q = \{(s + 1, x_2, x_3) : 0 \leqslant x_3 \leqslant x_2 \leqslant s + 1\},$$

are determined by W_s. We partition Q into

$$Q = (P_{11} \cup P_{12} \cup P_{13} \cup P_{14}) \ \cup\ (P_{21} \cup P_{22}) \ \cup\ P_3,$$

where

$$\begin{aligned} P_{11} &= \{(s + 1, x_2, x_3) : 0 \leqslant x_3 \leqslant x_2 \leqslant s - 1\}, \\ P_{12} &= \{(s + 1, s, x_3) : 0 \leqslant x_3 \leqslant s - 2\}, \\ P_{13} &= \{(s + 1, s, s)\}, \\ P_{14} &= \{(s + 1, s, s - 1)\}, \\ P_{21} &= \{(s + 1, s + 1, x_3) : 0 \leqslant x_3 \leqslant s - 1\}, \\ P_{22} &= \{(s + 1, s + 1, s)\}, \\ P_3 &= \{(s + 1, s + 1, s + 1)\}. \end{aligned}$$

We show that all cells in each of these sets are determined.

P_{11} is 1-determined. Note that cells (s, x_2, x_3) with $0 \leqslant x_3 \leqslant x_2 \leqslant s - 1$ are in W_s. These cells have six nearest neighbours: $(s \pm 1, x_2, x_3)$, $(s, x_2 \pm 1, x_3)$, and $(s, x_2, x_3 \pm 1)$. Except for the cell $(s+1, x_2, x_3)$, all other nearest neighbours of these cells are in W_s. Hence, all cells $(s + 1, x_2, x_3)$ are 1-determined.

P_{12} *is 2-determined.* Note that cells $(s, s-1, x_3)$ with $0 \leqslant x_3 \leqslant s-2$ are in W_s. These cells have 12 next nearest neighbours whose coordinates are:

$$\begin{array}{ll} (s \pm 1, (s-1) \pm 1, x_3), & (s \pm 1, (s-1) \mp 1, x_3), \\ (s \pm 1, s-1, x_3 \pm 1), & (s \pm 1, s-1, x_3 \mp 1), \\ (s, (s-1) \pm 1, x_3 \pm 1), & (s, (s-1) \pm 1, x_3 \mp 1). \end{array}$$

Except for $(s+1, s, x_3)$, all other next nearest neighbours are in $W_s \cup P_{11}$ (or in one of its symmetric images). Thus, all cells $(s+1, s, x_3)$ are 2-determined.

P_{13} *is 3-determined.* The set P_{13} has one cell $c = (s+1, s, s)$. Note that $d = (s, s-1, s-1)$ is in W_s and has c as its next next nearest neighbour. Thus the distance between c and d is $\sqrt{3}$. Since the coordinates of d satisfy (5.1), it follows that, except for $(s+1, s, s)$, all other next next nearest neighbours of d are in $W_s \cup P_{11} \cup P_{12}$ (or in one of its symmetric images). Indeed, $(s, s-1, s-1) + (1, 1, 1) = (s+1, s, s) \notin W_s \cup P_{11} \cup P_{12}$ (or one of its symmetric images) and it is a next next nearest neighbour of d. Hence P_{13} is 3-determined.

P_{14} *is 4-determined.* The set P_{14} has one cell $c = (s+1, s, s-1)$. Note that $d = (s, s-1, s-2)$ is in W_s and has c as its next next nearest neighbour. This implies that the distance between c and d is $\sqrt{3}$. Since the coordinates of d satisfy (5.1), it follows that, except for $(s+1, s, s-1)$, all other next next nearest neighbours of d are in $W_s \cup P_{11} \cup P_{12} \cup P_{13}$ (or in one of its symmetric images). Hence, c is 4-determined.

P_{21} *is 5-determined.* Let $c = (s+1, s+1, x_3)$ where $0 \leqslant x_3 \leqslant s-1$. Note that $d = (s, s, x_3)$ is in W and has c as its next nearest neighbour. Thus, the distance between c and d is $\sqrt{2}$. Since the coordinates of d satisfy 5.1, it follows that, except for $(s+1, s+1, x_3)$, all other next nearest neighbours of d are in $W_s \cup P_{11} \cup P_{12} \cup P_{13} \cup P_{14}$ (or in one of its symmetric images). Hence, P_{21} is 5-determined.

P_{22} *is 6-determined.* The set P_{22} has one cell $c = (s+1, s+1, s)$. Note that $d = (s, s, s-1)$ is in W_s and has c as one of its next next nearest neighbours. Moreover, except for $(s+1, s+1, s)$, all other next next nearest neighbours of d are in $W_s \cup P_{11} \cup P_{12} \cup P_{13} \cup P_{14} \cup P_{21}$ (or in one of its symmetric images). Hence, c is 6-determined.

P_3 *is 7-determined.* The set P_3 has one cell $c = (s+1, s+1, s+1)$. Note that $d = (s, s, s)$ is in W_s and has c as one of its next next nearest neighbours. Moreover, except for $(s+1, s+1, s+1)$, all other next next nearest neighbours of d are in $W_s \cup P_{11} \cup P_{12} \cup P_{13} \cup P_{14} \cup P_{21} \cup P_{22}$ (or in one of its symmetric images). Hence, P_3 is 7-determined.

This concludes the proof that W_s determines its boundary for all $s \geqslant 3$. □

Corollary 5.2. *Let $G_{\mathcal{L}}$ be the standard cubic lattice network with nearest, second nearest and third NN. Then for each $k > 0$ the network $G_{\mathcal{L}}$ admits only a finite number of balanced k-colourings, each of which is spatially triply-periodic.*

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