Geometry of resonance tongues

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1. Introduction

Resonance tongues arise in bifurcations of discrete or continuous dynamical systems undergoing bifurcations of a fixed point or an equilibrium satisfying certain resonance conditions. They occur in several different contexts, depending, for example, on whether the dynamics is dissipative, conservative, or reversible. Generally, resonance tongues are domains in parameter space, with periodic dynamics of a specified type (regarding period of rotation number, stability, etc.). In each case, the tongue boundaries are part of the bifurcation set. We mention here several standard ways that resonance tongues appear.

1.1. *Various contexts*

Hopf bifurcation from a fixed point. Resonance tongues can be obtained by Hopf bifurcation from a fixed point of a map. This is the context of Section 2, which is based on Broer, Golubitsky, Vegter.[7] More precisely, Hopf bifurcations of maps occur when eigenvalues of the Jacobian of the map at a fixed point cross the complex unit circle away from the strong resonance points $e^{2\pi pi/q}$ with $q \leq 4$. Instead, we concentrate on the *weak* *resonance points* corresponding to roots of unity $e^{2\pi pi/q}$, where p and q are coprime integers with $q \geq 5$ and $|p| < q$. *Resonance tongues* themselves are regions in parameter space near the point of Hopf bifurcation where periodic points of period q exist and *tongue boundaries* consist of points in parameter space where the q-periodic points disappear, typically in a saddle-node bifurcation. We assume, as is usually done, that the critical eigenvalues are simple with no other eigenvalues on the unit circle. Moreover, usually just two parameters are varied; The effect of changing these parameters is to move the eigenvalues about an open region of the complex plane. In the non-degenerate case a *pair of q-periodic orbits* arises or disappears as a single complex parameter governing the system crosses the boundary of a resonance tongue. Outside the tongue there are no q-periodic orbits. In the degenerate case there are two complex parameters controlling the evolution of the system. Certain domains of complex parameter space correspond to the existence of zero, two or even *four q-periodic orbits*.

Lyapunov-Schmidt reduction is the first main tool used in Section 2 to reduce the study of q-periodic orbits in families of planar diffeomorphisms to the analysis of zero sets of families of $\mathbb{Z}_q$-*equivariant* functions on the plane. Equivariant Singularity Theory, in particular the theory of equivariant contact equivalence, is used to bring such families into low-degree polynomial normal form, depending on one or two complex parameters. The discriminant set of such polynomial families corresponds to the resonance tongues associated with the existence of q-periodic orbits in the original family of planar diffeomorphisms.

Hopf bifurcation and birth of subharmonics in forced oscillators Let

$$\frac{dX}{dt} = F(X)$$

be an autonomous system of differential equations with a periodic solution $Y(t)$ having its Poincaré map P centered at $Y(0) = Y_0$. For simplicity we take $Y_0 = 0$, so $P(0) = 0$. A Hopf bifurcation occurs when eigenvalues of the Jacobian matrix $(dP)_0$ are on the unit circle and resonance occurs when these eigenvalues are roots of unity $e^{2\pi pi/q}$. *Strong resonances* occur when $q < 5$. This is one of the contexts we present in Section 3. Except at strong resonances, Hopf bifurcation leads to the existence of an invariant circle for the Poincaré map and an invariant torus for the autonomous system. This is usually called a Naimark-Sacker bifurcation. At weak resonance points the flow on the torus has very thin regions in parameter space (between the tongue boundaries) where this flow consists of a *phase-locked* periodic solution that winds around the torus q times in one direction (the direction approximated by the original periodic solution) and p times in the other. Section 3.1 presents a Normal Form Algorithm for continuous vector fields, based on the method of Lie series. This algorithm is applied in Section 3.2 to obtain the results summarized in this paragraph. In particular, the analysis of the Hopf normal form reveals the birth or death of an invariant circle in a non-degenerate Hopf bifurcation.

Related phenomena can be observed in periodically forced oscillators. Let

$$\frac{dX}{dt} = F(X) + G(t)$$

be a periodically forced system of differential equations with 2π-periodic forcing $G(t)$. Suppose that the autonomous system has a hyperbolic equilibrium at $Y_0 = 0$; That is, $F(0) = 0$. Then the forced system has a 2π-periodic solution $Y(t)$ with initial condition $Y(0) = Y_0$ near 0. The dynamics of the forced system near the point Y_0 is studied using the *stroboscopic map* P that maps the point X_0 to the point $X(2\pi)$, where $X(t)$ is the solution to the forced system with initial condition $X(0) = X_0$. Note that $P(0) = 0$ in coordinates centered at Y_0. Again resonance can occur as a parameter is varied when the stroboscopic map undergoes Hopf bifurcation with critical eigenvalues equal to roots of unity. Resonance tongues correspond to regions in parameter space near the resonance point where the stroboscopic map has q-periodic trajectories near 0. These q-periodic trajectories are often called *subharmonics of order* q. Section 3.2 presents a Normal Form Algorithm for such periodic systems. The Van der Pol transformation is a tool for reducing the analysis of subharmonics of order q to the study of zero sets of $\mathbb{Z}_q$-equivariant polynomials. In this way we obtain the $\mathbb{Z}_q$-equivariant Takens Normal Form[37] of the Poincaré time-2π-map of the system. After this transformation, the final analysis of the resonance tongues corresponding to the birth or death of these subharmonics bears strong resemblance to the approach of Section 2.

Coupled cell systems. Finally, in Section 4 we report on work in progress by presenting a case study, focusing on a feed-forward network of three coupled cells. Each cell satisfies the same dynamic law, only different choices of initial conditions may lead to different kinds of dynamics for each cell. To tackle such systems, we show how a certain class of dynamic laws may give rise to time-evolutions that are equilibria in cell 1, periodic in cells 2, and exhibting the Hopf-Neĭmark-Sacker phenomenon in cell 3. This kind of dynamics of the third cell, which is revealed by applying the theory Normal Form theory presented in Section 3, occurs despite the fact that the dynamic law of each individual cell is simple, and identical for each cell.

1.2. *Methodology: generic versus concrete systems.*

Analyzing bifurcations in generic families of systems requires different tools than analyzing a concrete family of systems and its bifurcations. Furthermore, if we are only interested in restricted aspects of the dynamics, like the emergence of periodic orbits near fixed points of maps, simpler methods might do than in situations where we are looking for complete dynamic information, like normal linear behavior (stability), or the coexistence of periodic, quasi-periodic and chaotic dynamics near a Hopf-Neĭmark-Sacker bifurcation. In general, the more demanding context requires more powerful tools.

This paper illustrates this 'paradigm' in the context of local bifurcations of vector fields and maps, corresponding to the occurrence of degenerate equilibria or fixed points for certain values of the parameter. These degenerate features are encoded by a semi-algebraic stratification of the space of jets (of some fixed order) of local vector fields or maps. Ideally, each stratum is represented by a 'simple model', or *normal form*, to which all other systems in the stratum can be reduced by a coordinate transformation (or, normal form transformation). These 'simple models' are usually low-degree polynomial systems, equivalent to either the full system or some jet of sufficiently high order. Moreover, generic unfoldings of such degenerate systems also have simple polynomial normal forms. The guiding idea is that the interesting features of the system are much more easily extracted from the normal form than from the original system.

Singularity Theory provides us with algebraic algorithms that compute such simple polynomial models for *generic (families of) functions*. In Section 2 we apply Equivariant Singularity Theory in this way to determine resonance tongues corresponding to bifurcations of periodic orbits from fixed points of maps. However, before Singularity Theory can be applied we have to use the Lyapunov-Schmidt method to reduce the study of bifurcating periodic orbits to the analysis of zero sets of equivariant families of functions on the plane. In this reduction we loose all other information on the dynamics of the system.

To overcome this restriction, we apply Normal Form algorithms in the context of flows[4,36] yielding simple models of generic families of vector fields (possibly up to terms of high order), without first reducing the system according to the Lyapunov-Schmidt approach. Therefore, all dynamic information is present in the normal form. We follow this approach to study the Hopf-Neĭmark-Sacker phenomenon in *concrete systems*, like the feedforward network of coupled cells.

1.3. *Related work*

The geometric complexity of resonance domains has been the subject of many studies of various scopes. Some of these, like the present paper, deal with quite universal problems while others restrict to interesting examples. As opposed to this paper, often normal form theory is used to obtain information about the nonlinear dynamics. In the present context the normal forms automatically are $\mathbb{Z}_q$-equivariant.

Chenciner's degenerate Hopf bifurcation. Chenciner[17–19] considers a 2-parameter unfolding of a degenerate Hopf bifurcation. Strong resonances to some finite order are excluded in the 'rotation number' ω_0 at the central fixed point. Chenciner[19] studies corresponding periodic points for sequences of 'good' rationals p_n/q_n tending to ω_0, with the help of $\mathbb{Z}_{q_n}$-equivariant normal form theory. For a further discussion of the codimension k Hopf bifurcation compare Broer and Roussarie.[12]

The geometric program of Peckam *et al.* The research program reflected in Peckam *et al.*[30,31,33,35] views resonance 'tongues' as projections on a 'traditional' parameter plane of (saddle-node) bifurcation sets in the product of parameter and phase space. This approach has the same spirit as ours and many interesting geometric properties of 'resonance tongues' are discovered and explained in this way. We note that the earlier result Peckam and Kevrekidis[34] on higher order degeneracies in a period-doubling uses $\mathbb{Z}_2$ equivariant singularity theory.

Particularly we like to mention the results of Peckam and Kevrekidis[35] concerning a class of oscillators with doubly periodic forcing. It turns out that these systems can have coexistence of periodic attractors (of the same period), giving rise to 'secondary' saddle-node lines, sometimes enclosing a flame-like shape. In the present, more universal, approach we find similar complications of traditional resonance tongues, compare Figure 1 and its explanation in Section 2.4.

Related work by Broer *et al.* Broer *et al.*[15] an even smaller universe of annulus maps is considered, with Arnold's family of circle maps as a limit. Here 'secondary' phenomena are found that are similar to the ones discussed presently. Indeed, apart from extra saddle-node curves inside tongues also many other bifurcation curves are detected.

We like to mention related results in the reversible and symplectic settings regarding parametric resonance with periodic and quasi-periodic forcing terms by Afsharnejad[1] and Broer *et al.*[5,6,8–11,13,14,16] Here the methods use Floquet theory, obtained by averaging, as a function of the parameters.

Singularity theory (with left-right equivalences) is used in various ways. First of all it helps to understand the complexity of resonance tongues in the stability diagram. It turns out that crossing tongue boundaries, which may give rise to instability pockets, are related to Whitney folds as these occur in 2D maps. These problems already occur in the linearized case of Hill's equation. A question is whether these phenomena can be recovered by methods as developed in the present paper. Finally, in the nonlinear cases, application of $\mathbb{Z}_2$- and $\mathbb{D}_2$-equivariant singularity theory helps to get dynamical information on normal forms.

2. Bifurcation of periodic points of planar diffeomorphisms

2.1. *Background and sketch of results*

The types of resonances mentioned here have been much studied; we refer to Takens,[37] Newhouse, Palis, and Takens,[32] Arnold[2] and references therein. For more recent work on strong resonance, see Krauskopf.[29] In general, these works study the complete dynamics near resonance, not just the shape of resonance tongues and their boundaries. Similar remarks can be made on studies in Hamiltonian or reversible contexts, such as Broer and Vegter[16] or Vanderbauwhede.[39] Like in our paper, in many of these references some form of singularity theory is used as a tool.

The problem we address is how to find resonance tongues in the general setting, without being concerned by stability, further bifurcation and similar dynamical issues. It turns out that contact equivalence in the presence of $\mathbb{Z}_q$ symmetry is an appropriate tool for this, when first a Liapunov-Schmidt reduction is utilized, see Golubitsky, Schaeffer, and Stewart.[23,25] The main question asks for the number of q-periodic solutions as a function of parameters, and each tongue boundary marks a change in this number. In the next subsection we briefly describe how this reduction process works. Using equivariant singularity theory we arrive at equivariant normal forms for the reduced system in Section 2.3. It turns out that the standard, nondegen-

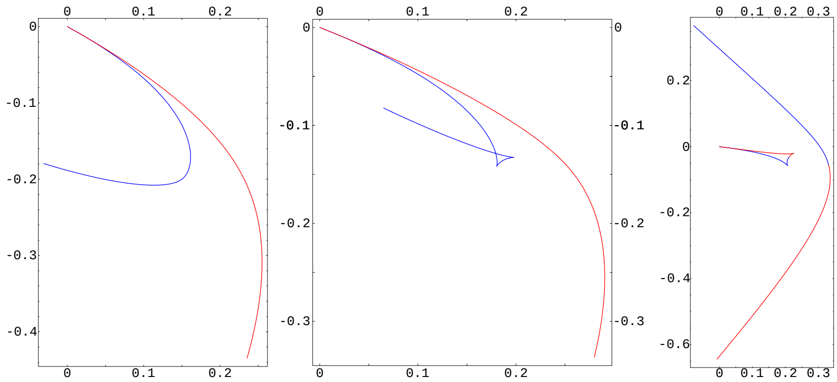

Fig. 1. Resonance tongues with pocket- or flame-like phenonmena near a degenerate Hopf bifurcation through $e^{2\pi ip/q}$ in a family depending on two complex parameters. Fixing one of these parameters at various (three) values yields a family depending on one complex parameter, with resonance tongues contained in the plane of this second parameter. As the first parameter changes, these tongue boundaries exhibit cusps (middle picture), and even become disconnected (rightmost picture). The small triangle in the rightmost picture encloses the region of parameter values for which the system has four q-periodic orbits.

erate cases of Hopf bifurcation[2,37] can be easily recovered by this method. When $q \geq 7$ we are able to treat a degenerate case, where the third order terms in the reduced equations, the 'Hopf coefficients', vanish. We find pocket- or flame-like regions of four q-periodic orbits in addition to the regions with only zero or two, compare Figure 1. In addition, the tongue boundaries contain new cusp points and in certain cases the tongue region is blunter than in the nondegenerate case. These results are described in detail in Section 2.4.

2.2. *Reduction to an equivariant bifurcation problem*

Our method for finding resonance tongues — and tongue boundaries — proceeds as follows. Find the region in parameter space corresponding to points where the map P has a q-periodic orbit; that is, solve the equation $P^q(x) = x$. Using a method due to Vanderbauwhede (see[39,40]), we can solve for such orbits by Liapunov-Schmidt reduction. More precisely, a q-periodic orbit consists of q points $x_1, \ldots, x_q$ where

$$P(x_1) = x_2, \ldots, P(x_{q-1}) = x_q, P(x_q) = x_1.$$

Such periodic trajectories are just zeroes of the map

$$\widehat{P}(x_1,\ldots,x_q)=(P(x_1)-x_2,\ldots,P(x_q)-x_1).$$

Note that $\widehat{P}(0)=0$, and that we can find all zeroes of $\widehat{P}$ near the resonance point by solving the equation $\widehat{P}(x)=0$ by Liapunov-Schmidt reduction. Note also that the map $\widehat{P}$ has $\mathbb{Z}_q$ symmetry. More precisely, define

$$\sigma(x_1,\ldots,x_q)=(x_2,\ldots,x_q,x_1).$$

Then observe that

$$\widehat{P}\sigma=\sigma\widehat{P}.$$

At 0, the Jacobian matrix of $\widehat{P}$ has the block form

$$J=\begin{pmatrix} A & -I & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & -I & 0 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & A & -I \\ -I & 0 & 0 & 0 & \cdots & 0 & A \end{pmatrix}$$

where $A=(dP)_0$. The matrix J automatically commutes with the symmetry σ and hence J can be block diagonalized using the isotypic components of irreducible representations of $\mathbb{Z}_q$. (An *isotypic component* is the sum of the $\mathbb{Z}_q$ isomorphic representations. See[25] for details. In this instance all calculations can be done explicitly and in a straightforward manner.) Over the complex numbers it is possible to write these irreducible representations explicitly. Let ω be a q^{th} root of unity. Define V_ω to be the subspace consisting of vectors

$$[x]_\omega=\begin{pmatrix} x \\ \omega x \\ \vdots \\ \omega^{q-1}x \end{pmatrix}.$$

A short calculation shows that

$$J[x]_\omega=[(A-\omega I)x]_\omega.$$

Thus J has zero eigenvalues precisely when A has q^{th} roots of unity as eigenvalues. By assumption, A has just one such pair of complex conjugate q^{th} roots of unity as eigenvalues.

Since the kernel of J is two-dimensional — by the simple eigenvalue assumption in the Hopf bifurcation — it follows using Liapunov-Schmidt reduction that solving the equation $\widehat{P}(x) = 0$ near a resonance point is equivalent to finding the zeros of a reduced map from $\mathbb{R}^2 \to \mathbb{R}^2$. We can, however, naturally identify $\mathbb{R}^2$ with $\mathbb{C}$, which we do. Thus we need to find the zeros of a smooth implicitly defined function

$$g : \mathbb{C} \to \mathbb{C},$$

where $g(0) = 0$ and $(dg)_0 = 0$. Moreover, assuming that the Liapunov-Schmidt reduction is done to respect symmetry, the reduced map g commutes with the action of σ on the critical eigenspace. More precisely, let ω be the critical resonant eigenvalue of $(dP)_0$; then

$$g(\omega z) = \omega g(z). \tag{1}$$

Since p and q are coprime, ω generates the group $\mathbb{Z}_q$ consisting of all q^{th} roots of unity. So g is $\mathbb{Z}_q$-equivariant.

We propose to use $\mathbb{Z}_q$-equivariant singularity theory to classify resonance tongues and tongue boundaries.

2.3. $\mathbb{Z}_q$ *singularity theory*

In this section we develop normal forms for the simplest singularities of $\mathbb{Z}_q$-equivariant maps g of the form (1). To do this, we need to describe the form of $\mathbb{Z}_q$-equivariant maps, contact equivalence, and finally the normal forms.

The structure of $\mathbb{Z}_q$-equivariant maps. We begin by determining a unique form for the general $\mathbb{Z}_q$-equivariant polynomial mapping. By Schwarz's theorem[25] this representation is also valid for C^∞ germs.

Lemma 2.1. *Every $\mathbb{Z}_q$-equivariant polynomial map $g : \mathbb{C} \to \mathbb{C}$ has the form*

$$g(z) = K(u,v)z + L(u,v)\overline{z}^{q-1},$$

where $u = z\overline{z}$, $v = z^q + \overline{z}^q$, and K, L are uniquely defined complex-valued function germs.

$\mathbb{Z}_q$ contact equivalences. Singularity theory approaches the study of zeros of a mapping near a singularity by implementing coordinate changes that transform the mapping to a 'simple' normal form and then solving the normal form equation. The kinds of transformations that preserve the zeros of a mapping are called contact equivalences. More precisely, two $\mathbb{Z}_q$-equivariant germs g and h are $\mathbb{Z}_q$-*contact equivalent* if

$$h(z) = S(z)g(Z(z)),$$

where $Z(z)$ is a $\mathbb{Z}_q$-equivariant change of coordinates and $S(z) : \mathbb{C} \to \mathbb{C}$ is a real linear map for each z that satisfies

$$S(\gamma z)\gamma = \gamma S(z)$$

for all $\gamma \in \mathbb{Z}_q$.

Normal form theorems. In this section we consider two classes of normal forms — the codimension two standard for resonant Hopf bifurcation and one more degenerate singularity that has a degeneracy at cubic order. These singularities all satisfy the nondegeneracy condition $L(0,0) \neq 0$; we explore this case first.

Theorem 2.1. *Suppose that*

$$h(z) = K(u,v)z + L(u,v)\overline{z}^{q-1}$$

where $K(0,0) = 0$.

(1) Let $q \geq 5$. *If* $K_u L(0,0) \neq 0$, *then* h *is* $\mathbb{Z}_q$ *contact equivalent to*

$$g(z) = |z|^2 z + \overline{z}^{q-1}$$

with universal unfolding

$$G(z,\sigma) = (\sigma + |z|^2)z + \overline{z}^{q-1}. \tag{2}$$

(2) Let $q \geq 7$. *If* $K_u(0,0) = 0$ *and* $K_{uu}(0,0)L(0,0) \neq 0$, *then* h *is* $\mathbb{Z}_q$ *contact equivalent to*

$$g(z) = |z|^4 z + \overline{z}^{q-1}$$

with universal unfolding

$$G(z,\sigma,\tau) = (\sigma + \tau|z|^2 + |z|^4)z + \overline{z}^{q-1}, \tag{3}$$

where $\sigma, \tau \in \mathbb{C}$.

Remark. Normal forms for the cases $q = 3$ and $q = 4$ are slightly different. See[7] for details.

2.4. *Resonance domains*

We now compute boundaries of resonance domains corresponding to universal unfoldings of the form

$$G(z) = b(u)z + \overline{z}^{q-1}. \tag{4}$$

By definition, the *tongue boundary* is the set of parameter values where local bifurcations in the number of period q points take place; and, typically, such bifurcations will be saddle-node bifurcations. For universal unfoldings of the simplest singularities the boundaries of these parameter domains have been called tongues, since the domains have the shape of a tongue, with its tip at the resonance point. Below we show that our method easily recovers resonance tongues in the standard least degenerate cases. Then, we study a more degenerate singularity and show that the usual description of tongues needs to be broadened.

Tongue boundaries of a $p : q$ resonance are determined by the following system

$$\begin{aligned} \overline{z}G &= 0 \\ \det(dG) &= 0. \end{aligned} \tag{5}$$

This follows from the fact that local bifurcations of the period q orbits occur at parameter values where the system $G = 0$ has a singularity, that is, where the rank of dG is less than two. Recalling that

$$u = z\overline{z} \quad v = z^q + \overline{z}^q \quad w = i(z^q - \overline{z}^q),$$

we prove the following theorem, which is independent of the form of $b(u)$.

Theorem 2.2. *For universal unfoldings (4), equations (5) have the form*

$$\begin{aligned} |b|^2 &= u^{q-2} \\ b\overline{b}' + \overline{b}b' &= (q-2)u^{q-3} \end{aligned}$$

To begin, we discuss weak resonances $q \geq 5$ in the nondegenerate case corresponding to the situation of Theorem 2.1, part 1. where a $\frac{q}{2} - 1$ cusp forms the tongue-tip and where the concept of resonance tongue remains unchallenged. Using Theorem 2.2, we recover several classical results on the geometry of resonance tongues in the present context of Hopf bifurcation. Note that similar tongues are found in the Arnold family of circle maps,[2] also compare Broer, Simó and Tatjer.[15]

We find some new phenomena in the case of weak resonances $q \geq 7$ in the mildly degenerate case corresponding to the situation of Theorem 2.1, part 2. Here we find 'pockets' in parameter space corresponding to the occurrence of *four* period-q orbits.

The nondegenerate singularity when $q \geq 5$. We first investigate the nondegenerate case $q \geq 5$ given in 2. Here

$$b(u) = \sigma + u$$

where $\sigma = \mu + i\nu$. We shall compute the tongue boundaries in the (μ, ν)-plane in the parametric form $\mu = \mu(u), \nu = \nu(u)$, where $u \geq 0$ is a local real parameter. Short computations show that

$$\begin{aligned} |b|^2 &= (\mu + u)^2 + \nu^2 \\ b\overline{b}' + \overline{b}b' &= 2(\mu + u). \end{aligned}$$

Then Theorem 2.2 gives us the following parametric representation of the tongue boundaries:

$$\begin{aligned} \mu &= -u + \frac{q-2}{2} u^{q-3} \\ \nu^2 &= u^{q-2} - \frac{(q-2)^2}{4} u^{2(q-3)}. \end{aligned}$$

In this case the tongue boundaries at $(\mu, \nu) = (0, 0)$ meet in the familiar $\frac{q-2}{2}$ cusp

$$\nu^2 \approx (-\mu)^{q-2}.$$

See also Figure 2. It is to this and similar situations that the usual notion

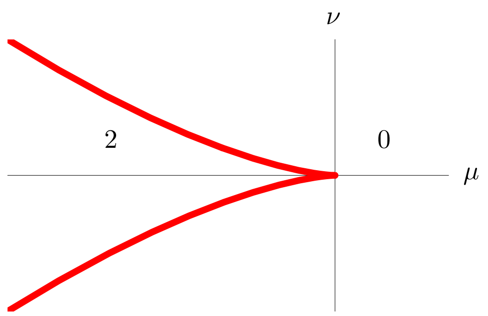

Fig. 2. Resonance tongue in the parameter plane. A pair of q-periodic orbits occurs for parameter values inside the tongue.

of resonance tongue applies: inside the sharp tongue a pair of period q orbits exists and these orbits disappear in a saddle-node bifurcation at the boundary.

Tongue boundaries in the degenerate case. The next step is to analyze a more degenerate case, namely, the singularity

$$g(z) = u^2 z + \overline{z}^{q-1},$$

w hen $q \geq 7$. We recall from 3 that a universal unfolding of g is given by $G(z) = b(u)z + \overline{z}^{q-1}$, where

$$b(u) = \sigma + \tau u + u^2.$$

Here σ and τ are complex parameters, which leads to a real 4-dimensional parameter space. As before, we set $\sigma = \mu + i\nu$ and consider how the tongue boundaries in the (μ, ν)-plane depend on the complex parameter τ. Broer, Golubitsky and Vegter[7] find an explicit parametric representation of the tongue boundaries in (σ, τ)-space for $q = 7$. Cross-sections of these resonance tongues of the form $\tau = \tau_0$ are depicted in Figure 1 for several constant values of τ_0. A new complication occurs in the tongue boundaries for certain τ, namely, cusp bifurcations occur at isolated points of the fold (saddle-node) lines. The interplay of these cusps is quite interesting and challenges some of the traditional descriptions of resonance tongues when $q = 7$ and presumably for $q \geq 7$.

3. Subharmonics in forced oscillators

As indicated in the introduction, subharmonics of order q ($2q\pi$-periodic orbits) correspond to q-periodic orbits of the Poincaré time-2π-map. However, since the Poincaré map is not known explicitly, applying the method of Section 2 directly is at best rather cumbersome, if not completely infeasible in most cases, especially since we are after a method for computing resonance tongues in *concrete* systems. Therefore we follow an other, more direct approach by introducing a Normal Form Algorithm for time dependent periodic vector fields. This method is explicit, and in principle computes a Normal Form up to any order.

3.1. *A Normal Form Algorithm*

First we present the Normal Form Algorithm in the context of autonomous vector fields. Our approach is an extension of the well-known methods introduced in,[36] and aimed at the derivation of an implementable algorithm. This procedure transforms the terms of the vector field in 'as simple a form as possible', up to a user-defined order. It does so via iteration with respect to the total degree of these terms. In concrete systems we determine this normal form exactly, i.e., making the dependence on the coefficients of the input system explicit. To this end we have to compute the transformed system explicitly up to the desired order. The method of Lie series turns out to be a powerful tool in this context. We first present the key property of the Lie series approach, that allows us to computate the transformed system in a rather straightforward way, up to any desired order.

Lie series expansion. For nonnegative integers m we denote the space of vector fields of total degree m by $\mathcal{H}_m$, and the space of vector fields with vanishing derivatives up to and including order m at $0 \in \mathbb{C}$ by $\mathcal{F}_m$. Note that $\mathcal{F}_m = \prod_{k \geq m} \mathcal{H}_k$.

Proposition 3.1. *Let X and Y be vector fields on $\mathbb{C}$, where X is of the form*

$$X = X^{(1)} + X^{(2)} + \cdots + X^{(N)} \mod \mathcal{F}_{N+1}, \tag{6}$$

with $X^{(n)} \in \mathcal{H}_n$, and $Y \in \mathcal{H}_m$, with $m \geq 2$. Let Y_t, $t \in \mathbb{R}$, be the one-parameter group generated by Y, and let $X^t = (Y_t)_*(X)$. Then

$$X^t = X + \sum_{k=1}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \operatorname{ad}(Y)^k(X) \mod \mathcal{F}_{N+1} \tag{7}$$

$$= X + \sum_{n=1}^{N} \sum_{k=1}^{\lfloor \frac{N-n}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \operatorname{ad}(Y)^k(X^{(n)}) \mod \mathcal{F}_{N+1}. \tag{8}$$

Proof. We follow the approach of Takens[36] and Broer et.al.[3,4] to obtain the Taylor series of X^t with respect to t in $t = 0$ using the basic identity

$$\frac{\partial}{\partial t} X^t = [X^t, Y] = -\operatorname{ad}(Y)(X^t).$$

Using this relation, we inductively prove that:

$$\frac{\partial^k}{\partial t^k} X^t = (-1)^k \operatorname{ad}(Y)^k(X^t).$$

Using the latter identity for $t = 0$, we obtain the formal Taylor series

$$\begin{aligned} X^t &= \sum_{k \geq 0} \frac{1}{k!} t^k \left. \frac{\partial^k}{\partial t^k} \right|_{t=0} X^t \\ &= \sum_{k \geq 0} \frac{(-1)^k}{k!} t^k \operatorname{ad}(Y)^k(X). \end{aligned}$$

Since $Y \in \mathcal{H}_m$, the operator $\mathrm{ad}(Y)^k$ increases the degree of each term in its argument by $k(m-1)$. Since the terms of lowest order in X are linear, we see that

$$\mathrm{ad}(Y)^k(X) = 0 \mod \mathcal{F}_{N+1},$$

if $1 + k(m-1) > N$. Therefore,

$$\begin{aligned} X^t &= \sum_{k=0}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \,\mathrm{ad}(Y)^k(X) \mod \mathcal{F}_{N+1} \\ &= X + \sum_{k=1}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \,\mathrm{ad}(Y)^k(X) \mod \mathcal{F}_{N+1}, \end{aligned}$$

which proves (7). In view of (6) the latter identity expands to

$$X^t = \sum_{n=1}^{N} \sum_{k=0}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \,\mathrm{ad}(Y)^k(X^{(n)}) \mod \mathcal{F}_{N+1}. \tag{9}$$

Since $\mathrm{ad}(Y)^k(X^{(n)}) \in \mathcal{H}_{n+k(m-1)}$, we see that

$$\mathrm{ad}(Y)^k(X^{(n)}) = 0 \mod \mathcal{F}_{N+1},$$

for $k > \frac{N-n}{m-1}$. Therefore, for fixed index n, the inner sum in (9) can be truncated at $k = \lfloor \frac{N-n}{m-1} \rfloor$, which concludes the proof of (8). □

The Normal Form Algorithm. Consider a vector field X having a singular point with semisimple linear part S. Our goal is to design an iterative algorithm bringing X into normal form, to some prescribed order N.

Lemma 3.1. ***(Normal Form Lemma[36])***
The vector field X can be brought into the normal form

$$X = S + G^{(2)} + \cdots + G^{(m)} \mod \mathcal{F}_{m+1},$$

for any $m \geq 2$, where $G^{(i)} \in \mathcal{H}_i$ belongs to $\mathrm{Ker}\,\mathrm{ad}(S)$.

Proof. Assume that X is of the form

$$X = S + G^{(2)} + \cdots + G^{(m-1)} + X^{(m)} \mod \mathcal{F}_{m+1}, \tag{10}$$

where $X^{(m)} \in \mathcal{H}_m$, and $G^{(i)} \in \mathcal{H}_i$ belongs to $\mathrm{Ker}\,\mathrm{ad}(S)$. If $Y \in \mathcal{H}_m$ and $X^t = (Y_t)_*(X)$, then

$$X^t = X - t\,\mathrm{ad}(Y)(X^{(1)}) \mod \mathcal{F}_{m+1}. \tag{11}$$

This is a direct consequence of Proposition 3.1. See also Takens[36] and Broer et.al.[3,4] Since S is semisimple, we know that

$$\mathcal{H}_m = \operatorname{Ker}\operatorname{ad}(S) + \operatorname{Im}\operatorname{ad}(S),$$

so we write $X^{(m)} = G^{(m)} + B^{(m)}$, where $G^{(m)} \in \operatorname{Ker}\operatorname{ad}(S)$ and $B^{(m)} \in \operatorname{Im}\operatorname{ad}(S)$. If the vector field Y satisfies the *homological equation*

$$\operatorname{ad}(S)(Y) = -B^{(m)}, \tag{12}$$

it follows from (10) and (11) that X^1 is in normal form to order m, since

$$X^1 = S + G^{(2)} + \cdots + G^{(m-1)} + G^{(m)} \mod \mathcal{F}_{m+1}. \qquad \square$$

Our final goal, namely bringing X into normal form to order N, is achieved by repeating this algorithmic step with X replaced by the *transformed vector field* X^1, bringing the latter vector field into normal form to order $m + 1$. Since the homological equation involves the homogeneous terms of X^1 of order $m+1$, we use identity (8) to compute these terms. Furthermore, we enforce uniqueness of the solution Y of (12) by imposing the condition $Y \in \operatorname{Im}\operatorname{ad}(S)$. However, computing just the homogeneous terms of X^1 of order $m + 1$ is not sufficient if $m + 1 < N$, since subsequent steps of the algorithm access the terms of even higher order in the transformed vector field. Therefore, we use (8) to compute these higher order terms.

These steps are then repeated until the final transformed vector field is in normal form to order $N + 1$. This procedure is expressed more precisely in the normal form algorithm in Figure 3.

3.2. *Applications of the Normal Form Algorithm*

The Hopf bifurcation occurs in one-parameter families of planar vector fields having a nonhyperbolic equilibrium with a pair of pure imaginary eigenvalues with nonzero imaginary part. In this bifurcation a limit cycle emerges from the equilibrium as the parameters of the system push the eigenvalues off the imaginary axis. See also Figure 4.

In this context it is easier to express the system in coordinates $z, \overline{z}$ on the complex plane. The linear part of the vector field at the point of bifurcation is then $\dot{z} = i\omega z$. To apply the Normal Form Algorithm, we first derive an expression for the Lie brackets of real vector fields with in these coordinates.

Algorithm (Normal Form Algorithm)
Input: N, S, $X[2..N]$, satisfying
1. S is a semisimple linear vector field
2. $X = S + X[2] + \cdots + X[N] \mod \mathcal{F}_{N+1}$,
   with $X[n] \in \mathcal{H}_n$

   (∗ X is in normal form to order 1 ∗)

for $m = 2$ **to** N **do**
  (∗ bring X into normal form to order m ∗)
  determine $G \in \operatorname{Ker}\operatorname{ad}(S) \cap \mathcal{H}_m$ and $B \in \operatorname{Im}\operatorname{ad}(S) \cap \mathcal{H}_m$ such that
    $X[m] = G + B$
  determine Y, with $Y \in \operatorname{Im}\operatorname{ad}(S) \cap \mathcal{H}_m$, such that
    $\operatorname{ad}(S)(Y) = -B$
  (∗ compute terms of order $m+1, \ldots, N$ of transformed vector field ∗)
  for $n = 1$ **to** N **do**
    for $k = 1$ **to** $\lfloor \frac{N-n}{m-1} \rfloor$ **do**
$$X[n + k(m-1)] := X[n + k(m-1)] + \frac{(-1)^k}{k!}\operatorname{ad}(Y)^k(X[n])$$

Fig. 3. The Normal Form Algorithm.

The Lie-subalgebra of real vector fields. We identify $\mathbb{R}^2$ with $\mathbb{C}$, by associating the point (x_1, x_2) in $\mathbb{R}^2$ with $x_1 + ix_2$ in $\mathbb{C}$. The real vector field X, defined on $\mathbb{R}^2$ by

$$X = Y_1 \frac{\partial}{\partial x_1} + Y_2 \frac{\partial}{\partial x_2},$$

corresponds to the vector field

$$X = Y\,\frac{\partial}{\partial z} + \overline{Y}\,\frac{\partial}{\partial \overline{z}}, \tag{13}$$

on $\mathbb{C}$, where $Y = Y_1 + iY_2$.

Example. Taking $Y(z, \overline{z}) = cz^{k+1}\overline{z}^k$, with c a complex constant, the vector field X given by (13) is SO(2)-equivariant. Writing $c = a + ib$, with $a, b \in \mathbb{R}$, and $z = x_1 + ix_2$, we obtain its real form via a straightforward computation:

$$X = (x_1^2 + x_2^2)^k \big(a(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) + b(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2})\big).$$

In particular, the real vector field $\omega_N(-x_2 \dfrac{\partial}{\partial x_1} + x_1 \dfrac{\partial}{\partial x_2})$ corresponds to the complex vector field $S = i\omega_N(z \dfrac{\partial}{\partial z} - \overline{z} \dfrac{\partial}{\partial \overline{z}})$.

We denote the $\dfrac{\partial}{\partial z}$-component of a real vector field X by $X_{\mathbb{R}}$, so:

$$X = X_{\mathbb{R}} \frac{\partial}{\partial z} + \overline{X_{\mathbb{R}}} \frac{\partial}{\partial \overline{z}}.$$

The real vector fields form a Lie-subalgebra of the algebra of all vector fields on $\mathbb{C}$. The following result justifies this claim.

Lemma 3.2. *Let X and Y be real vector fields on $\mathbb{C}$, and let $f : \mathbb{C} \to \mathbb{C}$ be a smooth function. Then*

$$\overline{X(f)} = X(\overline{f}), \tag{14}$$

and

$$[X, Y] = \langle X, Y\rangle \frac{\partial}{\partial z} + \overline{\langle X, Y\rangle} \frac{\partial}{\partial \overline{z}},$$

where the bilinear antisymmetric form $\langle\cdot,\cdot\rangle$ is defined by

$$\langle X, Y\rangle = X(Y_{\mathbb{R}}) - Y(X_{\mathbb{R}}).$$

Derivation of the Hopf Normal Form. To derive the Hopf Normal Form, we apply the Normal Form Algorithm to a vector field with linear part

$$S = i\omega_N (z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}}).$$

The adjoint action of S on the Lie-subalgebra of real vector fields is given by:

$$\mathrm{ad}(S)(X) = \langle S, X\rangle \frac{\partial}{\partial z} + \overline{\langle S, X\rangle} \frac{\partial}{\partial \overline{z}},$$

with

$$\langle S, X\rangle = i\omega_N (z \frac{\partial X_{\mathbb{R}}}{\partial z} - \overline{z} \frac{\partial X_{\mathbb{R}}}{\partial \overline{z}} - X_{\mathbb{R}}).$$

In particular, if $Y = Y_{\mathbb{R}} \dfrac{\partial}{\partial z} + \overline{Y_{\mathbb{R}}} \dfrac{\partial}{\partial \overline{z}}$ with $Y_{\mathbb{R}} = z^k \overline{z}^l$, then

$$\langle S, Y\rangle = i\omega_N \,(k - l - 1)\, z^k \overline{z}^l.$$

Therefore, the adjoint operator $\mathrm{ad}(S) : \mathcal{H}_m \to \mathcal{H}_m$ has non-trivial kernel for m odd. If $m = 2k + 1$, this kernel consists of the monomial vector field Y with $Y_{\mathbb{R}} = z|z|^{2k}$. These observations lead to the following Normal Form.

Corollary 3.1. *If a vector field on $\mathbb{C}$ has linear part $S = i\omega_N(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}})$, then the Normal Form Algorithm brings this vector field into the form*

$$\dot{z} = i\omega z + \sum_{k=1}^{m} c_k z|z|^{2k} + O(|z|^{2m+3}). \tag{15}$$

The nondegenerate Hopf bifurcation. An other application of the Normal Form Algorithm is the computation of the first Hopf coefficient c_1 in (15). Let X be given by

$$\dot{z} = i\omega z + a_0 z^2 + a_1 z\overline{z} + a_2\overline{z}^2 + b_0 z^3 + b_1 z^2\overline{z} + b_2 z\overline{z}^2 + b_3\overline{z}^3 + O(|z|^4).$$

The Normal Form Algorithm computes the following normal form for this system:

$$\dot{z} = i\omega z + \big(b_1 - \frac{i}{3\omega}(3a_0a_1 - 3|a_1|^2 - 2|a_2|^2)\big) z^2\overline{z} + O(|z|^4).$$

This result can also be obtained by a tedious calculation. See, for example, [26, page 155]*

To analyze the emergence of limit cycles we rewrite the Hopf Normal Form

$$\dot{w} = i\omega_N w + wb(|w|^2, \mu) + O(n+1)$$

in polar coordinates as:

$$\begin{aligned} \dot{r} &= r\ \mathrm{Re}\, b(r^2, \mu) + O(n+1) \\ \dot{\varphi} &= \omega_N + \mathrm{Im}\, b(r^2, \mu) + O(n+1) \end{aligned}$$

Limit cycles are obtained by solving $r = r(\mu)$ from the equation $\mathrm{Re}\, b(r^2, \mu) = 0$. The frequency of the limit cycle is then of the form $\omega(\mu) = \omega_N + \mathrm{Im}\, b(r(\mu)^2, \mu)$.

A non-degenerate Hopf bifurcation occurs if the first Hopf coefficient c_1 in (15) is nonzero. Consider, e.g., the simple case $b(u, \mu) = \mu + u$. Putting $\mu = a + i\delta$ we see that the limit cycle corresponds to the trajectory w

$$w_{a,\delta}(t) = \sqrt{-a}\, e^{i(\omega_N+\delta)t} \qquad (a \leq 0)$$

This limit cycle exists for $a < 0$, and is repulsive in this case.

*The term $|h_{ww}|^2$ in identity (3.4.26) of [26, page 155] should be replaced by $|h_{\overline{w}\overline{w}}|^2$.

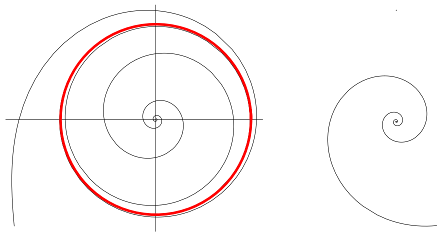

Fig. 4. Birth or death of a limit cycle via a Hopf bifurcation.

Hopf-Neĭmark-Sacker bifurcations in forced oscillators. We now study the birth or death of subharmonics in forced oscillators depending on parameters. In particular, we consider 2π-periodic systems on $\mathbb{C}$ of the form

$$\dot{z} = F(z, \overline{z}, \mu) + \varepsilon G(z, \overline{z}, t, \mu), \tag{16}$$

obtained from an autonomous system by a small 2π-periodic perturbation. Here ε is a real perturbation parameter, and $\mu \in \mathbb{R}^k$ is an additional k-dimensional parameter. Subharmonics of order q may appear or disappear upon variation of the parameters if the linear part of F at $z = 0$ satisfies a $p : q$-resonance condition which is appropriately detuned upon variation of the parameter μ.

The Normal Form Algorithm of Section 3.1 can be adapted to the derivation of the Hopf-Neĭmark-Sacker Normal form of periodic systems. Consider a 2π-periodic forced oscillator on $\mathbb{C}$ of the form

$$\dot{z} = X_{\mathbb{R}}(z, \overline{z}, t, \mu),$$

where

$$X_{\mathbb{R}}(z, \overline{z}, t, \mu) = i\omega_N z + (\alpha + i\delta)z + zP(z, \overline{z}, \mu) + \varepsilon Q(z, \overline{z}, t, \mu). \tag{17}$$

Here $\mu \in \mathbb{R}^k$, and ε is a small real parameter. Furthermore we assume that P and Q contain no terms that are independent of z and $\overline{z}$ (i.e., $P(0, 0, \mu) = 0$ and $Q(0, 0, t, \mu) = 0$), and that Q does not even contain terms that are linear in z and $\overline{z}$. Any system of the form (16) with linear part $\dot{z} = i\omega_N z$ can be brought into this form after a straightforward initial transformation. See[3] for details. Subharmonics of order q are to be expected if the linear part satisfies a $p : q$-resonance condition, in other words, if the normal frequency ω_N is equal to $\dfrac{p}{q}$ (with p and q relatively prime).

Theorem 3.1. ***(Normal Form to order*** q***)***
The system (17) has normal form

$$\dot{z} = i\omega_N z + (\alpha + i\delta)z + zF(|z|^2, \mu) + d\,\varepsilon\,\overline{z}^{q-1}\,e^{ipt} + O(q+1), \tag{18}$$

where $F(|z|^2, \mu)$ *is a complex polynomial of degree* $q-1$ *with* $F(0, \mu) = 0$*, and* d *is a complex constant.*

Proof. To derive a normal form for the system (17) we consider 2π-periodic vector fields on $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$ of the form

$$X = X_{\mathbb{R}}\,\frac{\partial}{\partial z} + \overline{X_{\mathbb{R}}}\,\frac{\partial}{\partial \overline{z}} + \frac{\partial}{\partial t},$$

with 'linear' part

$$S = i\omega_N(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}) + \frac{\partial}{\partial t}.$$

For nonnegative integers m we denote the space of 2π-periodic vector fields of total degree m in $(z, \overline{z}, \mu)$ with vanishing $\dfrac{\partial}{\partial t}$-component by $\mathcal{H}_m$. As before $\mathcal{F}_m = \prod_{k \geq m} \mathcal{H}_k$.

The adjoint action of S on the Lie-subalgebra of real 2π-periodic vector fields with zero $\dfrac{\partial}{\partial t}$-component is given by:

$$\mathrm{ad}(S)(X) = \langle S, X\rangle_{\mathbb{R}}\frac{\partial}{\partial z} + \overline{\langle S, X\rangle_{\mathbb{R}}}\frac{\partial}{\partial \overline{z}},$$

with

$$\langle S, X\rangle_{\mathbb{R}} = i\omega_N(z\frac{\partial X_{\mathbb{R}}}{\partial z} - \overline{z}\frac{\partial X_{\mathbb{R}}}{\partial \overline{z}} - X_{\mathbb{R}}) + \frac{\partial X_{\mathbb{R}}}{\partial t}.$$

If $X_{\mathbb{R}} = \mu^{\sigma} z^k \overline{z}^l e^{imt}$, with $|\sigma| + k + l = n$, then

$$\langle S, X\rangle_{\mathbb{R}} = (i\omega_N(k - l - 1) + im)\,X_{\mathbb{R}}.$$

Therefore, the normal form contains time-independent rotationally symmetric terms corresponding to $k = l + 1$ and $m = 0$. Since for $\varepsilon = 0$ the system is in Hopf Normal Form, all non-rotationally symmetric terms contain a factor ε, so $|\sigma| > 0$ for these terms. It is not hard to see that for $n \leq q$ and $|\sigma| > 0$ the only non-rotationally symmetric term corresponds to $k = 0$, $l = q - 1$, $m = p$, and $|\sigma| = 1$. Therefore, this non-symmetric term is of the form

$$d\,\varepsilon\,\overline{z}^{q-1}\,e^{ipt},$$

for some complex constant d. □

3.3. *Via covering spaces to the Takens Normal Form*

Existence of $2\pi q$-periodic orbits. The Van der Pol transformation. Subharmonics of order q of the 2π-periodic forced oscillator (17) correspond to q-periodic orbits of the Poincaré time 2π-map $P : \mathbb{C} \to \mathbb{C}$. These periodic orbits of the Poincaré map are brought into one-one correspondence with the zeros of a vector field on a q-sheeted cover of the phase space $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$ via the *Van der Pol transformation*, cf.[16] This transformation corresponds to a q-sheeted covering

$$\begin{aligned} \Pi : \mathbb{C} \times \mathbb{R}/(2\pi q\mathbb{Z}) \to \mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z}), \\ (z,t) \mapsto (z\mathrm{e}^{itp/q}, t \pmod{2\pi\mathbb{Z}}) \end{aligned} \tag{19}$$

with cyclic Deck group of order q generated by

$$(z,t) \mapsto (z\mathrm{e}^{2\pi i p/q}, t - 2\pi).$$

The Van der Pol transformation $\zeta = ze^{-i\omega_N t}$ lifts the forced oscillator (16) to the system

$$\dot{\zeta} = (\alpha + i\delta)\zeta + \zeta P(\zeta e^{i\omega_N t}, \overline{\zeta} e^{-i\omega_N t}, \mu) + \varepsilon Q(\zeta e^{i\omega_N t}, \overline{\zeta} e^{-i\omega_N t}, t, \mu) \tag{20}$$

on the covering space $\mathbb{C} \times \mathbb{R}/(2\pi q\mathbb{Z})$. The latter system is $\mathbb{Z}_q$-equivariant. A straightforward application of (20) to the normal form (18) yields the following normal form for the lifted forced oscillator.

Theorem 3.2. ***(Equivariant Normal Form of order*** q*)*
On the covering space, the lifted forced oscillator has the $\mathbb{Z}_q$-equivariant normal form:

$$\dot{\zeta} = (\alpha + i\delta)\zeta + \zeta F(|\zeta|^2, \mu) + d\,\varepsilon\,\overline{\zeta}^{q-1} + O(q+1), \tag{21}$$

where the $O(q+1)$ terms are $2\pi q$-periodic.

Resonance tongues for families of forced oscillators. Bifurcations of q-periodic orbits of the Poincaré map P on the base space correspond to bifurcations of fixed points of the Poincaré map $\tilde{P}$ on the q-sheeted covering space introduced in connection with the Van der Pol transformation (19). Denoting the normal form system (18) on the base space by $\mathcal{N}$, and the normal form system (21) of the lifted forced oscillator by $\tilde{\mathcal{N}}$, we see that

$$\Pi_* \tilde{\mathcal{N}} = \mathcal{N}.$$

The Poincaré mapping $\tilde{P}$ of the normal form on the covering space now is the $2\pi q$-period mapping

$$\tilde{P} = \tilde{\mathcal{N}}^{2\pi q} + O(q+1),$$

where $\tilde{\mathcal{N}}^{2\pi q}$ denotes the $2\pi q$-map of the (planar) vector field $\tilde{\mathcal{N}}$. Following the Corollary to the Normal Form Theorem of [16, page 12], we conclude for the original Poincaré map P of the vector field X on the base space that

$$P = R_{2\pi\omega_N} \circ \tilde{\mathcal{N}}^{2\pi} + O(q+1),$$

where $R_{2\pi\omega_N}$ is the rotation over $2\pi\omega_N = 2\pi p/q$, which precisely is the Takens Normal Form[37] of P at $(z, \mu) = (0, 0)$.

Our interest is with the q-periodic points of P_μ, which correspond to the fixed points of $\tilde{P}_\mu$. This fixed point set and the boundary thereof in the parameter space $\mathbb{R}^3 = \{a, \delta, \varepsilon\}$ is approximately described by the discriminant set of

$$(a + i\delta)\zeta + \zeta\tilde{F}(|\zeta|^2, \mu) + \varepsilon d\overline{\zeta}^{q-1},$$

which is the truncated right hand side of (21). This gives rise to the bifurcation equation that determine the boundaries of the resonance tongues. The following theorem implies that, under the conditions that $d \neq 0 \neq F_u(0, 0)$, the order of tangency at the tongue tips is generic. Here $F_u(0, 0)$ is the partial derivative of $F(u, \mu)$ with respect to u.

Theorem 3.3. ***(Bifurcation equations modulo contact equivalence)*** *Assume that $d \neq 0$ and $F_u(0, 0) \neq 0$. Then the polynomial (21) is $\mathbb{Z}_q$-equivariantly contact equivalent with the polynomial*

$$G(\zeta, \mu) = (a + i\delta + |\zeta|^2)\zeta + \varepsilon\overline{\zeta}^{q-1}. \tag{22}$$

The discriminant set of the polynomial $G(\zeta, \mu)$ is of the form

$$\delta = \pm\varepsilon(-a)^{(q-2)/2} + O(\varepsilon^2). \tag{23}$$

Proof. The polynomial (22) is a universal unfolding of the germ $|\zeta|^2\zeta + \varepsilon\overline{\zeta}^{q-1}$ under $\mathbb{Z}_q$ contact equivalence. See[7] for a detailed computation. The tongue boundaries of a $p : q$ resonance are given by the bifurcation equations

$$\begin{aligned} G(\zeta, \mu) &= 0, \\ \det(dG)(\zeta, \mu) &= 0. \end{aligned}$$

As in [7, Theorem 3.1] we put $u = |z|^2$, and $b(u, \mu) = a + i\delta + u$. Then $G(\zeta, \mu) = b(u, \mu)\zeta + \varepsilon\overline{\zeta}^{q-1}$. According to (the proof of) [7, Theorem 3.1], the system of bifurcation equations is equivalent to

$$\begin{aligned} |b|^2 &= \varepsilon^2 u^{q-2}, \\ b\overline{b}' + \overline{b}b' &= (q-2)\varepsilon^2 u^{q-3}, \end{aligned}$$

where $b' = \dfrac{\partial b}{\partial u}(u, \mu)$. A short computation reduces the latter system to the equivalent

$$(a+u)^2 + \delta^2 = \varepsilon^2 u^{q-2},$$
$$a + u = \frac{1}{2}(q-2)\varepsilon^2 u^{q-3}.$$

Eliminating u from this system of equations yields expression (23) for the tongue boundaries. □

The discriminant set of the equivariant polynomial (22) forms the boundary of the resonance tongues. See Figure 5. At this surface we expect

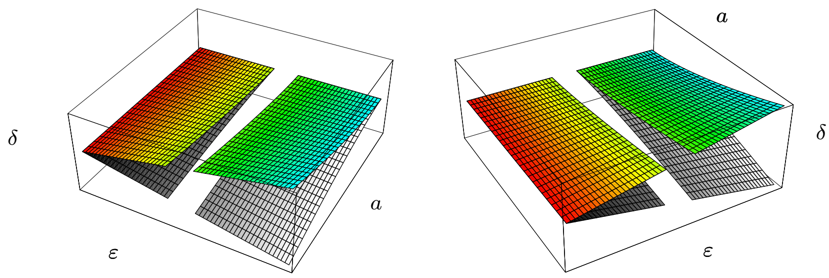

Fig. 5. Resonance zones for forced oscillator families: the Hopf-Neĭmark-Sacker phenomenon.

the Hopf-Neĭmark-Sacker bifurcation to occur; here the Floquet exponents of the linear part of the forced oscillator cross the complex unit circle. This bifurcation gives rise to an invariant 2-torus in the 3D phase space $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$. Resonances occur when the eigenvalues cross the unit circle at roots of unity $e^{2\pi i p/q}$. 'Inside' the tongue the 2-torus is phase-locked to subharmonic periodic solutions of order q.

4. Generic Hopf-Neĭmark-Sacker bifurcations in feed forward systems?

Coupled Cell Systems. A coupled cell system is a network of dynamical systems, or cells, coupled together. This network is a finite directed graph with nodes representing cells and edges representing couplings between these cells. See, e.g., Golubitsky, Nicol and Stewart.[22]

We consider the three-cell feed-forward network in Figure 6, where the first cell is coupled externally to itself. The network has the form of a *coupled*

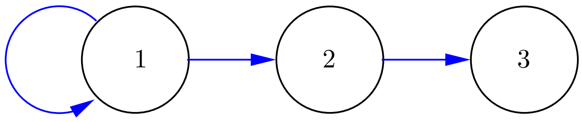

Fig. 6. Three-cell linear feed-forward network

cell system

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_1)\\ \dot{x}_2 &= f(x_2, x_1)\\ \dot{x}_3 &= f(x_3, x_2)\end{aligned}$$

with $x_j \in \mathbb{R}^2$.

Under certain conditions these networks have time-evolutions that are equilibria in cell 1 and periodic in cells 2 and 3. Elmhirst and Golubitsky[20] describe a curious phenomenon: the amplitude growth of the periodic signal in cell 3 is to the power $\frac{1}{6}$ rather than to the power $\frac{1}{2}$ with respect to the bifurcation parameter in the Hopf bifurcation. See also Section 3.2.

For technical reasons we assume that the function f, describing the dynamics of each cell, is $\mathbb{S}^1$-symmetric in the sense that

$$f(e^{i\theta} z_2, e^{i\theta} z_1) = e^{i\theta} f(z_2, z_1), \tag{24}$$

for all real θ. Here we identify the two-dimensional phase space of each cell with $\mathbb{C}$ by writing $z_j = x_{j1} + i x_{j2}$. Identity (24) is a special assumption, that we will try to relax in future research. However, Elmhirst and Golubitsky[20] verify that this symmetry condition holds to third order after a change of coordinates. We also assume that the dynamics of each cell depends on external parameters λ, μ, to be specified later on.

Dynamics of the first and second cell. The $\mathbb{S}^1$-symmetry (24) implies that $f_{\lambda,\mu}(0,0) = 0$. Note that from now on we make the dependence of f on the parameters explicit in our notation. Assume that the linear part of $f_{\lambda,\mu}(z_1, z_1)$ at $z_1 = 0$ has eigenvalues with negative real part. Then *the first cell has a stable equilibrium at* $z_1 = 0$.

The second cell has dynamics

$$\dot{z}_2 = f_{\lambda,\mu}(z_2, z_1) = f_{\lambda,\mu}(z_2, 0),$$

where we use that the first cell is in its stable equilibrium. Golubitsky and Stewart[24] introduce a large class of functions $f_{\lambda,\mu}$ for which the *second cell undergoes a Hopf bifurcation.* For this class of cell dynamics, and for linear feed-forward networks of increasing length, there will be 'repeated Hopf' bifurcation, reminiscent of the scenarios named after Landau-Lifschitz and Ruelle-Takens.

To obtain more precise information on the Hopf bifurcation in the dynamics of the second cell we consider a special class of functions $f_{\lambda,\mu}$ satisfying (24). In particular, we require that

$$f_{\lambda,\mu}(z_2, 0) = (\lambda + i - |z_2|^2)\, z_2, \tag{25}$$

giving a supercritical Hopf bifurcation at $\lambda = 0$. The stable periodic solution, occurring for $\lambda > 0$, has the form

$$z_2(t) = \sqrt{\lambda} e^{it}.$$

Dynamics of the third cell. The main topic of our research is the generic dynamics of the *third cell*, given simple time-evolutions of the first two cells. Here we like to know what are the correspondences and differences with the general ODE setting. In particular this question regards the coexistence of periodic, quasi-periodic and chaotic dynamics.

In co-rotating coordinates the dynamics of the third cell becomes time-independent. To see this, set $z_3 = e^{it}y$, and use the $\mathbb{S}^1$-symmetry to derive

$$\begin{aligned} ie^{it}y + e^{it}\dot{y} = &= f_{\lambda,\mu}(e^{it}y, \sqrt{\lambda}\, e^{it}) \\ &= e^{it}\, f_{\lambda,\mu}(y, \sqrt{\lambda}). \end{aligned}$$

Therefore, the dynamics of the third cell is given by

$$\dot{y} = -iy + f_{\lambda,\mu}(y, \sqrt{\lambda}). \tag{26}$$

Equation (26) is autonomous, so the present setting might exhibit Hopf bifurcations, but it is still too simple to produce resonance tongues. Indeed, all (relative) periodic motion in (26) will lead to parallel (quasi-periodic) dynamics and the Hopf-Neĭmark-Sacker phenomenon. Therefore, we now perturb the basic function $f = f_{\lambda,\mu}(z_2, z_1)$, to

$$F_{\lambda,\mu,\varepsilon}(z_2, z_1) := f_{\lambda,\mu}(z_2, z_1) + \varepsilon P(z_2, z_1).$$

In cells 1 and 2 any choice of the perturbation term $P(z_2, z_1)$ gives the dynamics

$$\begin{aligned} \dot{z}_1 &= F_{\lambda,\mu,\varepsilon}(z_1, z_1) = f_{\lambda,\mu}(z_1, z_1) + \varepsilon P(z_1, z_1), \\ \dot{z}_2 &= F_{\lambda,\mu,\varepsilon}(z_2, 0), \end{aligned}$$

with the same conclusions as before, namely a steady state $z_1 = 0$ in cell 1 and a periodic state $z_2 = \sqrt{\lambda} e^{it}$ in cell 2 (when $\lambda > 0$). For these two conclusions it is sufficient that

$$P(z_2, 0) \equiv 0.$$

Turning to the third cell we again put $y = e^{-it} z_3$, and so get a perturbed reduced equation

$$\dot{y} = -iy + f_{\lambda,\mu}(y, \sqrt{\lambda}) + \varepsilon e^{-it} P(y\, e^{it}, \sqrt{\lambda}\, e^{it}). \tag{27}$$

The third cell therefore has forced oscillator dynamics with driving frequency 1. The question about generic dynamics regards the possible coexistence of periodic and quasi-periodic dynamics. We aim to investigate (27) for Hopf-Neĭmark-Sacker bifurcations, which are expected along curves $\mathcal{H}_\varepsilon$ in the (λ, μ)-plane of parameters. We expect to find periodic tongues (See also Figure 5) and quasiperiodic hairs, like in Broer et al.[15] This is the subject of ongoing research. The machinery of Section 3.2 should provide us with sufficiently powerful tools to investigate this phenomenon for a large class of coupled cell systems.

5. Conclusion and future work

We have presented several contexts in which bifurcations from fixed points of maps or equilibria of vector fields lead to the emergence of periodic orbits. For each context we present appropriate normal form techniques, illustrating the general paradigm of 'simplifying the system before analyzing it'. In the context of generic families we apply generic techniques, based on Lyapunov-Schmidt reduction and $\mathbb{Z}_q$-equivariant contact equivalence. In this way we recover standard results on resonance tongues for nondegenerate maps, but also discover new phenomena in unfoldings of mildly degenerate systems. Furthermore, we present an algorithm for bringing concrete families of dynamical systems into normal form, without losing information in a preliminary reduction step, like the Lyapunov-Schmidt method. An example of such a concrete system is a class of feedforward networks of coupled cell systems, in which we expect the Hopf-Neĭmark-Sacker-phenomenon to occur.

With regard to further research, our methods can be extended to other contexts, in particular, to cases where extra symmetries, including time reversibility, are present. This holds both for Lyapunov-Schmidt reduction and $\mathbb{Z}_q$ equivariant singularity theory. In this respect Golubitsky, Marsden, Stewart, and Dellnitz,[21] Knobloch and Vanderbauwhede,[27,28] and Vanderbauwhede[38] are helpful.

Furthermore, there is the issue of how to apply our results to a concrete family of dynamical systems. Golubitsky and Schaeffer[23] describe methods for obtaining the Taylor expansion of the reduced function $g(z)$ in terms of the Poincaré map P and its derivatives. These methods may be easier to apply if the system is a periodically forced second order differential equation, in which case the computations again may utilize parameter dependent Floquet theory. We also plan to turn the Singularity Theory methods of Section 2 into effective algorithms, along the lines of our earlier work.[9]

Finally, in this paper we have studied only degeneracies in tongue boundaries. It would also be interesting to study low codimension degeneracies in the dynamics associated to the resonance tongues. Such a study will require tools that are more sophisticated than the singularity theory ones that we have considered here.

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