Two-Color Patterns of Synchrony in Lattice Dynamical Systems

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Abstract

Using the theory of coupled cell systems developed by Stewart, Golubitsky, Pivato, and Török, we consider patterns of synchrony in four types of planar lattice dynamical systems: square lattice and hexagonal lattice differential equations with nearest neighbor coupling and with nearest and next nearest neighbor couplings. *Patterns of synchrony* are flow-invariant subspaces for all lattice dynamical systems with a given network architecture that are formed by setting coordinates in different cells equal. Such patterns can be formed by symmetry (through fixed-point subspaces), but many patterns cannot be obtained in this way. Indeed, Golubitsky, Nicol, and Stewart present patterns of synchrony on square lattices that are not predicted by symmetry. The general theory shows that finding patterns of synchrony is equivalent to finding balanced equivalence relations on the set of cells. In a two-color pattern one set of cells is colored white and the complement black. Two-color patterns in lattice dynamical systems are *balanced* if the number of white cells connected to a white cell is the same for all white cells and the number of black cells connected to a black cell is the same for all black cells. In this paper, we find all two-color patterns of synchrony of the four kinds of lattice dynamical systems, and show that all of these patterns, including spatially complicated patterns, can be generated from a finite number of distinct patterns. Our classification shows that all balanced two-colorings in lattices systems with both nearest and next nearest neighbor couplings are spatially doubly periodic. We also prove that equilibria associated to each such two-color pattern can be obtained by codimension one synchrony-breaking bifurcation from a fully synchronous equilibrium.

1 Introduction

Stewart, Golubitsky, Pivato, and Török [8, 7] formalize the concept of a *coupled cell system.* A *cell* is a system of ODE. A *coupled cell system* consists of cells whose equations are coupled. Stewart et al. define the architecture of coupled cell systems and develop the theory that shows how network architecture leads to synchrony. The architecture of a coupled cell system is a graph that indicates which cells have same phase space, which cells are coupled to which, and which couplings are the same.

The *input set* $I(c)$ of a cell c consists of all cells coupled to c. Two input sets are *isomorphic* if there is a bijection between the input sets that preserves coupling types. A coupled cell network is *homogeneous* if the input sets of all cells are isomorphic.

A lattice dynamical system is a homogeneous coupled cell system with cells indexed by a lattice $\mathcal{L}$. Such a system has the form

$$\dot{x}_c = g(x_c, x_{I(c)}) \qquad c \in \mathcal{L} \tag{1.1}$$

where $x_c \in \mathbf{R}^n$, $I(c) = \{c_1, \ldots, c_k\}$, $x_{I(c)} = (x_{c_1}, \ldots, x_{c_k}) \in (\mathbf{R}^n)^k$ and $g : (\mathbf{R}^n)^{k+1} \to \mathbf{R}^n$.

Specifically, a *square lattice dynamical system with nearest neighbor coupling* has the form

$$\dot{x}_{i,j} = g(x_{i,j},\ \overline{x_{i+1,j}, x_{i-1,j}, x_{i,j+1}, x_{i,j-1}}) \tag{1.2}$$

where $(i,j) \in \mathbf{Z}^2$, $x_{i,j} \in \mathbf{R}^n$, and g is invariant under the permutations of the variables under the bar. A *square lattice dynamical system with nearest and next nearest neighbor couplings* has the form

$$\dot{x}_{i,j} = g(x_{i,j},\ \overline{x_{i+1,j}, x_{i-1,j}, x_{i,j+1}, x_{i,j-1}},\ \overline{x_{i+1,j+1}, x_{i-1,j+1}, x_{i+1,j-1}, x_{i-1,j-1}}) \tag{1.3}$$

See Figure 1 (a) a picture of the cells that are coupled to cell (i,j).

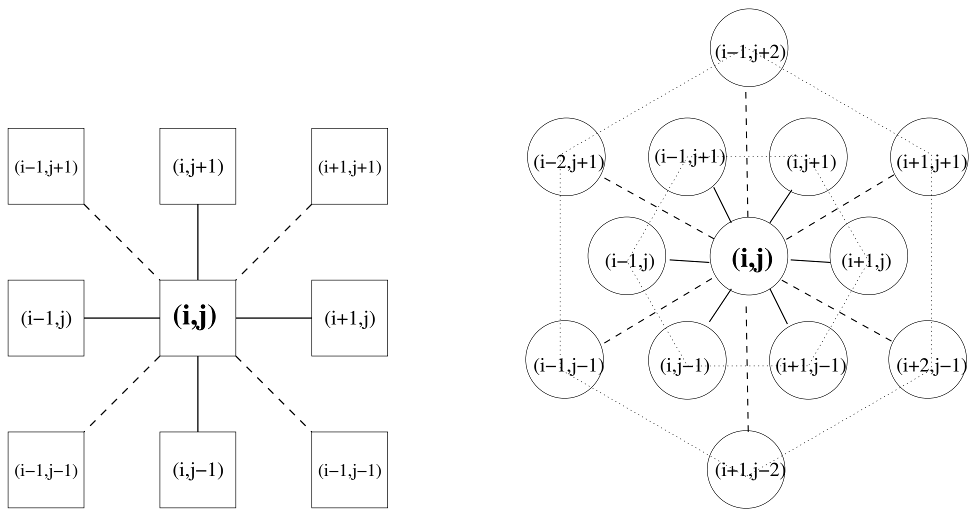
Let the hexagonal lattice be $\{il_1 + jl_2 : i,j \in \mathbf{Z}\}$, where $l_1 = (0,1)$ and $l_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. A *hexagonal lattice dynamical system with nearest neighbor coupling* has the form

$$\dot{x}_{i,j} = g(x_{i,j},\ \overline{x_{i-1,j}, x_{i+1,j}, x_{i,j-1}, x_{i,j+1}, x_{i-1,j+1}, x_{i+1,j-1}}) \tag{1.4}$$

where $x_{i,j}$ denotes the coordinate of the cell at the $il_1 + jl_2$ lattice site. The form of a *hexagonal lattice dynamical system with nearest and next nearest neighbor couplings* is given by

$$\begin{aligned}\dot{x}_{i,j} = g(x_{i,j},\ &\overline{x_{i-1,j}, x_{i+1,j}, x_{i,j-1}, x_{i,j+1}, x_{i-1,j+1}, x_{i+1,j-1}},\\ &\overline{x_{i-1,j+2}, x_{i-2,j+1}, x_{i-1,j-1}, x_{i+1,j-2}, x_{i+2,j-1}, x_{i+1,j+1}})\end{aligned} \tag{1.5}$$

See Figure 1 (b) a picture of the cells that are coupled to cell (i,j).



(a) (b)

Figure 1: The nearest neighbor cells of cell (i, j) are the cells that are connected with the cell by solid lines, and next nearest neighbor cells are the cells that are connected with the cell by dashed lines: (a) square lattice; (b) hexagonal lattice. Note that the dotted lines in (b) only indicate the hexagonal structure.

A *polydiagonal* is a subspace of the phase space of a coupled cell network that is defined by equality of cells coordinates. A polydiagonal is *robustly polysynchronous* if it is flow-invariant for every coupled cell system with the given network architecture. Robustly polysynchronous polydiagonals are identified with *patterns of synchrony*. Stewart et al. [8, Theorem 6.1] prove that a polydiagonal is robustly polysynchronous if and only if the coloring given by coloring cells that have the same coordinates with the same color is balanced. Following this result, we see that classifying robustly polysynchronous polydiagonals is equivalent to classifying balanced colorings. In a homogeneous network a two-coloring is balanced if each black cell receives the same number of inputs from black cells of each coupling type and each white cell receives the same number of inputs from white cells of each coupling type.

Some patterns of synchrony can be predicted by symmetry, namely, those that correspond to fixed-point subspaces of the group of network symmetries. However, not all patterns can be obtained in this way, and some of these nonsymmetric patterns are quite interesting. Golubitsky et al. [4] give an infinite class of two-color patterns of synchrony on square lattice dynamical systems with nearest neighbor coupling. We extend this result by presenting, up to symmetry, all possible two-color patterns of synchrony of the four kinds of lattice differential equations. These classification results are stated in Theorems 1.1-1.4. The proofs of these results are given in Sections 3-5. It follows from these theorems that with both nearest and next nearest neighbor couplings all balanced two-colorings are periodic.

We also prove that equilibria associated to each such pattern can be obtained by a codimension one synchrony-breaking bifurcation from a fully synchronous equilibrium. See Section 2. Recall that the group of symmetries of a lattice is generated by translations within the lattice and rotations and reflections that preserve the lattice.

Theorem 1.1 *There are eight two-color periodic patterns of synchrony of square lattice differential equations with nearest neighbor coupling shown in Figure 2. There are two infinite families of two-color patterns of synchrony that are generated from the periodic patterns in Figure 3 by interchanging black and white on diagonals along which black and white cells alternate. Up to symmetry, these are all of the two-color patterns of synchrony.*

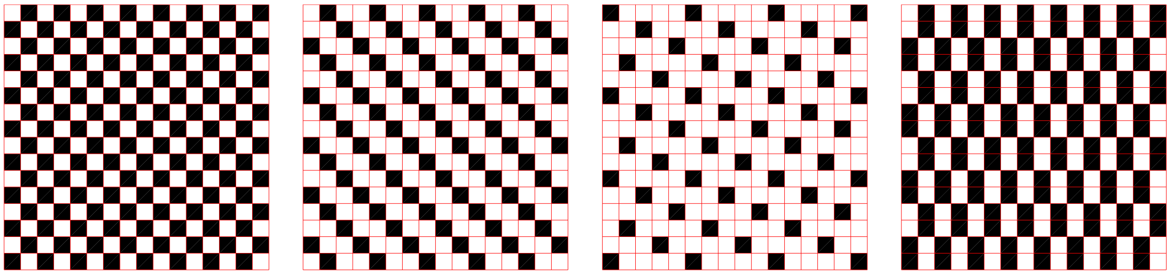
Remark: All patterns in Figure 2 are doubly-periodic and the repeating patterns associated to these patterns are given in Figure 4. The patterns marked by the boxes in this section cannot be expected from symmetry. Figure 5 contains some examples that can be obtained from the patterns in Figure 3 by interchanging black and white along diagonals. In particular, patterns (b) and (d) are spatially complicated.

Theorem 1.2 *Up to symmetry, there are twelve two-color patterns of synchrony in square lattice differential equations with nearest and next nearest neighbor couplings: the seven patterns in Figure 2 except for pattern (f), Figure 3 (a), and the four patterns in Figure 6.*

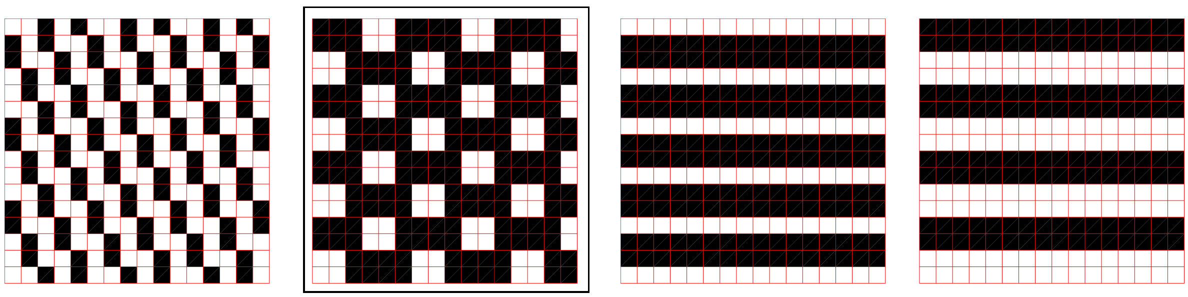
Remark: All twelve patterns are doubly-periodic. Those patterns in Figure 6 can be generated from Figure 3 (a) by interchanging black and white along diagonals on which black and white cells alternate.

Theorem 1.3 *There are ten two-color patterns of synchrony in hexagonal lattice differential equations with nearest neighbor coupling shown in Figure 7. There are three infinite families of two-color patterns of synchrony generated from the patterns in Figure 8 by interchanging white and black on diagonals along which white and black cells alternate. Up to symmetry, these are all of the two-color patterns of synchrony.*

Remark: Except for Figure 7 case 44a, all other patterns in Figure 7 are doubly-periodic and the repeating patterns associated to these patterns are given in Figure 9.

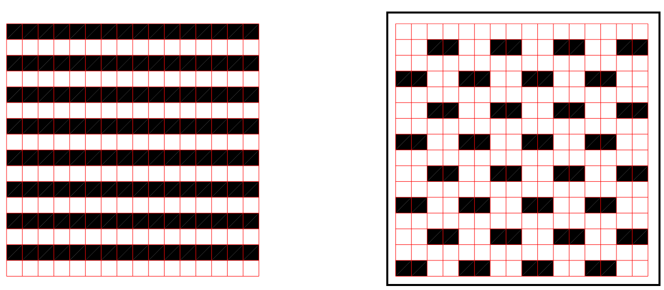


(a) case 44 (b) case 42 (c) case 41 (d) case 33



(e) case 32 (f) case 12a (g) case 12b (h) case 11

Figure 2: Illustrations of patterns of synchrony of finite classes in 16×16 periodic array.



(a) case 22 (b) case 31

Figure 3: Illustrations of patterns of synchrony of infinite classes in 16×16 periodic array.

Theorem 1.4 *Up to symmetry, there are thirteen two-color patterns of synchrony in hexagonal lattice differential equations with nearest and next nearest neighbor couplings: the nine patterns*

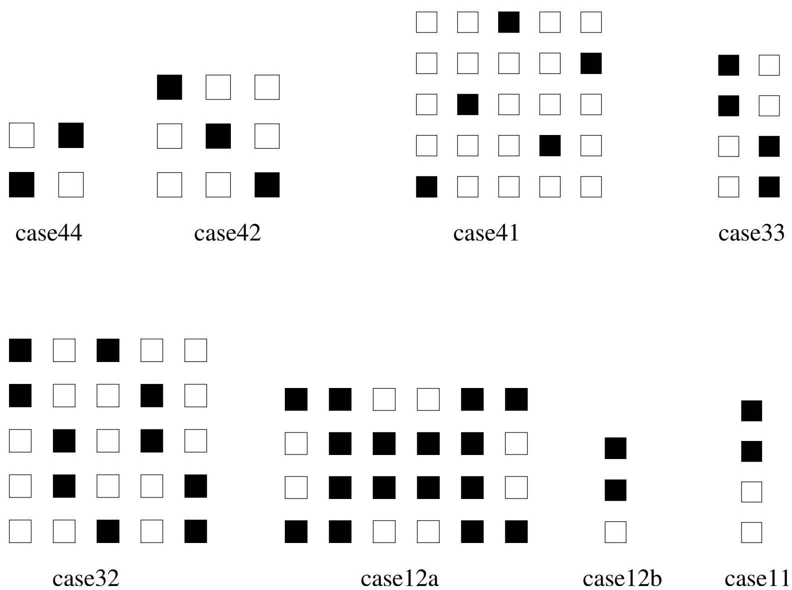


Figure 4: Repeating patterns for finite classes of patterns of synchrony shown in Figure 2.

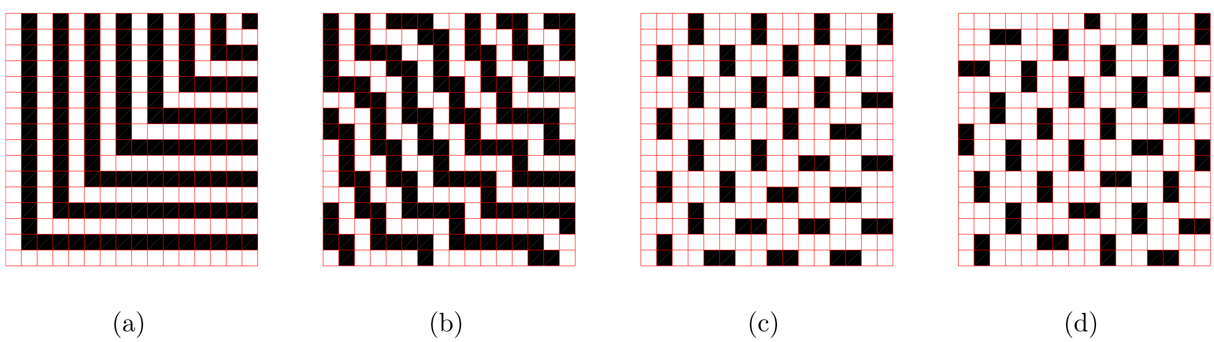
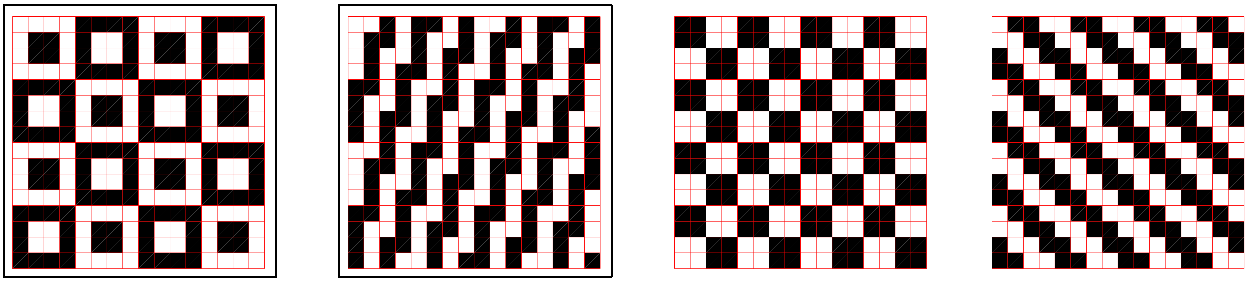


Figure 5: The patterns of synchrony that cannot be expected by symmetry.

in Figure 7 except for pattern (j), Figure 8 (a,c), and the two patterns in Figure 10.

Remark: All thirteen patterns are doubly-periodic. Those patterns in Figure 10 can be generated from Figure 8 (c) by interchanging black and white along diagonals on which black and white cells alternate.



(a) (b) (c) (d)

Figure 6: The four patterns in this figure and Figure 3 (a) are NN-balanced in infinite classes.

2 Bifurcation

In this section, we state and prove Theorem 2.1.

Theorem 2.1 *For each balanced two-coloring of a homogeneous system, there exists a codimension one synchrony-breaking bifurcation from a homogeneous equilibrium that leads to a branch of equilibria corresponding to the given two-coloring.*

Proof It is sufficient to prove this theorem for architectures with a single kind of coupling. Suppose a homogeneous coupled cell system has a balanced two-color equivalence relation on the cells, then the system can be reduced to a two-cell quotient system [7] (the coupled cell system restricted to the synchrony subspace) with the network pictured in Figure 11. Note that l_j is the number of times that cell j is coupled to itself, and m_j is the number of times that the other cell is coupled to cell j. Since the original system is homogeneous, so is the quotient. Hence

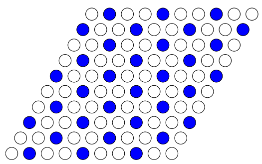
$$k \equiv l_1 + m_1 = l_2 + m_2 \tag{2.1}$$

where $l_1, l_2, m_1, m_2 \geq 0$.

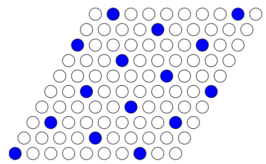
The quotient system has the form

$$\begin{aligned} \dot{x_1} &= f(x_1, \overline{\underbrace{x_1, \cdots, x_1}_{l_1 \text{ times}}, \underbrace{x_2, \cdots, x_2}_{m_1 \text{ times}}}, \lambda) \\ \dot{x_2} &= f(x_2, \overline{\underbrace{x_2, \cdots, x_2}_{l_2 \text{ times}}, \underbrace{x_1, \cdots, x_1}_{m_2 \text{ times}}}, \lambda) \end{aligned} \tag{2.2}$$

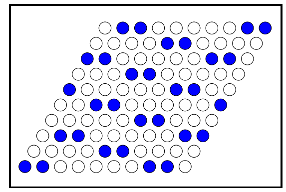
where $x_i \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is a parameter, and f is invariant under the permutations of the variables under the bar. Since the original system is homogeneous, the space formed by setting all cells



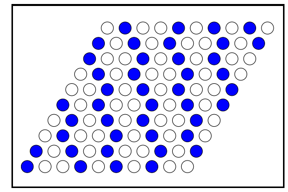
(a) case 63



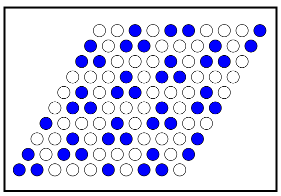
(b) case 61



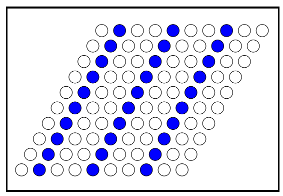
(c) case 52



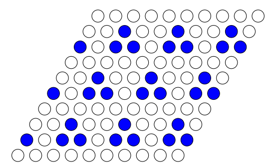
(d) case 43a



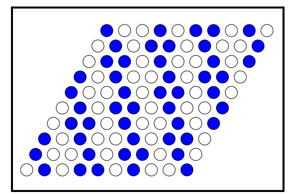
(e) case 43b



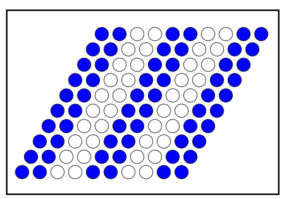
(f) case 42a



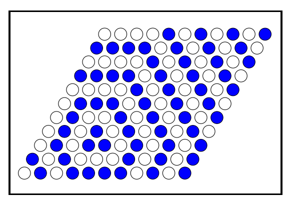
(g) case 42b



(h) case 33



(i) case 22



(j) case 44a

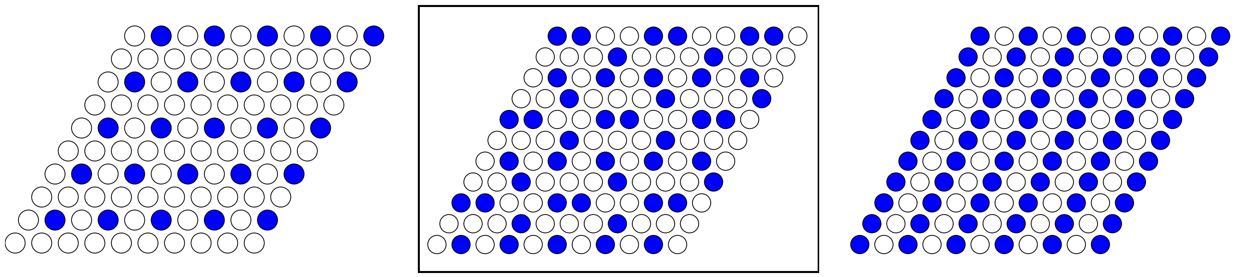
Figure 7: Illustrations of patterns of synchrony of finite classes in 10×10 periodic array.

equal is a polysynchronous subspace. We assume that there is a trivial branch of synchronous equilibria which has the form $x_1 = x_2$ in the quotient network. Without loss of generality, we assume it is at $(0, 0, \lambda)$. It follows from (2.2) that the Jacobian matrix at $(0, 0, \lambda)$ has the form

$$J = \begin{pmatrix} A + l_1 B & m_1 B \\ m_2 B & A + l_2 B \end{pmatrix} \tag{2.3}$$

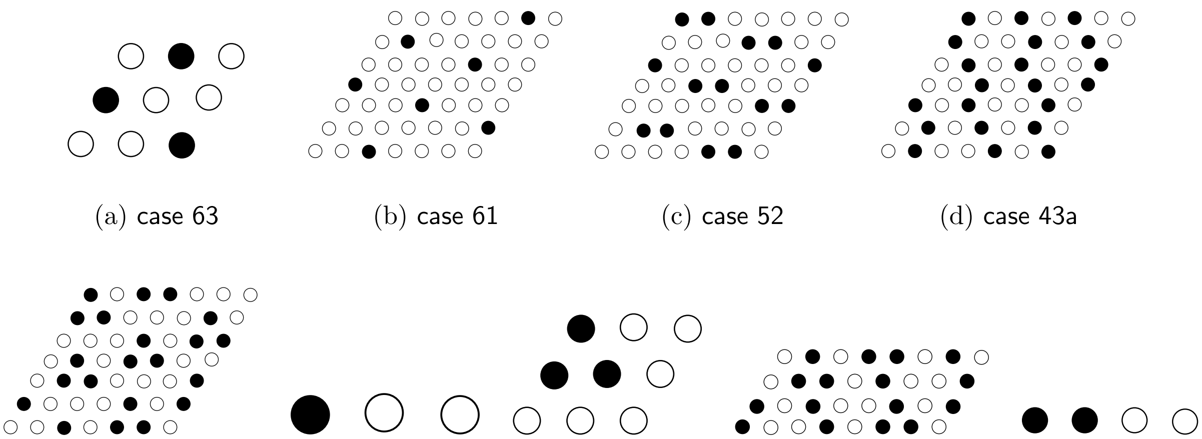
where $A = A(\lambda)$ is the matrix of linearized internal dynamics, and $B = B(\lambda)$ is the linearized coupling matrix.

We claim that the $2n$ eigenvalues of J (including multiplicity) are the eigenvalues of the $n \times n$ matrices $A + kB$ and $A + (l_1 + l_2 - k)B$. To verify this point, let μ be an eigenvalue of $A + kB$ (*resp.* $A + (l_1 + l_2 - k)B$) with eigenvector v. Then a straightforward calculation



(a) case 62 (b) case 53 (c) case 44

Figure 8: Illustrations of patterns of synchrony of infinite classes in 10×10 periodic array.



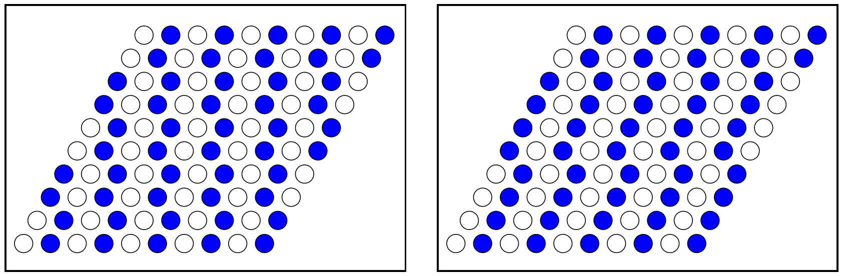
(a) case 63 (b) case 61 (c) case 52 (d) case 43a

(e) case 43b (f) case 42a (g) case 42b (h) case 33 (i) case 22

Figure 9: The repeating patterns associated to the figures in Figure 7.

shows that $(v, v)^t$ (*resp.* $(m_1 v, -m_2 v)^t$) is an eigenvector of J with eigenvalue μ. In fact, the corresponding statement about eigenvalue multiplicity is valid.

Generically, $A + (l_1 + l_2 - k)B$ can have a simple real eigenvalue crossing zero with nonzero speed as λ varies, and with no other eigenvalue of $A + (l_1 + l_2 - k)B$ and $A + kB$ on the imaginary axis at the same value of λ. It then follows from Crandall and Rabinowitz [2] that there exists a unique smooth branch of nontrivial solutions to $f = 0$. Moreover, since the eigenvector $(m_1 v, -m_2 v)$ is not in a synchronous direction, the nontrivial solution satisfies $x_1 \neq x_2$. This means synchrony-breaking bifurcation occurs. $\square$



(a) case 4422 (b) case 4433

Figure 10: The two patterns in this figure and two patterns cases 62 and 44 in Figure 8 are NN-balanced patterns in infinite classes.

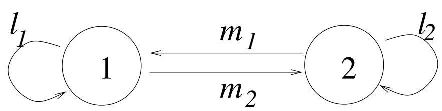
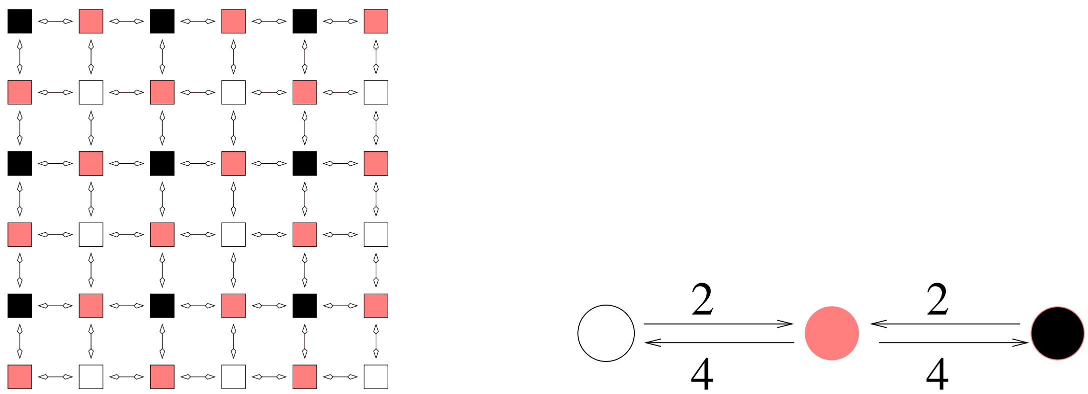


Figure 11: The network of the quotient system.

Theorem 2.1 is an analog of the Equivariant Branching Lemma [6] for balanced two-colorings. We note that the analogue of Theorem 2.1 for balanced three-colorings is only sometimes valid. We present two examples: the first where the analog is valid and the second where it is not.

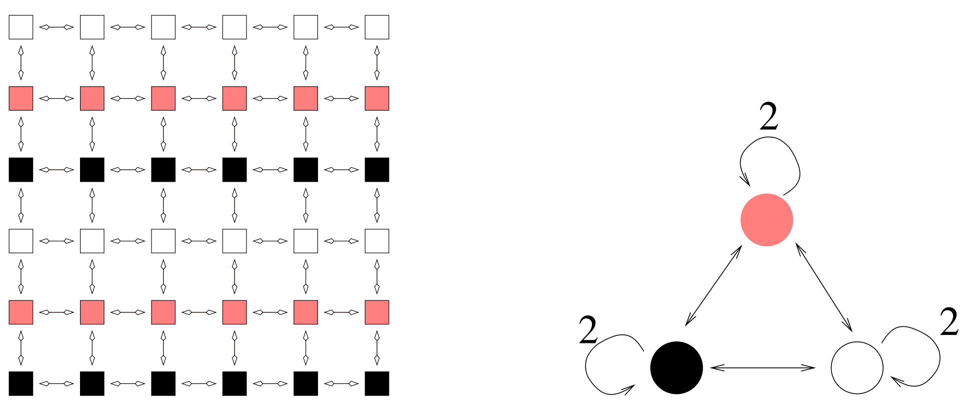
1. The balanced three-coloring Figure 12 (a) has the quotient network Figure 12 (b). Note that the Figure 12 (b) has $\mathbf{Z}_2$-symmetry. $\mathbf{Z}_2$-equivariant bifurcation theory shows that Figure 12 (a) can be expected from codimension one symmetry-breaking bifurcation from a homogeneous equilibrium.

2. The balanced three-coloring Figure 13 (a) has the $\mathbf{D}_3$-symmetric quotient network Figure 13 (b). $\mathbf{D}_3$-equivariant bifurcation theory shows that $\mathbf{D}_3$ symmetry-breaking leads to branches of equilibria with $\mathbf{Z}_2$-symmetry. See [6, p. 14] for details. So Figure 13 (a) cannot be expected from codimension one symmetry-breaking bifurcation from a homogeneous equilibrium.



(a) (b)

Figure 12: Pattern (a) can be obtained from synchrony-bifurcation.



(a) (b)

Figure 13: Pattern (a) cannot be obtained from synchrony-bifurcation.

3 Square lattice with nearest neighbor coupling

In this section we prove Theorem 1.1. Consider an infinite square lattice with bidirectional nearest neighbor coupling (horizontal and vertical only). Our object is to find all balanced two-colorings. Denote the two colors by black and white. We enumerate balanced two-colorings by quotient network. Denote each case by a pair of integers: the number of white cells coupled to black cells and the number of black cells coupled to white cells. We recall that *balanced coloring* means that each black cell receives the same number of inputs from black cells and from white cells. Similarly, each white cell receives the same number of inputs from black cells and from white cells. Since each cell in the array is either white or black and each has four inputs, the possible coloring of the input set of the cell can only be one of the five types listed in Table 1.

black cells	white cells
4	0
3	1
2	2
1	3
0	4

Table 1: Possible coloring of the input set of a cell.

Following our approach, case ij ($0 \leq$ i,j≤ 4) represent the case that each black cell has i white and 4-i black inputs, and each white cell has j black and 4-j white inputs. Note that suppose that the input set of each black cell consists of four black cells, then the balanced planar pattern must consist of black cells. Similarly for white cells. So we do not need to consider these cases. In addition, note that case ij and case ji consist of the same patterns since swapping the colors of all cells of a pattern in case ij obtains a pattern in case ji. So we need only to consider case ij ($1 \leq$ j $\leq$ i ≤ 4). All cases that we need to consider are listed in Table 2.

We define a *local pattern* to be a pattern enclosed by a polygon. We show that all patterns that fill the plane can be obtained by two processes: periodicity and interchange colors along diagonals. We begin by determining, on a case by case basis, the local patterns that can be extended uniquely to the whole plane for finite classes, then prove result for infinite classes.

Finite classes

Case 44 It is straightforward to check the pattern associated to this case. On each horizontal and vertical lines white and black cells alternate. It follows that the checkerboard pictured as Figure 2 case 44 is the only possibility.

Case 43 We will show that no pattern of synchrony exists in this case. Suppose there exists a pattern. Then the pattern must contain the local pattern Figure 14 (b). First, since each black cell is surrounded by four white cells, we must have the local pattern of square cells in

cell of array	Black cell		White cell		Number of pattern(s)
input sets	White	Black	White	Black	
case 44	4	0	0	4	1
case 43	4	0	1	3	0
case 42	4	0	2	2	1
case 41	4	0	3	1	1
case 33	3	1	1	3	1
case 32	3	1	2	2	1
case 31	3	1	3	1	infinity
case 22	2	2	2	2	infinity
case 12	1	3	2	2	2
case 11	1	3	3	1	1

Table 2: Classification of balanced two-colorings. The last numbers are the numbers of patterns in corresponding cases.

Figure 14 (a). We claim that the two cells on positions (1) and (2) in Figure 14 (a) must be black. The reason is that each white cell has 3 black inputs; so the two cells cannot be white. Since the input set of a black cell consists of 4 white cells, we get the local pattern of square cells pictured in Figure 14 (b).

Now we analyze the cell on position (3) in Figure 14 (b). Suppose that cell is black, then the circled white cell in Figure 14 (b) has 4 black inputs. This contradicts the input assumption on white cells. Suppose the cell on position (3) is a white cell, then that white cell has at least 3 white inputs. This contradicts again the input assumption on white cells. So no possible pattern exists in this case.

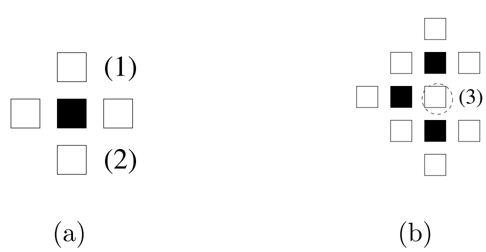


Figure 14: No pattern exists in case 43.

Case 42 We begin with a black cell. By assumption all of the nearest neighbor cells of this cell are white. So the local pattern around the black cell is pictured in Figure 15 (a). The circled white cell in Figure 15 (a) should have two black inputs; one of them is the black cell in Figure 15 (a). The other black cell can be any of the other three nearest neighbors. The possible local patterns are pictured in Figure 15 (I), (II), (III), which we consider in order. We show that Figure 15 (I), (II) determine the same pattern uniquely. Figure 15 (III) is impossible.

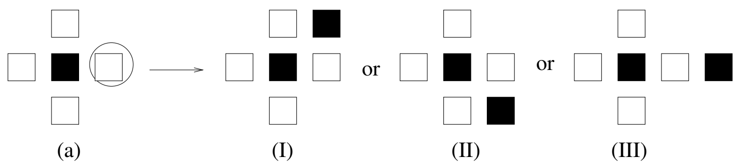


Figure 15: (Case 42) The possible local patterns.

I By assumption the black cell on the upper right of Figure 15 (I) has four white inputs, so Figure 15 (I) determines uniquely Figure 16 (a). The circled white cell in Figure 16 (a) already has two black inputs, so the other two neighbor cells must be white (since each white cell has two white and two black inputs). The pattern is pictured in Figure 16 (b). Again by assumption on input sets of circled cells, Figure 16 (b) determines Figure 16 (c), and Figure 16 (c) determines in Figure 16 (d).

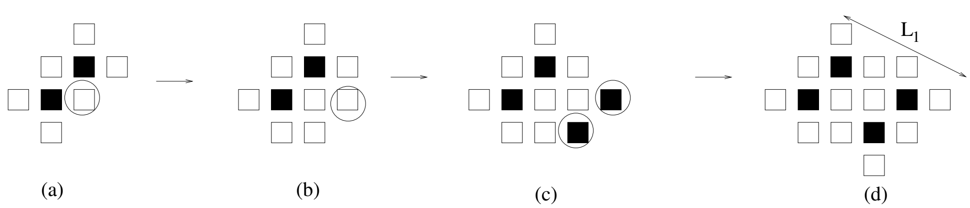


Figure 16: (Case 42) Pattern (a) is repeated in direction L_1.

Now observe that Figure 16 (d) consists of two copies of Figure 16 (a). Thus Figure 16 (a) can be repeated periodically along direction L_1 (both down to the right and up to the left).

Now show that the stripe, obtained by repeating Figure 16 (a) along line L_1, can be extended to the whole plane. By assumption on the circled cells of Figure 17 (a), Figure 17 (a) determines Figure 17 (b). We can see that the local pattern enclosed by the dashed polygon can be repeated periodically along direction L_2. Therefore Figure 15 (I) determines the pattern as in Figure 2 case 42.

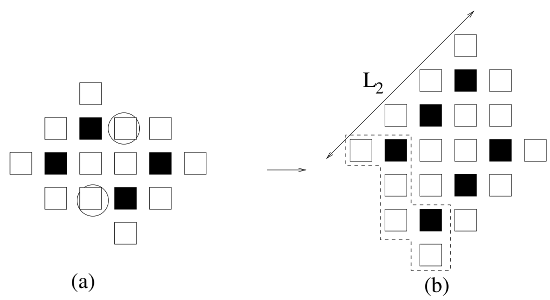


Figure 17: (Case 42) The strip is repeated in direction L_2.

II Observe that Figure 15 (II) can be obtained by reflecting Figure 15 (I) about the horizontal midline. So Figure 15 (II) also determines Figure 2 case 42.

III Now we show that Figure 15 (III) is impossible. Since the black cell on the right side of Figure 15 (III) has four white cell inputs, we have the pattern as in Figure 18. It is easy to observe that the circled cell cannot be black nor white. Either way causes a contradiction with the assumption on input sets. So this pattern cannot exist.

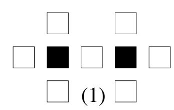


Figure 18: (Case 42) Impossible local pattern.

Case 41 Up to rotation, the local pattern around a white cell is pictured as Figure 19 (a). By assumption on the input set of the circled black cell, Figure 19 (a) determines Figure 19 (b) since black cells have four white inputs. Figure 19 (b) determines Figure 19 (c) uniquely since white cells have only one black neighbor. Now consider the circled white cell in Figure 19 (c). It must have one black input and that cell will be in either position (1) or (2). Observe that positions (1) and (2) are symmetric about the line in Figure 19 (c). So, without loss of generality, we can put a black cell in position (2) and a white cell in position (1) arriving at Figure 19 (d). We claim that the local pattern enclosed by the dashed polygon in Figure 19 (d) determines the planar pattern obtained by repeating Figure 19 (a) periodically in two different directions.

By assumption on the input sets of the circled cells, we can see that Figure 20 (a) determines Figure 20 (d). That means that Figure 20 (a) determines the whole strip that is obtained by repeating Figure 19 (a) in the direction of line L.

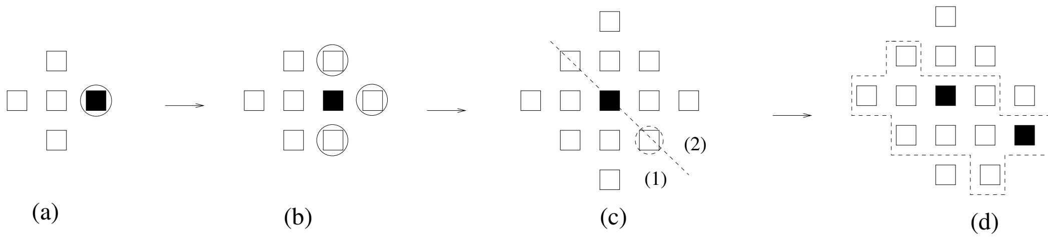


Figure 19: (Case 41) The possible local patterns.

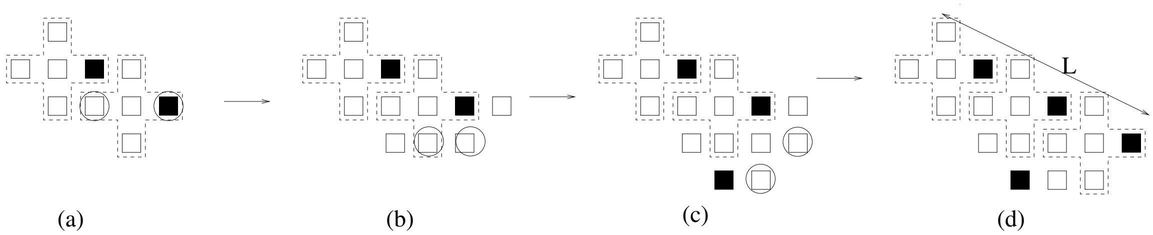


Figure 20: (Case 41) Pattern (a) determines a strip.

In addition, by assumption on the input sets of circled cells in Figure 21, we can see that Figure 21 (a) determines Figure 21 (d) uniquely. Thus, the strip can be extended in the direction of line L_1 in Figure 21 (d). So Figure 20 (a) determines the planar pattern pictured as Figure 2 Case 41.

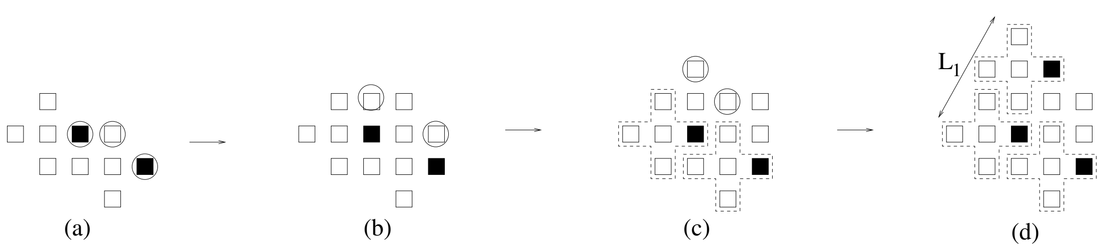


Figure 21: (Case 41) The strip determines the whole planar pattern.

Case 33 Up to rotation, we may assume that the local pattern of a black cell is pictured as Figure 22 (a). Figure 22 (b) follows from Figure 22 (a), since black cells are surrounded by three white cells. By assumption on the input sets of the circled cells in Figure 22 (b), this pattern determines Figure 22 (c). We claim that the planar pattern is determined uniquely by the local rectangle enclosed by the dashed polygon in Figure 22 (c). It is sufficient to show that this pattern can repeat itself in vertical and horizontal directions, which we now prove.

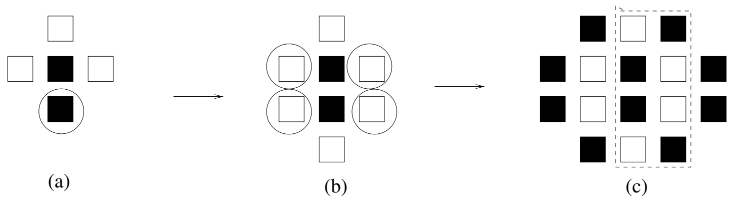


Figure 22: (Case 33) The local pattern enclosed by dashes determines a planar pattern.

In Figure 23, by assumption on the input sets of the circled cells, each pattern determines the next one uniquely. Since Figure 23 (a) determines Figure 23 (d), we can see that

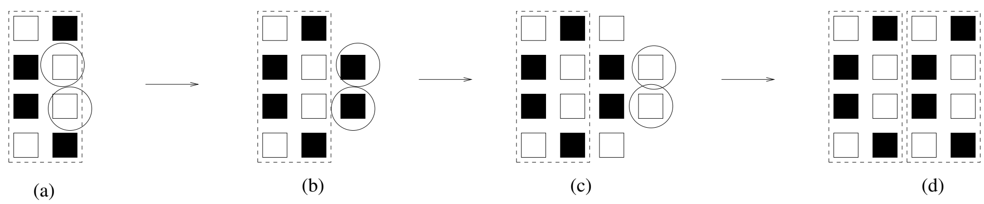


Figure 23: (Case 33) Pattern (a) repeats itself in horizontal direction.

Figure 23 (a) can be extended to the whole horizontal strip.

On the other hand, we see that Figure 24 (a) determines Figure 24 (d). Thus, the horizontal strip can be extended to the whole plane by periodically repeating the basic pattern vertically. So there is only one possible planar pattern, and it is the one pictured in Figure 2 Case 33.

Case 32 By rotation we may assume that the local pattern around a black cell is pictured as Figure 25 (a). By assumption on input set of the circled black cell, Figure 25 (a) determines

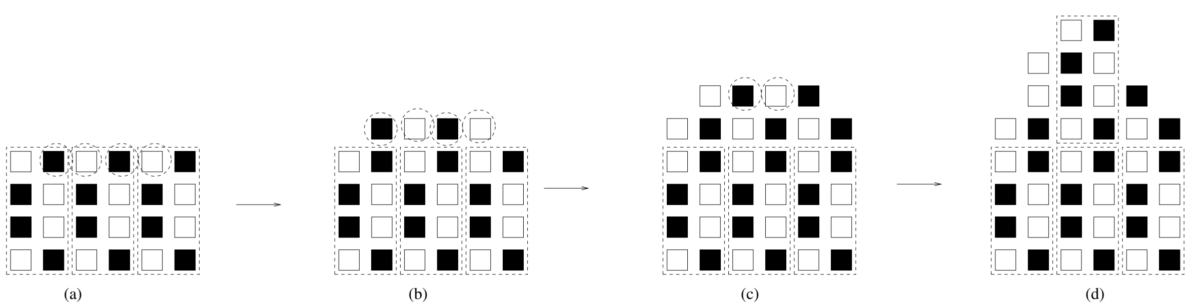


Figure 24: (Case 33) The horizontal strip repeats itself in vertical direction.

Figure 25 (b). Now consider the circled white cell in Figure 25 (b). The white cell already has one black inputs, so the other black input cell must be one of the other two neighbors. The possible local patterns are Figure 25 (I, II), which we will consider in order. We show that Figure 25 (I, II) determine the same planar pattern up to symmetry.

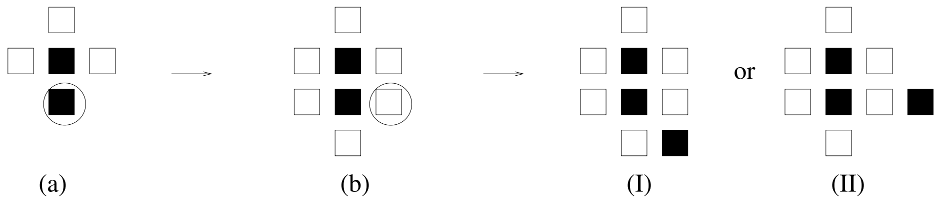


Figure 25: (Case 32) Possible local patterns.

I In Figure 26, by assumption on the input sets of the circled cells, each pattern determines the next one uniquely. We claim that the local pattern, enclosed by the dashed polygon in Figure 26 (d), determines the planar pattern.

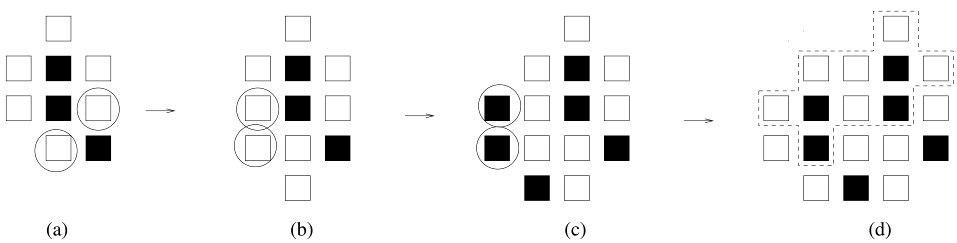


Figure 26: (Case 32) Local pattern around Figure 25 (I).

Note that the region enclosed by dashes in Figure 26 (d) is just two copies of Figure 25 (a). This is not sufficient information to show that Figure 25 (a) can be repeated periodically in the direction L, since we arrived at Figure 26 (d) by using assumption (I). We now assert that the dashed region can be extended uniquely to three copies of Figure 25 (a) and hence to the whole strip. By assumption on input sets of circled cells, we see that Figure 27 (a) determines Figure 27 (d), which proves our assertion.

Now we need to see whether this strip can be extended to the whole plane uniquely. Observe that Figure 28 (a) determines Figure 28 (d) uniquely. Thus, the strip can be extended to the whole plane and we obtain Figure 2 Case 32 .

II By assumption on input sets of circled cells, Figure 29 (a) determines Figure 29 (c). Observe that flipping Figure 29 (c) is Figure 27 (b). So the planar pattern in this case is same as the pattern found in I.

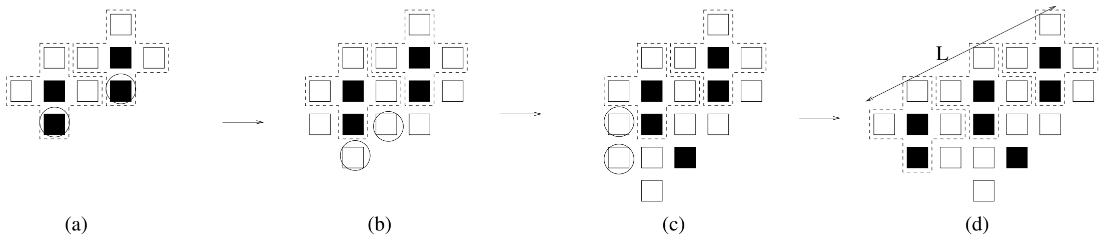


Figure 27: (Case 32) Pattern (a) determines a strip.

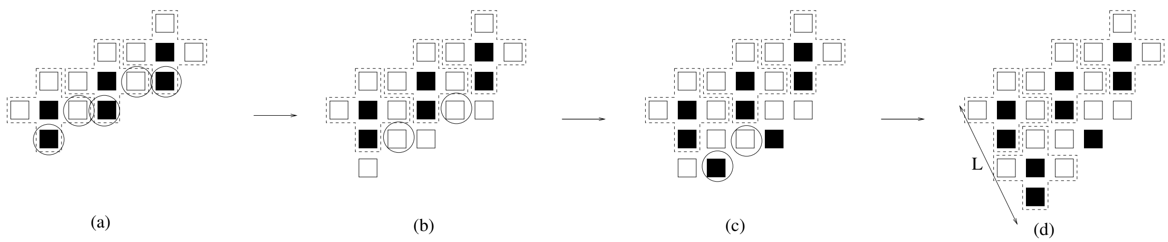


Figure 28: (Case 32) The strip can be extended to the whole plane.

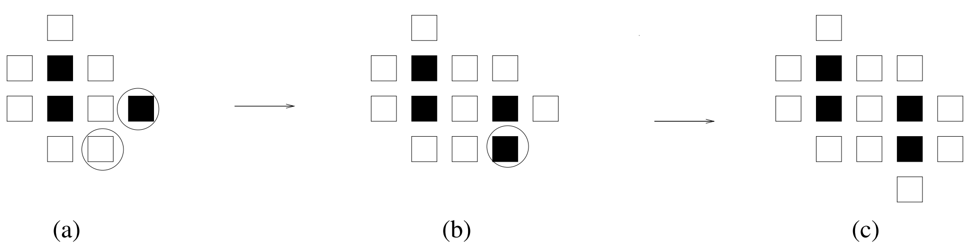


Figure 29: (Case 32) Pattern (a) determines the same pattern as the pattern obtained in (I).

Case 12 We can assume that a black cell is surround by its input cells as in Figure 30 (a). By assumption on the input set of the circled white cell in Figure 30 (a), the possible local patterns are as in Figure 30 (I, II, III). Observe that by reflecting Figure 30 (II), we get Figure 30 (III). So we need only study Figure 30 (I, II) which we consider in order.

I We claim that the pattern enclosed by the dashed polygon in Figure 30 (I) determines a planar pattern uniquely. By assumption on the input sets of circled cells, Figure 31 (a) determines Figure 31 (b). Thus, Figure 31 (a) determines the horizontal strip formed by repeating Figure 31 (a) periodically in the horizontal direction.

Note that Figure 32 (a) determines Figure 32 (d). So the horizontal strip can be extended to the plane by periodic repetition in the vertical direction and we obtain Fig-

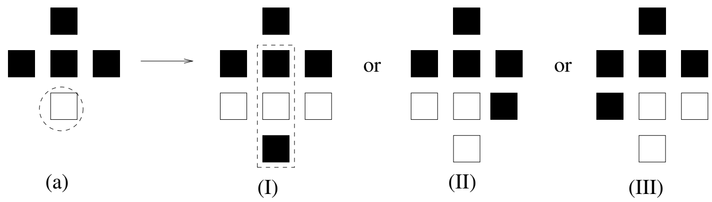


Figure 30: (Case 12) Possible local patterns.

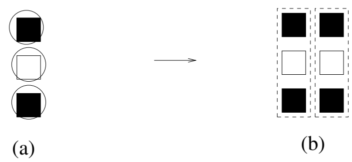


Figure 31: (Case 12) Figure 30 (I) determines a pattern.

ure 2 case 12b.

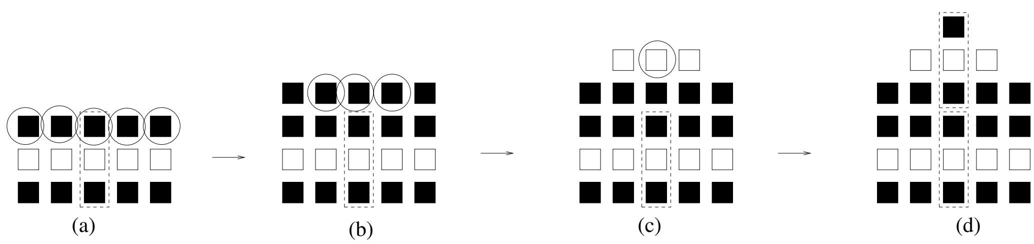


Figure 32: (Case 12) The horizontal strip can be extended to the whole space.

II Now consider Figure 30 (II) (Figure 33 (a)). We claim that cell 1 in Figure 33 (a) must be white. Suppose cell 1 is black, then the black cell at least has two white inputs. This contradicts the assumption on the input set of the black cell. By assumption on input sets of the circled cells, Figure 33 (a) determines Figure 33 (c). Now consider the circled black cell in Figure 33 (c), it already has two black cell inputs in Figure 33 (c). Since a black cell must have three black inputs, the third black input of the circled cell can be on position (1) or (2). However these two positions are symmetric about line L. Thus, without loss of generality, we can put a black cell in position (1) and a white cell in position (2) arriving at Figure 34 (a).

It is easy to check that Figure 34 (a) determines Figure 34 (c). We claim that the pattern enclosed by the dashed polygon in Figure 34 (c) determines a planar pattern uniquely.

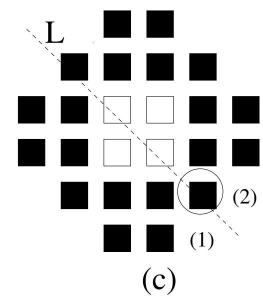


Figure 33: (Case 12) Figure 30 (II) determines a pattern.

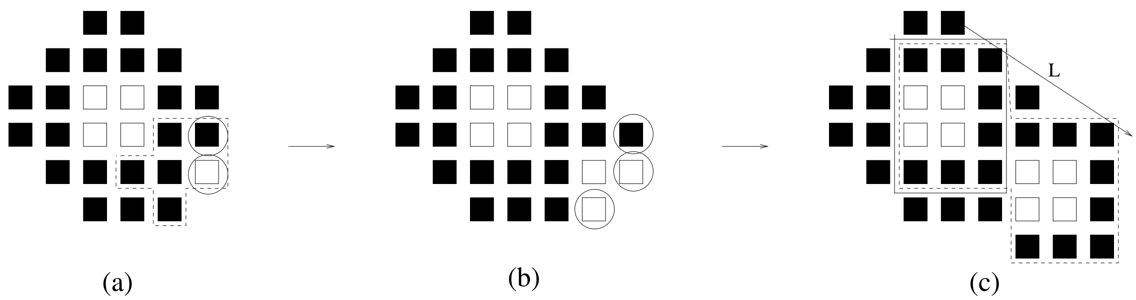


Figure 34: (Case 12) The local pattern enclosed by dashes in (c) determines a pattern.

In Figure 35 and Figure 36, by assumption on input sets of circled cells and cells in the dashed polygons, it is easy to see that each pattern determines the next pattern. Thus, Figure 35 (a) determines Figure 36 (e). Thus, Figure 35 (a) determines a pattern obtained by repeating periodically the pattern, enclosed by the solid polygon in Figure 34 (c), in the direction L_1 and vertical direction. We obtain Figure 2 case 12b.

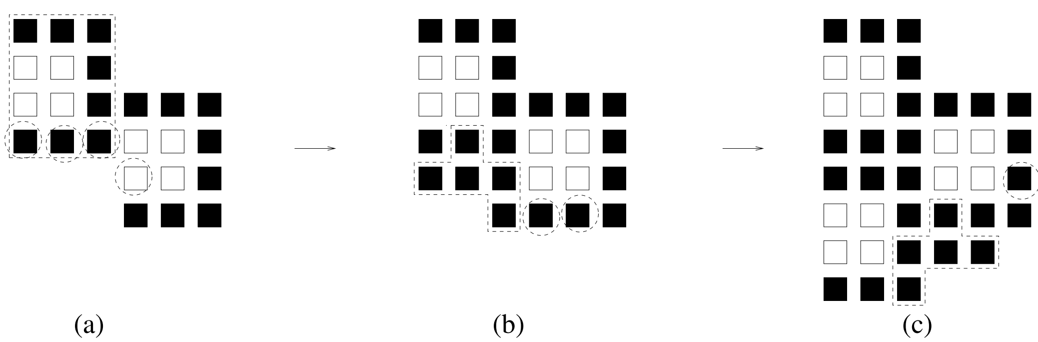


Figure 35: (Case 12) Pattern (a) determines Figure 36 (d).

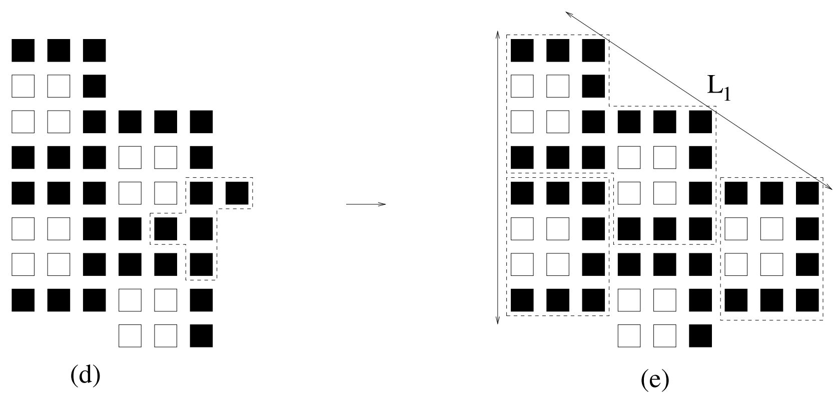


Figure 36: (Case 12) Figure 35 (a) determines pattern (d).

Case 11 After rotation we may assume the local pattern around a black cell is Figure 37 (a). By assumption on input set of a white cell, Figure 37 (a) determines Figure 37 (b). We claim that the local pattern enclosed by dashes determines the planar pattern of this case.

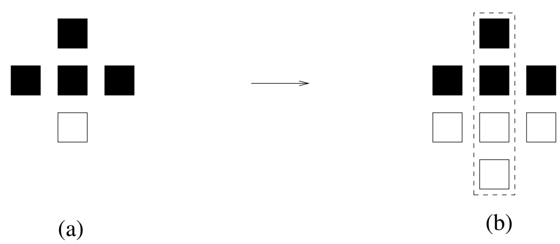


Figure 37: (Case 11) The local pattern enclosed by dashes determines a planar pattern.

By assumption on input sets of the circled cells, Figure 38 (a) determines Figure 38 (c). Thus, Figure 38 (a) determines the horizontal strip that is obtained by repeating Figure 38 (a) periodically in the horizontal direction.

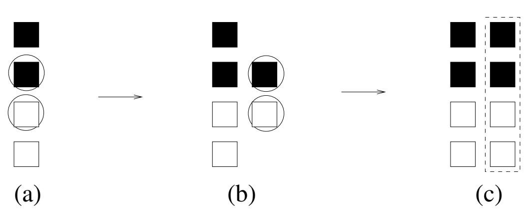


Figure 38: (Case 11) Pattern (a) determines a horizontal strip.

Observe that Figure 39 (a) determines Figure 39 (c). So the horizontal strip can be extended to the whole plane by repeating itself in the vertical direction and we obtain Figure 2 case 11.

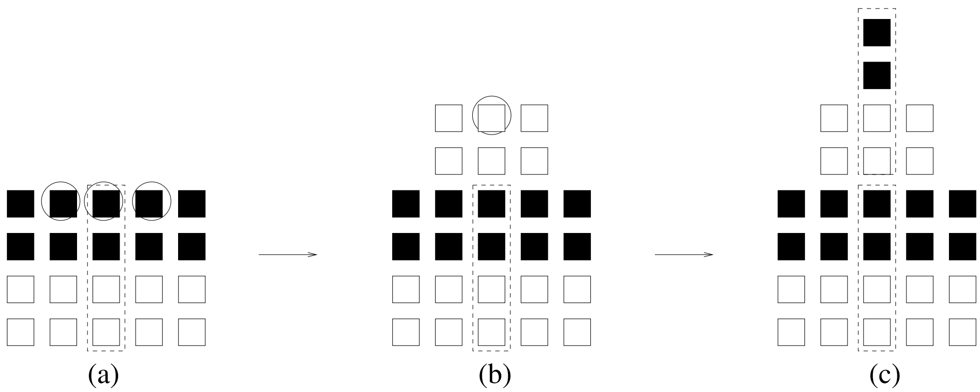


Figure 39: (Case 11) The horizontal strip determines the planar pattern.

Infinite classes

By a *diagonal* we mean a slope ± 1 line of cells in the square lattice. We define an *alternating diagonal* to be a diagonal on which white cells and black cells alternate. Note that diagonals are either parallel or perpendicular. In the remaining two cases, Case 22 and Case 31, cells of both colors have the same input sets. The two cases share a property that is described in [4] and proved here.

Lemma 3.1 *Interchanging color along an alternating diagonal of a balanced pattern gives a new balanced pattern.*

Proof A cell c influences the balanced relation of a pattern in two ways. First, cell c has an input set. Second, c is in the input set of its nearest neighbors. Since black cells and white cells have the same input sets, we will not change the balanced relation in the first way if we interchange color along an alternating diagonal. So we need only show that interchanging color along an alternating diagonal also does not change the balanced relation of the pattern in the second way. In Figure 40, we see that a cell in an alternating diagonal L only influences the cells in two diagonals L_1 and L_2. The alternating diagonal L supplies one white and one black cell for every cell in L_1 or L_2. When we interchange color of L and arrive at L', L' still is alternating and supplies every cell in L_1 or L_2 with one white and one black input. Thus, interchanging color of an alternating diagonal does not change the balanced relation. □

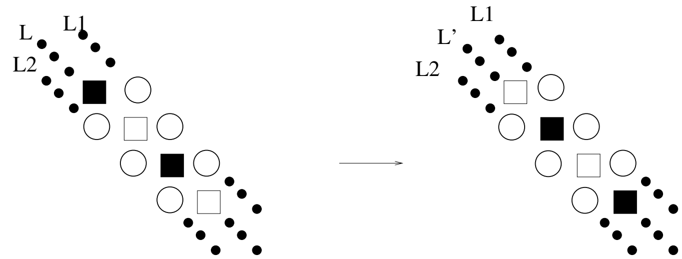


Figure 40: The common property of the two infinite cases.

Case 22 We prove that each pattern in Case 22 can be transformed into Figure 3 (a), a pattern that consists of alternating black and white horizontal stripes, by interchanging black and white along a (perhaps infinite) set of alternating diagonals.

We first verify two properties concerning patterns in Case 22.

1. If the diagonal L in Figure 41 is alternating, then the parallel diagonal L_1, two squares to the left of L, is also alternating.

   Reason Suppose L_1 is not alternating. Then two consecutive cells a and b in L_1 have the same color. Without loss of generality, assume they are white. Next, consider cell c in Figure 41, whose four neighboring cells are a, b, and two cells in line L. Cell c has one black and three white inputs. This contradicts the assumption on input sets of cells. Thus, L_1 is an alternating diagonal. □

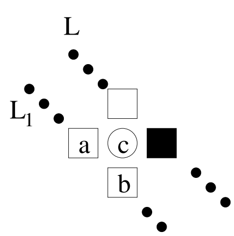


Figure 41: (Case 22) L_1 must be alternating.

   Property 1 implies: if a diagonal is alternating, then every second parallel diagonal is also alternating.

2. If a diagonal L in Figure 42 (a) is not alternating, then there is a pair of consecutive perpendicular alternating diagonals through a pair of consecutive cells on L.

Reason Since L is not alternating, there are two consecutive cells a and b on L that have the same color. Without loss of generality, assume the two cells are white. See Figure 42 (a).

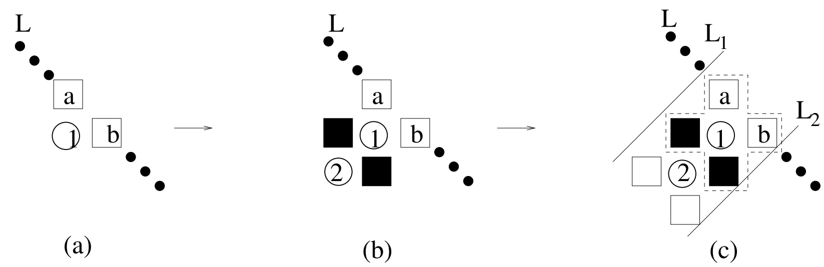


Figure 42: (Case 22) L is not alternating, then L_1 and L_2 must be alternating.

By assumption on the input sets of the cells in the numbered positions, we see that Figure 42 (a) determines Figure 42 (c). Hence, the pattern enclosed by the dashed polygon in Figure 42 (c), not including cell 1, is repeated periodically in the perpendicular direction L_1 to L. So the pair of perpendicular diagonals through a and b must be alternating. □

Next, we use these properties to transform the pattern to Figure 3 (a) by interchanging black and white along diagonals.

By Property 2, there must be an alternating diagonal. Without loss of generality, assume that alternating diagonal is the diagonal L in Figure 43. By Property 1 and its remark, beginning from L, every other diagonal must be alternating. By Lemma 3.1, we can interchange color along some of the diagonals so that all of those alternating diagonals are as in Figure 43.

Next, consider the diagonal L_1 marked by the line in Figure 43. If L_1 is alternating, then every second diagonal beginning from L_1 is alternating. If L_1 is not alternating, then Property 2 implies that there is a perpendicular alternating diagonal L_2 through a cell in L_1. By Property 1, beginning from L_2, every other diagonal is alternating. Finally, proceed as in the first half of the argument to transform the pattern to the desired striped pattern.

Case 31 We prove that any pattern in Case 31 can be transformed into Figure 3 (b) by interchanging black and white along a (perhaps infinite) set of alternating diagonals.

We first verify two properties concerning patterns in Case 31. For convenience, we call a diagonal that consists only of white cells a *white diagonal.*

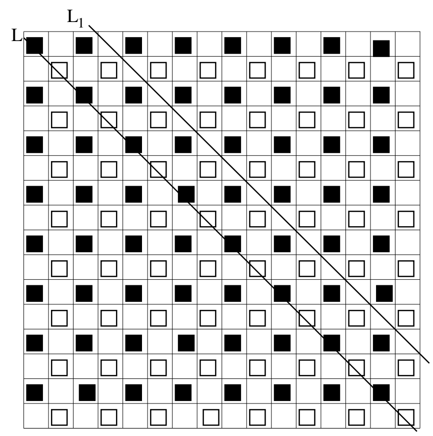


Figure 43: (Case 22) Generating process.

1. Suppose the diagonal L in Figure 44 is an alternating diagonal. Then, alternating diagonals and white diagonals alternate every second diagonal.

   Reason Since every cell in the diagonal P_1 in Figure 44 (a) has one white input and one black input from L, the other two inputs must be white. Therefore, the diagonal two squares to the right of L must be a white diagonal. See Figure 44 (b). The assumption on the input sets of the cells on diagonal P_2 in Figure 44 (b) implies that the diagonal L_2 (two squares to the right of L_1) must be alternating. Thus, white diagonals and alternating diagonals alternate in the perpendicular direction to L. □

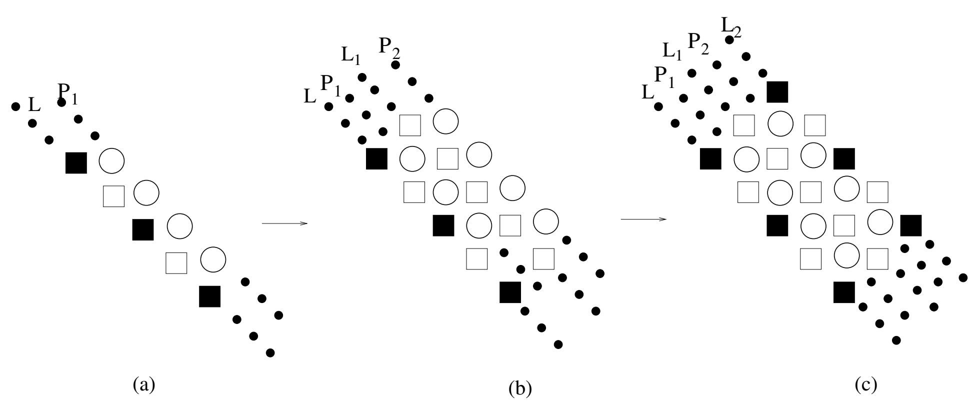


Figure 44: (Case 31) Alternating diagonal and white diagonal alternate every second diagonal.

Similarly, if L is a white diagonal, then the result is also true.

2. If the diagonal L is neither alternating nor white, then there exists a perpendicular alternating diagonal through a cell on L.

   Reason We may assume, after rotation, that L has slope -1. Since L is neither alternating nor white, there exists a segment on L pictured in Figure 45 (a). (Note that the consecutive same color cells must be white.) Now consider the cell numbered 1. It has one white and one black input supplied by L, so the other two inputs must be white; and we arrive at Figure 45 (b).

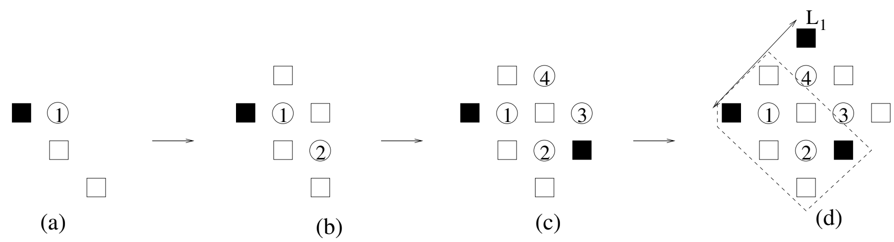
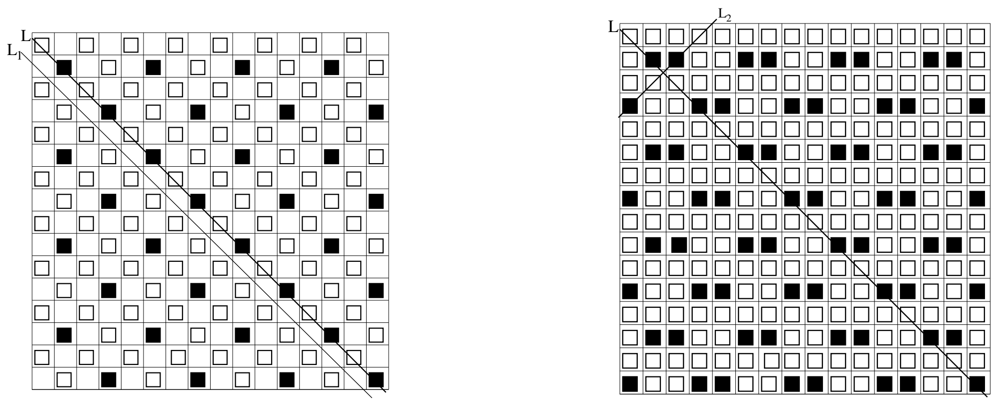


Figure 45: (Case 31) The existence of alternating diagonal.

   Cell 2 in Figure 45 (b) already has three white inputs shown in Figure 45 (b), therefore the last input must be black. So Figure 45 (b) determines Figure 45 (c). Similarly, Figure 45 (c) determines Figure 45 (d) by assumption of the inputs sets of Cells 3 and 4. Observing Figure 45 (d), we can see that the local pattern in Figure 45 (d) enclosed by the dashed polygon, not including Cells 1 and 2, is repeated periodically along the line L_1. So the diagonals through the black cells in Figure 45 (d) along the direction of L_1 must be alternating. □

Next, we use these properties to transform the pattern to Figure 3 (b) by interchanging black and white along diagonals. By Property 2, there must be an alternating diagonal. Without loss of generality, assume that alternating diagonal is the diagonal L in Figure 46. By Property 1 and its remark, beginning from L, alternating diagonals and white diagonals alternate every other diagonal. By Lemma 3.1, we can interchange color along some of the diagonals so that those alternating diagonals are as in Figure 46.

Next, consider the diagonal L_1 in Figure 46 (a). If L_1 is alternating or white, proceed as the first half of the argument to transform the pattern to the desired pattern. If L_1 is neither white nor alternating, by Property 2, there is a perpendicular alternating diagonal L_2 through a cell in L_1. Without loss of generality, assume L_2 as in Figure 46 (b). By the Lemma 3.1, we also can transform the original pattern to Figure 3 (b).



(a) (b)

Figure 46: (Case 31) All patterns in case 31 can be transformed to the pattern (b).

4 Square lattice with nearest and next nearest neighbor coupling

In this section, we prove Theorem 1.2. The patterns of synchrony of square lattices with nearest and next nearest neighbor coupling are the patterns in Theorem 1.1 which are also balanced with next nearest neighbor coupling. We say that a pattern in Theorem 1.1 is *NN-balanced* if it is balanced with nearest and next nearest neighbor coupling. Thus, in order to prove Theorem 1.2, we need only to check which patterns in Theorem 1.1 are balanced with the additional coupling. It is easy to check that all patterns in Figure 2, except for pattern (f), are NN-balanced. We use the phrase *NN-inputs of a cell* to mean the next nearest neighbors of the cell.

To prove Theorem 1.2, we need to determine which balanced patterns associated to Cases 22 and 33 are NN-balanced. We prove that there are precisely five patterns in Case 22 are NN-balanced, and that none in Case 31 that are NN-balanced.

Case 22 Begin with the white cell c. We showed in Section 3 that at least one diagonal through c is alternating. Therefore, we need to consider two cases: both diagonals are alternating; only one of the two diagonals is alternating.

I Suppose both diagonals through c are alternating as pictured in Figure 47 (a). By Property 1 of Section 3 Case 22, the half pattern that contain c must be Figure 47 (b).

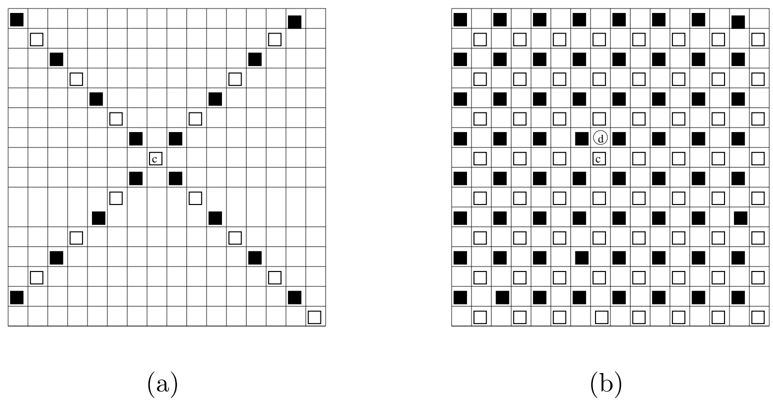


Figure 47: If both diagonals through c are alternating, there only one possible half pattern.

In Figure 47 (b), observe that each black cell has four white NN-inputs and each white cell has four black NN-inputs. So all NN-balanced patterns containing (b) must satisfy the same conditions on the NN-input sets. Therefore, if cell d in Figure 47 (b) is black, then the planar pattern is Figure 3 (a); if d is white, then the planar pattern is the pattern by rotating Figure 3 (a) with 90°.

II Suppose one diagonal L through c is not alternating. It follows that there are two adjacent cells c_1, c_2 in L that have the same color. Without loss of generality, assume c_1, c_2 are white. By Property 1 in Section 3 Case 22, the two perpendicular diagonals L_1, L_2 through c_1, c_2 respectively must be alternating.

Next, we consider Cell c_3 which can be either white and black. Each of these two choice determines uniquely a half pattern as we show now.

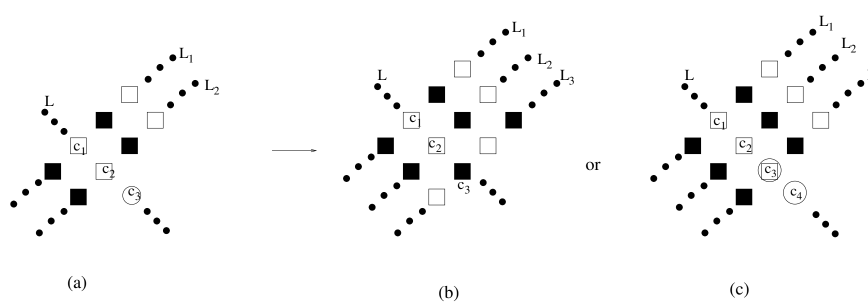
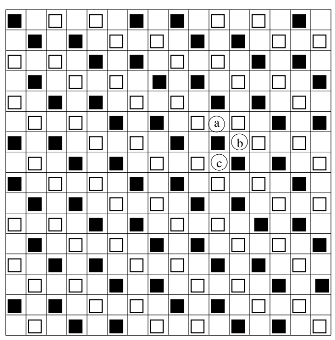
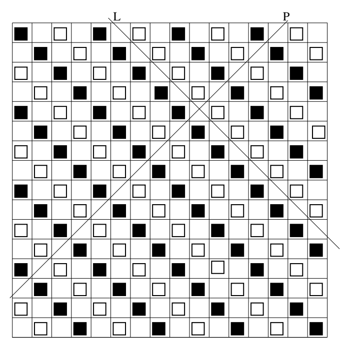


Figure 48: If one diagonal is not alternating, then there only two possible half patterns.

1. If c_3 is black, by Property 1 again, the alternating diagonal through c_3 must be pictured as L_3 in Figure 48 (b). From Figure 48 (b), we can see that all NN-balanced patterns containing Figure 48 (b) must satisfy that each black cell has one black and three white NN-inputs, and each white cell have one white and three black NN-inputs. With the same argument as Section 3 Case 33, the possible half pattern containing Figure 48 (b) can only be Figure 49 (a) up to symmetry. By assumption on the input sets, there must exist two adjacent black cell in the second half pattern. In Figure 49 (a), if cells a and b in Figure 49 (a) are black, then we get Figure 6 (a); if cells b and c are black, then we get Figure 6 (b). It is easy to check that up to symmetry, these two patterns are all the possible patterns with half patterns as pictured in Figure 49 (a).



(a)



(b)

Figure 49: Figures 48 (b) and (c) determine uniquely a half pattern respectively.

2. If c_3 is white, then the alternating diagonal L_3 through c_3 must be as pictured in Figure 48 (c). It follows that each cell in all NN-balanced patterns must have two white and two black NN-inputs. Next, consider cell c_3 in Figure 48 (c). That cell already has two black NN-inputs, so cell c_4 must be white. It follows that L is a white diagonal. By Property 1 in Section 3 Case 22 again, the NN-balanced half pattern containing (c) is formed by alternating white diagonals and black diagonals every second diagonal. See Figure 49 (b).

   According to the arguments above in this section, we can see that the second half pattern must also be formed by alternating white diagonals and black diagonals every second diagonal. Otherwise, the second half pattern cannot keep the same NN-input sets condition. In Figure 49 (b), if the diagonal P is black, then we get

Figure 6 (c); if the diagonal L is black, then we get Figure 6 (d). It is easy to check that up to symmetry, these are only two possible patterns containing Figure 49 (a).

Case 31 In this case, we claim that no pattern is NN-balanced. Recall that each pattern in the infinite family consists of two half patterns. Each half pattern is formed from repeating a pair of diagonals: one white and one alternating. So all patterns in this case include the pattern in Figure 50. Note that the white cell a has two white and two black NN-inputs, while the white cell b has at least three white NN-inputs. This implies no NN-balanced pattern exists in the infinite family.

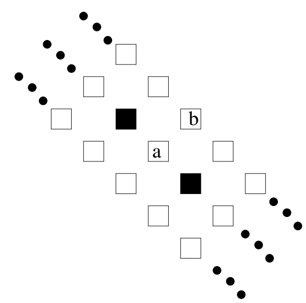


Figure 50: No pattern in the infinite family is NN-balanced.

5 Hexagonal lattice with nearest neighbor coupling

In this section, we prove Theorem 1.3. proceeding as in Section 3. We enumerate balanced two-colorings by a pair of integers: the number of white cells coupled to black cells and the number of black cells coupled to white cells. For hexagonal lattice, the possible coloring of input sets of cells are listed in Table 3.

Following our approach, case ij ($0 \leq$ i,j≤ 6) represents the case that each black cell has i white and 6-i black inputs, and each white cell has j black and 6-j white inputs. Note that case ij and case ji consist of the same patterns since swapping the colors of all cells in a pattern of case ij obtains a pattern in case ji. In addition, the only possible patterns of case i0 ($0 \leq$i≤ 6) is the pattern consisting of white cells since all input cell of white cells are white. Thus, we need only to consider case ij ($0 <$ j $\leq$ i≤ 6) that are listed in Table 4.

We show that all planar patterns except one can be obtained using two ideas: periodicity and interchange of color along lines of cells. We begin by determining, on a case by case basis, the local patterns that can be extended uniquely to the planar patterns for each finite class.

black cells	white cells
6	0
5	1
4	2
3	3
2	4
1	5
0	6

Table 3: Possible coloring of the input set of a cell.

Then prove the infinite classes. In this section, we use circles represent the known cells and squares represent the cells just added.

5.1 Finite classes

Cases 66, 65, 64 First, we prove that no pattern exists in case 66 and case 65. Each two-color pattern of synchrony in case 6j ($0 \leq j \leq 6$) must contain the pattern in Figure 51 (a). Note that each white cell in the pattern has at least two white inputs. Therefore, no pattern exists in case 66 and case 65. Next prove that no pattern exists in case 64. Suppose there exists a

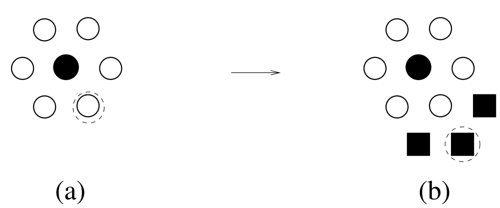


Figure 51: (Case 6j) No pattern exists in Cases 66, 65, 64.

pattern. Then the pattern must contain Figure 51 (a). By assumption on the input set of the circled white cell, Figure 51 (a) determines Figure 51 (b). Note that the circled black cell in (b) has at least two black inputs. This contradicts the assumption that the black cell has no black input. So no pattern exists in case 64.

Case 63 We begin with the local pattern Figure 52 (a). Consider the circled white cell c. Since c already has one black and two white inputs, the other three neighbors must be one white and two black. The possible local patterns are shown in Figure 52 (I, II, III). Since each

cell of array	Black cell		White cell		Number of patterns
input sets	White	Black	White	Black	
case 66	6	0	0	6	0
case 65	6	0	1	5	0
case 64	6	0	2	4	0
case 63	6	0	3	3	1
case 62	6	0	4	2	infinity
case 61	6	0	5	1	1
case 55	5	1	1	5	0
case 54	5	1	2	4	0
case 53	5	1	3	3	infinity
case 52	5	1	4	6	1
case 51	5	1	5	1	0
case 44	4	2	2	4	infinity
case 43	4	2	3	3	2
case 42	4	2	4	2	2
case 41	4	2	5	1	0
case 33	3	3	3	3	1
case 32	3	3	4	2	0
case 31	3	3	5	1	0
case 22	4	2	4	2	1
case 21	2	4	5	1	0
case 11	1	3	3	1	0

Table 4: Classification of balanced two-colorings. The last numbers are the numbers of patterns in corresponding cases.

black cell has no black input, patterns (II, III) contradict the assumption. So we need only to consider pattern (I). We show that pattern (I) determines uniquely a planar pattern, which is obtained by repeating the local pattern enclosed by the solid polygon along two different directions.

Again by assumption on the input sets of the circled cells, Figure 53 (a) determines Figure 53 (c). Thus, Figure 53 (a) determines the whole strip that is obtained by repeating pattern (a) periodically in the direction L.

Next, we prove that the strip determines uniquely a planar pattern. Observe that Figure 54 (a) determines Figure 54 (c). Thus, the strip can be extended to the whole plane by

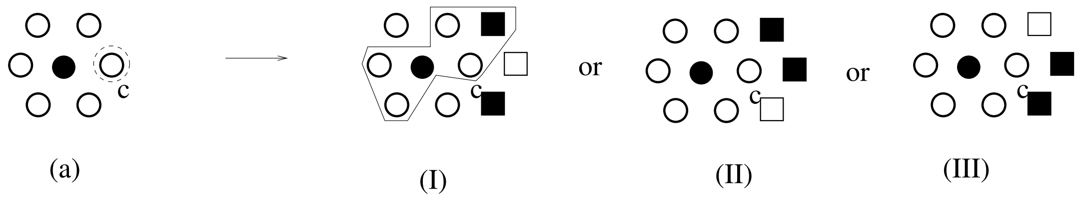


Figure 52: (Case 63) Possible local patterns.

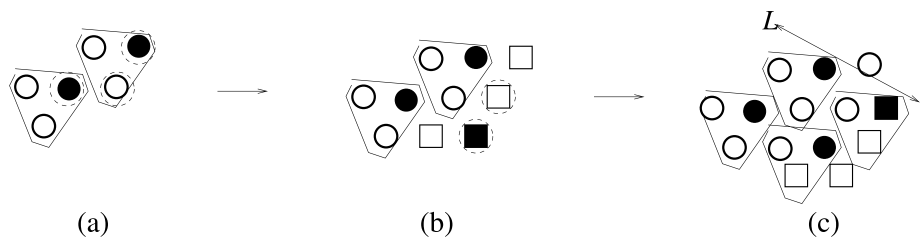


Figure 53: (Case 63) Pattern (a) repeats itself in the direction L.

repeating itself periodically in the direction L_1. Then we obtain Figure 7 (a).

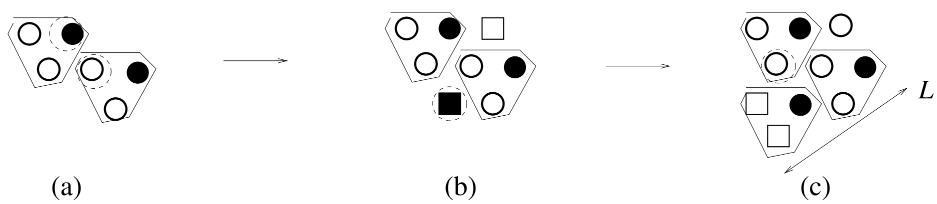


Figure 54: (Case 63) The strip repeats itself in the direction L_1.

Case 61 Begin with Figure 55 (a). By assumption on the input set of the circled cell, Figure 55 (a) determines Figure 55 (b) . Consider the circled white cell in Figure 55 (b). The white cell already has four white inputs, so the other two neighboring cells must be black and white. The possible local patterns are shown in Figure 55 (I, II). Observe that flipping Figure 55 (I) about L obtains Figure 55 (II). So we need only to consider Figure 55 (I).

By assumption on the input set of the circled black cell, Figure 56 (a) determines Figure 56 (b). We claim that the local pattern enclosed by the polygon in Figure 56 (b) determines a planar pattern.

By assumption on the input sets of the circled cells, Figure 57 (a) determines Figure 57 (d). Thus, Figure 57 (a) determines the strip formed by repeating itself in the direction L.

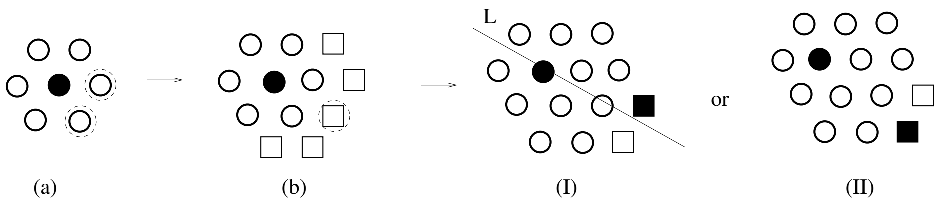


Figure 55: (Case 61) Possible local patterns.

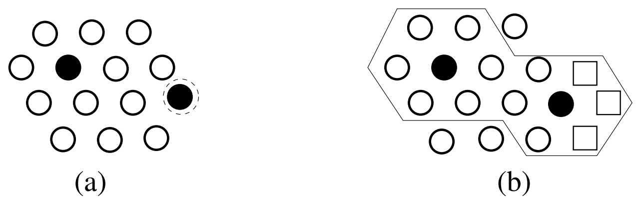


Figure 56: (Case 61) Pattern (a) determines a planar pattern.

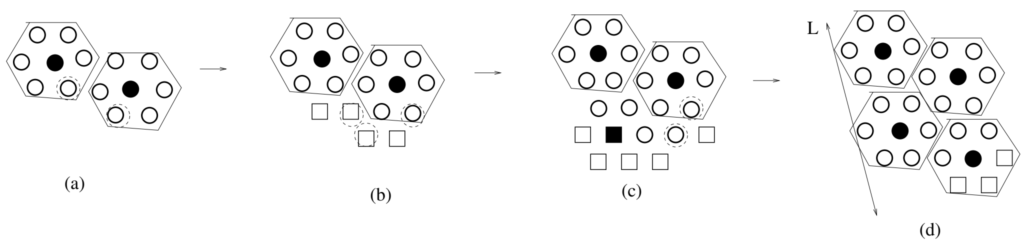


Figure 57: (Case 61) The local pattern determines a strip.

By assumption on the input sets of the circled cells, Figure 58 (a) determines Figure 58 (d). Thus, the strip determines the planar pattern obtained by repeating itself in the direction L. The planar pattern is shown in Figure 7 (b). □

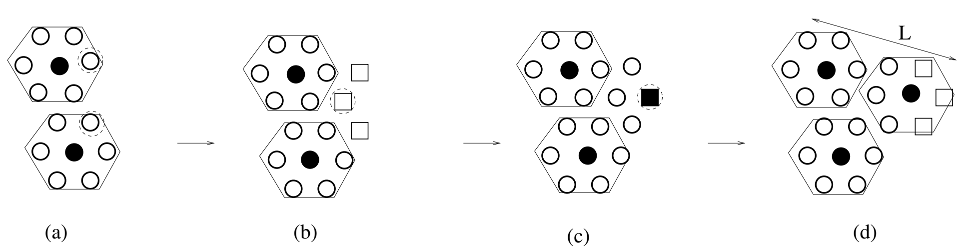


Figure 58: (Case 61) The strip determines the planar pattern.

Cases 55, 54, 51 First, we prove that no pattern exists in Case 55, Case 51. By assumption on the input sets of black cells, up to symmetry, all patterns in Case 5j ($0 < j \leq 5$) contain Figure 59 (a). Note that the circled white cell 1 in the pattern (a) has at least two white and two black inputs. This means that each white cell must have at least two white and two black inputs. Therefore, no pattern exists in Cases 55, 51.

Next we show that no pattern exists in case 54. Otherwise, by assumption on the input sets of the circled cells, Figure 59 (a) determines Figure 59 (b). Observe that the circled black cell in Figure 59 (b) has at least two black inputs. This contradicts the assumption that the black cell has only one black input.

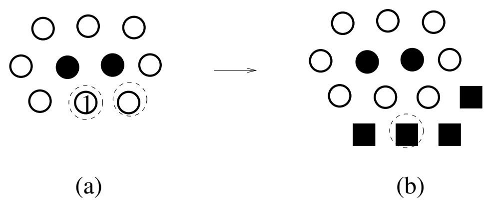


Figure 59: (Case 5j) No pattern exists in Case 54.

Case 52 By assumption on the input set of the circled cell, Figure 60 (a) determines Figure 60 (b) . Now consider the circled white cell in Figure 60 (b). That cell already has one black and three white inputs. By assumption on the input set, the other two neighboring cells of the white cell are white and black. The possible local patterns are shown in Figure 60 (I, II). We show that, up to symmetry, the two local patterns determine the same planar pattern.

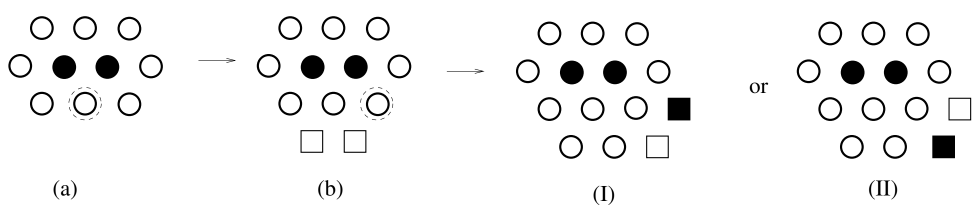


Figure 60: (Case 52) Possible local patterns.

I Begin with Figure 61 (a) (Figure 60 (I)). By assumption on the input sets of the circled cells, Figure 61 (a) determines Figure 61 (d) . Note that the local pattern enclosed by the dotted polygon in Figure 61 (a) determines Figure 61 (a). So, we can see that the the

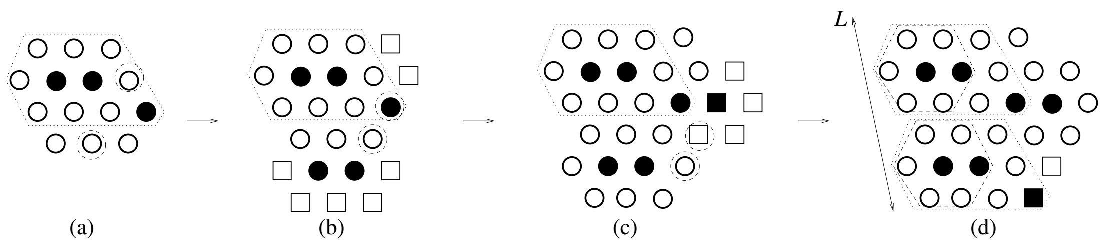


Figure 61: (Case 52) The pattern in the dashed polygon is repeated in the direction L.

local pattern enclosed by the dashed polygons in Figure 61 (d) are repeated periodically in the direction L, forming a strip.

Next, we prove that the strip determines the planar pattern. By assumption on the input sets of circled cells, Figure 62 (a) determines Figure 62 (c) . Note that the local pattern enclosed by the dashed polygon in Figure 62 (c) is the same pattern as Figure 61 (a). By the argument in the last paragraph, we know that Figure 62 (c) determines Figure 62 (d). Thus, the strip can be expanded to a planar pattern , arrives at Figure 7 case 52. □

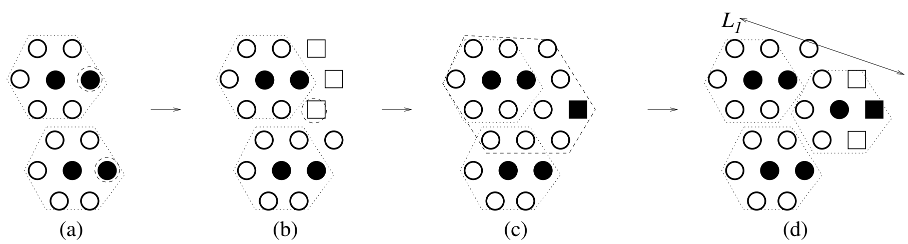


Figure 62: (Case 52) The strip determines a planar pattern.

II Begin with Figure 63 (a) (Figure 60 (II)). Consider the circled white cell in Figure 63 (a). The cell already has three white and one black inputs, so the other two neighboring cells are white and black. The possible local patterns are shown in Figure 63 (1,2). Note that the local pattern enclosed by the dashed polygon in Figure 63 (1) is the same pattern as the local pattern enclosed by the dotted polygon in Figure 61 (a) up to symmetry. Thus, Figure 63 (1) also determines Figure 7 case 52. So we need only to consider Figure 63 (2). We show that Figure 63 (2) is impossible.

In Figure 64 and Figure 65, we consider Figure 63 (2). By assumption on the input sets of the circled cells and the cells in the dashed polygon, Figure 64 (a) determines Figure 65 (d). By considering the input set of the circled black cell in Figure 65 (d), cells

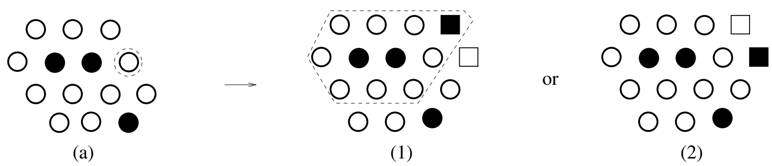


Figure 63: (Case 52) Pattern (1) determines a planar pattern, and pattern (2) is impossible.

1 and 2 must be one white and one black. Note that Figure 65 (d) is symmetric about the dashed line. So, without loss of generality, we assume cell 1 is white, and cell 2 is black arriving at Figure 65 (e). It is easy to see that Figure 65 (e) determines Figure 65 (f). Observe that the local pattern enclosed by dashes in Figure 65 (f) is the same pattern as

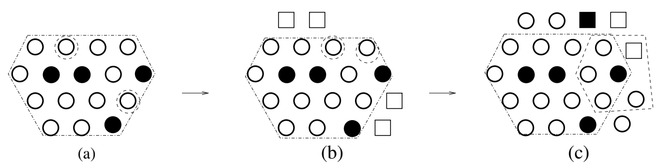


Figure 64: (Case 52) Pattern (a) determines pattern (f).

Figure 64 (a). Since Figure 64 (a) determines Figure 65 (d) , Figure 64 (a) determines the cells marked by arrow in Figure 64 (d) to be black. Thus, Figure 65 (f) forces the circled cell in Figure 65 (f) to be black, which is a contradiction. □

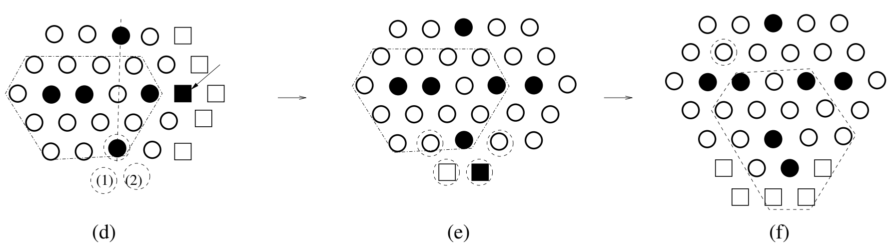


Figure 65: (Case 52) The circled white cell in pattern (f) is a contradiction.

Case 41 By assumption on the input sets of black cells, up to symmetry, the possible local patterns around a black cell in case 4j can only be Figure 66 (I, II, III). Note that the circled white cells in Figure 66 have at least two black inputs. Thus, no pattern exists in Case 41. □

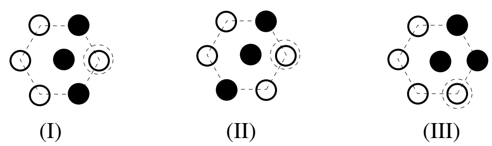


Figure 66: (Case 4j) Possible local patterns around a black cell in case 4j ($1 \leq j \leq 4$).

Case 43 We show that, up to symmetry, each possible local pattern around a black cell in Figure 66 determines a planar pattern which we will prove in order, and patterns (I, II) determine Figure 7 case 43a up to symmetry.

I Begin with Figure 67 (a) (or Figure 66 (I)), by assumption on the input sets of the circled cells, Figure 67 (a) determines Figure 67 (d). Note that the circled white cell in Figure 67 (d) has two black and two white inputs. So the other two neighboring cells are white and black. Also note that Figure 67 (d) is symmetric about the vertical midline. So, without loss of generality, we can put a black cell in position 1 and a white cell in position 2 arriving at Figure 68 (a).

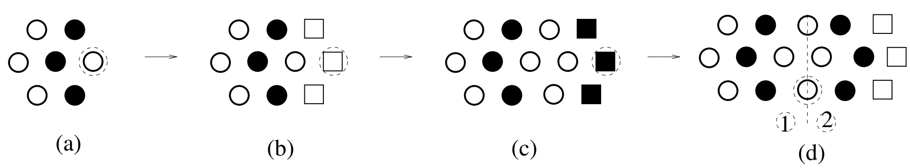


Figure 67: (Case 43) Possible patterns around Figure 66 (I).

By assumption on the input sets of the circled cells again, Figure 68 (a) determines Figure 68 (c). We claim that the local pattern enclosed by dashes determines a planar pattern, which is obtained by repeating the pattern enclosed by the solid polygon in two different directions.

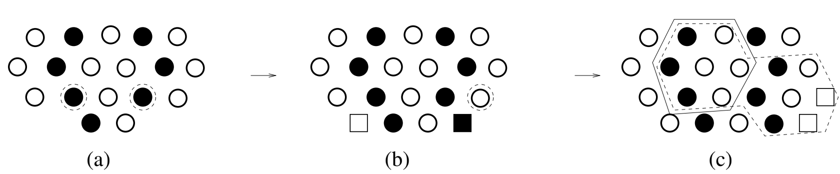


Figure 68: (Case 43) Pattern (a) determines a planar pattern.

We prove our claim in Figure 69 and Figure 70. By assumption on the input sets of the circled cells, Figure 69 (a) determines Figure 70 (d). Note that in Figure 70 (d), the local pattern enclosed by dashes is the same as Figure 67 (a) up to symmetry. Then we get the new cells in the left part of Figure 70 (e). The new cells on the right part of the pattern (e) is because of the assumption on the input sets of the circled cells in Figure 70 (d). So Figure 69 (a) determines Figure 70 (e). Thus, Figure 69 (a) determines the planar pattern obtained by repeating the pattern enclosed by the solid polygon in Figure 68 (c) in the directions L_1 and L_2.

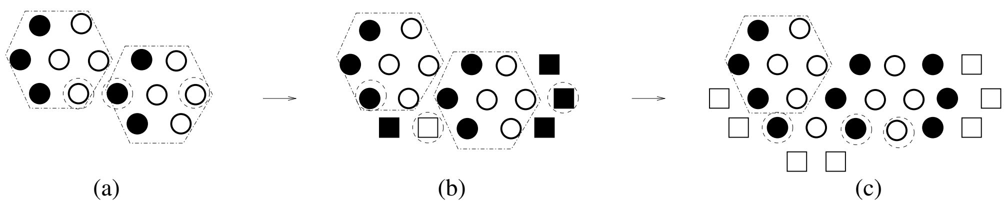


Figure 69: (Case 43) The pattern (a) determines the planar pattern.

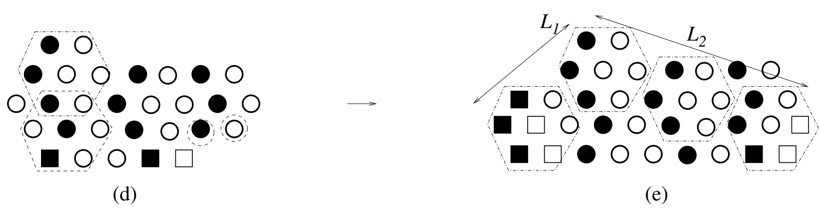


Figure 70: (Case 43) Continuum of Figure 69.

II Begin with Figure 71 (a) (or Figure 66 (II)). Observe that the circled black cell has one black and two white inputs. So the possible local patterns are Figure 71 (1, 2, 3). Note that, up to symmetry, Figure 71 (1) and (2) are the same. Moreover, the local pattern enclosed by dashes in pattern (1) is the same pattern as Figure 66 (I). By the argument in I, we know that pattern (1) also determines Figure 7 case 43a. So we need only to consider Figure 71 (3). We show that pattern (3) is impossible.

Now consider cell c in Figure 71 (3). If cell c is white then the local pattern in the dashed polygon must be Figure 71 (1) or (2). But note that Figure 7 case 43a determined by Figure 71 (1) does not contain a local pattern as Figure 71 (3). So cell c must be black. Then we can see that the lines of cells: L_1, L_2 and L_3 in Figure 71 (3) must be white, black and white respectively.

Next we show that no pattern in this case contains the strip consisting of the three lines. By assumption on the input sets of the white cells on L in Figure 72 (a), the neighboring

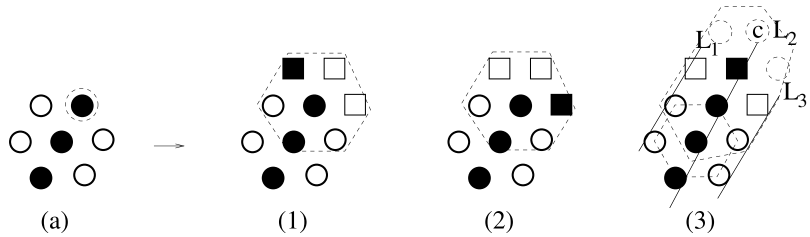


Figure 71: (Case 43) Pattern (a) determines the same pattern as Figure 66 (I).

line right to L have to be alternating. Without loss of generality, we can assume the line pictured as the line L_1 in Figure 72 (b). By assumption on the input sets of the black cells, the neighboring line right to L_1 must be black. See Figure 72 (c). Observe that the circled white cell in Figure 72 (c) has four black inputs. This contradicts the assumption on the input set of the white cell. Hence, Figure 71 (3) is impossible.

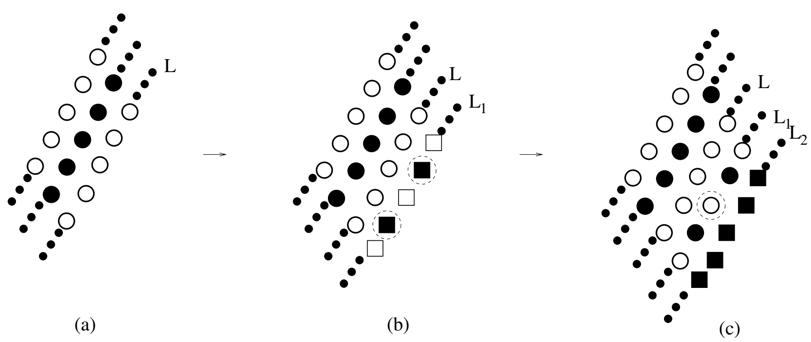


Figure 72: (Case 43) No pattern contains Figure 71 (3).

III Begin with Figure 73 (a) (or Figure 66 (III)). Note that the other two possible local patterns around a black cell Figure 66 (I, II) lead to Figure 7 case 43a that does not contain a local pattern as Figure 73 (a). This means that *in a planar pattern containing Figure 73 (a), the only possible local pattern around a black cell must be Figure 73 (a) up to symmetry*. We will use this property to prove that Figure 73 (a) determines Figure 7 case 43b.

By assumption on the input set of the circled cell, Figure 73 (a) determines Figure 73 (b). Considering the input set of the circled white cell in Figure 73 (b), the possible local patterns are Figure 73 (1,2). Observe that flipping pattern (1) about the vertical midline obtains pattern (2). So we need only to consider pattern (1).

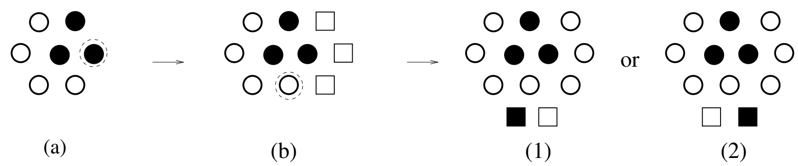


Figure 73: (Case 43) Possible local patterns around Figure 66 (III).

In Figure 74 and Figure 75, we prove that Figure 73 (1) determines Figure 7 case 43b. By assumption on the input sets of the circled cells and the property about the local pattern around a black cell, Figure 74 (a) determines Figure 75 (f) (Checking each step by beginning with considering the solid circled cell if there is a cell marked by solid circled). From Figure 75 (e) and Figure 75 (f), we can see that the local patterns enclosed by the dashed polygons are repeated in the directions L_1 and L_2, arriving at Figure 7 case 43b. □

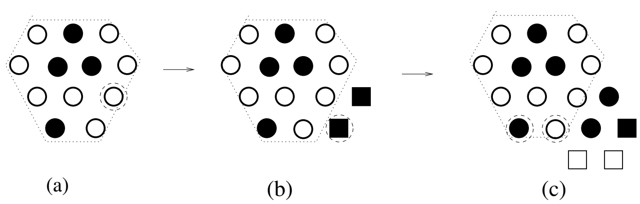


Figure 74: (Case 43) Pattern (a) determines the planar pattern.

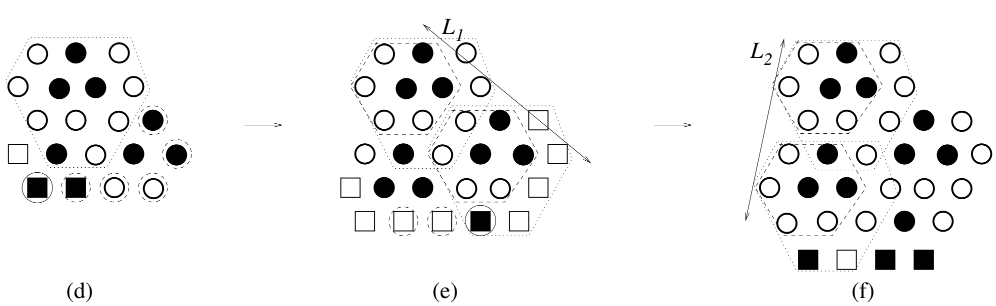


Figure 75: (Case 43) The local patterns enclosed by dashes are repeated in two different directions.

Case 42 The possible local patterns around a black cell can only be Figure 66 (I, II, III). Note that the circled white cell in pattern (I) has at least three black inputs. So Figure 66 (I) is impossible in this case. We show that pattern (II) determines Figure 7 case 42a and pattern(III) determines Figure 7 case 42b.

I We consider Figure 66 (II). By assumption on the input sets of the circled cells, Figure 76 (a) determines Figure 76 (c). Note that the local patterns enclosed by the dashed polygons in Figure 76 (c) are the same patterns as Figure 76 (a). So Figure 76 (a) can be extended uniquely in direction L to a strip that consists of three lines that alternate white, black, white.

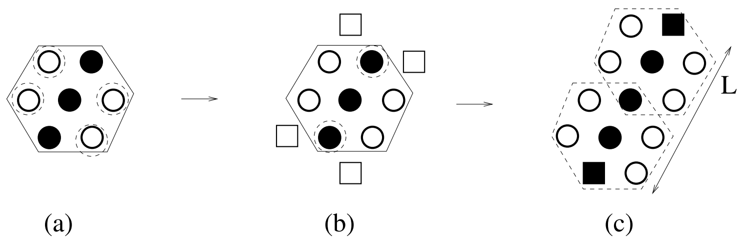


Figure 76: (Case 42) Pattern (a) determines a strip.

Next we prove that the strip determines Figure 7 case 42a. By assumption on the input sets of the circled cells, Figure 77 (a) determines Figure 77 (d). Note that the local pattern enclosed by the dashed polygon in Figure 77 (d) is the same pattern as Figure 76 (a). By the argument in the previous paragraph, we can see that the strip determines Figure 7 case 42a. □

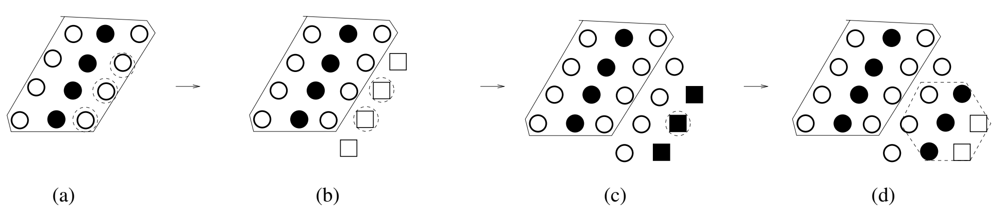


Figure 77: (Case 42) The strip determines the whole pattern.

II We consider Figure 66 (III). By assumption on the input sets of the circled cells, Figure 78 (a) determines Figure 78 (b). Considering the input set of the circled white cell in Figure 78 (b), the possible local patterns are pictured as Figure 78 (1, 2). We show that pattern (2) determines a planar pattern, whereas pattern (1) is impossible.

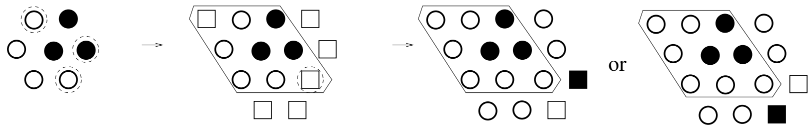


Figure 78: (Case 42) Possible local patterns containing pattern (a).

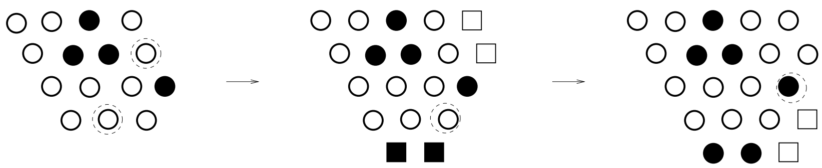


Figure 79: (Case 42) Pattern (a) is impossible.

1. We show that Figure 78 (1) is impossible. By assumption on input sets of the circled cells, Figure 79 (a) determines Figure 79 (c). Note that the circled black cell in Figure 79 (c) has at least five white inputs. This contradicts the assumption that each black cell has four white inputs. So Figure 78 (a) determines Figure 78 (2).
2. We prove that Figure 78 (2) determines Figure 7 case 42b by repeating the local pattern enclosed by the solid polygon in Figure 78 (b) in two different directions. In Figures 80 and 81, we prove that point. We have already proved that Figure 80 (a) implies Figure 80 (b), since Figure 78 (1) is impossible. By assumption on the input sets of the circled cells, Figure 80 (b) determines Figure 81 (e). Hence, Figure 78 (a) determines the whole planar pattern by repeating itself in two different directions L_1, L_2 in Figure 80 (e). □

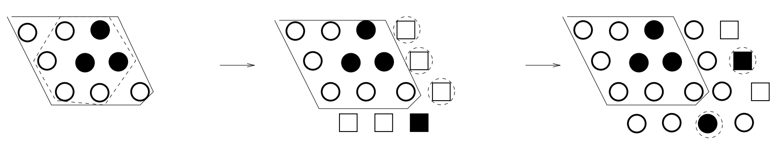


Figure 80: (Case 42) Pattern (a) is repeated in two different directions.

Cases 32, 31 By assumption on the input sets of black cells, up to symmetry, the possible local patterns of case 3j ($0 < \mathsf{j} \leq 3$) around a black cell can only be Figure 82 (I, II, III). Note

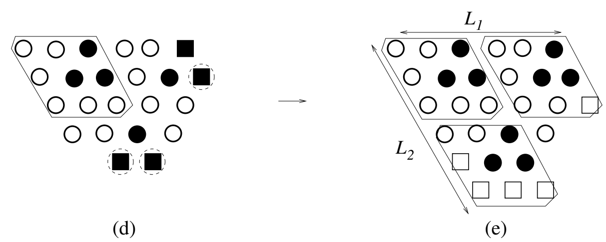


Figure 81: (Case 42) Pattern (a) in Figure 80 determines a whole planar pattern.

that the circled cells in Figure 82 have at least two black inputs. Thus, no pattern exists in Case 31.

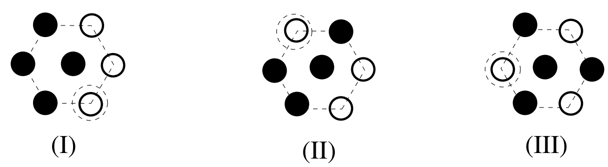


Figure 82: (Case 3j) Possible local patterns around a black cell in case 3j.

Next, we prove that no pattern exists in case 32. Note that the only possible local pattern around a black cell in this case is Figure 82 (I), since the circled white cells in Figure 82 (II, III) have at least three black cells. We assert that Figure 82 (I) also is impossible.

By assumption on the input sets of the circled cells, Figure 83 (a) determines Figure 83 (c). Observe that the circled black cell in Figure 83 (c) has two black and four white inputs. This contradicts the assumption on the input set of the black cell. □

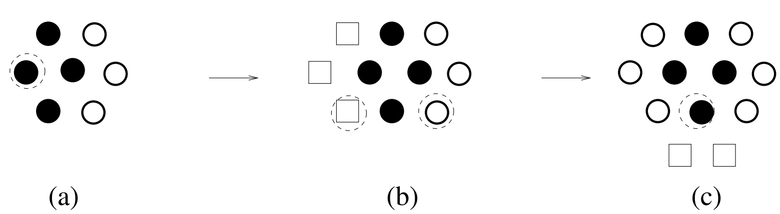


Figure 83: (Case 32) No pattern exists.

Case 33 We prove that, up to symmetry, each local pattern in Figure 82 determines Figure 7 case 33.

I We consider Figure 82 (I). By assumption on the input set of the circled cell, Figure 84 (a) determines Figure 84 (b). Note that the circled cell in Figure 84 (b) already has two white and two black inputs. So the other two neighboring cells must be black and white. Also note that Figure 84 (b) is symmetric about the vertical midline. Hence, we can put a white cell on position 1 and a black cell on position 2 arriving at Figure 85 (a).

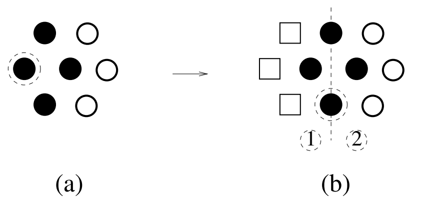


Figure 84: (Case 33) Pattern (b) is symmetric about the vertical midline.

In Figure 85 and Figure 86, by assumption on the input sets of the circled cells, Figure 85 (a) determines Figure 86 (g). From Figure 86 (f) and Figure 86 (g), we can see that that the local patterns enclosed by the dashed polygons are repeated in directions L_1 and L_2, arriving at Figure 7 (h).

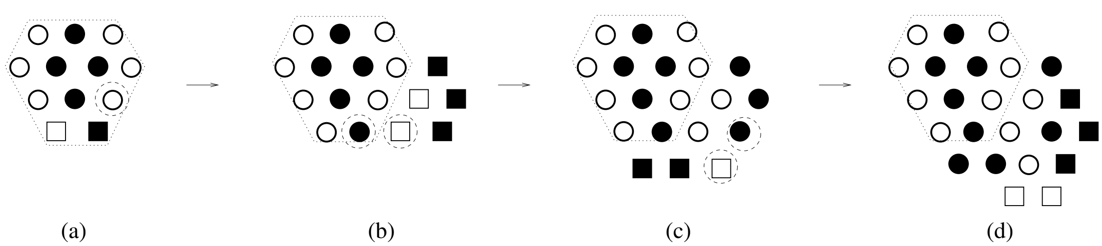


Figure 85: (Case 33) Pattern (a) determines a planar pattern.

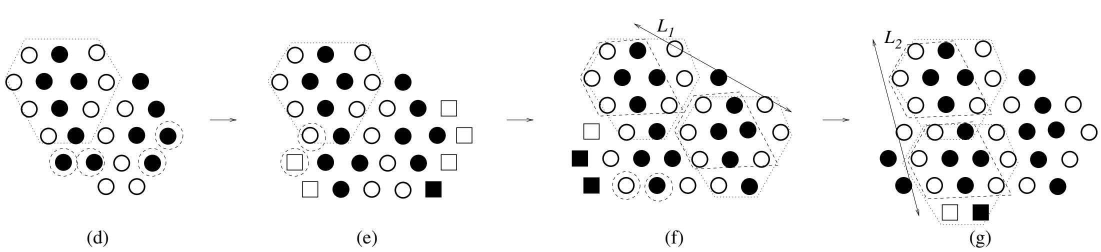


Figure 86: (Case 33) Continuum of Figure 85.

II We consider Figure 82 (II). By assumption on the input set of the circled white cell, Figure 87 (a) determines Figure 87 (b). Note that flipping around a vertical line the local pattern enclosed by the dotted polygon in Figure 87 (b), we get Figure 84 (a) (after interchanging the colors of cells). Also note that black cells and white cells have input cells with the same set of colors. So Figure 87 (a) determines the same planar pattern as the one obtained in (I).

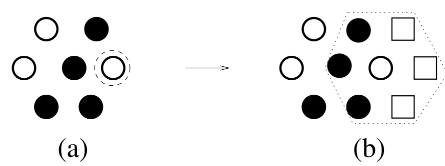


Figure 87: (Case 33) Pattern (a) determines Figure 7 (h).

III By assumption on the input set of the circled white cell, Figure 88 (a) determines Figure 88 (b). Note that the local pattern enclosed by the dashed polygon in pattern (b) also is the same as Figure 84 (a). So it also determines the same planar pattern as the one obtained in (I). □

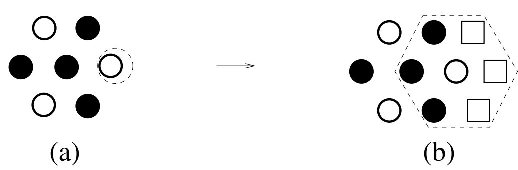


Figure 88: (Case 33) Pattern (a) determines Figure 7 (h).

Case 22 Up to symmetry, the possible local patterns around a black cell are Figure 89 (I, II, III). Observe that the circled white cells in (II,III) have at least three black inputs. So patterns (II,III) are impossible in this case. We assert that pattern (I) determines Figure 7 (i).

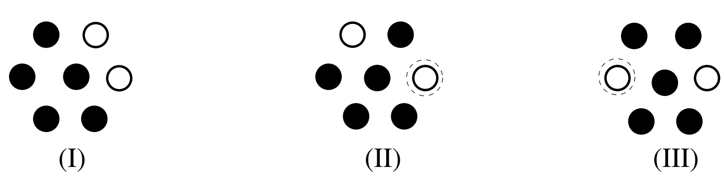


Figure 89: (Case 22) Only pattern (I) is possible in this case.

Now we begin with Figure 89 (I). By assumption on the input sets of the circled cells, Figure 90 (a) determines Figure 90 (c). Note that the local patterns enclosed by the dashed polygons in Figure 90 (c) are the same patterns as Figure 90 (a). So we can see that Figure 90 (a) determines a strip consisting of two white and two black lines of cells by expanding Figure 90 (a) in direction L.

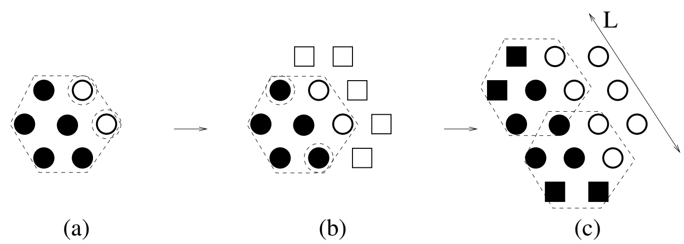


Figure 90: (Case 22) Pattern (a) determines a strip consisting of two white and two black lines of cells.

Next we prove that the strip determines the whole pattern. In Figure 91, by assumption on the input sets of the circled cells, pattern (a) determines pattern (d). The local pattern enclosed by the dashed polygon in pattern (d) is the same pattern as Figure 90 (a). Since Figure 90 (a) determines a strip consisting of two white and two black lines of cells, we can see that the strip determines the whole pattern by repeating the strip periodically in the horizontal direction. □

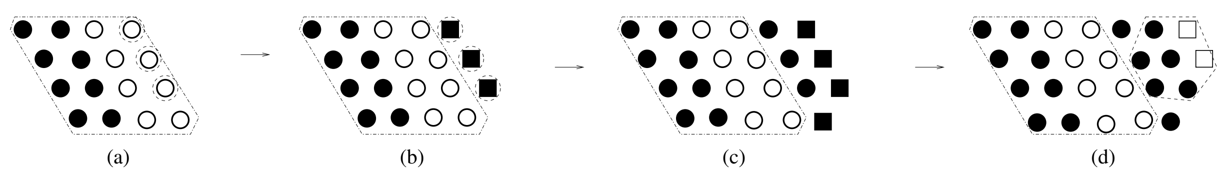


Figure 91: (Case 22) The strip determines a planar pattern by repeating itself periodically in the horizontal direction.

Cases 21, 11 We prove that no pattern exists in the two cases. Up to symmetry, each two-color synchrony pattern in case i1 ($1 \leq \mathsf{i} \leq 6$) must contain Figure 92. Observe that the black cell in the figure has at least three white inputs. So no pattern exists in cases 21 and 11.

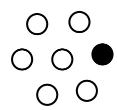


Figure 92: (Case i1) Each pattern in Case i1 contains this local pattern.

5.2 Infinite classes

In a hexagonal lattice, we call a line of cells of slope 0 or $\pm\frac{\sqrt{3}}{2}$ *a diagonal*. In the remaining three cases: Cases 62, 53, and 44, we prove Theorem 1.3 by beginning with a local pattern, showing that the local pattern determines a strip, and then showing that the strip determines the whole family. Note that all three cases satisfy: the number of black inputs of each white cell is equal to the number of black inputs of each black cell plus two. A result similar to Lemma 3.1 holds.

Lemma 5.1 *If the number of black inputs of each white cell is equal to the number of black inputs of each black cell plus two, then interchanging color along an alternating diagonal of a balanced pattern gives a new balanced pattern.*

Proof Suppose each white cell has m ($2 \leq m \leq 4$) black and $6 - m$ white inputs, and each black cell has $m - 2$ black and $6 - (m - 2)$ white inputs. Let the diagonal L in Figure 93 be alternating. Since balanced relations are determined by constraints on input sets, each cell c on L influences a balanced relation of a pattern in two ways. First, cell c has an input set. Second, cell c is in the input sets of its nearest neighbors. So the cells on L influence balanced relation for the cells on L, and the cells on the two neighboring diagonals L_1 and L_2. Table 5 shows the distribution of inputs of the cells on L. From Table 5, we can see that L_1 and L_2

cells on L	distribution of inputs			
	L		L_1 and L_2	
white cell	2 black	0 white	$m-2$ black	$6-m$ white
black cell	0 black	2 white	$m-2$ black	$6-m$ white

Table 5: The distribution of inputs of cells on L.

supply the same number of white inputs and the same number of black inputs for each cell on L. So the balanced relation for the cells on L is not changed when we interchange color along L. Note that L supplies one white and one black input to each cell on L_1 and L_2. So the balanced relation for the cells on L_1 and L_2 is also not changed when we interchange color along L. Thus, the balanced relation of a balanced pattern is not changed when interchanging color along an alternating diagonal. □

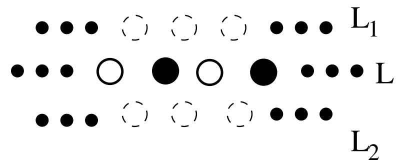


Figure 93: The common property of infinite classes.

Case 62 We first verify a property concerning patterns in Case 62.

Proposition 5.2 *Let L be an alternating diagonal. Then parallel diagonals alternate between all white and alternating.*

Reason By assumption on the input sets of the cells in L in Figure 94 (a), the neighboring diagonal L_1 in Figure 94 (b) must be a white diagonal. By assumption on the input sets of the white cells on L_1, diagonal L_2 must contribute one white and one black input to each cell on L_1. Hence L_2 must be alternating. Thus, parallel diagonals alternate between all white and alternating. □

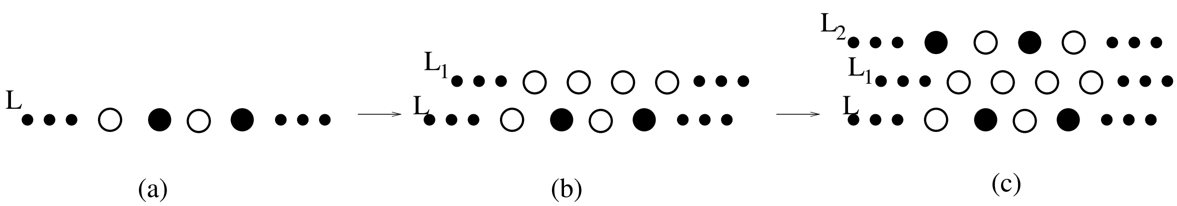


Figure 94: (Case 62) White diagonals and alternating diagonals alternate.

Next we prove that there always exists an alternating diagonal. Begin with the local pattern Figure 95 (a). Consider the input set of cell c. That cell has a black input in Figure 95 (a), so the other three neighboring cells must be two white and one black. The possible local patterns are Figure 95 (I, II, III). Observe that flipping pattern (I) about the horizontal midline is pattern (III). Therefore, we need only to consider patterns (I) and (II).

I Here we consider Figure 95 (I). By assumption on the input set of the circled cell, Figure 96 (a) determines Figure 96 (b). Next considering the input set of the circled white cell in Figure 96 (b), the possible local patterns are Figure 96 (1, 2). We assert that pattern (1) is impossible, and pattern (2) determines a strip.

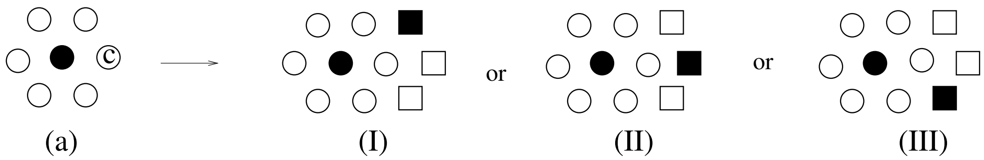


Figure 95: (Case 62) Possible local patterns of case 62.

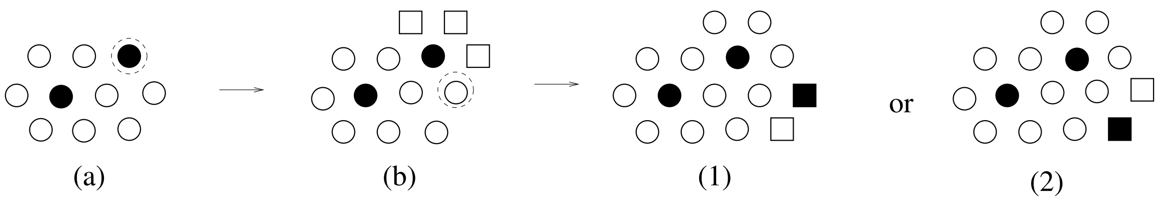


Figure 96: (Case 62) Possible local patterns around Figure 95 (I).

1. We consider Figure 96 (1). By assumption on the input set of the circled cell, Figure 97 (a) determines Figure 97 (b). Observe that the circled black cell in Figure 97 (b) has a black input. This contradicts the assumption that the black cell has no black input. Thus, Figure 96 (1) cannot appear in the patterns of this case. This means that Figure 96 (a) determines Figure 96 (2).

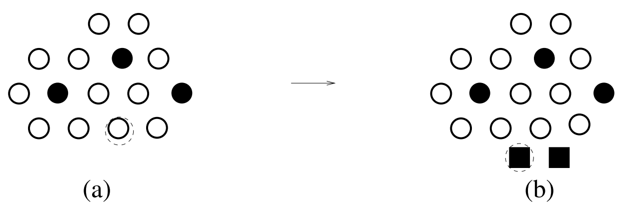


Figure 97: (Case 62) Impossible local patterns around Figure 96 (a).

2. Next we prove that Figure 96 (2) determines a strip. By assumption on the input sets of the circled cells, Figure 98 (a) determines Figure 98 (c). Note that the local patterns enclosed by the solid polygons in Figure 98 (c) are the same patterns as Figure 96 (a). Since Figure 96 (a) determines Figure 96 (2), we can see that the local patterns enclosed by the dotted polygons in Figure 98 (c) are repeated in the direction L . Then we can see that Figure 98 (a) determines the strip obtained by expanding Figure 98 (a) in the direction L. obviously, the strip contains two alternating diagonals.

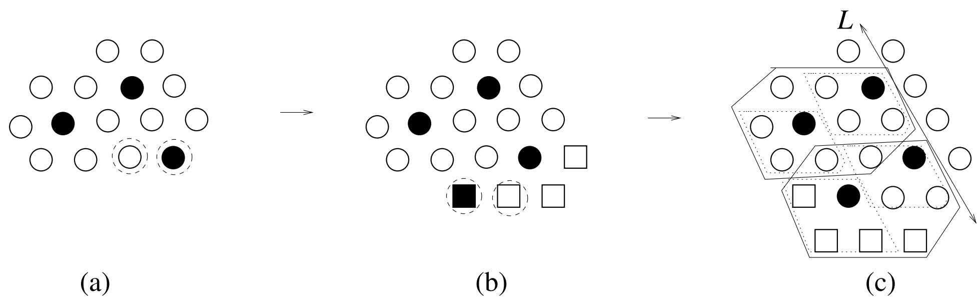


Figure 98: (Case 62) Figure 98 (a) determines a strip containing two alternating diagonals.

II Here we consider Figure 95 (II). We assert that if the horizontal diagonal through the black cells in Figure 95 (II) is not alternating, then the patterns containing Figure 95 (II) must contain local pattern Figure 95 (I) or (III). Suppose the diagonal is not alternating, then it must contain up to symmetry a segment pictured in Figure 99 (a). By assumption on the input sets of the circled cells, Figure 99 (a) determines Figure 99 (b). The possible local patterns enclosed by dashed polygon can only be Figure 95 (I) or (III). □

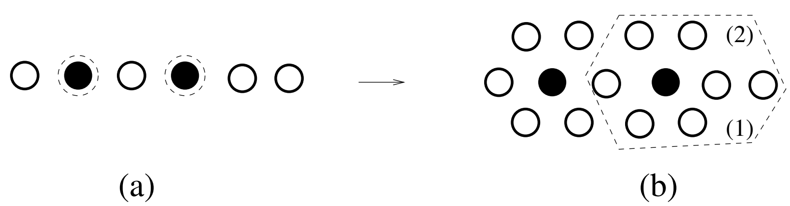


Figure 99: (Case 62) If L is not alternating, then the patterns must contain Figure 95 (I) or (III).

Following the above arguments, there must be an alternating diagonal. By Property 5.2, each pattern in this case consists of parallel diagonals which alternate between all white and alternating. That is, each pattern in this case can be obtained from Figure 8 (case 62). □

Case 53 We claim that *up to symmetry, each pattern in this case has the form of the pattern in Figure 100.* In this pattern, the strips enclosed by the dashed polygons are formed by repeating the local patterns enclosed by the solid polygons in the horizontal direction. The strips are repeated periodically in the direction l. Moreover, the lines of dashed circles between the strips are alternating diagonals. It follows that by Lemma 5.1, we always can put the pattern into the form of Figure 8 (case 53) by interchanging color along alternating diagonals.

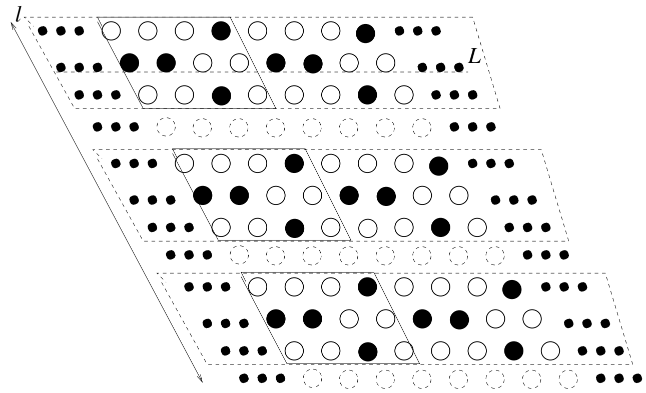


Figure 100: (Case 53) Up to symmetry, each pattern in this case has the form of this pattern.

We define a diagonal to be *double alternating* if two adjacent white and two adjacent black cells alternate along the diagonal. It is easy to see that line L in Figure 100 is double alternating. We prove our claim in four steps. The first three steps prove that L (or any double alternating diagonal) implies L_1 to L_5 in Figure 100. It follows that L implies Figure 100. Step four proves that each pattern in this case must contain a double alternating diagonal.

Step 1. Double alternating L implies L_1 and L_2. By assumption on the input sets of the cells in L in Figure 101 (a), starting from the circled black cell, it is easy to check that L_1 and L_2 must be the diagonals that are obtained by repeating the segments enclosed by the dashed polygons in the horizontal direction.

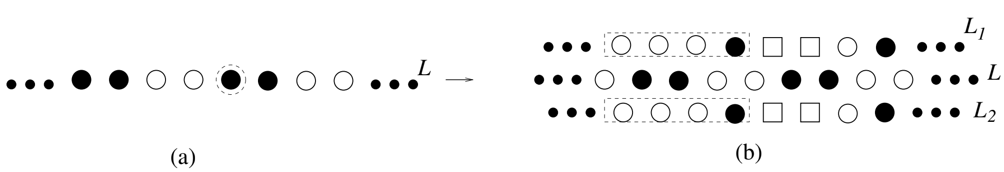


Figure 101: (Case 53) L determines L_1 and L_2.

Step 2. L_3 must be alternating. Consider the cells in L_2. Since each white cell on L_2 in Figure 101 already has two white and two black inputs and each black cell has four white inputs, each cell in L_2 must receive one black and one white input from L_3. So L_3 must be alternating.

Step 3. L_4 and L_5 are determined. Since L_3 is alternating, L_3 has two choices shown in Figure 102 (a ,b). We show that either choice implies L_4 and L_5.

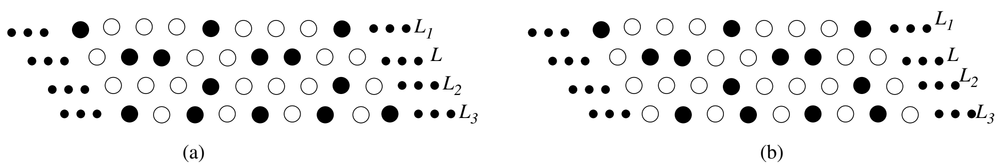


Figure 102: (Case 53) L_3 is alternating.

1. First, we consider Figure 103 (a) (or Figure 102 (a)). By assumption on the input sets of the cells in L_3, starting from the circled cell, we see that the diagonal L_4 is a translation of L_1 in the direction l. By assumption on the input sets of the cells in L_4 in Figure 103 (b),

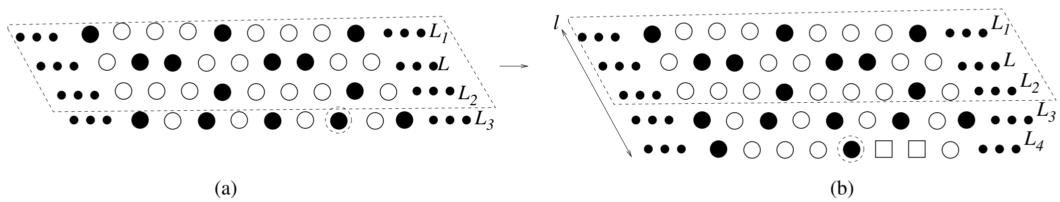


Figure 103: (Case 53) L_3 implies L_4.

starting from the circled cell, we obtain L_5 in Figure 104 (c) that is double alternating and is a translation of L in the direction of l.

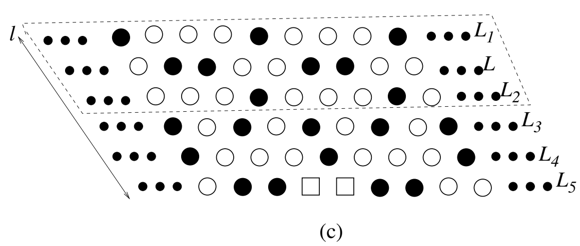


Figure 104: (Case 53) L_3 implies L_5.

2. Now we consider Figure 105 (a) (or Figure 102 (b)). With a similar argument as before, we see that Figure 105 (a) determines Figure 106 (c), where L_4 and L_5 in Figure 106 (c) are translations of L_1 and L in the direction l.

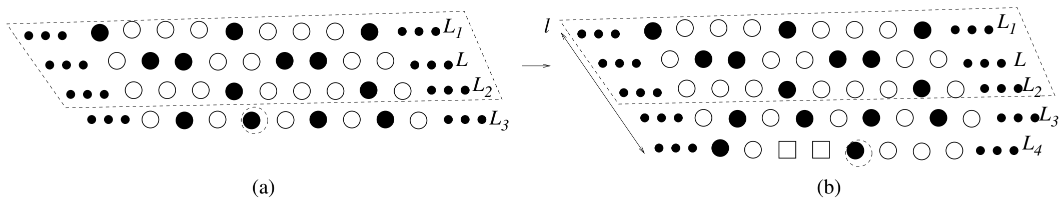


Figure 105: (Case 53) L_4 is a translation diagonal of L_1 in direction l.

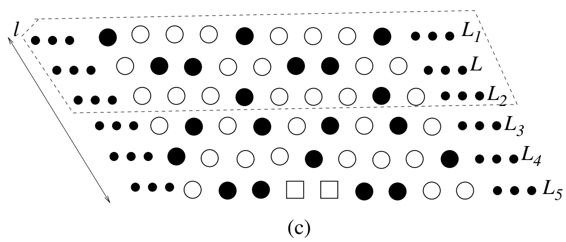


Figure 106: (Case 53) L_5 is a translating diagonal of L in direction l.

So both choices of L_3 imply L_4 and L_5. Thus, a double alternating diagonal implies that double alternating diagonals appear every four lines. It follows that a pattern containing a double alternating diagonal must has the form of Figure 100.

Step 4. Each pattern in this case contains a double alternating diagonal. We prove this point by beginning with the local pattern around a black cell Figure 107 (a). Considering the input set of the circled white cell in pattern (a), the possible local patterns around pattern (a) are Figure 107 (I, II, III). Note that the horizontal flip of Figure 107 (I) is Figure 107 (III). So we need only consider patterns (I, II). We prove that pattern (I) is impossible, and pattern (II) leads to the existence of a double alternating diagonal.

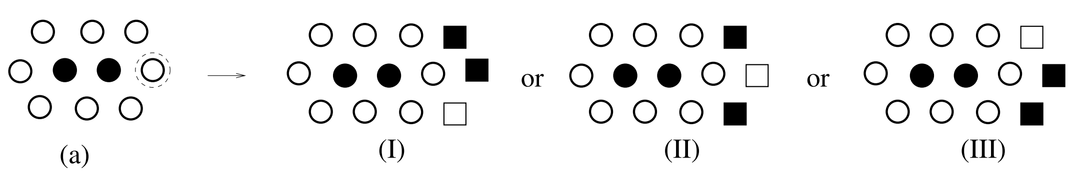


Figure 107: (Case 53) Possible local patterns in case 53.

I We consider Figure 108 (a) (or Figure 107 (I)). By assumption on the input sets of the circled cells, Figure 108 (a) determines Figure 108 (c). Observe that the circled white cell in Figure 108 (c) has four white inputs. This contradicts that the white cell has three white inputs. So Figure 107 (I) is impossible. That means Figure 107 (a) determines Figure 107 (II). Observe that the segment in the dashes in Figure 107 (I) determines Figure 107 (I). That means that the segment is impossible in this case.

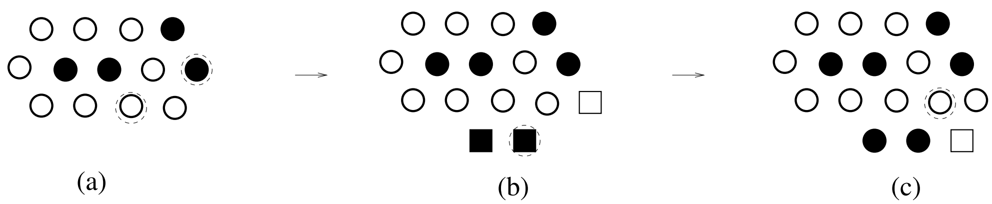


Figure 108: (Case 53) Pattern (a) is impossible.

II Since two adjacent black cells determine Figure 107 (a), following the result from I, two adjacent black cells determine Figure 107 (II). Also since Figure 107 (a) is symmetric about the vertical midline, two adjacent black cells determines Figure 109 (a). Next we prove that Figure 109 (a) leads to the existence of a double alternating diagonal.

Consider the circled white cell in Figure 109 (a). That white cell already has two black inputs. By assumption on the input set of the white cell, the possible local patterns of pattern (a) are Figure 109 (1, 2, 3). Observe that the horizontal flip of Figure 109 (3) is Figure 109 (2). So we need only consider Figure 109 (1, 2).

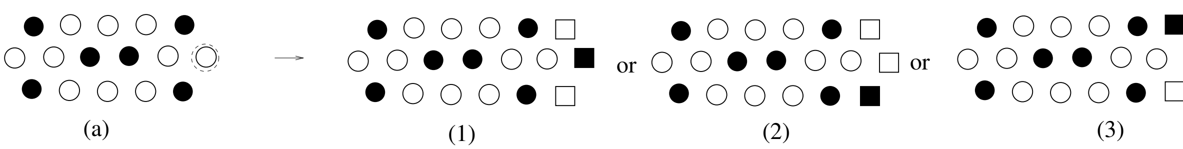


Figure 109: (Case 53) Pattern (2) is obtained by flipping pattern (3) about the horizontal midline.

1. We consider Figure 110 (1) (or Figure 109 (1)). By assumption on the input set of the circled black cell, the local patterns around Figure 110 (1) are Figure 110 (a, b, c). Note that flipping Figure 110 (a) about the horizontal midline is Figure 110 (c). So we need only consider Figure 110 (a, b). Since two adjacent black cells determine Figure 109 (a), it is easy to check that if the horizontal diagonal L in Figure 110 (b) is not double alternating, then the three-lines strip obtained by expanding pattern (b) in the horizontal direction must contain Figure 110 (a) or Figure 109 (2) up

to symmetry. So it is sufficient to prove that Figure 110 (a) and Figure 109 (2) determine the existence of a double alternating diagonal.

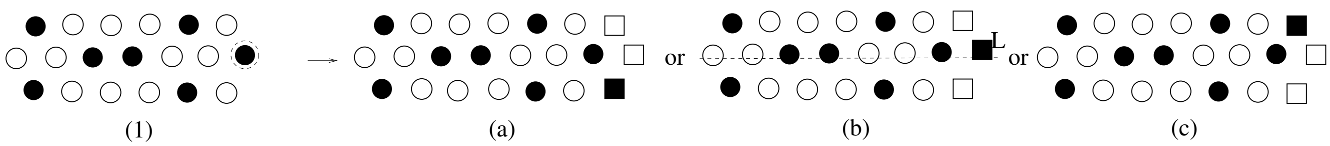


Figure 110: (Case 53) Pattern (a) is obtained by flipping pattern (c) about the horizontal midline.

In Figure 111, Figure 112, Figure 113, and Figure 114, we prove that Figure 110 (a) determines a strip that contains a double alternating diagonal. Considering the input sets of the cells in the dashed polygons and the circled cells, starting from the circled cells, Figure 111 (a) determines Figure 114 (k). Note that the local patterns enclosed by the dotted polygons in Figure 114 (k) contain Figure 111 (a). That means the local patterns enclosed by the dotted polygons can be expanded to the rugged strip by repeating themselves in the direction L. We see that diagonal L_1 in Figure 114 (k) is double alternating. So Figure 110 (a) determines the existence of a double alternating diagonal.

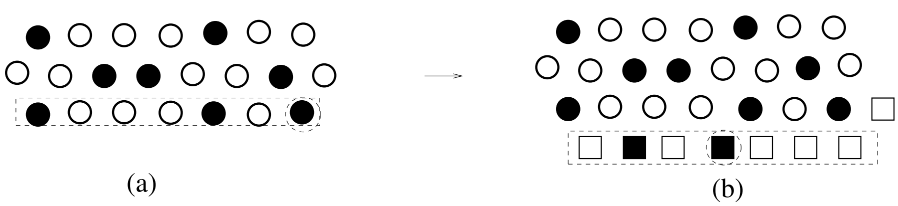


Figure 111: (Case 53) Checking that pattern (a) determines pattern (b) by starting from the circled cell.

2. Beginning with Figure 115 (a) (or Figure 109 (2)). Consider cell 1 in Figure 115 (a). If that cell is black, then the segment in the dashes is the same as the segment in Figure 107 (I) which we showed is impossible. So the cell must be white. See Figure 115 (b). Considering the input sets of the circled cells, Figure 115 (b) determines Figure 115 (c). Note that the segment in the dashes in Figure 115 (c) is the same as the segment in the dashes in Figure 109 (1), and that segment determines Figure 109 (1). By the previous result, we know that Figure 115 (a) also leads to the existence of a double alternating diagonal.

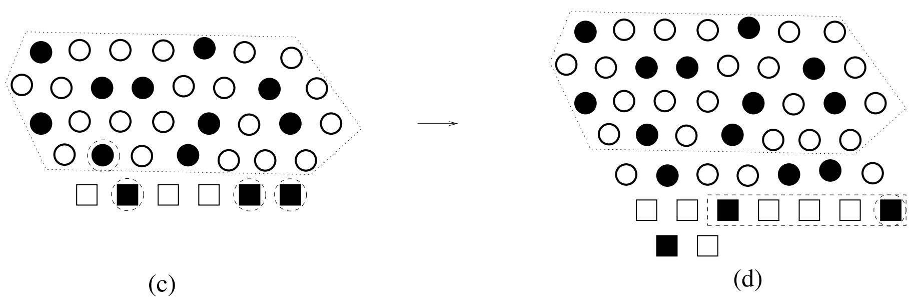


Figure 112: (Case 53) Pattern (c) determines (d) since the two adjacent black cells determines Figure 110 (a).

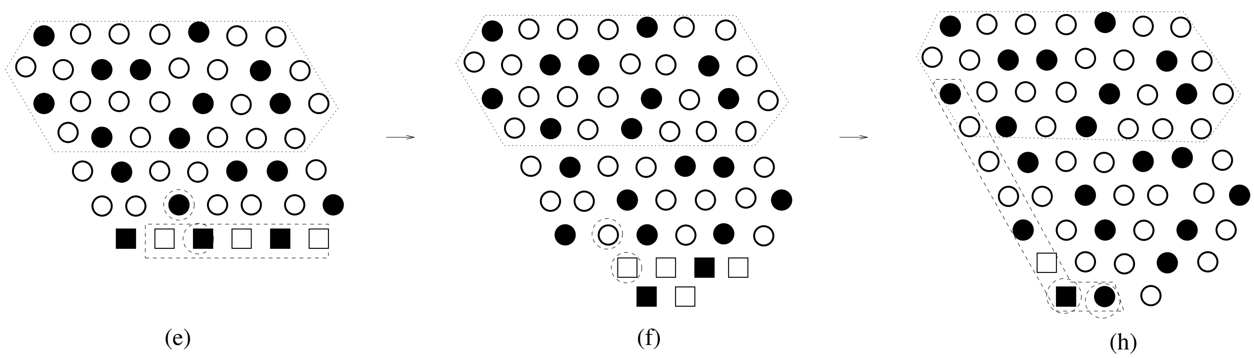


Figure 113: (Case 53) Figure 112 (d) determines pattern (h).

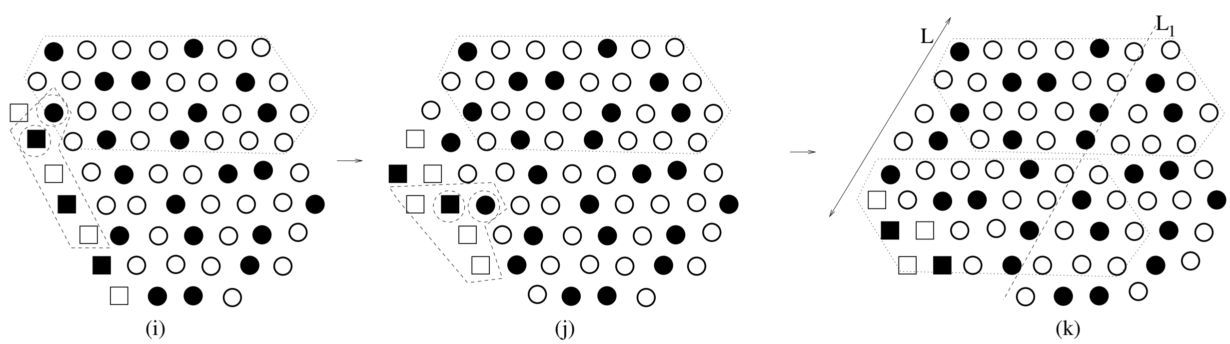


Figure 114: (Case 53) Figure 113 (h) determines pattern (k).

So each pattern in this case must contain a double alternating diagonal. It follows that up to symmetry, each pattern in this case can be pictures as Figure 100.

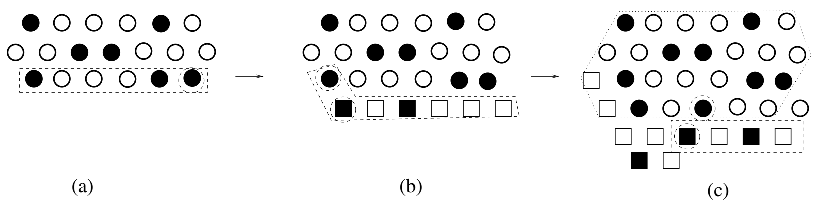


Figure 115: (Case 53) Cell 1 must be white.

Case 44 The possible local patterns around a black cell in this case are Figure 66 (I, II, III). We will prove Figure 66 (III) determines Figure 7 case 44a. We also show that a planar pattern that contains Figure 66 (I) or (II) is either Figure 7 case 44a or a pattern generated from Figure 8 case 44 by interchanging color along alternating diagonals.

First, we prove that Figure 66 (III) determines Figure 7 case 44a. By the assumption on the input sets of the circled cells, Figure 116 (a) determines Figure 116 (b). Since each white cell on the outer layer of Figure 116 (b) already has two white inputs, all cells on neighboring layers must be black as in Figure 116 (c). We see that white layers and black layers alternate in the planar pattern. Then we obtain Figure 7 case 44a. □

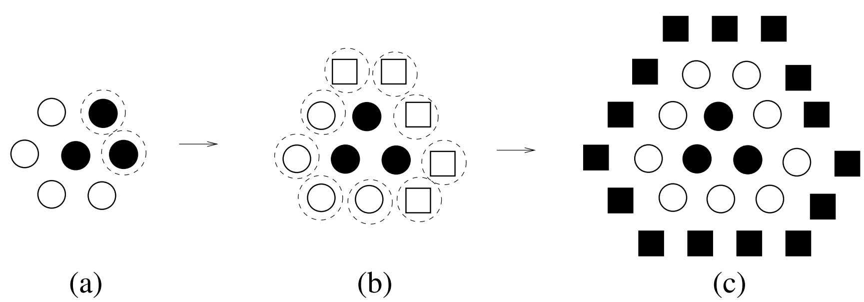


Figure 116: (Case 44) Pattern (a) determines Figure 7 case 44a.

Remark Figure 116 (a) is determined by the three black cells which are adjacent to each other. That means three black cells adjacent to each other determines Figure 7 case 44a. Since we are in case 44, three white cells adjacent to each other determines Figure 7 case 44a up to symmetry.

Next we prove two properties concerning patterns in this case.

1. If two adjacent parallel diagonals are alternating, then all parallel diagonals are alternating.

   Reason Let L_1, L_2 in Figure 117 be two adjacent parallel alternating diagonals. Note that each white cell in L_2 already has three black inputs and one white input, and each black cell in L_2 already has three white inputs and one black input. So the parallel diagonal L_3 must be alternating. Continuing the same procedure, we see that all parallel diagonals must be alternating. □

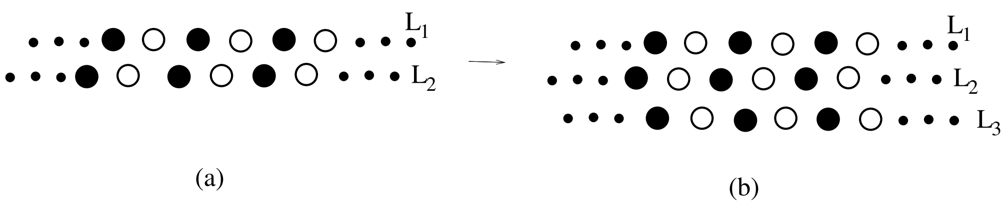


Figure 117: (Case 44) All parallel diagonal of two adjacent alternating are alternating.

2. If a planar pattern contains Figure 66 (I) or (II), and does not contain Figure 66 (III), then the pattern contains two adjacent parallel alternating diagonals.

   Reason

   (a) We consider Figure 66 (I). By assumption on the input sets of circled cells, Figure 118 (a) determines Figure 118 (c). Note that Figure 118 (a) is determined by the local pattern enclosed by the dashed polygon in Figure 118. So Figure 118 (a) determines a half strip that is obtained by repeating the local patterns enclosed by the dashed polygons in the direction L.

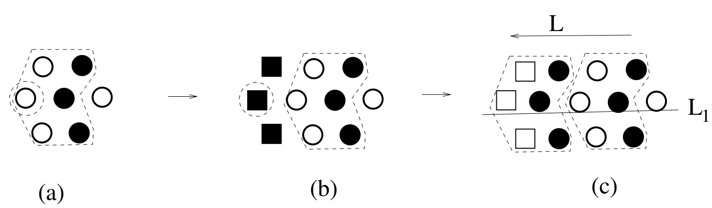


Figure 118: (Case 44) Pattern (a) determines a half strip.

   Next we prove that the diagonal L_1 in Figure 118 (c) must be alternating. Suppose L_1 is not alternating, then the strip containing Figure 118 (c) must contain Figure 119 (a) or (b) up to symmetry. However, either case contains Figure 66 (III). So the diagonal L_1 in Figure 118 (c) is alternating. Then, we can see easily that the upper and the lower diagonals in Figure 118 (c) are also alternating.

   (b) We consider Figure 66 (II). By assumption on the input set of the circled black cell, the possible local pattern are Figure 120 (I, II, III). Note that Figure 120 (I)

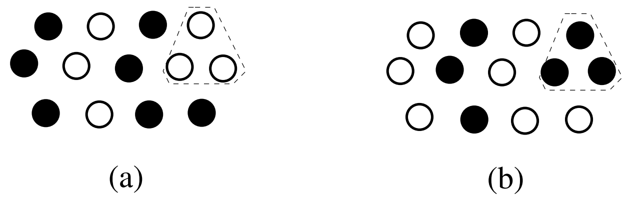


Figure 119: (Case 44) Both local patterns determine Figure 7 case 44a.

and (III) are symmetric. Also note that the local pattern enclosed by the dashed polygon in Figure 120 (I) is the same pattern as Figure 66 (I) up to symmetry. So we need only to consider Figure 120 (II). Note that the local patterns enclosed by

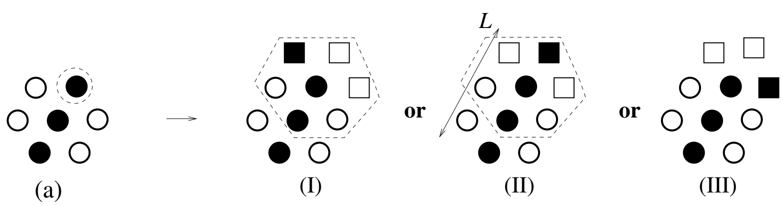


Figure 120: (Case 44) Possible local patterns around (a).

the dashed polygons in Figure 120 (II) are the same patterns as Figure 120 (a). So if we expand pattern (II) in direction L, then the strip must contain Figure 120 (I) or (III), or consists of three lines of cells that alternate white, black, and white. Since Figure 120 (I, III) lead to the existence of two adjacent parallel alternating diagonals. So we need only to show that a strip, which consists of three lines of cells that alternate white, black and white, also determines the existence of two adjacent parallel alternating diagonals. In fact, the strip determines a planar pattern.

Since each cell on L_1 in Figure 121 (a) already has two white and two black inputs, the diagonal L_2 in Figure 121 (b) must be black. Since each cell on L_2 in Figure 121 (b) already has two white and two black inputs, the diagonal L_3 in Figure 121 (c) must be white. We can see that the strip determines the planar pattern obtained by alternating white diagonals and black diagonals. Obviously, each horizontal diagonal in this pattern is alternating. □

From Properties 1 and 2, we can see that a two-color synchrony pattern in this case is either Figure 7 case 44a or can be generated from Figure 8 case 44 by interchanging color along alternating diagonals.

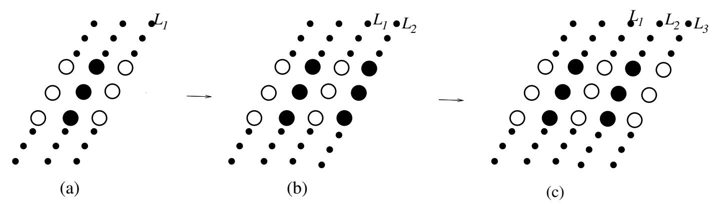


Figure 121: (Case 44) The strip determines a planar pattern.

6 Hexagonal lattice with nearest and next nearest neighbor coupling

In this section, we prove Theorem 1.4. The patterns of synchrony of hexagonal lattices with nearest and next nearest neighbor coupling are the patterns in Theorem 1.3 which are also NN-balanced. Thus, in order to prove Theorem 1.4, we need only to check which patterns in Theorem 1.3 are NN-balanced. It is easy to check that all patterns in Figure 7, except for pattern (j), are NN-balanced.

Next we need to determine which balanced patterns associated to Cases 62, 53, and 44 are NN-balanced. We prove that there is one pattern in Case 62, three patterns in Case 44, and none in Case 53 that are NN-balanced.

Case 62 By Property 5.2 of Case 62 in Section 5, each pattern consists of parallel diagonals that alternate between alternating and all white. So, up to symmetry, each pattern in this case contains the strip in Figure 122 (a) that consists of two white diagonals L_1 and L_3 and an alternating diagonal L_2. Since L_4 is alternating, there are two possible patterns shown in Figure 122 (I, II). In Figure 122 (I), note that the white cell 1 has at most four white NN-

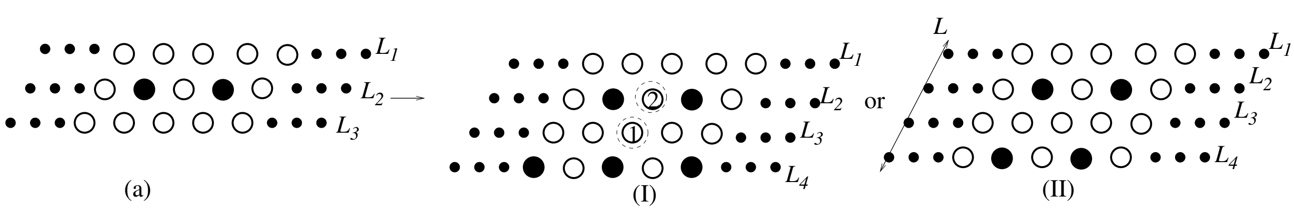


Figure 122: (Case 62) The pattern is NN-balanced in case 62.

inputs, and the white cell 2 has at least five white NN-inputs. So Figure 122 (I) is impossible in an NN-balanced pattern. It is easy to check that Figure 122 (II) is possible in an NN-balanced pattern. Thus, Figure 122 (a) is repeated in direction L obtaining Figure 8 (a).

Case 53 We have proved that, up to symmetry, each pattern in this family contains the strip in Figure 123, where l_2 and l_6 are alternating. Now consider cells a and b. Both of them receive four white NN-inputs from l_3 and l_5, and receive the other two NN-inputs from l_1 and l_2. Since l_2 and l_6 are alternating, cells 1 and 3, cells 2 and 4 have different colors. In order to keep the pattern NN-balanced, cells 1 and 2, and cells 2 and 4 must have different colors. That means if the pattern is NN-balanced, then each white cell must have five white NN-inputs and one black NN-input. However, the white cell c on l_3 receives two black NN-inputs: one from l_1, and the other from l_2 (since l_2 is alternating.) This contradicts the previous argument. So no pattern in this family is NN-balanced. □

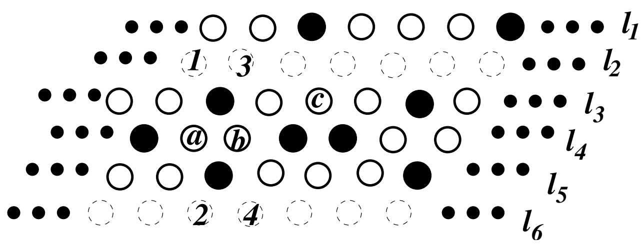


Figure 123: (Case 53) No pattern in case 53 is NN-balanced.

Case 44 We have proved that up to symmetry, except for Figure 7 case 44a, each pattern in this case, consists of parallel alternating diagonals. So up to symmetry, we assume that each pattern contains two adjacent alternating diagonals as in Figure 124. Note that each cell on an alternating diagonal must receive two white and two black NN-inputs from the parallel nearest neighboring diagonals. So each white cell can have two, three or four black NN-inputs. We show that each choice determines a planar pattern.

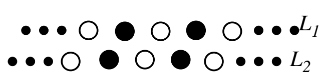


Figure 124: (Case 44) Three possible patterns in this case.

1. Suppose each white cell has two black and four white NN-inputs. Then the two NN-inputs from the next nearest neighboring horizontal diagonals must be white. By this assumption of NN-inputs of the circled cells 1 and 2 in Figure 125 (a), cells 3 and 4 must be white. Since all horizontal diagonals are alternating, cells 3 and 4 determine L_3 and L_4 respectively. Thus, we see that Figure 125 (a) determines the planar pattern by repeating Figure 125 (a) in vertical direction, arriving at Figure 10 case 4422.

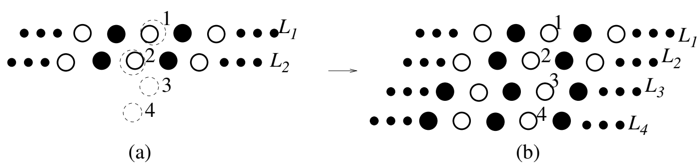


Figure 125: (Case 44) Each white cell has two black and four white NN-inputs.

2. Suppose each white cell has four black and two white NN-inputs. Then the two NN-inputs from the next nearest neighboring horizontal diagonals must be black. By this assumption of NN-inputs of the circled cell on L_1 in Figure 126 (a), cell c must be black. Since L_3 is alternating, Figure 126 (a) determines Figure 126 (b). Thus, we see that Figure 125 (a) determines the planar pattern by repeating Figure 125 (a) in direction L, arriving at Figure 8 case 44.

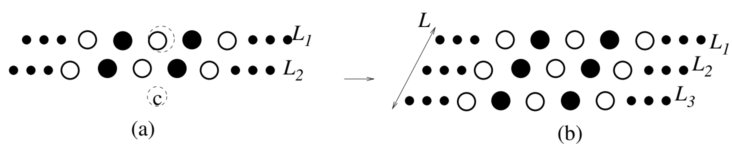


Figure 126: (Case 44) white cell has four black and two white NN-inputs.

3. Suppose each white cell has three black and three white NN-inputs. Note that under this condition, each black cell must have three white and three black NN-inputs. Also note that, beginning from a cell, by next nearest neighbor coupling relation in the lattice, we can expand the cell to a hexagonal lattice that is one third of the original one. By case 33 in Section 5 that, up to symmetry, there is only one pattern in this case. Now, starting from cell c, we use next nearest neighbor coupling to fill in an hexagonal lattice that is obtained by next nearest neighbor coupling. See Figure 127. Since all horizontal diagonal are alternating, we see that Figure 127 determines the planar pattern Figure 10 case 4433. □

Acknowledgment

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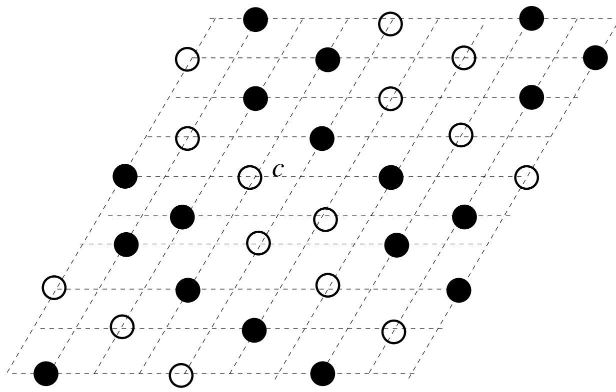

Figure 127: (Case 44) Each white cell has three black and three white NN-inputs.

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