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Flow Invariant Subspaces for Lattice Dynamical Systems

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Abstract. Stewart *et al.* have shown that flow invariant subspaces for coupled networks are equivalent to a combinatorial notion of a balanced coloring. Wang and Golubitsky have classified all balanced two colorings of planar lattices with either nearest neighbor (NN) or both nearest neighbor and next nearest neighbor coupling (NNN). This classification gives a rich set of patterns and shows the existence of many nonspatially periodic patterns in the NN case. However, all balanced two-colorings in the NNN case on the square and hexagonal lattices are spatially periodic. We survey these and new results showing that all balanced k-colorings in the NNN case on square lattices are spatially periodic.

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1 Coupled Cell Systems

A wide variety of physical and biological systems can naturally be modelled by networks of nonlinear dynamical systems, see Wang [7], Stewart [5]. The theoretical understanding of such systems is also under intensive development. Networks of differential equations possess additional structure, namely, canonical observables — the dynamical behavior of the individual nodes [3]. These observables can be compared, revealing such features as synchrony or specified phase-relations in periodic solutions, and these features are important in many applications. Any theoretical treatment of network dynamics must take this additional structure into account. In particular, the topology (or 'architecture') of the network imposes constraints on the dynamics, with the result that many new phenomena become 'generic' for a given architecture, see for example Golubitsky *et al.* [2].

Stewart, Golubitsky, Pivato, and Török [6, 4] formalize the concept of a *coupled cell network*, where a *cell* is a system of ordinary differential equations (ODEs) and a *coupled cell system* consists of cells whose equations are coupled. Stewart *et al.* define the architecture of coupled cell networks and develop a theory that shows how network architecture leads to synchrony. The architecture of a coupled cell network is a graph that indicates which cells have the same phase space, which cells are coupled to which, and which couplings are the same.

In its simplest form the *input set* $I(c)$ of a cell c consists of all cells coupled to c. Two input sets are *isomorphic* if there is a bijection between the input sets that preserves coupling types. A coupled cell network is *homogeneous* if the input sets of all cells are isomorphic. Suppose we color the cells in a homogeneous network. That coloring is *balanced* if for each pair of cells c and d with the same color, there is a color and coupling type preserving bijection from $I(c)$ to $I(d)$.

A lattice dynamical system is a homogeneous coupled cell system with cells indexed by a lattice $\mathcal{L}$. Such a system has the form

$$\dot{x}_c = g(x_c, x_{I(c)}) \qquad c \in \mathcal{L} \tag{1.1}$$

where $x_c \in \mathbf{R}^n$, $I(c) = \{c_1, \ldots, c_k\}$, $x_{I(c)} = (x_{c_1}, \ldots, x_{c_k}) \in (\mathbf{R}^n)^k$ and $g : (\mathbf{R}^n)^{k+1} \to \mathbf{R}^n$.

A *polydiagonal* is a subspace of the phase space of a coupled cell network that is defined by equality of cells coordinates. A polydiagonal is *robustly polysynchronous* if it is flow-invariant for every coupled cell system with the given network architecture. Robustly polysynchronous polydiagonals are identified with *patterns of synchrony*. Stewart *et al.* [6, Theorem 6.1] prove that a polydiagonal is robustly polysynchronous if and only if the coloring given by coloring cells that have the same coordinates with the same color is balanced. Thus, classifying robustly polysynchronous polydiagonals is equivalent to the combinatorial question of classifying balanced colorings.

Some patterns of synchrony can be predicted by symmetry, namely, those that correspond to fixed-point subspaces of the group of network symmetries. However, not all patterns can be obtained in this way, and some of these nonsymmetric patterns are quite interesting. Golubitsky *et al.* [2] give an infinite class of two-color patterns of synchrony on square lattice dynamical systems with nearest neighbor coupling. Wang and Golubitsky [8] classify all possible two-color patterns of synchrony of square lattice differential equations with two different architectures —

nearest neighbor coupling (NN) and both nearest neighbor and next nearest neighbor coupling (NNN). These classification results are stated in Theorems 3.1-3.2. It follows from these theorems that with NNN architecture balanced two-colorings are finite in number and spatially doubly-periodic. Thus, there is a profound difference between balanced two-colorings in the NN and NNN cases: one classification is finite, the other is infinite; one set has spatially periodic and nonperiodic colorings, the other has only periodic colorings.

In this note we prove that all balanced colorings of a one-dimensional lattice are spatially periodic (Theorem 2.4) and that for each $k \geq 3$ there are exactly four balanced k colorings (Theorem 2.5). The proof that balanced k colorings are periodic and finite in number uses a notion of *window* (a small finite patch of the network that determines the full pattern associated to any balanced k-coloring) and that proof generalizes to higher dimensions. In Section 3 we discuss balanced colorings in square arrays with nearest and next nearest neighbor coupling. We begin by summarizing the classification of balanced 2-colorings by Wang and Golubitsky [8]. Then we state the generalization of the results in Section 2 to square lattices and end with a brief discussion of some generalizations. Proofs and exact statements of these results appear in Antoneli *et al.* [1].

2 Linear Arrays with Nearest Neighbor Coupling

Let G be a one-dimensional lattice network with nearest neighbor coupling. An *interval* of a one-dimensional lattice network G is a finite sequence of consecutive cells $c_1 \dots c_n$. The *interior* of this interval is the cell sequence $c_2 \dots c_{n-1}$.

Definition 2.1 Given a one-dimensional lattice network G and a balanced k-coloring, we say that an interval is a *window* for the balanced coloring if its *interior* contains all k colors.

Note that the part of any balanced k-coloring that lies inside a window extends uniquely to the whole lattice. Since the k-coloring is balanced and each of the k colors has its nearest neighbors inside the interval, the colors of the cells on the right and left ends of the interval are determined. Proceeding recursively, the coloring can be extended uniquely to the whole lattice.

Lemma 2.2 *Given a balanced coloring of G with ℓ colors. Then every interval of length $2k-1$, where $k \leq \ell$, contains k colors.*

Proof Certainly every interval of length 1 contains one color. By induction, assume that the statement is valid for $k-1$. An interval of length $2k-1$ has the form $ac_1 \dots c_{2k-3}b$. By induction the interval $c_1 \dots c_{2k-3}$ contains at least $k-1$ colors. If that subinterval contains k colors, we are done. Similarly, if a or b is a k^{th} color, then we are done. Suppose not. Since the coloring is balanced and each of the $k-1$ colors has its nearest neighbors inside the interval, the coloring uniquely extends to the whole lattice and the extended coloring has fewer than k colors, a contradiction. □

Proposition 2.3 *Any interval of size $2k+1$ is a window for every balanced k-coloring of the linear lattice.*

Proof This follows directly from the definition of window and Lemma 2.2. □

Theorem 2.4 *For every k there exists a finite number of balanced k-colorings in one-dimensional lattice dynamical systems with nearest neighbor coupling. Moreover, all balanced colorings are periodic.*

Proof First, given a window W, there is a finite number of ways to color the cells in W with k colors. In particular, there is a finite number of balanced k-colorings inside any window. Therefore, the number of balanced k-colorings on the lattice is finite.

Given a balanced k-coloring of G, by Proposition 2.3, any interval of size $2k+1$ is a window that determines it uniquely. Let W be one such window. Consider a covering of the lattice by disjoint translates of W. Since there is an infinite number of such translates but only a finite number of balanced colorings, there must be at least two translates W_1 and W_2 exhibiting exactly the same coloring and thus these two windows determine the same balanced k-coloring. Now observe that any translation of a balanced k-coloring is again a balanced k-coloring. It follows that the translation that takes W_1 to W_2 leaves the k-coloring invariant and so the coloring is periodic. $\square$

Theorem 2.5 *Fix k and let $A_1, A_2, \ldots, A_k$ be k distinct colors. Then, every balanced k-coloring of the one-dimensional lattice with nearest neighbor coupling has one the following four forms.*

(i) No reflection; period k

$$\cdots \mathbf{A_1 A_2} \cdots \mathbf{A_k}\ \mathbf{A_1 A_2} \cdots \mathbf{A_k}\ \cdots$$

(ii) Two reflections both without fixed cells; period $2k$

$$\cdots \mathbf{A_1 A_2} \cdots \mathbf{A_k A_k} \cdots \mathbf{A_2 A_1}\ \mathbf{A_1 A_2} \cdots \mathbf{A_k A_k} \cdots \mathbf{A_2 A_1}\ \cdots$$

(iii) Two reflections one with fixed cell and one without fixed cell; period $2k-1$

$$\cdots \mathbf{A_1 A_2} \cdots \mathbf{A_k} \cdots \mathbf{A_2 A_1}\ \mathbf{A_1 A_2} \cdots \mathbf{A_k} \cdots \mathbf{A_2 A_1}\ \cdots$$

(iv) Two reflections both with fixed cells; period $2k-2$

$$\cdots \mathbf{A_1 A_2} \cdots \mathbf{A_k} \cdots \mathbf{A_2 A_1 A_2} \cdots \mathbf{A_k} \cdots \mathbf{A_2 A_1}\ \cdots$$

All of these balanced colorings are distinct when $k \geq 3$; balanced colorings (i) and (iv) are the same when $k = 2$; and all balanced colorings are identical when $k = 1$.

Proof The case $k = 1$ is straightforward and we assume $k \geq 2$. We fix a balanced k-coloring and let $m \geq 0$ be the smallest number of cells that occur between two cells of the same color. We claim that the balanced coloring has a reflection symmetry that fixes no cells when $m = 0$; a reflection that fixes a cell when $m = 1$; and is of type (i) when $m \geq 2$.

When $m = 0$ the balanced coloring contains two neighboring cells of the same color. Thus the balanced pattern contains $\cdots AA \cdots$. If color B is the left neighbor of the first A, then balanced implies that B is also the right neighbor of the second A and the pattern has the form $\cdots BAAB \cdots$. Inductively, balanced implies that the coloring has a reflection symmetry that fixes no cells.

When $m = 1$ the balanced coloring contains the pattern $\cdots ABA \cdots$. If color C is the left neighbor of the first A, then balanced implies that C is also the right neighbor of the second A and the pattern has the form $\cdots CABAC \cdots$. Inductively, balanced implies that the coloring has a reflection symmetry that fixes a cell.

When $m \geq 2$, the balanced coloring contains the pattern

$$\cdots AB_1B_2\cdots B_mA\cdots$$

where, from the definition of m, the colors of A and all of the B_j are distinct. It follows from balanced that the color to the left of the first A must be B_m and the color to the right of the second A must be B_1; that is, the pattern contains

$$\cdots B_mAB_1B_2\cdots B_mAB_1\cdots$$

Inductively, balanced implies that the pattern has the form (i) (and $m = k-1$).

We can now assume that the coloring has a reflection symmetry. All lattice reflections have the form

$$r_j(i) = j - i$$

where j is an integer. Note that r_j is the reflection about the point $\frac{j}{2}$ and is a reflection with a fixed cell when j is even and a reflection without a fixed cell when j is odd.

Theorem 2.4 states that every balanced k-coloring is periodic and hence has a translation symmetry $i \mapsto i + p$ where p is the minimum period of the coloring. It follows that the balanced pattern must have two independent reflections; just compose translation by p with r_j to obtain

$$r_j(i+p) = j - (i+p) = r_{j-p}(i)$$

The point of reflection in the second reflection is distance $\frac{p}{2}$ from the point of reflection of the first. Moreover, the composition of two reflections r_j and r_q is a translation by $j - q$; just compute

$$r_j(r_q(i)) = r_j(q-i) = j - (q-i) = i + (j-q)$$

It follows that a pattern with minimum period p cannot have reflections whose points of reflection are closer than $\frac{p}{2}$. Hence these patterns have exactly two reflections in a minimum period. Note also that if p is even either both reflections have a fixed cell or both do not. If p is odd there is one reflection of each type. Thus, there are three cases to consider: p even and reflections with a fixed cell; p even and reflections without fixed cells; and p odd and one reflection of each type. We claim that these cases lead to the patterns (iv), (ii), and (iii), respectively.

We now consider the case where p is even and the coloring has two reflections without fixed cells and show that this case leads to case (ii). It follows by reflection about the midpoint between two A cells, as in the case $m = 0$, that the pattern has the form

$$\cdots B_0B_0B_1\cdots B_\ell B_{\ell+1}B_{\ell+1}\cdots$$

where $\ell = \frac{p}{2} - 2$. Reflecting about $B_{\ell+1}$ leads to the periodic pattern

$$\cdots B_0B_0B_1\cdots B_\ell B_{\ell+1}B_{\ell+1}B_\ell\cdots B_1B_0B_0\cdots$$

We claim that balanced implies that the colors are $B_0, \ldots, B_{\ell+1}$ are all distinct. Suppose that B_{j+1} is the first color in this sequence that is the same as one of the colors $B_0, \ldots, B_j$. If the color of B_{j+1} is the same as the color of B_j, then, as in the case $m = 0$ above, the pattern has an additional reflection. If the color of B_{j+1} is the same as the color of B_{j-1}, then, as in the case $m = 1$ above, the pattern has an additional reflection. If the color of B_{j+1} is the same as one of the previous colors, then the coloring is not balanced. The claim is verified and the number of colors in this pattern is $k = \ell + 2$ and the period is $p = 2\ell + 4 = 2k$, as desired.

The remaining two cases are proved similarly by considering the appropriate reflections. □

3 Balanced Colorings on Square Arrays

In this section we review results on balanced colorings on square arrays. In particular we note that balanced two-colorings on square arrays need not be spatially periodic when only nearest neighbor coupling is assumed, whereas they are spatially periodic if both nearest and next nearest neighbor coupling are assumed. Indeed, the results of Section 2 can be extended to square arrays; namely, balanced k-colorings are spatially periodic when nearest and next nearest neighbor coupling are assumed.

Theorem 3.1 ([8]) *There are eight two-color periodic patterns of synchrony of square lattice differential equations with nearest neighbor coupling shown in Figure 1. There are two infinite families of two-color patterns of synchrony that are generated from the periodic patterns in Figure 2 by interchanging black and white on diagonals along which black and white cells alternate. Up to symmetry, these are all of the two-color patterns of synchrony.*

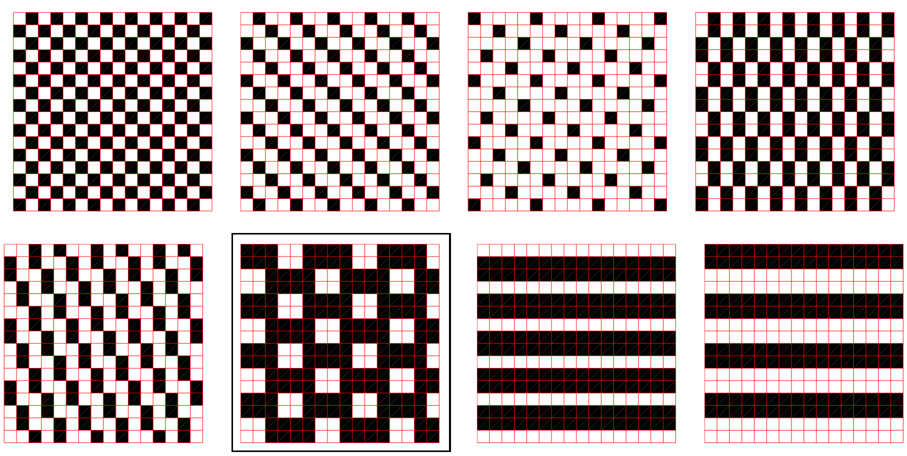

Figure 1 Illustrations of patterns of synchrony of finite classes [8].

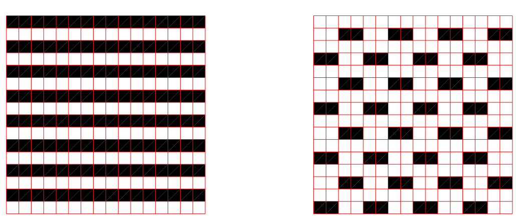

Figure 2 Illustrations of patterns of synchrony of infinite classes [8].

Some examples of the infinite number of nonperiodic balanced two-colorings are given in Figure 3.

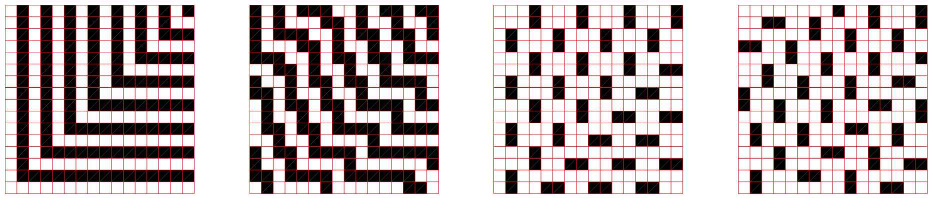

Figure 3 Patterns of synchrony that are not spatially periodic [2, 8].

Theorem 3.2 ([8]) *Up to symmetry, there are twelve two-color patterns of synchrony in square lattice differential equations with nearest and next nearest neighbor couplings: the seven patterns in Figure 1 except for the boxed pattern, Figure 2 (a), and the four patterns in Figure 4.*

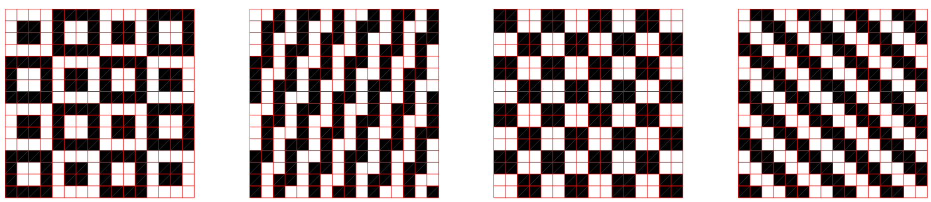

Figure 4 The four patterns in this figure and Figure 2 (left) are NNN-balanced [8].

Remark 3.3 All twelve patterns are doubly-periodic. Those patterns in Figure 4 can be generated from Figure 2 (a) by interchanging black and white along diagonals on which black and white cells alternate.

Indeed more is true once next nearest neighbor coupling is assumed.

Theorem 3.4 ([1]) *For any integer $k > 0$, there are only a finite number of k-color patterns of synchrony of square lattice dynamical systems with nearest and next nearest couplings. All of these patterns are periodic.*

The proof of Theorem 3.4 proceeds as in the case of one-dimensional arrays. Given a k-coloring of a two-dimensional square lattice network G, a bounded square subarray of G is called a *window* if its interior contains all k colors. We then prove that windows determine uniquely the whole pattern and that every square of size $2k+1$ is a window. The proof of the theorem follows from these observations.

We call a lattice *Euclidean* if it is generated by vectors of the same length. It is true that if enough different kinds of coupling are present, then all balanced k-colorings of any planar Euclidean lattice dynamical system are spatially periodic. See [1] for details. We conjecture that this result is valid in all dimensions.

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