SYMMETRY OF ATTRACTORS AND THE KARHUNEN-LOÈVE DECOMPOSITION

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Symmetry of Attractors and the Karhunen–Loève Decomposition

Michael Dellnitz, Martin Golubitsky, and Matthew Nicol

4.1 Introduction

Recent fluid dynamics experiments [13, 10, 4] have shown that the symmetry of attractors can manifest itself through the existence of spatially regular patterns in the time average of an appropriate observable such as the intensity of transmitted light in the Faraday experiment. In this chapter we discuss how the symmetry of attractors can be detected numerically in solutions of symmetric PDEs and how symmetry considerations affect the appropriateness of a popular method for computing asymptotic dynamics in PDEs—the Karhunen–Loève decomposition.

To motivate our discussion we first describe the observed phenomenon. In the Rayleigh–Bénard experiment a fluid layer is heated from below and the transition from pure conduction to convection as the temperature gradient is increased is investigated. It is well known that the initial transition to convection is accompanied by the appearance of (almost) regular patterns. As the temperature gradient is further increased, more complicated dynamics that are both temporally and spatially chaotic appear. Pierre Bergé made the following observation (as reported in a survey by David Campbell [9]): When operating his experiment in a chaotic regime and in a rectangular container, the time average of the observed fluid velocities had a well-defined rectangularly symmetric pattern, even though none of the time instantaneous velocity fields had this symmetry. No explanation of how this might happen was given. In addition, the experiment, although very suggestive, was not conclusive, as the time average was taken over only a rather short (scaled) time interval.

Dellnitz, Golubitsky and Melbourne [11] observed that symmetries of attractors of PDEs in phase space should manifest themselves as symmetry invariants of the time average of the solution. This possibility was verified in certain numerically computed solutions of the Brusselator and the complex Ginzburg–Landau equations—both reaction–diffusion systems defined on the unit interval.

In addition, Gluckman et al. [13] have investigated the Faraday surface wave model, looking for indications of pattern on average. This experiment showed that pattern on average is a physically observable phenomenon. The experiment is performed by vibrating a fluid layer at a fixed amplitude and frequency. As in Rayleigh–Bénard convection, it is also well known that a trivial state loses stability as the physical stress parameter of the system—in this case, the frequency of vibration—is increased. In the Faraday experiment the flat surface of the fluid layer begins to deform and forms surface waves as this frequency is increased. It is also well known that if the frequency of vibration is further increased, the fluid surface begins to vary chaotically both in space and time. Observations can be made on this system by measuring the intensity of light transmitted through the fluid layer. In the experiment this intensity is time-averaged at each point in space. The experiment was performed in both square and circularly symmetric containers, and in both cases the time-average turned out to reflect the symmetry of the apparatus. See Figure 1.

In another direction, numerical simulations of planar discrete dynamical systems with symmetry have illustrated the symmetry properties of attractors [10, 12, 15] and the possibility that the symmetry of attractors can change—through symmetry increasing bifurcations—as parameters are varied. (It has also been observed by the computation of approximate invariant measures on these attractors and by the illustration of these measures through the use of color that striking images may be found [12].)

With these various manifestations of the symmetry of attractors in mind, the question of how to compute numerically the symmetry of attractors for maps, ODEs and PDEs becomes important. In [3], a method, based on the notion of *detectives*, was developed to answer this question. The idea behind detectives is to transfer the question of determining the symmetry of a set (the attractor) in phase space to the problem of determining the symmetry of a point in some auxiliary space determined by the symmetry of the dynamical system. The way this method works is to thicken the attractor A to an open set $\hat{A}$ (preserving the symmetries) and then to integrate (with respect to Lebesgue measure) a certain (equivariant) observable over the thickened attractor. This technique was then proved to give the correct symmetry for open sets—at least generically—and was also implemented in [3] to show that it could work in practice.

There was, however, a difficulty concerning the use of detectives for computing the symmetry of attractors, which was not noted in [3]. The difficulty surrounds the notion of genericity used. The proof of the detective theorem relies on having points of trivial isotropy in the set $\hat{A}$—which is automatically valid for open sets. However, if A itself has no points of trivial isotropy, then $\hat{A}$ will, in general, not be generic in the sense used in the detective theorem.

It is worth noting that the scientific interest in the symmetries of attractors will be most directly understood in processes that are modeled

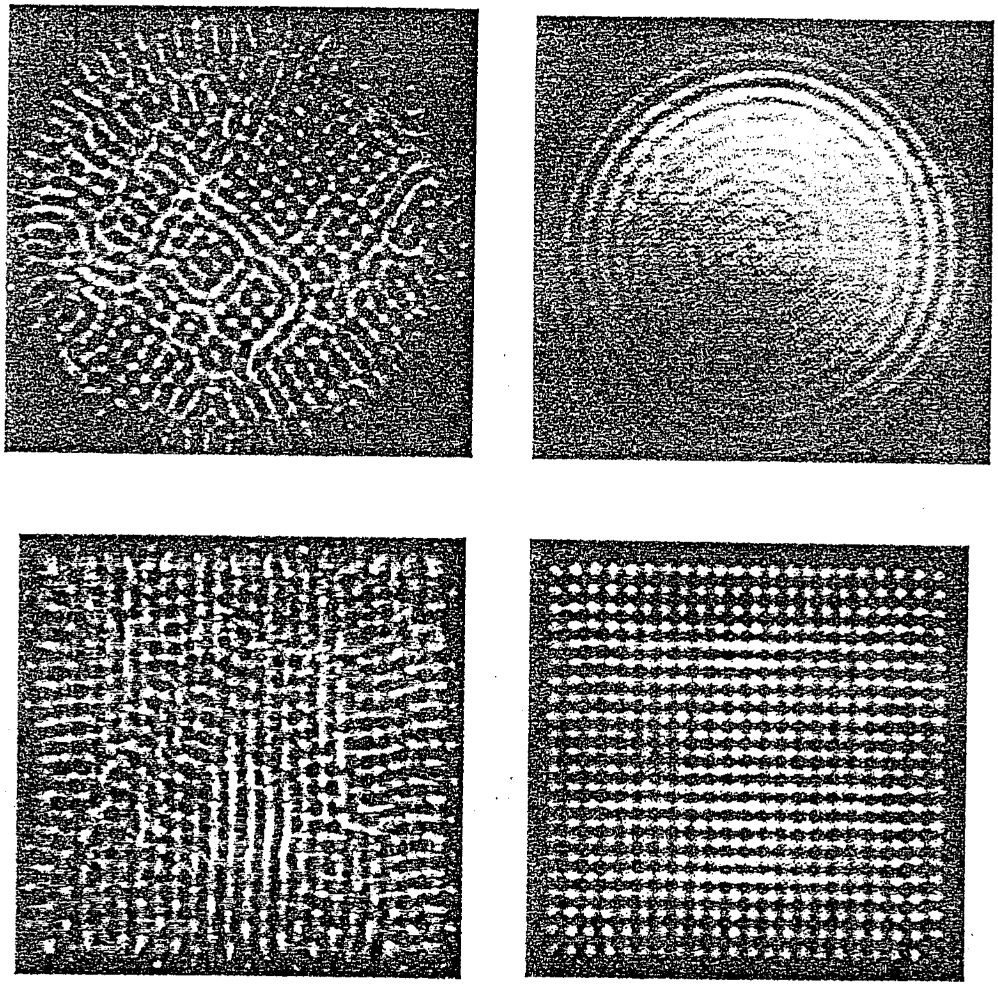


FIGURE 1. Pictures of instantaneous time and time-averaged intensities of transmitted light in the Faraday experiment in both circular and square geometries. Courtesy of J.P. Gollub.

by PDEs. One reason for this is that one needs space variables to support the symmetries and a time variable to support the dynamics. It is well known, both for equilibria and for time-periodic states, how to transfer symmetries in phase space to symmetries (or patterns) in physical space. For more complicated states the question of how to transfer the symmetry of attractors in phase space to a meaningful quantity in physical space is less clear and this issue for chaotic dynamics has received less attention. The numerical and fluid dynamics experiments suggest that this relationship can and should be made through time-averages. In Sections 4.2 and 4.3 we will show how, with the assumption of a Sinai–Bowen–Ruelle (SBR) measure on the attractor in phase space, one can prove these statements about the time average.

As described in [3] the method of integrating over thickened attractors is not practical in dimensions of more than moderate size—say in dimensions greater than six. An alternate approach to detectives where (Lebesgue) integration is replaced by ergodic sums was suggested in [3] and numerical comparisons between the two methods made (in low dimensions). In Section 4.3 we prove that detectives based on ergodic sums also generically predict the correct symmetries of attractors.

As mentioned previously there is a popular method for computing the long-term dynamics of a system of PDEs and of constructing sets of model ODEs for these dynamics and, indeed, for any time series. The Karhunen–Loève decomposition, also known as the proper orthogonal decomposition and by other aliases, proceeds by finding an orthogonal set of eigenfunctions that is well suited to the data—eigenfunctions that capture in decreasing order most of the "kinetic energy" of the system. The data is then expanded in terms of these eigenfunctions at each moment in time and the time variation of the coefficients describes the dynamics. To obtain a system of ODEs the eigenfunction expansion is truncated at some finite order, thus obtaining a sophisticated Galerkin-type approximation to the dynamics. See [23, 6].

The importance of symmetry for the Karhunen–Loève decomposition was emphasized in the work of Sirovich [22]. This theme has been expanded in recent work [2, 6, 7]. We have investigated how well the symmetry of the attractor used to generate the Karhunen–Loève decomposition is reflected in the end result. In Section 4.6 we will show that the Karhunen–Loève operator is equivariant with respect to the symmetry group of the underlying attractor. With this result we generalize the recent results in [7], where only Abelian symmetry groups are considered. But we have also found that the symmetry property of the Karhunen–Loève decomposition does not always exactly reflect the symmetry properties of the underlying attractor. In fact, in some important cases, there is more symmetry introduced into the reduced system of ODEs than is present in the data. For instance, we will show that an $\mathbf{SO}(2)$ symmetric attractor of a scalar PDE on the line with periodic boundary conditions automatically leads to a reduced system which has $O(2)$ symmetry. (This point will be discussed in more detail in Example 6.4.) We then suggest an extension of the Karhunen–Loève decomposition that is guaranteed to have the correct symmetry properties and show that this extension can be viewed as a construction of a detective for this case. Our results suggest that the method for constructing an appropriate reduced system via a Karhunen–Loève decomposition should always be combined with the computation of the symmetry type of the underlying attractor using detectives.

We now outline the structure of this chapter. In Section 4.2 we discuss the results on detectives given in [3] and introduce SBR measures. In Section 4.3 we prove that ergodic sums also provide a method for constructing detectives. We then interpret these results for systems of PDEs in Section 4.4.

The remainder of the chapter discusses symmetry aspects of the Karhunen–Loève decomposition. In Section 4.5 we describe the standard Karhunen–Loève decomposition and in Section 4.6 we show how the symmetries of an attractor for a PDE system are inherited by the Karhunen–Loève decomposition. We note that this decomposition has at least the symmetries of the PDE attractor; as noted previously it may have more. In Section 4.7 we show how to modify this method so that it will produce the correct symmetries. This technique is based on the theory of detectives of Section 4.3. In the last two sections we discuss the symmetry of the reduced (Galerkin type) system of ODEs produced by the Karhunen–Loève decomposition (Section 4.8) and present an example—the Kuramoto–Sivashinsky equation (Section 4.9).

4.2 Detectives and SBR Measures

In this section we introduce the notion of detectives and recall the main theorem of [3] which presents one method for constructing detectives based on Lebesgue integration. The main aim of detectives was to find a method for the (numerical or experimental) computation of the symmetry types of attractors. We also recall a second method based on ergodic sums. In the next section we will prove that this method also yields detectives—at least under the assumption that the attractor has an SBR measure. With this in mind we also introduce SBR measures in this section.

Detectives

Let Γ be a finite group acting orthogonally on $\mathbf{R}^n$ and let A be an open subset of $\mathbf{R}^n$ with piecewise smooth boundary. We discuss a method for determining the symmetries of the set A—defined as follows:

$$\Sigma(A) = \{\gamma \in \Gamma : \gamma A = A\}.$$

We assume that A satisfies the dichotomy $\gamma A = A$ or $\gamma A \cap A = \emptyset$ for all $\gamma \in \Gamma$. (This dichotomy is natural (see [10], [18]) when the set A is a "thickened attractor" (see [3]) for a continuous Γ-equivariant discrete dynamical system.) We will denote this class of open sets by $\mathcal{A}$.

We find the symmetry of sets A by transferring the question to that of finding the symmetry of a point in some observation space W. This is done by use of observables, which we now define. Let W be a finite-dimensional representation space of Γ.

Definition 2.1. An *observable* is a C^1 Γ-equivariant mapping $\phi : \mathbf{R}^n \to W$. The *observation* of A is

$$K_\phi(A) = \int_A \phi \, d\nu,$$

where ν is Lebesgue measure.

Note that the observation $K_\phi(A)$ is a vector in the space W since the observation is just the integral of a W-valued function. Thus, $K_\phi : \mathcal{A} \to W$.

It can easily be verified that $K_\phi(A)$ always possesses at least the symmetry properties of the set A. More precisely, the isotropy subgroup of the observation $K_\phi(A)$ in W,

$$\Sigma_\phi(A) = \{\gamma \in \Gamma : \gamma K_\phi(A) = K_\phi(A)\},$$

always contains $\Sigma(A)$. Observables ϕ which generically yield equality of $\Sigma(A)$ and $\Sigma_\phi(A)$ are called *detectives.*

Definition 2.2. The observable ϕ is a *detective* if for each subset $A \in \mathcal{A}$, an open dense subset of near identity Γ-equivariant diffeomorphisms $\psi : \mathbf{R}^n \to \mathbf{R}^n$ satisfy $\Sigma_\phi(\psi(A)) = \Sigma(A)$.

Before stating the main theorem we introduce the notion of lattice equivalence.

Definition 2.3. Two representation spaces V and W of Γ are *lattice equivalent* if there exists a linear isomorphism $L : V \to W$ such that

$$L(\mathrm{Fix}_V(\Sigma)) = \mathrm{Fix}_W(\Sigma)$$

for every subgroup $\Sigma \subset \Gamma$.

Let $W_1, \ldots, W_s$ be, up to lattice equivalence, all the nontrivial irreducible representations of Γ and define

$$W(\Gamma) = W_1 \oplus \cdots \oplus W_s. \tag{2.1}$$

The following theorem is the main result of [3]:

Theorem 2.4. *Let W contain $W(\Gamma)$ and let $W = W(\Gamma) \oplus W^0$. Let $\phi : \mathbf{R}^n \to W$ be a polynomial observable where $\phi = (\phi_1, \ldots, \phi_s, \phi^0)$ in coordinates adapted to the decomposition of W. Suppose that $\phi_j \neq 0$ for all $1 \leq j \leq s$. Then ϕ is a detective.*

It was noted in [3] that detectives based on ergodic sums provides an alternative method to computing symmetries that is more effective in many instances. We shall prove an analogue to Theorem 2.4 for ergodic sums—but we will have to make explicit our assumption that attractors have points of trivial isotropy.

When speaking of the ergodic sum we will assume that the set A is an *attractor* (that is, A is the ω-limit set of a point $x \in \mathbf{R}^n$ for a continuous mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ and there is an open neighborhood $U \supset A$ in which all points have ω-limit sets contained in A) rather than an open set. For these attractors A, the ergodic sum is defined by

$$K_\phi^E(A) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(f^j(x)). \tag{2.2}$$

For this definition of $K_\phi^E(A)$ to be useful, the right-hand side of (2.2) must be largely independent of x in a sense that we will make precise later.

The advantage of using the ergodic sum is particularly evident in PDE systems where typically approximations in high-dimensional spaces are taken. In the next section we show that the ergodic sum also transforms observables ϕ into detectives under the same conditions on ϕ and W that work for the method of integrating with respect to Lebesgue measure over thickened attractors. However, an extra assumption must be imposed on the type of attractor A for which the method will work; in particular, we must assume that a Sinai–Bowen–Ruelle measure exists on A. We now discuss why we need this assumption and why it is a reasonable assumption to make.

Sinai–Bowen–Ruelle Measures

For the ergodic sum method to be useful, the limit in (2.2) should be independent of the point $x \in \mathbf{R}^n$. What is actually required is that Lebesgue a.e. $x \in U$ give the same limiting sum. It is well known that for any continuous transformation of a compact metric space, there is always an ergodic invariant measure [19]. Note that $\overline{U}$ is a compact f-invariant set. Then the Birkhoff ergodic theorem states: If ρ is an f-invariant ergodic measure on $\overline{U}$ and if $\phi \in L^2(\overline{U})$, then for ρ a.e. $x \in \overline{U}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(f^j(x)) = \int_{\overline{U}} \phi d\rho. \tag{2.3}$$

However, ρ is typically a singular measure and so this result has little physical relevance in terms of justifying the arbitrariness of the choice of x. In particular, in order for this method to be useful, we need to be able to choose almost any $x \in U$ and get the same answer on the left-hand side of (2.3). Thus, for the ergodic sum method to work, one needs a physically relevant measure, a Sinai–Bowen–Ruelle measure. The consequence of the existence of such a measure is that the ergodic sum is then constant Lebesgue a.e. in a neighborhood of the attractor and the constant is equal to the integral of ϕ with respect to the SBR measure over the attractor. This equality allows us to construct detectives in a way similar to the method outlined previously.

There is a general feeling that SBR measures "usually" exist in physical dynamical systems [25]—although rigorous results along these lines are scarce. Thus, the assumption of an SBR measure, even though it is not usually verifiable, appears to be a reasonable one.

First, we discuss in more detail the properties of SBR measures and list the systems for which the existence of SBR measures has been established. Then, in Sections 4.3 and 4.4, we prove the existence of detectives, using the ergodic sum method, in both discrete dynamical systems and PDE systems.

In the literature there are two main definitions of SBR measures. One stresses the statistical behavior of the system and the other stresses the technical conditions that guarantee this behavior. For most systems the definitions are essentially equivalent. The consequence of either definition is that the average of an observable on the orbit of a point is constant Lebesgue almost everywhere on a neighborhood of the attractor, and this constant is determined by a measure (often singular) on the attractor. For our purposes we will only consider the definition based on statistical properties.

Definition 2.5. An *SBR measure* for a mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ with an attractor A is an ergodic measure μ with support equal to A and with the property that there exists an open neighborhood $U \supset A$ such that for every continuous function $\phi : \mathbf{R}^n \to \mathbf{R}$ and for Lebesgue a.e. $x \in U$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(f^j(x)) = \int_A \phi d\mu. \tag{2.4}$$

Typically, we apply this definition to Γ-equivariant $\phi : \mathbf{R}^n \to W$ where W is a vector space on which Γ acts. This definition is sometimes weakened to requiring that (2.4) holds for a set of positive Lebesgue measure and requiring that $\mathrm{supp}(\mu) \subset A$. We call this the *weak form* of the SBR measure in order to contrast it to the stronger form where (2.4) holds Lebesgue a.e. Note that one consequence of Definition 2.5 is that for Lebesgue a.e. x near A,

$$K_\phi^E(A) = \int_A \phi d\mu.$$

We define an *SBR attractor* to be an attractor with an SBR measure.

Numerical experiments provide justification for the belief that SBR measures are a "common" phenomenon [24]. Their existence is often assumed in physics and numerical experiments. The existence of SBR measures in Axiom A systems has been proven in the works of Bowen, Ruelle, and Sinai [8, 20, 21]. Axiom A systems are those systems for which the nonwandering set is uniformly hyperbolic and whose periodic points are dense.

Although it has been conjectured that many other attractors admit SBR measures (and this conjecture has been supported by numerical results) very few nonuniformly hyperbolic attractors have been shown to possess SBR measures. In fact, maps of Henon and Lozi type are so far the only nonuniformly hyperbolic systems in which SBR measures have been established [25]. Henon maps have the form

$$T_{a,b}(x, y) = (1 - ax^2 + y, bx).$$

SBR measures have been established only for small values of the parameter b. More precisely, Benedicks and Young [5] have shown that there is a set S of positive Lebesgue measure in parameter space such that for $(a, b) \in S$ the map $T_{a,b}$ has an attractor A which is the support of a measure μ such that

$$K_\phi^E(A) = \int_A \phi d\mu$$

holds for a set of positive measure in any sufficiently small neighborhood of A.

Young [24] also has results which show that certain types of Lozi maps have a weak SBR measure; however, the support of this measure is the whole attractor. Lozi maps have the form

$$T_{a,b}(x, y) = (1 - a|x| + by, x).$$

The techniques used to prove these results depend in an essential way upon the hyperbolicity of the system. The main technical difficulty is to obtain absolutely continuous conditional measures on unstable manifolds; then one proves some form of ergodicity which implies that the orbits of points on different unstable manifolds behave the same way statistically.

4.3 Detectives for the Ergodic Sum Method

In this section we prove the existence of detectives using ergodic sums. When symmetry is present, we assume that attractors *have points of trivial isotropy*. The main result of this section is that ergodic sums lead to a detective for such SBR attractors.

Suppose the finite group Γ acts on $\mathbf{R}^n$ and suppose ρ is an SBR measure for the Γ-equivariant map f with attractor A. We denote the symmetry group of A by $\Sigma(A)$ and we denote the isotropy subgroup of the observation $K_\phi^E(A)$ by $\Sigma_\phi(A)$. Thus, there exists an open set U containing A such that (2.3) is valid for Lebesgue a.e. $x \in U$.

We may choose the open set U so that $\Sigma(U) = \Sigma(A)$. (This follows from the compactness of A which allows us to choose U to be an ϵ-neighborhood of A.) We call U the *future basin* and write the SBR attractor as (f, A, ρ, U). The following proposition is proved in the same way as the corresponding result in [3], Proposition 3.3 and for this reason its proof is omitted. It states that given a specific SBR attractor it is possible to find an observable which correctly distinguishes the symmetry of that attractor. A detective satisfies the stronger condition that generically it distinguishes the correct symmetry of SBR attractors.

Proposition 3.1. *Given an SBR attractor (f, A, ρ, U), there is a vector space W and an observable $\phi : \mathrm{R}^n \to W$ such that $\Sigma_\phi(A) = \Sigma(A)$.*

We begin with some preliminary observations. Let ψ be a C^1 Γ-equivariant diffeomorphism of $\mathbf{R}^n$ and let (f, A, ρ, U) be an SBR attractor. Note that $\psi(A)$ is an SBR attractor for the mapping $f_\psi = \psi f \psi^{-1}$. The SBR attractor is $(f_\psi, \psi(A), \rho_\psi, \psi(U))$ where ρ_ψ is the measure defined by $\rho_\psi(B) = \rho(\psi^{-1}(B))$.

Almost everywhere with respect to Lebesgue measure, the future average

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(f_\psi^j(x))$$

converges to $\int_A \phi \circ \psi d\rho$. This is seen by making a change of variables and the fact that ψ is a *nonsingular transformation* (that is, sets of positive Lebesgue measure are taken to sets of positive Lebesgue measure under ψ). Furthermore, if ψ is a C^1 Γ-equivariant diffeomorphism on $\mathbf{R}^n$, then the symmetry groups of A and $\psi(A)$ are identical, that is, $\Sigma(\psi(A)) = \Sigma(A)$.

We now define the notion of detective that is relevant for ergodic sums. Recall that SBR attractors are assumed to have points of trivial isotropy.

Definition 3.2. The observable ϕ is an *SBR detective* if for each SBR attractor (f, A, ρ, U) almost all near identity diffeomorphisms $\psi \in \mathrm{Diff}_\Gamma(\mathbf{R}^n)$

satisfy

$$\Sigma_\phi(\psi(A)) = \Sigma(A).$$

Theorem 3.3. *Let W contain $W(\Gamma)$ and let $W = W(\Gamma) \oplus W^0$. Let $\phi : \mathbf{R}^n \to W$ be a polynomial observable where $\phi = (\phi_1, \ldots, \phi_s, \phi^0)$ in coordinates adapted to the decomposition of W. Suppose that $\phi_j \neq 0$ for all $1 \leq j \leq s$. Then ϕ is an SBR detective.*

This theorem has a natural physical interpretation. It says that we may perturb an SBR attractor (f, A, ρ, U) by any element of an open dense set of near-identity Γ-equivariant diffeomorphisms ψ and the resulting SBR attractor, call it (f', A', ρ', U'), will satisfy

$$\Sigma_\phi(A') = \Sigma(A') = \Sigma(A).$$

Before proceeding with the proof of this theorem we note the following proposition. This proposition is of fundamental importance to the interpretation of symmetries of attractors as leading to patterns in the time-average of experiments. See Remark 4.2 for a discussion of the continuous time version of this proposition. To facilitate the statement of this proposition we make precise our definition of the ergodic sum or time-average of a continuous function.

The *ergodic sum* of a continuous function ϕ is the function $\overline{\phi}$ given by

$$\overline{\phi}(x) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(f^j(x))$$

where this limit exists. When $\overline{\phi}(x)$ is independent of x and ϕ is an observable we may use the notation of Section 2 and write $K_\phi^E(A)$ instead of $\overline{\phi}(x)$. Given a measure ρ and a symmetry σ we define the measure ρ_σ by

$$\rho_\sigma(B) = \rho(\sigma^{-1}(B)).$$

Proposition 3.4. *Suppose (f, A, ρ, U) is an SBR attractor and A is Σ-invariant, where Σ is a subgroup of Γ. Then,*

(a) *The SBR measure ρ is Σ-invariant, that is, $\rho_\sigma = \rho$.*

(b) *The ergodic sum of a continuous function $\phi : \mathbf{R}^n \to \mathbf{R}$ is Σ-invariant, that is, $\overline{\phi}(\sigma x) = \overline{\phi}(x)$ for Lebesgue a.e. $x \in U$.*

It is worth mentioning that Proposition 3.4(a) gives a reason why the color pictures of [12] appear to be symmetric.

Proof. a) To show the invariance of ρ we note that for each continuous function $g : \mathbf{R}^n \to \mathbf{R}$ and for Lebesgue a.e. $x \in U$, we have

$$\int_A g d\rho = \lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} g(f^j(x)) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} g(f^j(\sigma x)) = \int_A g\circ\sigma d\rho.$$

This is a consequence of the Σ-invariance of U, the equivariance of f and the fact that ρ is an SBR measure. Thus for all continuous functions $g : \mathbf{R}^n \to \mathbf{R}$

$$\int_A g d\rho = \int_A g\circ\sigma d\rho.$$

A change of variables and the Σ-invariance of A implies that for all continuous functions $g : \mathbf{R}^n \to \mathbf{R}$

$$\int_A g d\rho = \int_A g d\rho_\sigma .$$

Hence $\rho = \rho_\sigma$.

(b) As a consequence of the Σ-invariance of U and the fact that ρ is an SBR measure with basin U, we have for Lebesgue a.e. $x \in U$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} \phi(f^j(x)) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} \phi(f^j\sigma(x)) = \int_A \phi d\rho.$$

Thus $\overline{\phi}(x) = \overline{\phi}(\sigma x)$ for Lebesgue a.e. $x \in U$. Note that if ϕ is an observable then for Lebesgue a.e. $x \in U$, $\overline{\phi}(x)$ lies in $\mathrm{Fix}_W(\Sigma) \subset W$. ■

The following technical lemma will be needed in the proof of Theorem 3.3. It was given in [3] in the context of observations formed by integrating with respect to Lebesgue measure over thickened attractors. The proof of the analogous result in the present context of observations formed by taking an ergodic sum; so we omit the proof.

Let $\mathrm{Diff}_\Gamma(\mathbf{R}^n)$ denote the set of C^1 Γ-equivariant diffeomorphisms on $\mathbf{R}^n$. We define

$$T_A^\phi : \mathrm{Diff}_\Gamma(\mathbf{R}^n) \to \mathrm{Fix}_W(\Sigma(A))$$

by

$$T_A^\phi(\psi) = \int_A \phi\circ\psi d\rho.$$

Note that a change of variables argument shows that the image of T_A^ϕ is actually in $\mathrm{Fix}_W(\Sigma(A))$.

Lemma 3.5. *Let $\phi : \mathbf{R}^n \to W$ be an observable and assume that W contains $W(\Gamma)$. If for each SBR attractor (f, A, ρ, U) there exists an open neighborhood $\mathcal{N}$ of the identity in* $\mathrm{Diff}_\Gamma(\mathbf{R}^n)$ *such that the observations*

$T_A^\phi(\mathcal{N})$ cover an open neighborhood $\mathcal{O}$ of $K_\phi^E(A)$ in $\mathrm{Fix}_W(\Sigma(A))$, then ϕ is a detective.

Proof of Theorem 3.3. We begin by giving an outline of this proof. We first specify a polynomial observable ϕ. Next we verify that the conditions of Lemma 3.5 hold if a certain linear map constructed from ϕ,

$$L_A^\phi : C^1(\mathbf{R}^n, \mathbf{R}^n) \to W,$$

is onto $\mathrm{Fix}_W(\Sigma(A))$. This step is basically an application of the implicit function theorem. We then show that for an open dense set of C^1 Γ-equivariant diffeomorphisms ψ close to the identity, the corresponding linear maps $L_A^{\phi\circ\psi}$ are onto $\mathrm{Fix}_W(\Sigma(A))$. Thus, by Lemma 3.5, for an open dense set of near identity Γ-equivariant diffeomorphisms ψ, the groups $\Sigma_\phi(\psi(A))$ and $\Sigma(A)$ are identical. Hence, ϕ is an SBR detective.

Now we proceed with the details.

Let ϕ be a polynomial observable such that W_ϕ, the subspace generated by the vectors $\phi(x), x \in \mathbf{R}^n$, is equal to W where $W \supset W(\Gamma)$. It is shown in [3] how to obtain such an observable ϕ and vector space W. In particular, we may take W to be $W(\Gamma)$ and ϕ to be a polynomial observable with a nonzero component in each irreducible representation.

In light of Lemma 3.5 we need to show that for each SBR attractor (f, A, ρ, U) and for each element ψ of an open dense set of near identity Γ-equivariant diffeomorphisms, the map $T_A^{\phi\circ\psi}$ is onto a neighborhood of $\int_A \phi d\rho$. We do this by using the implicit function theorem.

The Lebesgue-dominated convergence theorem allows one to show that the map T_A^ϕ is smooth and to compute its derivative L_A^ϕ. More precisely, let ψ_t be a smooth one-parameter family of C^1 Γ-equivariant maps of $\mathbf{R}^n$, with ψ_0 the identity map. Let $X = \frac{d}{dt}\psi_t\big|_{t=0}$. Then

$$\begin{aligned} L_A^\phi(X) &= \frac{d}{dt}\bigg|_{t=0} \int_A \phi\circ\psi_t d\rho \\ &= \int_A D\phi(X) d\rho. \end{aligned} \tag{3.1}$$

Since $L_A^\phi(X)$ is the derivative of a mapping whose image lies in $\mathrm{Fix}_W(\Sigma(A))$, its image also lies in $\mathrm{Fix}_W(\Sigma(A))$. If L_A^ϕ is onto $\mathrm{Fix}_W(\Sigma(A))$, then as a consequence of the implicit function theorem and Lemma 3.5, ϕ is a detective.

We now show that L_A^ϕ may be perturbed to $L_A^{\phi\circ\psi}$ for an open dense set of Γ-equivariant $\psi \in \mathrm{Diff}_\Gamma(\mathbf{R}^n)$ so that $L_A^{\phi\circ\psi}$ is onto $\mathrm{Fix}_W(\Sigma(A))$. This will be sufficient to establish the theorem.

There are four steps in this proof:

1. First, we thicken A to $A^\epsilon \equiv \{x : d(x, A) < \epsilon\}$ by choosing ϵ sufficiently small so that the symmetry group of A is the same as the symmetry group

of A^ϵ. Let $P : W \to \mathrm{Fix}_W(\Sigma(A))$ be the orthogonal projection. Define the vector space $S^\epsilon_\phi \subset \mathrm{Fix}_W(\Sigma(A))$ by

$$S^\epsilon_\phi = \text{ span } \{P(D\phi)_x(X(x))\},$$

where $x \in A^\epsilon$ and $X \in C^1(\mathbf{R}^n, \mathbf{R}^n)$. We claim that $S^\epsilon_\phi = \mathrm{Fix}_W(\Sigma(A))$. We begin by noting that if $g : \mathbf{R}^n \to \mathbf{R}^m$ is smooth and if the images $(Dg)_x X(x)$ all lie in a proper subspace of $\mathbf{R}^m$, then, modulo a fixed constant vector, the image of g also lies in that subspace. We also note that since ϕ is a polynomial mapping and A^ϵ is an open subset of $\mathbf{R}^n$, the space W_ϕ is equal to the span of $\phi(x)$ for all $x \in A^\epsilon$. Applying the first comment to $g = P \circ \phi | A^\epsilon$, it follows that if the linear subspace S^ϵ_ϕ is a proper subspace of $\mathrm{Fix}_W(\Sigma(A))$, then $P(W_\phi)$ must lie in a proper subset of $\mathrm{Fix}_W(\Sigma(A))$, contradicting the assumption that $W_\phi = W$.

2. Since $S^\epsilon_\phi = \mathrm{Fix}_W(\Sigma(A))$, we may choose a finite number of points $x_i \in A^\epsilon$ and a finite number of vector fields $X_i \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ such that the set of vectors $\{P(D\phi)_{x_i}(X_i(x_i))\}$ is a basis for $\mathrm{Fix}_W(\Sigma(A))$. By continuity, this basis property holds for $y_i \in A^\epsilon$ sufficiently close to x_i.

3. In this step our goal is to show that the image of L^ϕ_A [see (3.1)] is onto S_ϕ, where the subspace S_ϕ is defined as

$$S_\phi = \text{ span } \{P(D\phi)_x(X(x))\},$$

for $x \in A$ and $X \in C^1(\mathbf{R}^n, \mathbf{R}^n)$. Certainly the image of L^ϕ_A is contained in S_ϕ. We prove the reverse inclusion, with the aid of the trace formula [3], which gives an explicit formula for the projection P defined by

$$P(v) = \frac{1}{|\Sigma(A)|} \sum_{\sigma \in \Sigma(A)} \sigma(v). \tag{3.2}$$

Choose points $z_i \in A$ and vector fields $X_i \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ so that $\{P(D\phi)_{z_i}(X_i(z_i))\}$ is a basis for S_ϕ. We now show that we may approximate each vector $\{P(D\phi)_{z_i}(X_i(z_i))\}$ arbitrarily well by a vector in the image of L^ϕ_A and, hence by linearity, we have that the image of L^ϕ_A contains S_ϕ. For concreteness, choose the vector $P(D\phi)_{z_1}(X_1(z_1))$. Let $B_\epsilon(z_1)$ be a small ball centered at z_1. Let X be a vector field such that

$$X(z) = \begin{cases} \frac{1}{|\Sigma(A)|\rho(B_\epsilon(z_1))} X_1(z_1) & \text{for } z \in B_\epsilon(z_1), \\ 0 & \text{off of a slightly larger set.} \end{cases}$$

Use the group action of Γ to extend X to a Γ-equivariant vector field on $\mathbf{R}^n$ supported on the balls $\gamma B_\epsilon(z_1)$ (which we can assume are either disjoint or equal if ϵ is small enough). Then,

$$\begin{aligned} L_A^\phi(X) &= \int_A D\phi(X)d\rho \\ &\approx \sum_{\sigma\in\Sigma(A)} \int_{B_\epsilon(\sigma z_1)} \frac{1}{|\Sigma(A)|\rho(B_\epsilon(\sigma z_1))}(D\phi)_x(X_1(x))d\rho \qquad (3.3) \\ &\underset{\epsilon\to 0}{\longrightarrow} \frac{1}{|\Sigma(A)|} \sum_{\sigma\in\Sigma(A)} (D\phi)_{\sigma z_1}(X_1(\sigma z_1)) \\ &= P(D\phi)_{z_1}(X_1(z_1)) \quad \text{by (3.2).} \end{aligned}$$

Thus, the image of L_A^ϕ equals S_ϕ. Note that in the last equality we needed to use the fact that $\rho(B_\epsilon(\sigma x)) = \rho(B_\epsilon(x))$, which follows from the $\Sigma(A)$-invariance of SBR measures. See Remark 3.4.

There is an error in (3.3) due to the truncation of the vector field X outside the ball $B(z_1, \epsilon)$, which we have ignored for ease of exposition. It is easy to see that this error can be controlled and we omit the proof.

4. Recall that S_ϕ^ϵ equals $\mathrm{Fix}_W(\Sigma(A))$. Choose a basis for $\mathrm{Fix}_W(\Sigma(A))$ of the form $\{P(D\phi)_{x_i}(X_i(x_i))\}$ where $x_i \in A^\epsilon$ and $X_i \in C^1(\mathbf{R}^n, \mathbf{R}^n)$. We may assume that the x_i's have disjoint orbits under Γ and have trivial isotropy. Note that if an attractor has one point of trivial isotropy, then it has a dense subset of points with trivial isotropy.

Now choose $a_i \in A$ close to x_i with trivial isotropy and map $x_i \to a_i$ under a Γ-equivariant diffeomorphism ψ. We now define

$$S_\psi = \text{ span } \{P(D\phi)_{\psi(x)}(X(\psi(x)))\},$$

where $x \in A$ and $X \in C^1(\mathbf{R}^n, \mathbf{R}^n)$. By our choice of ψ the sets S_ψ and $\mathrm{Fix}_W(\Sigma(A))$ are equal. But, as shown in step 3, S_ψ is the same as the image of $L_A^{\phi\circ\psi}$.

Hence, the image of $L_A^{\phi\circ\psi}$ is equal to $\mathrm{Fix}_W(\Sigma(A))$ which finishes the proof. ∎

4.4 Detectives for PDE Systems

In this section we show that detectives exist in the context of the ergodic sum method for revealing the symmetry of an attractor A in a PDE system. The proof of this result involves representing each function $U(x,t)$ with respect to a basis and then truncating the representation at a high enough dimension so that Γ acts faithfully on the resulting finite-dimensional space.

The Equations and Their Symmetry

We investigate the dynamical behavior of a system of partial differential equations of the form

$$\frac{\partial}{\partial t} U = G(U), \qquad (4.1)$$

where G is a differential operator and $U(x,t) = (U^1(x,t),\ldots,U^m(x,t))$ is a function of spatial coordinates x and time t. We consider this system on a domain $\Omega \subset \mathbf{R}^n$ and impose some homogeneous boundary conditions. There are three types of symmetry for this system that we will consider here.

(a) The symmetry $\mathbf{D}_p$ of the domain Ω.

(b) The symmetry of the boundary conditions.

(c) Range symmetries of the operator G.

We now discuss each of these symmetry types in turn. We assume that the partial differential equation (4.1) is invariant under all symmetries of the domain and we have denoted this group of symmetries by $\mathbf{D}_p$. We assume that the boundary conditions respect these domain symmetries. However, depending on the domain and the type of boundary conditions, there may be additional symmetries for the PDE. For example, when a differential operator defined on one space variable has translation symmetry, then periodic boundary conditions with translation symmetry yield $\mathbf{SO}(2)$ symmetry in (4.1). We combine all of these domain symmetries into the group Γ_d. Finally, there are symmetries that act on the range of the differential operator. For example, in reaction–diffusion equations there is a $\mathbf{Z}_2$ symmetry that appears when the reaction term is itself odd. We denote the range symmetries by Γ_r and observe that the actions of the groups Γ_d and Γ_r on (4.1) commute . Hence, the full group of symmetries that we consider is $\Gamma = \Gamma_r \times \Gamma_d$.

More precisely, these domain and range symmetries act on the domain and range function spaces on which the differential equation is defined. We now consider these spaces. We assume that an appropriate function space for (4.1) is a subspace $\mathcal{X}$ of $L^2(\Omega)$ and we denote by $\mathcal{X}_0$ the subspace of elements that satisfy the boundary conditions. This system can be considered as a dynamical system. There is a continuous one-parameter family of maps $S(t) : \mathcal{X}_0 \to \mathcal{X}_0$ where $S(t)(U_0(0)) = U_0(t)$ and $U_0(t) = U(x,t)$.

We now consider the action of Γ on function spaces. An element $\gamma_d \in \Gamma_d$ acts on $V \in \mathcal{X}_0$ by a coordinate transformation

$$\gamma_d V(x) = V(\gamma_d^{-1} x).$$

On the other hand, the range symmetries of G are symmetry operations on the components of V. An example for this is the gauge symmetry in the *complex Ginzburg–Landau equation* (see also Example 6.5). We assume that Γ_r consists of orthogonal matrices.

The previous discussion shows that it is natural to denote an element $\gamma \in \Gamma$ as a pair

$$\gamma = (\gamma_r, \gamma_d),$$

where γ_r is acting on the range and γ_d is acting on the domain. The action on $V \in \mathcal{X}_0$ is explicitly given by

$$\gamma V(x) = (\gamma_r, \gamma_d)V(x) = \gamma_r V(\gamma_d^{-1} x).$$

The symmetry group of a subset $A \subset L^2(\Omega)$ is defined by

$$\Sigma(A) = \{\gamma \in \Gamma : \gamma A = A\}.$$

As for isotropy subgroups of points, it is easy to verify that for $\gamma \in \Gamma$

$$\Sigma(\gamma A) = \gamma \Sigma(A) \gamma^{-1}. \tag{4.2}$$

The Attractor

Definition 4.1. The compact set $A \subset \mathcal{X}_0$ is a *PDE attractor* if the following properties are satisfied:

(a) There is a neighborhood $\mathcal{N} \supset A$ such that for all $U \in \mathcal{N}$, $S(t)U \to A$.

(b) $S(t)A \subset A$ for all $t \geq 0$.

(c) A has an $S(t)$-invariant measure μ, $S(t)|_A$ is ergodic, and for all U_0 in an open, dense Σ-symmetric subset $\mathcal{N}_0 \subset \mathcal{N}$ and for all continuous $\Phi : \mathcal{X}_0 \to \mathbf{R}$ we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(S(t)U_0)dt = \int_A \Phi d\mu.$$

We call the measure μ defined on the PDE attractor A an *SBR measure* for A.

Remarks 4.2.

1. *Let $\sigma \in \Sigma(A)$ and assume that A is a PDE attractor with SBR measure μ. Then μ is σ-invariant.* The proof of this is identical to the proof of the corresponding statement in Remark 3.4

2. *If Φ is a continuous function then $\overline{\Phi}(\sigma U_0) = \overline{\Phi}(U_0)$ for all $U_0 \in \mathcal{N}_0$.* The proof of this also is identical to the proof of the corresponding statement in Proposition 3.4(b).

3. *The four conditions of Definition* 4.1 *are satisfied whenever there is an inertial manifold M, conditions* (a) *and* (b) *hold and the set A is a finite-dimensional Axiom A attractor for $S(t)$.*

4. By the assumption on $\mathcal{N}_0 \subset \mathcal{N}$ we give a topological notion of a "large" or "physically relevant" set to avoid the technical difficulties associated with Gaussian or Lebesgue measures on infinite dimensional spaces.

Note that (2) states that the time-average can be symmetric, that is, have a well-defined "pattern", even though at each instant of time no symmetry exists.

The Basis Representation

Assume that the compact group Γ acts linearly on $\mathcal{X}$. Let $\{v_k\}$ be a basis for $\mathcal{X}$. We assume that $\mathcal{X} \supset W(\Gamma)$ and that

$$\mathcal{X} = \oplus V_p$$

where each V_p is a Γ-invariant subspace generated by basis functions. (This basis could be obtained by solving an eigenvalue problem and then V_p would correspond to a particular eigenspace.) Thus, we may write each vector $U \in \mathcal{N}$ in terms of this decomposition as

$$U(x,t) = \sum_{k=0}^{\infty} a_k(t) v_k(x). \tag{4.3}$$

We truncate the series at $\mathcal{V}_P = \oplus_{p=0}^{P} V_p$. There is a natural action of Γ on $\mathcal{V}_P$. We choose P large enough so that Γ acts faithfully on $\mathcal{V}_P$.

Let Π_P be the projection $\Pi_P : \mathcal{X} \to \mathcal{V}_P$. Then Π_P is Γ-equivariant. Let $\phi : \mathcal{V}_P \to W$ be a detective for Γ; in particular, we assume that $W \supset W(\Gamma)$ and that ϕ is a polynomial mapping which is nonzero on the irreducible subspaces of $W(\Gamma)$.

Theorem 4.3. *$\phi \circ \Pi_P$ is a detective for the PDE.*

Proof. For $U \in \mathcal{N}_0$ we define

$$K^E_{\phi \circ \Pi_P}(A) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi \circ \Pi_p U(x,t) dt.$$

This vector lies in W. By Definition 4.1(c), as $\phi \circ \Pi_p$ is continuous, for all $U \in \mathcal{N}_0$ the integral above equals $\int_A \phi \circ \Pi_p(U) d\mu(U)$. This integral is easily seen to be $\Sigma(A)$-invariant by the usual change of variables argument and $\Sigma(A)$-invariance of A.

Thus, $K^E_{\phi \circ \Pi_P}(A)$ lies in the fixed-point subspace of $\Sigma(A)$, $\mathrm{Fix}_W(\Sigma(A))$. We want to show that generically it lies in the fixed-point subspace of no larger subgroup; that is, the observable is a detective. In fact, we show that for an open dense set of near-identity invertible Γ-equivariant transformations $\Psi : \mathcal{X}_0 \to \mathcal{X}_0$, the observation

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi \circ \Pi_p \Psi \circ U(x,t) dt = \int_A \phi \circ \Pi_p(\Psi \circ U) d\mu(U)$$

has symmetry group $\Sigma(A)$. We consider a perturbation Ψ as acting on the components of the representation of a function $U(x,t)$; that is, if $U(x,t)$ has basis representation

$$U(x,t) = \sum_{k=0}^{\infty} a_k(t) v_k(x),$$

then

$$\Psi \circ U(x,t) = \sum_{k=0}^{\infty} a_k^{\Psi}(t) v_k(x).$$

Note that W has been chosen so that $W \supset W(\Gamma)$. We adopt with little modification the following lemma which was used in the proof of the existence of detectives for the ergodic sum method in the case of maps.

Lemma 4.4. *Let $\phi : \mathcal{V}_P \to W$ be an observable and assume that W contains $W(\Gamma)$. If for each PDE attractor $(S(t), A, \mu, \mathcal{N})$ there exists an open neighborhood $\mathcal{O}$ of the identity in the space of invertible Γ-equivariant maps $\psi : \mathcal{V}_P \to \mathcal{V}_P$ such that the observations $\int_A \phi \circ \psi \circ \Pi_P d\mu$ cover an open neighborhood $\mathcal{O}'$ of $\int_A \phi \circ \Pi_P d\mu$ in $\mathrm{Fix}_W(\Sigma(A))$, then $\phi \circ \Pi_P$ is a detective.*

Proof. The same proof given in [3] for the case of maps holds in this context. The only thing to note is that an open set of $\psi : \mathcal{V}_P \to \mathcal{V}_P$ extends to an open set of transformations $\Psi : \mathcal{X}_0 \to \mathcal{X}_0$ where $\Pi_P \circ \Psi = \psi \circ \Pi_P$. ∎

We return to the proof of the theorem. Let ψ_t be a one-parameter family of maps $\psi_t : \mathcal{V}_P \to \mathcal{V}_P$. Define $X = \frac{d}{dt}\big|_{t=0} \psi_t$. We form the linear map

$$L_A^{\phi}(X) = \frac{d}{dt}\bigg|_{t=0} \int_A \phi \circ \psi_t \circ \Pi_P d\mu. \tag{4.4}$$

Note that by the Lebesgue-dominated convergence theorem,

$$L_A^{\phi}(X) = \int_A (D\phi) X(\Pi_P U) d\mu(U),$$

and it is easy to see that $L_A^{\phi}(X)$ lies in $\mathrm{Fix}_W(\Sigma(A))$. If we show that L_A^{ϕ} is onto $\mathrm{Fix}_W(\Sigma(A))$, then as a consequence of the implicit function theorem and Lemma 4.4, ϕ is a detective.

However, since we have reduced to the finite-dimensional map $\phi : \mathcal{V}_P \to W$, the same proof given previously holds. Note that this argument uses just the differentiability of $\phi \circ \psi$ and the fact that ϕ is a Γ-equivariant polynomial map with a nonzero component in each irreducible representation of Γ (which gives $W_\phi = W$). Thus, we show in exactly the same manner as before that, for each element ψ in an open dense set of Γ-equivariant near identity diffeomorphisms, $L_A^{\phi \circ \psi}$ is onto $\mathrm{Fix}_W(\Sigma(A))$. Hence, ϕ is a detective. ∎

One interpretation of this fact is that given any basis for $\mathcal{X}_0$ for which the conditions above can be demonstrated to hold and a PDE system with attractor A, then for any of an open, dense set of near identity invertible transformations $\Psi : \mathcal{X}_0 \to \mathcal{X}_0$, the observation

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi \circ \Pi_P \circ \Psi(U(x,t)) dt \tag{4.5}$$

has the same symmetries as A for all $U \in \mathcal{N}_0$. Another interpretation is: Suppose we perturb $(S(t), A, \mathcal{N})$ with flow $S(t)$ where A is an attractor with basin $\mathcal{N}$ by any of an open dense set of invertible Γ-equivariant Ψ. Then we obtain the flow $\Psi \circ S(t) \circ \Psi^{-1}$ with attractor $\Psi(A)$ and basin $\Psi(\mathcal{N})$, and for these Ψ the symmetry group of our observation on $\Psi(A)$ is precisely $\Sigma(A)$.

4.5 The Karhunen–Loève Decomposition

We shall briefly outline the Karhunen–Loève decomposition and its basic properties. For a more detailed discussion the reader is referred to [6, 22].

The idea behind the Karhunen–Loève decomposition can be formulated as follows. Let us assume that (4.1) has a PDE attractor A, as in Definition 4.1. Suppose that U is a solution to the PDE in a neighborhood of A; for example, in a fluid PDE, U will be a velocity field. Then we want to find a direction ϕ_1 in phase space with $\|\phi_1\| = 1$, which has the most kinetic energy on average; that is, we want to maximize the expression

$$\lambda_1 = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T (\phi_1, U(\cdot,t))^2 dt \right),$$

where $(\cdot,\cdot)$ is the usual inner product on $L^2(\Omega)$. Next, we want to find ϕ_2 with $\|\phi_2\| = 1$ and orthogonal to ϕ_1, such that

$$\lambda_2 = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T (\phi_2, U(\cdot,t))^2 dt \right)$$

is maximal. Proceeding inductively this leads to an eigenvalue problem

$$\mathcal{K}_U \phi = \lambda \phi$$

with a (non-negative and compact) integral operator $\mathcal{K}_U$.

Let us be more precise. For a (bounded) function $V : [0,\infty) \to L^2(\Omega^k)$ we define the *temporal ensemble average* by

$$\langle V \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T V(t)\, dt.$$

Let $\left(L^2(\Omega^2)\right)^{m,m}$ be the $m \times m$ matrices with entries in $L^2(\Omega^2)$ and let $\mathcal{L} : L^2(\Omega) \to \left(L^2(\Omega^2)\right)^{m,m}$ be defined by

$$\mathcal{L}(V)(\xi_1, \xi_2) = V(\xi_1) \cdot V(\xi_2)^t ,$$

where the dot stands for matrix multiplication.

Using this notation we define the kernel K

$$K(U)(\xi_1, \xi_2) = \langle \mathcal{L}(U)(\xi_1, \xi_2) \rangle .$$

The kernel K appears to depend on the trajectory U; but, under the assumption that U lies in $\mathcal{N}_0$ (see Definition 4.1), it actually only depends on the attractor defined by the trajectory U. Note that $\mathcal{L}$ acts on U by acting on $U(\cdot, t)$ as a function on Ω for each fixed time t. For instance, for the special case $m = 2$ we obtain

$$K(U)(\xi_1, \xi_2) = \begin{pmatrix} \langle U^1(\xi_1, \cdot) U^1(\xi_2, \cdot) \rangle & \langle U^1(\xi_1, \cdot) U^2(\xi_2, \cdot) \rangle \\ \langle U^2(\xi_1, \cdot) U^1(\xi_2, \cdot) \rangle & \langle U^2(\xi_1, \cdot) U^2(\xi_2, \cdot) \rangle \end{pmatrix} .$$

Finally, we define the associated operator $\mathcal{K}_U$ by

$$(\mathcal{K}_U \phi)(x) = \int_\Omega K(U)(x, y) \phi(y) \, dy .$$

Let μ be the SBR measure on the PDE attractor A. The following equation then holds for all continuous functions $\mathcal{S} : \mathcal{X}_0 \to \mathbf{R}$ and for all $U \in \mathcal{N}_0$:

$$\langle \mathcal{S}(U) \rangle = \int_A \mathcal{S} \, d\mu .$$

In particular we have

$$K(U) = \langle \mathcal{L}(U) \rangle = \int_A \mathcal{L} \, d\mu . \tag{5.1}$$

Moreover, μ is invariant with respect to the symmetry transformations in $\Sigma(A)$ (see Remarks 4.2).

Under certain additional assumptions on the underlying invariant measure μ it can be shown (see, e.g. [1, 6, 19, 22, 23]) that $\mathcal{K}_U$ is non-negative and compact. Hence, its eigenfunctions generate a complete orthonormal set $\{\phi_k\}$. Moreover, the following *Karhunen–Loève decomposition* for U holds almost everywhere with respect to the invariant measure μ (see [1, 6]):

$$U(x, t) = \sum_{k=1}^{\infty} a_k(t) \phi_k(x) . \tag{5.2}$$

One can show (see [6]) that the a_k are uncorrelated, i.e.,

$$\langle a_i a_k \rangle = \delta_{ik} \lambda_k ,$$

where δ_{ik} is the Kronecker delta and the λ_k are the eigenvalues of the operator $\mathcal{K}_U$. If we order the eigenvalues so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0,$$

then it can be shown that the Karhunen–Loève decomposition is optimal in the sense that the first eigenmodes contain the most kinetic energy on average (for a proof, cf. [6, 22]):

Proposition 5.1. *Let $\{a_k(t)\}$ be defined as in (5.2). Let $\{\psi_k\}$ be an arbitrary orthonormal set such that*

$$U(x,t) = \sum_{k=1}^{\infty} b_k(t)\psi_k(x).$$

Then for every n, we have

$$\sum_{k=1}^{n}\langle a_k^2\rangle = \sum_{k=1}^{n}\lambda_k \geq \sum_{k=1}^{n}\langle b_k^2\rangle.$$

4.6 Symmetry in the Karhunen–Loève Decomposition

In the numerical use of the Karhunen–Loève decomposition one performs a Galerkin approximation which is based on the eigenfunctions of the operator $\mathcal{K}_U$. Therefore, it is of interest to study the symmetry properties of the Karhunen–Loève decomposition.

The main results of this section concern the symmetry properties of the kernel $K(U)$ (Proposition 6.1) and the operator $\mathcal{K}_U$ (Proposition 6.3). Essentially we show that both of them possess *at least* the symmetry $\Sigma(A)$ of the underlying PDE attractor A. These results extend those in [7], where abelian groups are considered.

Although we will prove that the operator $\mathcal{K}_U$ has the symmetry $\Sigma(A)$ of the underlying attractor, we will also see that in certain cases the symmetry of the Karhunen–Loève operator does not precisely reflect the symmetry property of the underlying attractor—$\mathcal{K}_U$ might possess more symmetry (see Example 6.4). This can lead to a choice of a basis for the Galerkin method which is not optimal. We will illustrate this by an example in Section 4.9.

Proposition 6.1. *Suppose that $\gamma = (\gamma_r, \gamma_d) \in \Sigma(A)$ and $U \in \mathcal{N}_0$. Then*

$$\gamma_r^t K(U)\gamma_r = K(\gamma_d U).$$

In particular,

$$\gamma_r K(U) = K(U)\gamma_r^t \quad \textit{for all } \gamma_r \in \Sigma(A); \tag{6.1}$$

that is, $K(U)$ is equivariant with respect to the elements in $\Gamma_r \cap \Sigma(A)$.

Remark 6.2. There is one immediate but interesting consequence of this proposition (see also [2]). Suppose that we impose periodic boundary conditions. As indicated above, this implies that Γ_d contains a p-torus T^p and—under the assumption that $T^p \subset \Sigma(A)$—Proposition 6.1 states that

$$K(U)(x,y) = K(U)(x-\theta, y-\theta) \quad \text{for all } \theta \in T^p.$$

Hence, in the case where p is equal to the dimension of the spatial domain, $K(U)$ is simply a periodic function of $z = x - y$.

Proposition 6.3. *If $U \in \mathcal{N}_0$, then $\mathcal{K}_U$ is $\Sigma(A)$-equivariant; that is,*

$$\sigma\mathcal{K}_U = \mathcal{K}_U\sigma \quad \textit{for all } \sigma \in \Sigma(A).$$

The following example illustrates that the symmetry property of the operator $\mathcal{K}_U$ is not exact in the sense that $\mathcal{K}_U$ might possess more than just $\Sigma(A)$-symmetry.

Example 6.4. We consider a system of (parabolic) partial differential equations on the line with periodic boundary conditions, that is, $U(x,t) = U(x+1,t)$ for all $t \geq 0$. Assuming that there is no additional range symmetry, the symmetry group of the problem is $\Gamma = \Gamma_d = \mathbf{SO}(2)$ [or even $\mathbf{O}(2)$ if G does not contain any odd powers of x-derivatives of odd order]. Here $\theta \in \mathbf{SO}(2)$ acts by

$$\theta U(x,t) = U(x-\theta, t)$$

and the reflection $\kappa \in \mathbf{O}(2)$ as

$$\kappa U(x,t) = U(1-x, t).$$

Let A be the ω-limit point set of the time series $\{U(\cdot,t) : 0 \leq t < \infty\}$; that is,

$$A = \cap_{s>0}\overline{\{U(\cdot,t) : t > s\}}.$$

Suppose that A is an $\mathbf{SO}(2)$ symmetric PDE attractor of the system and that the solution U lies in the set $\mathcal{N}_0$ of Definition 4.1. Then by Remark 6.2 the corresponding kernel $K(U)$ satisfies

$$K(U)(x,y) = K(U)(x-\theta, y-\theta) \quad \text{for all } \theta \in \mathbf{SO}(2)$$

and therefore can be written as a periodic function $\mathcal{F}$ of $x - y$

$$K(U)(x,y) = \mathcal{F}(x-y).$$

By construction of the kernel, we have

$$\mathcal{F}(x-y) = \mathcal{F}(y-x)^t.$$

This implies that for $\kappa \in \mathbf{O}(2)$,

$$\begin{aligned} K(\kappa U)(x,y) &= K(U)(1-x,1-y) \\ &= \mathcal{F}(y-x) \\ &= \mathcal{F}(x-y)^t \\ &= K(U)(x,y)^t. \end{aligned}$$

Hence, for a *scalar* equation, we conclude that the kernel K is invariant not only with respect to $\mathbf{SO}(2)$ but also with respect to $\mathbf{O}(2)$. Accordingly, the operator $\mathcal{K}$ is not just $\mathbf{SO}(2)$ but also $\mathbf{O}(2)$-equivariant, even if the system is just $\mathbf{SO}(2)$ symmetric. ∎

We give another example in which the Karhunen–Loève decomposition fails in the sense that the symmetry of the underlying attractor cannot be recovered in the related operator $\mathcal{K}$:

Example 6.5. We consider the complex *Ginzburg–Landau equation*,

$$\frac{\partial U}{\partial t} = q^2(i+c_0)\frac{\partial^2 U}{\partial x^2} + \rho U + (i-\rho)U|U|^2 .$$

The constants q, c_0, ρ are real, whereas U is a complex-valued function. If we impose periodic boundary conditions, this problem has the range symmetry S^1 given by $U \to e^{i\theta}U$ and the domain symmetry $\mathbf{O}(2)$. In the literature usually the following kernel is considered (see, e.g., [19]):

$$K_c(x,y) = \langle U(x)\overline{U(y)}\rangle$$

in which real and imaginary parts are not correlated separately. Obviously this kernel is *invariant* under symmetry transformations in S^1 and therefore it is not possible to distinguish between different range symmetries of attractors by the symmetry properties of the operator $\mathcal{K}$. Correspondingly, phenomena which are related to the S^1 symmetry cannot be seen in a reduced system which is obtained by use of this operator. ∎

In the remainder of this section we prove Propositions 6.1 and 6.3.

Lemma 6.6. *The kernel K is $\Sigma(A)$-invariant: If $U \in \mathcal{N}_0$, then*

$$K(\sigma U) = K(U) \quad \textit{for all } \sigma \in \Sigma(A).$$

Proof. Since $\sigma A = A$ for all $\sigma \in \Sigma(A)$ we compute using (5.1)

$$\begin{aligned} K(\sigma U) &= \langle \mathcal{L}(\sigma U) \rangle \\ &= \int_A \mathcal{L}(\sigma \cdot)\, d\mu \\ &= \int_{\sigma A} \mathcal{L}\, d\mu \qquad \text{(since } \mu \text{ is } \Sigma(A)\text{-invariant)} \\ &= \int_A \mathcal{L}\, d\mu \\ &= K(U)\,. \end{aligned}$$

∎

Proof of Proposition 6.1. We compute

$$\begin{aligned} K(\gamma U)(x,y) &= \langle \gamma_r U(\gamma_d^{-1} x)(\gamma_r U(\gamma_d^{-1} y))^t \rangle \\ &= \langle \gamma_r U(\gamma_d^{-1} x) U(\gamma_d^{-1} y)^t \gamma_r^t \rangle \\ &= \gamma_r \langle U(\gamma_d^{-1} x) U(\gamma_d^{-1} y)^t \rangle \gamma_r^t \\ &= \gamma_r K(\gamma_d U)(x,y) \gamma_r^t. \end{aligned}$$

Using the fact that the elements of Γ_r are orthogonal the result follows immediately by Lemma 6.6. ∎

We consider next the eigenvalue problem

$$\mathcal{K}_U \phi = \lambda \phi\,, \quad \phi \in \mathcal{X}_0\,. \tag{6.2}$$

The following result concerns the symmetry properties of the operator $\mathcal{K}_U$.

Proposition 6.7. *Let $\gamma \in \Gamma$. Then for $U \in \mathcal{N}_0$,*

$$\mathcal{K}_{\gamma U} = \gamma \mathcal{K}_U \gamma^{-1}\,. \tag{6.3}$$

Proof. Using Proposition 6.1 we compute for $\psi \in \mathcal{X}_0$

$$\begin{aligned} (\mathcal{K}_{\gamma U}\psi)(x) &= \int_\Omega K(\gamma U)(x,y)\psi(y)\, dy \\ &= \gamma_r \int_\Omega K(U)(\gamma_d^{-1} x, \gamma_d^{-1} y)\gamma_r^t \psi(y)\, dy \\ &= \gamma_r \int_\Omega K(U)(\gamma_d^{-1} x, y)\gamma_r^t \psi(\gamma_d y)\, dy \\ &= \gamma_r \int_\Omega K(U)(\gamma_d^{-1} x, y)(\gamma_r^t, \gamma_d^{-1})\psi(y)\, dy \\ &= (\gamma \mathcal{K}_U \gamma^{-1} \psi)(x) \end{aligned}$$

and the proposition is proved. ∎

Corollary 6.8. *Suppose that $\phi \in \mathcal{X}_0$, $U \in \mathcal{N}_0$ and $\lambda \in \mathbf{C}$. Then*

$$\mathcal{K}_U\phi = \lambda\phi \quad \Longleftrightarrow \quad \mathcal{K}_{\gamma U}\gamma\phi = \lambda\gamma\phi \quad \text{for all } \gamma \in \Gamma.$$

Proof. We set $\psi = \gamma\phi$. Then with (6.3)

$$\mathcal{K}_{\gamma U}\gamma\phi = \mathcal{K}_{\gamma U}\psi = \gamma\mathcal{K}_U\gamma^{-1}\psi = \gamma\mathcal{K}_U\phi = \lambda\gamma\phi\,. \qquad ∎$$

Proof of Proposition 6.3. The proof follows directly from (6.3) since by Lemma 6.6, $\mathcal{K}_{\sigma U} = \mathcal{K}_U$ for all $\sigma \in \Sigma(A)$. ∎

4.7 Finding the Symmetry of an Attractor

In this section we will indicate how detectives can be used to obtain a Karhunen–Loève decomposition which generically has the appropriate symmetry properties.

First let us investigate the relationship between the quantities in the Karhunen–Loève decomposition and the notion of observables and observations. A comparison of Definition 2.1 and (5.1) shows that $\mathcal{L}$ can be viewed as an observable ϕ and, accordingly, the kernel K represents the observation $K_\phi(A)$. The observation space W then is just the subspace of $(L^2(\Omega^2))^{m,m}$ given by

$$\{(h_{ij}(x,y))_{1\le i,j\le m} \in (L^2(\Omega^2))^{m,m} : h_{ij}(x,y) = h_{ji}(y,x)\}.$$

The open, dense set $\mathcal{N}_0$ can be viewed as the set from which solutions U can be expected to be sampled. The pair $(\gamma_r, \gamma_d) \in \Gamma$ is acting on $w \in W$ as

$$\gamma w = \gamma_r(\gamma_d w)\gamma_r^t\,.$$

In this notation it can be seen that Proposition 6.1 just states the result mentioned in Section 4.2: the observation carries at least the symmetry of the set A.

As pointed out previously, the Karhunen–Loève decomposition is useful when analyzing the dynamical behavior of a PDE as it approximates the dynamics by a low-dimensional ODE (cf. [16]). On the other hand, the discussion in Section 4.6 has shown that this method cannot be used for the detection of the symmetry type $\Sigma(A)$ of an attractor. The symmetry of the operator $\mathcal{K}$ might not exactly reflect the symmetry of the underlying PDE attractor (see Example 6.4) and, as a consequence, the reduced system might not possess the correct symmetry. In other words, the observable $\mathcal{L}$ cannot always be a detective.

One way to overcome this difficulty is to consider detectives for the dynamical system that is used in the numerical simulation of the underlying PDE. Using finite-difference schemes, we always end up having a finite symmetry group acting on a finite-dimensional space so that the results of Sections 4.2 and 4.3 can be applied. Moreover, those approximating dynamical systems always have the form of coupled oscillators with a certain symmetry. The difficulty with this approach is that the symmetry group then depends on the discretization.

For instance, for a system of PDEs on the line with periodic boundary conditions we obtain after the discretization a system of coupled oscillators with D_p-symmetry, where p is the number of points on the line used in the discretization. These are systems of the form

$$\dot{z}_j = f(z_{j-1}, z_j, z_{j+1}, \lambda), \quad (j = 1, \ldots, p), \tag{7.1}$$

where $z_j \in \mathbf{R}^m$ and $f(a,b,c,\lambda) = f(c,b,a,\lambda)$ for all $a,b,c \in \mathbf{R}^m$. In the following theorem a detective for such dynamical systems is presented.

Theorem 7.1 ([3]). *Assume that the number of cells p is at least three and the number of equations governing each cell m is at least two. Then the mapping*

$$\phi(z) = z \cdot z^t \tag{7.2}$$

is a detective.

Observe that in the discretized version of the PDE the kernel $K(U)$ of the Karhunen–Loève decomposition is precisely of the type (7.2). Hence, in the discretized problem, this kernel is a detective if and only if the number of PDEs in the system is greater than or equal to two—assuming that at least three spatial points are taken into account for the discretization. In fact, this also explains why the Karhunen–Loève decomposition does not distinguish between $\mathbf{SO}(2)$ and $\mathbf{O}(2)$ symmetry in Example 6.4. In the case of a scalar PDE on the line, the number of equations m governing each cell in the discretization is one, which is not sufficient by Theorem 7.1.

Instead of (7.2), let us consider the observable $(z, \varphi(z))^t \cdot (z, \varphi(z))$ with an equivariant polynomial function φ. We can view this construction as an artificial introduction of a second component in the oscillator and in the observable. By Theorem 7.1 we know that this extended observable is a detective—even in the case where $m = 1$—as long as φ is nontrivial. In particular, we may choose φ to be the right-hand side in (7.1), that is, we additionally consider correlations of z with its derivative $\dot{z}$.

These considerations suggest that in the PDE we should enlarge the observable $\mathcal{L}$ in the following way. Instead of just considering the time series of U one has to consider the pair (U, U_t) and to redefine $\mathcal{L}$ by

$$\mathcal{L}_e(V)(\xi_1, \xi_2) = (V(\xi_1), V_t(\xi_1))^t \cdot (V(\xi_2), V_t(\xi_2))$$

$$= \begin{pmatrix} V(\xi_1)V(\xi_2) & V(\xi_1)V_t(\xi_2) \\ V_t(\xi_1)V(\xi_2) & V_t(\xi_1)V_t(\xi_2) \end{pmatrix}.$$

Then $\mathbf{SO}(2)$ and $\mathbf{O}(2)$ can be distinguished by the symmetry properties of the corresponding operator $\mathcal{K}_e$. To see this we first observe that, whenever $\mathbf{SO}(2)$ is contained in $\Sigma(A)$, then also the extended kernel K_e has to have the form

$$K_e(U)(x,y) = \begin{pmatrix} \mathcal{F}_1(x-y) & \mathcal{F}_2(x-y) \\ \mathcal{F}_2(y-x) & \mathcal{F}_3(x-y) \end{pmatrix}$$

with periodic functions $\mathcal{F}_j$, $j = 1, 2, 3$ (see Remark 6.2).

Differentiation of $U(x,t)U(y,t)$ with respect to t and averaging shows that

$$\mathcal{F}_2(x-y) = \langle U(x,\cdot)U_t(y,\cdot)\rangle = -\langle U_t(x,\cdot)U(y,\cdot)\rangle = -\mathcal{F}_2(y-x).$$

In particular, for $\mathcal{F}_2$ to possess $\mathbf{O}(2)$ symmetry we must have

$$\begin{aligned} \mathcal{F}_2(x-y) &= \mathcal{F}_2(2\pi - x - (2\pi - y)) \\ &= \mathcal{F}_2(y-x) \\ &= -\mathcal{F}_2(x-y), \end{aligned}$$

i.e., $\mathcal{F}_2$ vanishes identically. Here we have assumed that the reflection in $\mathbf{O}(2)$ is acting by $x \to 2\pi - x$.

The following example illustrates that, as expected, $\mathbf{SO}(2)$ symmetry alone does not imply that $\mathcal{F}_2$ has to vanish:

Example 7.2. Assume that $U(x,t) = w(x-t)$ defines the $\mathbf{SO}(2)$ symmetric attractor. Then the modified kernel K_e becomes

$$K_e(U)(x,y) = \begin{pmatrix} \langle w(x-t)w(y-t)\rangle & \langle -w(x-t)w'(y-t)\rangle \\ \langle -w'(x-t)w(y-t)\rangle & \langle w'(x-t)w'(y-t)\rangle \end{pmatrix}.$$

Since, in general,

$$\langle w(x-t)w'(y-t)\rangle \neq 0$$

we can conclude that in that case $K_e(U)(x,y)$ is not $\mathbf{O}(2)$-invariant.

We will illustrate the usefulness of the extension of the kernel in Section 4.9 where we compute the symmetry types of attractors in the *Kuramoto–Sivashinsky equation.*

Remark 7.3. *Observe that* $\mathbf{Z}_k$*-symmetry and* $\mathbf{D}_k$*-symmetry of an attractor are distinguished by the standard Karhunen–Loève decomposition, even in the case of a scalar equation: discrete cyclic symmetry does not automatically lead to dihedral symmetry of the operator* $\mathcal{K}_U$.

The reason for this is related to the discussion surrounding Theorem 7.1. In the $\mathbf{Z}_k$-symmetric case, a k-dimensional observable would fail, since this would correspond to a coupled oscillator where each cell is governed by one equation. But the Karhunen–Loève kernel provides an infinite-dimensional observation and the (infinitely many) additional components again can be viewed as an extension of a k-dimensional observable.

4.8 The Reduced System and Its Symmetry

In applications, it is useful to employ the eigenfunctions ϕ_j of $\mathcal{K}_U$ (the Karhunen–Loève operator) in a Galerkin method. This approach provides a reasonable approximation to the dynamical behavior of (4.1), though no rigorous estimates stating the accuracy of this approximation are known. In this section we investigate the symmetry properties of the reduced system which is obtained by this Galerkin method.

We will assume that in the decomposition (5.2) the eigenfunctions ϕ_j are ordered according to the magnitude of their eigenvalues λ_j, that is,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0\,.$$

We make the Galerkin ansatz

$$U(x,t) \approx \sum_{k=1}^{M} a_k(t)\phi_k(x)\,,$$

and solve the equations

$$\left(\phi_j, \sum_{k=1}^{M} \dot{a}_k(t)\phi_k - G\left(\sum_{k=1}^{M} a_k(t)\phi_k\right)\right) = 0\,, \quad j = 1, \ldots, M, \tag{8.1}$$

where $(\cdot,\cdot)$ denotes the Γ-invariant inner product on $L^2(\Omega)$. This is an M-dimensional system of ordinary differential equations which we will refer to as the *reduced system.*

From now on we additionally assume that M is chosen in such a way that

$$\lambda_{M+1} < \lambda_M\,. \tag{8.2}$$

Then we can rewrite (8.1) in a coordinate-free manner. Let $X = \langle\phi_1, \ldots, \phi_M\rangle$. Since the operator $\mathcal{K}_U$ is $\Sigma(A)$-equivariant (cf. Proposition 6.3), we may decompose X into irreducible subspaces

$$X = \bigoplus_{j=0}^{N} X_j.$$

Let $\Pi_j : X \to X_j$ be a projection. According to the decomposition of X we denote $x \in X$ by $x = x_0 + \cdots + x_N$. Then (8.1) is equivalent to

$$\Pi_j \left(\sum_{k=0}^{N} \dot{b}_k(t) x_k - G(b_0(t)x_0 + \cdots + b_N(t)x_N) \right) = 0, \quad j = 0, \ldots, N. \tag{8.3}$$

Proposition 8.1. *The reduced dynamical system* (8.3) *is* $\Sigma(A)$*-equivariant.*

Proof. Using the $\Sigma(A)$-invariance of X_j and the Γ-equivariance of G we compute for $\sigma \in \Sigma(A)$

$$\begin{aligned} \sigma \Pi_j G(b_0(t)x_0 + \cdots + b_N(t)x_N) &= \Pi_j \sigma G(b_0(t)x_0 + \cdots + b_N(t)x_N) \\ &= \Pi_j G(\sigma(b_0(t)x_0 + \cdots + b_N(t)x_N)) \\ &= \Pi_j G(b_0(t)\sigma x_0 + \cdots + b_N(t)\sigma x_N). \end{aligned}$$

■

Remark 8.2. With Proposition 8.1 we generalize a result in [2] concerning the symmetry of a reduced system. There the authors considered the specific case of the *Kuramoto–Sivashinsky equation.*

We already know that for a range of parameter values there might exist simultaneously different attractors in the dynamical system (4.1) which are related by symmetry, that is, which are lying on the same group orbit of Γ and are conjugate to each other. However, during the process of collecting data for the approximation of the eigenfunctions ϕ_k of $\mathcal{K}_U$ we always stay in one of those conjugate attractors.

Proposition 8.3. *The reduced system* (8.1) *does not depend on the specific choice of the attractor on the group orbit* ΓA *of attractors.*

Proof. For $\gamma \in \Gamma$ we write

$$\gamma U(x,t) = \sum_{k=1}^{\infty} b_k(t) \psi_k(x),$$

where the ψ_k are eigenfunctions of $\mathcal{K}_{\gamma U}$. By Corollary 6.8 and (8.2) we may assume that $\psi_k = \gamma \phi_k$. By writing down the reduced system (8.1) for γU the result immediately follows since G is Γ-equivariant and since the inner product on $L^2(\Omega)$ is Γ-invariant. ■

It is immediate from Proposition 6.3 that eigenspaces of the eigenvalue problem (6.2) are $\Sigma(A)$-invariant. Therefore, the Γ action on $L^2(\Omega)$ induces naturally a $\Sigma(A)$ action on the reduced system (8.1). To identify this action we define the representation of $\Sigma(A)$ on $\mathbf{C}^m$ by $\sigma = (c_{kj})_{k,j=1,\ldots,M}$, where on $\{\phi_k\}_{k=1,\ldots,M}$

$$\sigma\phi_k = \sum_{j=1}^{M} c_{kj}\phi_j \,.$$

In coordinates, this is the action for which (8.1) is $\Sigma(A)$-equivariant.

Remark 8.4. As expected by Proposition 8.3 the transformation matrices (c_{kj}) induced by the $\Sigma(A)$ action on $\{\phi_k\}$ are the same as the transformation matrices induced by the $\Sigma_{\gamma A}$ action on $\{\gamma\phi_k\}$. To verify this, simply use (4.2).

Let us assume that we want to collect data numerically (for the computation of the kernel $K(U)$) at a specific parameter value where we know the symmetry $\Sigma(A)$ of the attractor A, for example, through the use of a detective. Then we know by Proposition 6.3 that the corresponding operator $\mathcal{K}_U$ is $\Sigma(A)$-equivariant. But the numerical procedure will, in general, lead to an approximation $\mathcal{N}_U$ of the operator $\mathcal{K}_U$ which just approximately possesses this symmetry property. We would expect $\mathcal{N}_U$ to be "almost" $\Sigma(A)$-equivariant.

We regain the $\Sigma(A)$-symmetry in this finite approximation by symmetrizing the operator $\mathcal{N}_U$; that is, we define

$$\mathcal{N}_U^{\Sigma} = \int_{\Sigma} \mathcal{N}_{\sigma U}$$

and consider the $\Sigma(A)$-symmetric eigenvalue problem

$$\mathcal{N}_U^{\Sigma}\phi = \lambda\phi\,, \quad \phi \in \mathcal{X}_0\,.$$

Now we can use the $\Sigma(A)$-symmetry and finally obtain a $\Sigma(A)$-equivariant reduced system (8.1) for the approximation of solutions of (4.1).

Observe that the numerical effort for computing the operators $\mathcal{N}_{\sigma U}$ is negligible. We simply make use of the relation (6.3).

4.9 Example

The *Kuramoto–Sivashinsky equation* can be written as (see, e.g., [14])

$$U_t + 4U_{xxxx} + \alpha\left(U_{xx} + \frac{1}{2}(U_x)^2\right) = 0.$$

We impose periodic boundary conditions, i.e.,

$$U(x,t) = U(x+2\pi, t) \quad \text{for all } t \in \mathbf{R}.$$

Hence, the symmetry group of the problem is $\Gamma = \mathbf{O}(2)$. A very impressive numerical work has been performed in [14], where the authors varied α between 0 and about 300. We will focus our attention on a small range in parameter space where very complicated dynamics have been found. Namely, we will focus on the region between $\alpha = 89$ and $\alpha = 93$ in [14], Figure 3. We denote by α_S and α_O two parameter values inside this region and indicate the corresponding temporal behavior in Figure 2. Even if we were looking at the solutions for a longer period of time, it would be impossible to read off symmetry types of the corresponding attractors.

We computed the standard kernel K for both of the parameter values and the result is shown in Figure 3. Their symmetry type reflects the fact that the two attractors have at least $\mathbf{SO}(2)$ symmetry. If we want to find out whether there is additionally an $\mathbf{O}(2)$ symmetry we have to check whether or not

$$\langle U(x,\cdot)U_t(y,\cdot)\rangle = 0.$$

In Table 1 we present numerical computations of the L^2 norm of $\langle U(x,\cdot)\ U_t(y,\cdot)\rangle$ for the two different parameter values.

TABLE 1. The L^2 norms of $\mathcal{F}_2$.

α	Norm of $\langle U(x,\cdot)U_t(y,\cdot)\rangle$
α_S	120.76
α_O	1.05

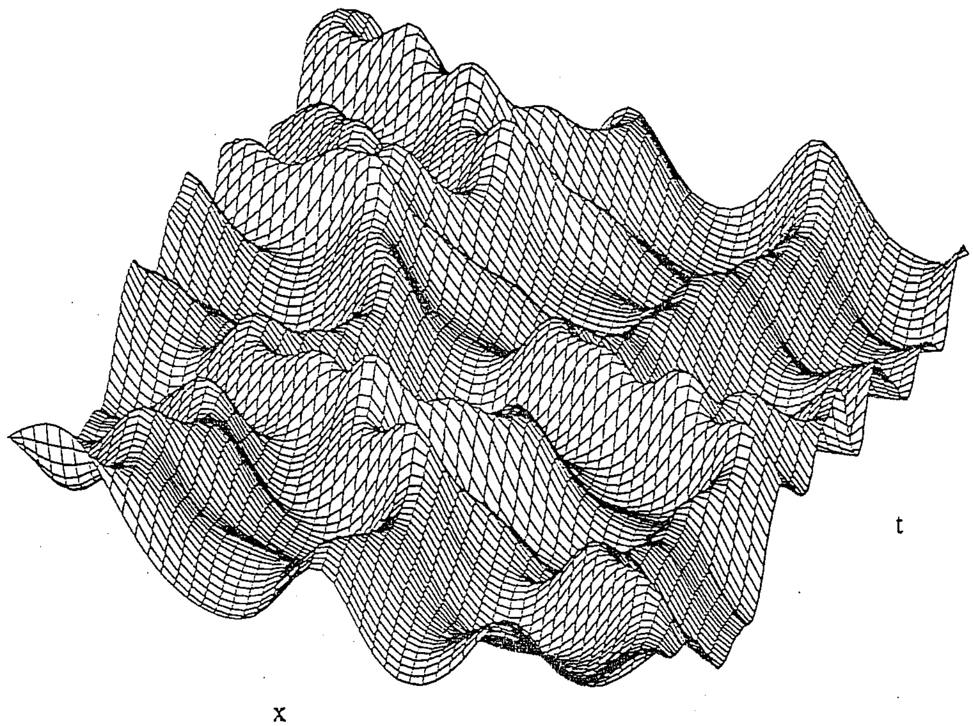
The jump in the norm indicates that for $\alpha = \alpha_O$ the attractor has $\mathbf{O}(2)$ symmetry, whereas for $\alpha = \alpha_S$ the attractor is just $\mathbf{SO}(2)$ symmetric. Thus, there is a symmetry increasing bifurcation between α_S and α_O [10], [12].

Finally, in Table 2, we illustrate how the distribution of energy over the eigenfunctions changes as α is varied. In particular, the significant increase in energy in the third mode is remarkable, indicating that this mode has become much more important after symmetry creation.

TABLE 2. Energies of the modes before and after the symmetry creation.

	Energy for	
Mode No.	α_S	α_O
1	0.581	0.500
2	0.228	0.195
3	0.097	0.183
4	0.073	0.091
5	0.011	0.020
6	0.008	0.008

Temporal behavior for alpha = alpha_S



Temporal behavior for alpha = alpha_O

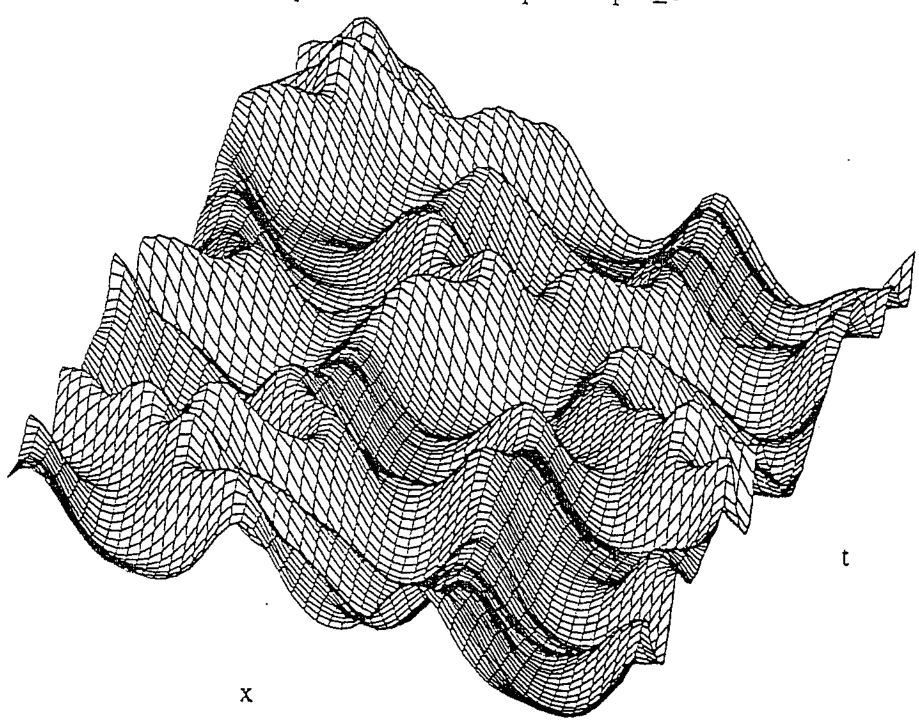


FIGURE 2. The temporal behavior of the solution for the parameter values (a) α_S and (b) α_O.

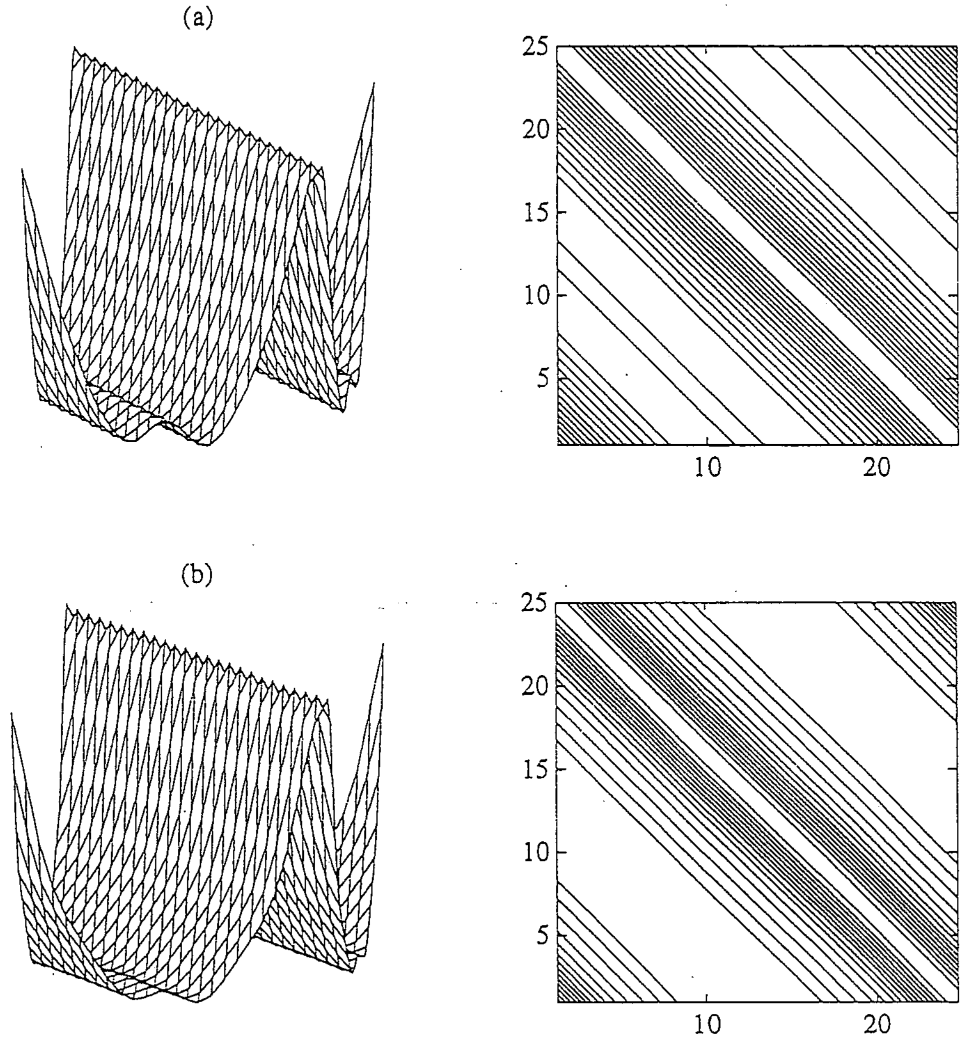


FIGURE 3. The standard kernel of the solution for the parameter values (a) α_S and (b) α_O together with a contour plot.

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