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ITERATES OF MAPS WITH SYMMETRY*

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Abstract. In this paper the elementary aspects of bifurcation of fixed points, period doubling, and Hopf bifurcation for iterates of equivariant mappings are discussed. The most interesting of these is an algebraic formulation of the hypotheses of Ruelle's theorem (D. Ruelle [1973], "Bifurcations in the presence of a symmetry group," *Arch. Rational Mech. Anal.*, 51, pp. 136–152) on Hopf bifurcation in the presence of symmetry.

In the last sections this result is used to show that Hopf bifurcation from standing waves in a system of ordinary differential equations with $O(2)$ symmetry can lead directly to motion on an invariant 3-torus; indeed, depending on the exact symmetry of the standing waves, one might expect to see three invariant 3-tori emanating from such a bifurcation. The unexpected third frequency comes from drift along the torus of standing waves whose existence is forced by the $O(2)$ symmetry.

Key words. symmetry, Hopf bifurcation, iterates of mappings

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Introduction. Symmetries change the types of bifurcation that may be expected in discrete dynamical systems. Typically, nonsymmetric systems generate unique branches of new solutions at points of bifurcation while symmetric systems generate multiple branches. Results of Vanderbauwhede [1980] and Golubitsky and Stewart [1985] on steady-state and Hopf bifurcation in continuous systems show that certain of these solution branches may be enumerated using only group theoretic techniques. The first task in this paper is the translation of these results to statements about bifurcation in the discrete dynamics of equivariant mappings. For further background, see Field [1980], [1986].

In § 1, we briefly describe the group theoretic results of Vanderbauwhede [1980] and Golubitsky and Stewart [1985]. In § 2, we apply Vanderbauwhede's result in a straightforward manner to enumerate certain branches of fixed points and branches of period two orbits for equivariant mappings. We also indicate how the simplest nontrivial symmetry ($\mathbb{Z}_2=\{\pm 1\}$ acting on $\mathbb{R}$) may be expected to affect period doubling cascades and lead naturally to mergings of attractors. In § 3, we adapt the results of Golubitsky and Stewart [1985] to enumerate branches of invariant curves stemming from Hopf bifurcation of equivariant mappings. This adaptation leans heavily on nontrivial results of Ruelle [1973]. Our contribution is really only to observe that there is an algebraic formulation for Hopf bifurcation of equivariant maps that satisfies the hypotheses of Ruelle's theorem.

The second task in this paper is to enumerate the number and type of tori that are produced when a periodic solution to an equivariant system of ordinary differential equations (ODEs) loses stability by having Floquet multipliers cross the unit circle in the complex plane. For example, we show in § 4 that (under certain hypotheses) standing wave solutions to $O(2)$ symmetric systems generate (generically) three branches of 3-tori at such a bifurcation. The existence of this extra frequency comes

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from the $O(2)$ symmetries and is based on observations of Iooss [1986] and Chossat [1986]. In § 5, we give a general setting for the example in § 4.

1. Background. Let $\Gamma \subset O(n)$ be a compact Lie group acting linearly on $\mathbb{R}^n$ and let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a one-parameter family of smooth mapping commuting with Γ, i.e.,

$$f(\gamma x, \lambda) = \gamma f(x, \lambda). \tag{1.1}$$

The equivariant branching lemma of Vanderbauwhede [1980] and Cicogna [1981] gives a simple algebraic condition for determining the existence of branches of steady-state solutions to the system of ODEs

$$\dot{x} = f(x, \lambda).$$

We assume that Γ acts absolutely irreducibly on $\mathbb{R}^n$, that is, that the only linear maps on $\mathbb{R}^n$ that commute with Γ are scalar multiples of the identity. For a subgroup Σ, we define

$$\operatorname{Fix}(\Sigma) = \{y \in \mathbb{R}^n : \sigma y = y \quad \forall \sigma \in \Sigma\}. \tag{1.2}$$

Applying the chain rule to (1.1) implies that

$$(df)_{0,\lambda}\gamma = \gamma (df)_{0,\lambda}.$$

Absolute irreducibility then implies that

$$(df)_{0,\lambda} = c(\lambda) I. \tag{1.3}$$

Also observe that

$$f: \operatorname{Fix}(\Sigma) \times \mathbb{R} \to \operatorname{Fix}(\Sigma) \tag{1.4}$$

since $\sigma f(y, \lambda) = f(\sigma y, \lambda) = f(y, \lambda)$ for all $\sigma \in \Sigma$, $y \in \operatorname{Fix}(\Sigma)$. In particular, irreducibility implies $\operatorname{Fix}(\Gamma) = \{0\}$, and hence by (1.4)

$$f(0, \lambda) = 0.$$

Thus, there is a "trivial" solution at $x = 0$.

EQUIVARIANT BRANCHING LEMMA. *Let $\Sigma \subset \Gamma$ be a subgroup. Assume that $c(0) = 0$, $c'(0) \neq 0$, and $\dim \operatorname{Fix}(\Sigma) = 1$. Then there exists a unique (nontrivial) branch of small amplitude steady states for $f|(\operatorname{Fix}(\Sigma) \times \mathbb{R}) = 0$.*

See Ihrig and Golubitsky [1984], Golubitsky, Swift, and Knobloch [1984], and Golubitsky, Stewart, and Schaeffer [1988] for applications of this result.

There is a similar result regarding Hopf bifurcation in symmetric systems. Here we assume that the system $\dot{x} = f(x, \lambda)$ is on the center manifold. In particular, we assume that $x \in \mathbb{R}^{2n}$ and that

$$L \equiv (df)_{0,0} = \begin{pmatrix} 0 & -\omega I_n \\ \omega I_n & 0 \end{pmatrix}.$$

There is the natural action of the circle group S^1 or $\mathbb{R}^{2n}$ given by

$$x \to \exp(tL)x. \tag{1.5}$$

We assume that the action of $\Gamma \times S^1$ on $\mathbb{R}^{2n}$ is irreducible. It then follows, as above, that $f(0, \lambda) \equiv 0$, i.e., that there is a "trivial" steady-state solution. It also follows that the eigenvalues of $(df)_{0,\lambda}$ are $\sigma(\lambda) \pm i\omega(\lambda)$, each of multiplicity n.

THEOREM 1.1. *Let* $\Sigma \subset \Gamma \times S^1$ *be a subgroup. Assume that* $\sigma(0)=0$, $\omega(0)\neq 0$, $\sigma'(0)\neq 0$, *and* $\dim \mathrm{Fix}(\Sigma)=2$. *Then there exists a unique (nontrivial) branch of small amplitude periodic trajectories with period near* $2\pi/\omega(0)$ *to* $\dot{x}=f(x,\lambda)$ *with symmetries* Σ.

Note $(\sigma,\theta)\in\Sigma\subset\Gamma\times S^1$ is a symmetry of a periodic solution $x(t)$ to $\dot{x}=f(x,\lambda)$ if

$$\gamma x(t)=x(t+\theta). \tag{1.6}$$

See Golubitsky and Stewart [1985], [1986b], Roberts, Swift, and Wagner [1986], and Golubitsky, Stewart, and Schaeffer [1988] for a proof and applications.

We remark that in certain instances it is possible to use invariant theory and group theory to compute the asymptotic stability of the steady-state and periodic solutions found using the results stated above. We refer to these references for examples of this process.

2. Fixed points and period doubling. Let $g:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ be a one-parameter family of Γ-equivariant mappings. We assume that Γ acts absolutely irreducibly on $\mathbb{R}^n$. It follows that $x=0$ is a "trivial" fixed point for g and that $(Dg)_{0,\lambda}=c(\lambda)I$. In this section, we briefly discuss the bifurcation of fixed points ($c(0)=+1$) and period doubling bifurcation ($c(0)=-1$).

LEMMA 2.1. *Let* $\Sigma\subset\Gamma$ *be a subgroup. Suppose that* $c(0)=1$, $c'(0)\neq 0$, *and* $\dim \mathrm{Fix}(\Sigma)=1$. *Then* $g(x,\lambda)$ *has a unique (nontrivial) branch of fixed points in* $\mathrm{Fix}(\Sigma)$.

Proof. Set $f(x,\lambda)=g(x,\lambda)-x$ and apply the Equivariant Branching Lemma. □

To eliminate trivialities, we assume that Σ is an isotropy subgroup, that is, there is an $x\in\mathbb{R}^n$ such that

$$\Sigma=\{\gamma\in\Gamma\colon \gamma x=x\}. \tag{2.1}$$

The largest subgroup of Γ that leaves $\mathrm{Fix}(\Sigma)$ invariant is $N(\Sigma)$, the normalizer of Σ in Γ (cf. Golubitsky [1983] or Golubitsky, Stewart, and Schaeffer [1988]). It follows that $g|\mathrm{Fix}(\Sigma)\times\mathbb{R}$ commutes with the action of $N(\Sigma)/\Sigma$. Now, if we assume that Σ is a maximal isotropy subgroup (a hypothesis that is satisfied when $\dim \mathrm{Fix}(\Sigma)=1$), then $N(\Sigma)/\Sigma$ acts fixed point free. It follows that when $\dim \mathrm{Fix}(\Sigma)=1$, either $N(\Sigma)=\Sigma$ or $N(\Sigma)/\Sigma\cong\mathbb{Z}_2$. In the latter case, the bifurcation of fixed points in Lemma 2.1 will be via a pitchfork bifurcation, with the two new bifurcating fixed points lying on the same group orbit (conjugacy being given by any $\gamma\in N(\Sigma)\sim\Sigma$). When $N(\Sigma)=\Sigma$, we expect each new fixed point to be on a distinct group orbit.

We now discuss the case of period doubling, i.e., $c(0)=-1$. As in the standard period doubling theorem (without symmetry), we observe that nonzero fixed points for the composite mapping g^2 correspond to period two points for g, since the implicit function theorem guarantees that there are no new fixed points for g. We apply Lemma 2.1 to g^2 to obtain the following lemma.

LEMMA 2.2. *Let* $\Sigma\subset\Gamma$ *be a subgroup. Suppose that* $c(0)=-1$, $c'(0)\neq 0$, *and* $\dim \mathrm{Fix}(\Sigma)=1$. *Then* $g(x,\lambda)$ *has a unique branch of period two points in* $\mathrm{Fix}(\Sigma)$.

Again, we have different interpretations for Lemma 2.2, depending on whether $N(\Sigma)=\Sigma$ or $N(\Sigma)/\Sigma\cong\mathbb{Z}_2$. In the first case, we expect a standard period doubling to occur, while in the second case, the equivariance of $g|(\mathrm{Fix}(\Sigma)\times\mathbb{R})$ with respect to $\mathbb{Z}_2$ implies that

$$g(x,\lambda)=-x. \tag{2.2}$$

To verify (2.2), note that $g(-x,\lambda)=-g(x,\lambda)$. Therefore, if x is a period two point for

g, then so is $-x$. Since x and $-x$ are in Fix (Σ) and the period two orbit obtained from Lemma 2.2 in Fix (Σ) is unique, it follows that $g(x, \lambda) = -x$.

Remark. Identity (2.2) states that this period two trajectory is a discrete analogue of a rotating wave; the same result is obtained by taking one timestep (iteration by g) or by acting by the group $(x \to -x)$.

We end this section with some speculations on period doubling sequences when $N(\Sigma)/\Sigma \cong \mathbb{Z}_2$. As a parameter is varied, we might expect the trivial fixed point to undergo a bifurcation to a nontrivial fixed point, as in Lemma 2.1. As we discussed above, when $N(\Sigma)/\Sigma \cong \mathbb{Z}_2$, this new fixed point is formed by a pitchfork bifurcation.

Suppose now that, as this parameter is varied, each of the nontrivial fixed points undergoes a period doubling sequence. The $\mathbb{Z}_2$ action forces the period doubling sequence to occur at the same parameter values for each nontrivial fixed point. The simplest such example is given by the cubic polynomial

$$g(x, \lambda) = \mu x - x^3, \qquad \mu > 0 \tag{2.3}$$

on Fix $(\Sigma) \times \mathbb{R}$. For $\mu > 0$, each of these period doubling sequences seems to behave like the simple logistic equation. This results in pairs of attractors (one for $x > 0$ and one for $x < 0$) consisting of single orbits filling up parts of the real line, say, for x in $[\alpha, \beta]$ and for x in $[-\beta, -\alpha]$.

As λ is increased, α decreases and eventually becomes negative (when $\lambda = 3\sqrt{3/2}$). This merging of attractors causes an interesting kind of chaotic behavior. Start with an initial point $x_0 > 0$ and form the iterated sequence $x_{n+1} = g(x_n, \lambda)$. Now form the symbol sequence of +'s and −'s where the nth element in the sequence is sgn (x_n). In effect, we see chaotic behavior on two time scales. There is the chaotic behavior on a fast time scale within each of the attractors ($[0, \beta]$ and $[-\beta, 0]$) and then there is the chaotic behavior on a slow time scale defined by the transitions between the remnants of the two attractors.

A detailed study of the related map

$$h(x, \lambda) = -(\mu x - x^3), \qquad \mu > 0$$

is given in Rogers and Whitley [1983]. There, however, the primary bifurcation of the fixed point $x = 0$ is a period doubling bifurcation, as discussed in Lemma 2.2.

3. Hopf bifurcation. In this section, we assume that the trivial fixed point for the Γ-equivariant mapping g loses stability by a pair of complex conjugate eigenvalues crossing the unit circle. Due to the presence of symmetry, the eigenvalues may have high multiplicity. We assume that $g: \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n}$ and that $(Dg)_{0,0}$ has eigenvalues $e^{\pm 2\pi i \theta}$, each with multiplicity n, where $\theta \neq 0, \frac{1}{2}$.

The standard Hopf bifurcation theorem for mappings $(n = 1)$ states that if $\theta \neq \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}$ and if the eigenvalues cross the unit circle with nonzero speed, then there exists a family of invariant circles for $g(\cdot, \lambda)$ emanating from the trivial fixed point $x = 0$. This theorem is proved using near identity changes of coordinates to put the terms of g up to order four in normal form. This truncated normal form actually has S^1 symmetry, and, because of this symmetry, we can easily find invariant circles using polar coordinates. Then we use scaling and normal hyperbolicity arguments to show that the invariant circles that are present at order four persist independently of the higher order terms in g. When resonances exist $(\theta = \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4})$, the normal form does not have this S^1-equivariance in the fourth-order truncated normal form (cf. Arnold [1977], [1983] and Iooss [1979]).

We obtain a simple generalization of the Hopf bifurcation theorem as follows: Let $\Sigma \subset \Gamma$ be a subgroup with dim Fix $(\Sigma) = 2$. Then there exists a branch of g-invariant circles in Fix (Σ), as long as $\theta \neq \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}$. Just apply the standard Hopf theorem to $g|\text{Fix}(\Sigma)$; the eigenvalues of $D(g|\text{Fix}(\Sigma))$ are constrained by dimension to be simple.

Remark. Assume that Σ is an isotropy subgroup with dim Fix $(\Sigma)=2$. The group $N(\Sigma)/\Sigma$ acts on Fix (Σ) by a fixed-point free action and $g|\text{Fix}(\Sigma)$ commutes with this action. (In fact, Σ is a maximal isotropy subgroup since the complex eigenvalues preclude the existence of one-dimensional fixed-point subspaces; cf. Golubitsky and Stewart [1985].) Fixed-point free actions on $\mathbb{R}^2$ exist only for the groups 1, $\mathbb{Z}_n$ $(n \geqq 2)$ or $SO(2)$. Observe that if $N(\Sigma)/\Sigma \cong \mathbb{Z}_n$ $(n \geqq 5)$ or $SO(2)$, then $g|\text{Fix}(\Sigma)$ automatically has a fourth-order truncated normal form with S^1 symmetry. In these cases, the assumption $\theta \neq \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}$ is *not* necessary, as the remainder of the proof of the standard Hopf theorem is still valid.

As in the case of Hopf bifurcation for systems of ODEs (Theorem 1.1), we can improve on this simple generalization by looking for subgroups of $\Gamma \times S^1$ with two-dimensional fixed-point subspaces. First, we define the action of S^1. Choose a matrix A with purely imaginary eigenvalues such that $e^A = (dg)_{0,0}$. The action of S^1 is then given by e^{tA}. Since $(dg)_{0,0}$ commutes with Γ, so does the action of S^1. In this way, we have defined an action of $\Gamma \times S^1$ on $\mathbb{R}^{2n}$.

THEOREM 3.1. *Let $\Sigma \subset \Gamma \times S^1$ be a subgroup such that* dim Fix $(\Sigma)=2$. *Assume $\theta \neq \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}$ and that the eigenvalues cross the unit circle with nonzero speed. Then generically there exists a unique branch of g-invariant circles emanating from the trivial fixed point $x=0$ and this branch is tangent to* Fix $(\Sigma) \subset \mathbb{R}^{2n} \times \mathbb{R}$ *at $x=0$.*

Proof. The truncated normal form h of g has symmetry group $\Gamma \times S^1$. Therefore, $h: \text{Fix}(\Sigma) \times \mathbb{R} \to \text{Fix}(\Sigma)$, and we can find invariant circles for h, as above. At this point, however, we cannot conclude directly from the proof of the standard Hopf bifurcation theorem that there is a family of g-invariant circles. The difficulty is that g itself need not leave invariant Fix (Σ) since g does not necessarily commute with S^1. However, Ruelle [1973, Thm. 3.1] does prove a theorem sufficient to conclude that g has a family of invariant circles, at least when certain assumptions, which are valid generically, hold.

The needed assumptions are the following:

(a) The third-order terms in h determine the direction of branching of the invariant circles of h in Fix (Σ).

(b) The invariant circles for h are normally hyperbolic in the sense that the eigenvalues of dh on the invariant circles, which are not forced by the $\Gamma \times S^1$ action to be unity, in fact lie off the unit circle.

In order for (b) to hold, it is often necessary to have truncated the normal form at some high order. This order depends on both $\Gamma \times S^1$ and the subgroup Σ. Once the invariant circles of h are normally hyperbolic, Ruelle's Theorem 3.1 is sufficient to prove that the higher order terms of g (which are not in normal form) will neither destroy the invariant circles nor change their stability. $\square$

Example 3.2. Consider $\Gamma = D_n$ $(n \geqq 3)$ acting absolutely irreducibly on $\mathbb{C}$ and by the diagonal action on $\mathbb{C}^2$. As was shown in Golubitsky and Stewart [1986b], there are three (conjugacy classes of) isotropy subgroups in $D_n \times S^1$ where the fixed-point subspaces are two-dimensional. Theorem 3.1 implies that for D_n-equivariant mappings, we may expect three families of g-invariant circles at such a Hopf bifurcation. We note that two of the isotropy subgroups are isomorphic to $\mathbb{Z}_2$ and one to $\mathbb{Z}_3$. The normal hyperbolicity of the $\mathbb{Z}_3$ circles are determined at third order, while the normal hyperbolicity of the $\mathbb{Z}_2$ branches are determined at order m where

$$m = \begin{cases} n, & n \text{ odd}, \\ (n+2)/2, & n \text{ even}. \end{cases}$$

4. Bifurcation of standing waves to 3-tori. For an $O(2)$ invariant system of ODEs, a symmetry-breaking Hopf bifurcation leads to two types of periodic solutions: rotating waves and standing waves (cf. Golubitsky and Stewart [1985]). We are interested here in the bifurcation of these periodic solutions to tori. Bifurcation from rotating waves has been considered by Rand [1982], Renardy [1982], and Iooss [1984]. By changing coordinates to a rotating frame, they show that rotating waves correspond to stationary solutions and that 2-tori may be found by standard Hopf bifurcation techniques for systems of ODEs. Moreover, the circular symmetry of the rotating waves forces the flow on the 2-torus to be linear. Standing waves, however, have no circular symmetry in their isotropy subgroup, and the technique of changing coordinates to a rotating frame does not apply. Using the techniques described in § 3 applied to a certain Poincaré map, we shall study here the bifurcation to tori from standing waves. In the next section, we give a unified discussion of these two techniques when $O(2)$ is replaced by a general group Γ.

Bifurcation to 2-tori from a branch of standing waves has been considered in the context of degenerate, symmetry-breaking, $O(2)$ Hopf bifurcations by a number of authors (Erneux and Matkowsky [1984], Knobloch [1986], and Golubitsky and Roberts [1987]). These authors decouple the normal form equations for $O(2)$ Hopf bifurcation (on $\mathbb{C}^2$) into phase-amplitude equations and find the 2-tori by steady-state bifurcation in the amplitude equations. Using the extra S^1 phase shift symmetry of normal form, it is easy to see that in normal form the flow on these 2-tori must also be linear. Chossat [1986] uses a Lyapunov–Schmidt reduction to prove that the flow on such 2-tori is linear even when the vector field is not assumed to be in normal form. His method is to assume that the flow on the 2-torus has the form $y(t)=R_{\eta t}x(t)$ where x is periodic, η is a real parameter, and R_θ denotes the action of θ in $SO(2)$. The original equation is then transformed by substitution of $y(t)$ and elimination of $R_{\eta t}$ to an equation for x. It is this equation to which the Lyapunov–Schmidt reduction is applied, and this idea we will use to analyze bifurcation to tori from standing waves. In § 5, we will show that, in principle, when considering bifurcation to tori from a branch of periodic solutions in a symmetric system, one of the two techniques described above always works. Which one will work depends on the symmetries of the periodic solution.

Consider the following system of ODEs:

$$\dot{y}=F(y,\lambda),\qquad F(0,\lambda)=0 \tag{4.1}$$

where $y\in\mathbb{R}^N$ and $F:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^N$ commutes with a linear action of $O(2)$ on $\mathbb{R}^N$. This action may not be faithful; we assume, however, that the kernel of the action is the cyclic group $\mathbb{Z}_n$, $n\geqq 1$.

Let $y(t)$ be a *standing wave* periodic solution to (4.1), that is, assume that the isotropy subgroup

$$\Sigma=\{y\in O(2):\gamma y(t)=y(t)\} \tag{4.2}$$

is discrete and contains a reflection in $O(2)$. Thus $\Sigma=D_n$. Note that standing waves lie on the invariant 2-torus

$$M=\{\gamma y(t):\gamma\in O(2)\}$$

foliated by periodic trajectories.

We now consider bifurcation of standing waves to tori. This bifurcation is detected by having a complex conjugate pair of Floquet multipliers cross the unit disk at $e^{\pm 2\pi i\theta}$ where $\theta\neq 1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{2}{3},\frac{3}{4}$. The eigenspaces corresponding to these Floquet multipliers are invariant under $\Sigma=D_n$, and generically we may assume that D_n acts irreducibly on the eigenspaces. The irreducible representations of D_n are either one-dimensional or, if $n \geqq 3$, two-dimensional.

We prove the following theorem.

THEOREM 4.1. *Let $x_\lambda(t)$ be a family of standing wave periodic solutions to the $O(2)$ symmetric system* (4.1) *with isotropy subgroup D_n. Assume that the periodic solution loses stability by having a pair of complex conjugate Floquet multipliers cross the unit disk with nonzero speed and assume that D_n acts irreducibly on the corresponding eigenspaces.*

(a) *If the Floquet multipliers are simple, then there exists a branch of* 3*-tori emanating from this bifurcation.*

(b) *If the Floquet multipliers are double* (*which may happen generically when $n \geqq 3$*), *then there exist three branches of* 3*-tori emanating from the bifurcation.*

Our proof consists of constructing a D_n-equivariant Poincaré map to which we can apply the results of § 3.

Remarks. (a) Such bifurcations to 3-tori occur in the interaction of two symmetry-breaking $O(2)$ Hopf bifurcations (see Chossat, Golubitsky, and Keyfitz [1986]) and in the interaction of $O(2)$ symmetry-breaking steady-state and Hopf bifurcations (see Golubitsky and Stewart [1986a]).

(b) Normally we would expect the bifurcation of a periodic solution to tori to produce an invariant 2-torus. The extra frequency comes from the $O(2)$ symmetry. As noted above, each standing wave $x(t)$ lies on the 2-torus M defined by $\gamma x(t)$ for $\gamma \in O(2)$. When bifurcation to tori occurs, we get two independent frequencies from the "Poincaré map" and a third independent (slow) frequency from flow transverse to $\gamma x(t)$ in the group generated 2-torus M. It is here that we use the ideas of Iooss [1986] and Chossat [1986], described above.

(c) Suppose that the standing waves are generated by Hopf bifurcation with $O(2)$ symmetry from an invariant steady state in (4.1). Then the bifurcation to tori we describe in Theorem 4.1 cannot occur in a system of differential equations posed only on the four-dimensional center subspace. Since the hypotheses of the theorem presume the existence of four nontrivial Floquet multipliers and periodic solutions always have one trivial Floquet multiplier (equal to unity), such a system cannot live on a four-dimensional manifold. In effect, the question we discuss here is: suppose that a standing wave with D_n symmetry is formed from a symmetry-breaking $O(2)$ Hopf bifurcation and suppose that we track this solution to finite amplitude; then how should we expect this standing wave to lose stability?

(d) In models of the Couette–Taylor apparatus where periodic boundary conditions in the axial direction are assumed, the transition from wavy vortices to modulated wavy vortices is an example of the bifurcation considered in Theorem 4.1.

Proof of Theorem 4.1. Let $x_\lambda(t)$ be the one-parameter family of standing-wave solutions to (4.1) with periods $2\pi/\omega_\lambda$. Write the Floquet equation

$$\frac{dy}{dt} = L_\lambda(t) \cdot y \tag{4.3}$$

where

$$L_\lambda(t) = (D_x F)_{x_\lambda(t),\,\lambda}.$$

We assume that (4.3) has a Floquet multiplier $\alpha(\lambda)$ of multiplicity two where $\alpha(0) = e^{2\pi i\theta}$ and $\theta \neq 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}$. We also assume

$$\frac{d}{d\lambda}|\alpha(\lambda)|\bigg|_{\lambda=0} \neq 0.$$

We know that $\dot{x}_\lambda(t)$ is always a solution to (4.3) yielding a Floquet multiplier equal to unity. Similarly, the $O(2)$ equivariance of F implies that $Jx_\lambda(t)$ is also a solution to (4.3) yielding another Floquet multiplier equal to unity where J is the infinitesimal generator of the $SO(2)$ action.

In order to eliminate this extra "trivial" Floquet multiplier, we look for solutions to (4.1) of the form

$$y(t) = R_{\eta t}x(t) \tag{4.4}$$

where R_ϕ denotes the action of $\phi \in SO(2)$ on $\mathbb{R}^N$, and η is a real parameter. As mentioned above, this trick is used in Chossat [1986] and, in a slightly different context, in Iooss [1986]. The system (4.1) now becomes

$$\dot{y} = F(y, \lambda) - \eta Jy. \tag{4.5}$$

Observe from (4.4) that periodic solutions of (4.5), $(y(t), \eta)$, correspond to quasiperiodic solutions of (4.1), $x(t)$.

Next we define our Poincaré map. Let $\phi_t(y_0, \lambda, \eta)$ denote the one-parameter group of solutions to (4.5) with initial condition y_0. Note that when $\eta = 0$, (4.5) is identical to (4.1). Recall that $x_0(t)$ is a $2\pi/\omega_0$-periodic solution to (4.1), and hence

$$x_0(0) = \phi_{2\pi/\omega_0}(x_0(0), 0, 0),$$

that is, $x_0(0)$ is a fixed point for the mapping $\phi_{2\pi/\omega_0}(\,\cdot\,, 0, 0)$.

Let $\zeta_1 = dx_0/dt(0)$ and $\zeta_2 = Jx_0(0)$. Since $x_0(t)$ is a nonconstant periodic solution to (4.1), we know that $x_0(0) \neq 0$ (since $F(0, \lambda) \equiv 0$). Thus, ζ_1 is tangent to the trajectory $x_0(t)$ and ζ_2 is tangent to the $O(2)$ group orbit through x_0. The hypothesis that $x_0(t)$ is a standing wave guarantees that ζ_1 and ζ_2 are linearly independent. Let $\langle\, ,\rangle$ denote an inner product on $\mathbb{R}^N$ and let $W = \operatorname{span}\{\zeta_1, \zeta_2\}^\perp$.

We now define the first return map of trajectories to (4.5) starting in the plane $W_0 = \{x_0(0) + y_0 : y_0 \in W\}$ close to $x_0(0)$. In order for such a trajectory to return to W_0 at time τ, it must satisfy the equations

$$f_j(y_0, \lambda, \eta, \tau) \equiv \langle \zeta_j, \phi_\tau(x_0(0) + y_0, \lambda, \eta) - x_0(0)\rangle = 0. \tag{4.6}$$

Now recall that if we set $y(t) = x_0(t) + z(t)$ in (4.5) and $z(0) = y_0$ is close enough to zero, then the integral form of (4.5) is

$$z(t) = S(t)y_0 + \int_0^t S(t-s)\hat{F}(z(s), \lambda, \eta)\,ds \tag{4.7}$$

where $S(t)$ is the monodromy operator associated with (4.3) and

$$\hat{F}(z, \lambda, \eta) = F(x_0 + z, \lambda) - F(x_0, \lambda) - L_0(t)z - \eta J(z).$$

Since $\phi_\tau(x_0(0) + y_0, \lambda, \eta) = x_0(\tau) + z(\tau)$ it can be seen from (4.7) that

$$\begin{aligned} &\text{(a)} \quad f_1(y_0, \lambda, \eta, \tau) = \tau - 2\pi/\omega_0 + \cdots, \\ &\text{(b)} \quad f_2(y_0, \lambda, \eta, \tau) = \eta + \cdots, \end{aligned} \tag{4.8}$$

where $\cdots$ indicates terms of the form

$$o(|\tau - 2\pi/\omega_0| + |\eta| + O(|\lambda| + |y_0|)).$$

Using the implicit function theorem, we can solve equations (4.6) for $\tau = \tau(y_0, \lambda)$ and $\eta = \eta(y_0, \lambda)$ when $\tau(0,0) = 2\pi/\omega_0$ and $\eta(0,0) = 0$. Observe that generically η itself is nonzero. The Poincaré map is now defined by

$$G_\lambda(y_0) = \phi_{\tau(y_0,\lambda)}(x_0(0) + y_0, \lambda, \eta(y_0, \lambda)) - x_0(0). \tag{4.9}$$

Note that $G_0(0) = \phi_{2\pi/\omega_0}(x_0(0), 0, 0) - x_0(0) = 0$.

A consequence of this construction is that, if G_λ undergoes a Hopf bifurcation at $y_0=0$, then we find an invariant 2-torus in (4.5) that corresponds using (4.4) to an invariant 3-torus in (4.1). It follows from (4.4) that one of the independent frequencies is η, which is small, but typically nonzero.

A second important consequence of the construction (4.8) is that G_λ is D_n-equivariant. We claim that

$$
\begin{aligned}
&\text{(a)}\quad \gamma\zeta_1=\zeta_1 \qquad \forall\gamma\in D_n;\\
\text{(4.10)}\qquad &\text{(b)}\quad \gamma\zeta_2=\zeta_2 \qquad \forall\gamma\in \mathbb{Z}_n, \quad\text{and}\quad \phi_t(\gamma y_0,\lambda,\eta)=\gamma\phi_t(y_0,\lambda,\eta);\\
&\text{(c)}\quad S\zeta_2=-\zeta_2 \qquad \forall S\in D_n\sim\mathbb{Z}_n, \quad\text{and}\quad \phi_t(Sy_0,\lambda,-\eta)=S\phi_t(y_0,\lambda,\eta).
\end{aligned}
$$

Using the identities (4.10) in (4.6) and uniqueness of solutions to the implicit function theorem allows us to conclude that

$$
\begin{aligned}
&\text{(a)}\quad \tau(\gamma y_0,\lambda)=\tau(y_0,\lambda) \qquad \forall\gamma\in D_n,\\
\text{(4.11)}\qquad &\text{(b)}\quad \eta(\gamma y_0,\lambda)=\eta(y_0,\lambda) \qquad \forall\gamma\in \mathbb{Z}_n, \quad\text{and}\\
&\text{(c)}\quad \eta(Sy_0,\lambda)=-\eta(y_0,\lambda) \qquad \forall S\in D_n\sim\mathbb{Z}_n.
\end{aligned}
$$

Using the definition of G_λ in (4.8), we now find it easy to check using (4.9) and (4.10) that G_λ commutes with D_n.

To verify (4.10)(a), recall that since $x_0(t)$ is a standing wave, we know that $\gamma x_0(t)=x_0(t)$ for all $\gamma\in D_n$. Now differentiate with respect to t. Next, observe that $\mathbb{Z}_n$ commutes with J while $SJ=-JS$ for all S in $D_n\sim\mathbb{Z}_n$. Now we prove (4.10)(b),(c) by invoking uniqueness of solutions to the initial value problems for systems of ODEs.

If the Floquet multipliers are simple, then this construction gives a unique invariant circle by the standard Hopf theorem for mappings. However, when the Floquet multipliers are double, we can invoke the discussion concerning Hopf bifurcation for D_n-equivariant mappings given at the end of § 3. We conclude that when G_λ undergoes a Hopf bifurcation, three families of invariant tori are produced from this bifurcation. Of course, the hypotheses of Theorem 4.1 imply that G_λ does undergo a Hopf bifurcation at $\lambda=0$. This completes our proof. □

Remarks. (a) The stability of these 3-tori can, in principle, be computed from the results in Golubitsky and Stewart [1986b]. The simplest statement of these results is: suppose the standing waves are stable when $\lambda<0$. Then generically, for any of the 3-tori to be stable, all those families must appear supercritically (for $\lambda>0$). If all three families are supercritical, then precisely one family is asymptotically stable.

(b) The reader may check that these results explain the existence of the invariant 3-tori found in the interaction of two symmetry-breaking $O(2)$ Hopf bifurcations from a branch of standing-wave solutions. See Chossat, Golubitsky, and Keyfitz [1986].

5. Bifurcation from periodic solutions. In this section, we generalize the discussion of bifurcation from a periodic solution of $O(2)$ symmetric systems of ODEs to bifurcation in systems invariant under a general compact Lie group Γ. Our formulation here is mainly geometric and may be contrasted with the analytic nature of the remarks in § 4.

Let $x(t)$ be a T-periodic solution to

$$\dot{x}=f(x) \tag{5.1}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$ is Γ-equivariant, $\Gamma\subset O(n)$. Let $\phi_t:\mathbb{R}^n\to\mathbb{R}^n$ be the flow associated with (5.1) and note that ϕ_t is also Γ-equivariant. We now define a Poincaré map associated with $x(t)$.

Define a local action of $\Gamma\times\mathbb{R}$ on $\mathbb{R}^n$ by

$$(\gamma, t)\cdot x=\gamma\phi_t(x). \tag{5.2}$$

Since the actions of γ and t commute (ϕ_t is Γ-equivariant), we see that (5.2) actually defines an action of $\Gamma\times\mathbb{R}$. Let $x_0=x(0)$. Since orbits of (smooth) Lie group actions are immersed submanifolds (cf. Bredon [1972]), we see that

$$M=(\Gamma\times\mathbb{R})\cdot x_0 \tag{5.3}$$

is an immersed submanifold of $\mathbb{R}^n$. However, since $x(t)$ is periodic, it follows that M is compact, and hence a submanifold of $\mathbb{R}^n$. Moreover, M is foliated by the periodic solutions $\{\gamma x(t):\gamma\in\Gamma\}$.

Remark. Let S^1 be identified with the interval $[0, T)$. Then we can define

$$\Sigma_{x_0}=\{(\gamma,\theta)\in\Gamma\times S^1:(\gamma,\theta)x_0=x_0\}. \tag{5.4}$$

Σ_{x_0} is the isotropy subgroup of $x(t)$ and M is diffeomorphic to $(\Gamma\times S^1)/\Sigma_{x_0}$.

Since M is compact and Γ-invariant, there exists an open Γ-invariant tubular neighborhood of M in $\mathbb{R}^n$. More precisely, there exists a vector bundle $N\xrightarrow{\pi}M$ and a smooth Γ-equivariant diffeomorphism $\sigma: N\to\mathbb{R}^n$ defined on an open neighborhood of M and N such that $\operatorname{Im}\sigma$ is an open neighborhood of M in $\mathbb{R}^n$ and $\sigma|M$ is the identity (see Bredon [1972, p. 306]). Via σ we can pull back the vector field f to N and discuss the bifurcation of the periodic orbit in N. The advantage of these coordinates is that Γ acts linearly on the fibers of N and these fibers are orthogonal to M.

Next we define the manifold

$$P_\gamma=\pi^{-1}(\{\gamma\cdot x(t)\}); \tag{5.5}$$

that is, P_γ is just that part of the vector bundle N that lies over the periodic trajectory $\gamma\cdot x(t)$ in M. It is possible to write (cf. Vanderbauwhede, Krupa, and Golubitsky [1988] or Krupa [1988])

$$f(y)=f_P(y)+f_T(y) \tag{5.6}$$

where $f_P(y)$ is tangent to P_γ for all $y\in P_\gamma$ and $f_T(y)$ is tangent to the group orbit Γy. Moreover, both f_P and f_T are Γ-equivariant.

Next, observe that Γ-equivariance implies that f_T is a linear vector field on Γy. Hence, the flow of f_T is given by $\exp(t\eta)$ for some $\eta\in L(\Gamma)$, the Lie algebra of Γ. (In fact, if we define

$$\Gamma_{x_0}=\{\gamma\in\Gamma:\gamma x_0=x_0\} \tag{5.7}$$

and we choose a vector space complement U to $\mathscr{L}(\Gamma_{x_0})$ in $\mathscr{L}(\Gamma)$, then we can assume η is uniquely defined in U.) Note that $\mathscr{L}(O(2))=\mathbb{R}$ and that the η defined in § 4 may be thought of as residing in the Lie algebra of $O(2)$. Also, in § 4, we solved implicitly for $\eta=\eta(y_0,\lambda)$. This discussion allows us to write explicitly the first-order terms of η. We have not set up such an explicit algorithm here. Nevertheless, we know that generically η is nonzero, which is all we need. Similarly, since f_P is Γ-equivariant, the dynamics of f_P are determined by the dynamics of $f_P|P_1$. We just transport the flow from P_1 to P_γ using multiplication by γ, which acts orthogonally.

It now follows that the flow of f may be understood as composing the flow of f_P on P_1 with linear flow on orbits Γy. In particular, if W is an invariant set under the flow of f_P, then

$$\hat{W}=\bigcup_{y\in W}\Gamma y\subset N$$

is invariant under f. Moreover, W is asymptotically stable under f_P if and only if the invariant set $\hat{W}$ is asymptotically stable under f. Since $f_T|M \equiv 0$, we may think of the flow of f on $\hat{W}$ as being the composition of the flow of f_P on W with a slow drift along the group orbits of Γ in $\hat{W}$. For the example $O(2)$, connected components of the group orbits are just circles (being diffeomorphic to $SO(2)$) and the flow for f_T along the group orbit is periodic. If, in addition, we assume that W is a 2-torus, then $\hat{W}$ will be a 3-torus with the flow along the group orbit having a small frequency.

Thus, bifurcation of the periodic orbit $x(t)$ for f is determined by bifurcation of the periodic orbit $x(t)$ for $F = f_P|P_1$. Note that, since f_P is Γ-equivariant, it follows that F is Γ_{x_0}-equivariant where Γ_{x_0} is the isotropy subgroup defined in (5.7). We assume henceforth that f, and hence F, depend on a real parameter λ.

Recall now the isotropy subgroup Σ_{x_0} of $x(t)$ in $\Gamma \times S^1$, which was defined in (5.4). We call $x(t)$ a *rotating wave* if there is a loop

$$(\gamma(\theta), \theta) \in \Sigma_{x_0}, \qquad \gamma(0) = 1 \tag{5.8}$$

and a *standing wave* otherwise. The bifurcation analysis for rotating waves proceeds along the lines of the Renardy-Rand approach. The assumption (5.8) implies that

$$x(t) = \gamma(t)^{-1} \cdot x(0).$$

Now transform the equation $\dot{y} = f_P(y)$ by looking for solutions of the form

$$y(t) = \gamma(t) z(t)$$

and obtain the system

$$\dot{z}(t) = f_P(z(t), \lambda) - \dot{\gamma}(0) z(t). \tag{5.9}$$

In this system, $x(t)$ corresponds to the steady-state solution $z(t) = x_0$ and bifurcation to tori for $x(t)$ is determined by a Hopf bifurcation in (5.9).

For standing waves, we use another approach, which is also valid for rotating waves. Let S be a cross section to $x(t)$ in the fiber of N over x_0. Since $x(t)$ is periodic, the flow of $F(\cdot, 0)$ returns to x_0 after time T. Thus, we can define the Poincaré map

$$\psi : S \times \mathbb{R} \to S$$

with $\psi(x_0, 0) = x_0$, and since F commutes with Γ_{x_0}, so does ψ. We can now study Hopf bifurcation of ψ with symmetry Γ_{x_0} using the techniques of § 3. Of course in the discussion of § 4, $\Gamma_{x_0} = D_n$ and that specific case represents an example of the general approach described here.

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