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Network periodic solutions: full oscillation and rigid synchrony

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Abstract

We prove two results about hyperbolic periodic solutions in networks of systems of ODEs. First, we show that generically hyperbolic periodic solutions of network admissible systems of differential equations oscillate in each node if and only if the network is transitive. We can associate a polydiagonal $\Delta(Z(t))$ with each hyperbolic periodic solution $Z(t)$ as follows. The cell coordinates of a point in $\Delta(Z(t))$ are equal if the corresponding cell coordinates of $Z(t)$ are equal for all t; that is, the outputs from the two cells are synchronous. Second, we prove that $\Delta(Z(t))$ is rigid (unchanged by small admissible perturbations) if and only if it is flow-invariant for all admissible vector fields.

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1. Introduction

In this paper we prove two main results about hyperbolic periodic solutions in networks of systems of ODEs. First, we prove that such hyperbolic periodic solutions are generically fully oscillatory (oscillating in each node) if and only if the network is transitive (see theorem 2.2). We associate a colouring of nodes to a hyperbolic periodic solution by assigning the same colour to any pair of synchronous nodes. Second, we prove rigid synchrony: this colouring is rigid if and only if it is balanced (see theorem 6.1). These results have been conjectured previously by Josić and Török [3] and Stewart and Parker [5], and are discussed by Aldis [1]. In this introduction we define terms and give an overview of our approach, an approach that is common to the two results. The first result begins to address the question of when one can reconstruct the dynamics of a network of differential equations just by looking at the output from one node. The second result provides one step towards the proof of a general conjecture by Stewart and Parker that rigid phase shifts between the outputs of two nodes for a periodic solutions are forced by symmetry—but symmetry in a quotient network [5, 6].

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A review of network issues. A *coupled cell network* (see [2,4] for details) is a graph that consists of a finite set of cells (or nodes) divided into cell types and a finite set of directed arrows or edges divided into edge types. Arrows indicate which cells are connected to which. The *input set* of a cell c is the set of arrows that terminate at cell c. Two cells are *input equivalent* if there exists a bijection between the input sets of the cells that preserves coupling type.

Let $\mathcal{G}$ be a network with n nodes. We associate a phase space $\boldsymbol{R}^{k_i}$ with each cell i and assume that cells of the same type have the same phase space. Then

$$\mathcal{P}_{\mathcal{G}} = \boldsymbol{R}^{k_1} \times \cdots \times \boldsymbol{R}^{k_n}$$

is the *phase space* of the coupled cell network $\mathcal{G}$. Suppose that cell j receives signals from the m_j cells $\sigma_j(1), \ldots, \sigma_j(m_j)$. Then an *admissible system* of ODEs associated with this network has the form

$$\dot{z}_j = f_j(z_j, z_{\sigma_j(1)}, \ldots, z_{\sigma_j(m_j)}) \tag{1.1}$$

for $j = 1, \ldots, n$. Moreover, if the arrows from cells $\sigma_j(p)$ and $\sigma_j(q)$ to cell j are equivalent, then f_j is assumed to be invariant under the transposition of coordinates $z_{\sigma_j(p)}$ and $z_{\sigma_j(q)}$. If cells i and j are input equivalent, then $f_i = f_j$.

Definition 1.1. Let $Z(t) = (z_1(t), \ldots, z_n(t))$ be a closed path in $\mathcal{P}_{\mathcal{G}}$. The *oscillating set* $\mathcal{O}_Z$ of Z is the set of cells i such that $z_i(t)$ is not constant. $Z(t)$ is *fully oscillatory* if $\mathcal{O}_Z$ contains all cells.

Fully oscillatory periodic solutions. Suppose that $Z_0(t) = (z_1^0(t), \ldots, z_n^0(t))$ is a hyperbolic periodic solution to (1.1). It follows from hyperbolicity that if we perturb the f_j slightly, the perturbed admissible system will have a unique periodic solution that is near $Z_0(t)$.

Definition 1.2. The property 'fully oscillatory' is *generic* for a fixed network if every hyperbolic periodic solution to an admissible vector field (1.1) for that network is the limit of fully oscillatory periodic solutions to small admissible perturbations of (1.1).

Equation (1.1) shows that there are arrows from cells $\sigma_j(1), \ldots, \sigma_j(m_j)$ to cell j. A network is *transitive* or *path connected* if there is a sequence of arrows in the graph that connect cell i to cell j for each pair i, j. If a network is not transitive, then we call it *feed-forward*. A standard example of a transitive network is the *all-to-all* coupled network where $m_j = n-1$ and the indices $\sigma_j(1), \ldots, \sigma_j(n-1)$ enumerate all cells not equal to j. Theorem 2.2 proves that fully oscillatory is a generic property for a fixed network if and only if that network is transitive.

It is straightforward to show that fully oscillatory is not generic in feed-forward networks. In these networks we can divide the cells into X cells and Y cells, where Y cells may couple only to Y cells and X cells may couple to either X cells or Y cells. It follows that in a feed-forward network, every admissible vector field (1.1) can be written in the skew-product form

$$\begin{aligned} \dot{X} &= F(X), \\ \dot{Y} &= G(X, Y). \end{aligned} \tag{1.2}$$

Let $(DF)_0 = -I$ so that the $X = 0$ is a stable equilibrium for the $\dot{X}$ equation. Suppose we can choose G so that equation $\dot{Y} = G(0, Y)$ has a hyperbolic periodic solution. It follows that any small perturbation of (1.2) yields a hyperbolic equilibrium in the $\dot{X}$ equation that is near the origin and that fully oscillatory is not generic for this feed-forward network. If any of the Y cells have a phase space that is at least two-dimensional (which we are free to assume when constructing a counterexample), then G can be constructed with a periodic solution (away from the origin using Hopf bifurcation)[3].

It is also straightforward to prove that fully oscillatory is generic for the all-to-all network in which all arrows are different. In this all-to-all network every system of ODEs $\dot{Z} = F(Z)$ on phase space is admissible. It follows that changes of coordinates of admissible systems are admissible. Let $Z_0(t)$ be a hyperbolic periodic solution to (1.1) and let Φ be a diffeomorphism on phase space. Then $\Phi(Z_0(t))$ is a hyperbolic periodic solution for the admissible system (1.1) in changed coordinates, namely

$$\dot{Z} = (\mathrm{d}\Phi^{-1})_Z F(\Phi(Z)). \tag{1.3}$$

We can find a near identity linear map $\Phi = I + \varepsilon A$ such that $\Phi(Z_0(t))$ is fully oscillatory. It follows that (1.3) has the desired fully oscillatory perturbation.

The proof that fully oscillatory is a generic property for transitive networks turns out to be surprisingly difficult. Specifically, the difficulty in proving theorem 2.2 is in identifying a large enough class of *admissible perturbations* of the given admissible system for which one can control how the periodic solution perturbs. As noted in the all-to-all example, without the network structure restriction, it is straightforward to perturb the original periodic solution to be fully oscillatory by use of a near identity change of coordinates. However, most such changes of coordinates do not retain the network structure because the jth equation does not in general depend on all of the other phase space variables. We next discuss the admissible perturbations that we use. The proof of theorem 2.2 proceeds as follows: if cell j is coupled to cell i and if cell j is oscillating, then generically cell i is also oscillating. See theorem 2.1.

The basic idea behind the proof of theorem 2.1 is to show the existence of an admissible perturbation that forces cell i to oscillate if cell j is already oscillating. Here we use a standard singularity theory/Floquet theory style argument by perturbing the admissible system of differential equations and then understanding how the periodic solution moves—at least to linear order. The trick to making this argument work is to exhibit a sufficiently rich class of admissible perturbations for which one can control the perturbation of the periodic solution—at least to linear order.

The admissible perturbations. As noted in [2], a useful class of coordinate changes that preserves network structure for all networks is the class of strongly admissible changes of coordinates. A map $\Phi : \mathcal{P}_\mathcal{G} \to \mathcal{P}_\mathcal{G}$ is *strongly admissible* if for each i its ith coordinate φ_i is a function only of z_i (that is, $(\Phi(Z))_i = \varphi_i(z_i)$) and $\varphi_i = \varphi_j$ for every pair of input equivalent cells. The following is a useful remark noted in [2, lemma 7.3]. Let $F : \mathcal{P}_\mathcal{G} \to \mathcal{P}_\mathcal{G}$ be admissible and let $\Phi : \mathcal{P}_\mathcal{G} \to \mathcal{P}_\mathcal{G}$ be strongly admissible. Then $\Phi \circ F$ and $F \circ \Phi$ are admissible. The proof is a straightforward calculation, but that calculation does require a detailed discussion of the definition of admissibility. This result is important because the composition of two admissible maps is generally not admissible.

However, strongly admissible changes of coordinates alone cannot transform a general periodic solution to a fully oscillatory one, since constant cells remain constant under such transformations. In our proofs we make explicit use of linear combinations of maps Φ and $\Psi_1 A \Psi_2$, where Ψ_1, Ψ_2 and Φ are strongly admissible and A is linear and admissible. We note that linear admissible matrices can be derived from network adjacency matrices (there is one such adjacency matrix for each coupling type). We suspect that these perturbations are sufficient to prove our results, but we do not assert that.

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[3] Indeed, if the network has a nontrivial transitive component, then even if the Y cells are all one-dimensional, a G can be constructed with a periodic solution. The only difficulty with this construction occurs if the Y cell network is tree-like and the Y cell dimensions are one. Then the $\dot{Y} = G(0, Y)$ equation cannot have periodic solutions.

Balanced colouring and rigidity. A *polydiagonal* is a subspace of $\mathcal{P}_{\mathcal{G}}$ defined by equality of some subsets of cell coordinates. Note that every closed path $Z(t) = (z_1(t), \ldots, z_n(t))$ in $\mathcal{P}_{\mathcal{G}}$ leads to a polydiagonal

$$\Delta(Z(t)) = \{X = (x_1, \ldots, x_n) \in \mathcal{P}_{\mathcal{G}} : \ x_i = x_j \text{ if } z_i(t) = z_j(t) \text{ for all } t\}.$$

In addition, every polydiagonal leads to a colouring of the network nodes in which two nodes i and j have the same colour if and only if the node coordinates for every point in the polydiagonal are equal. We can also colour network arrows so that two arrows have the same arrow-colour if and only if their coupling types are the same and the nodes from which the arrows emanate (their *tail* cells) have the same node-colour. The node-colouring is called *balanced* if there exists an arrow-colour preserving bijection between the input sets for each pair of nodes with the same node-colour.

It is proved in [2] (see also [4]) that polydiagonals are flow-invariant with respect to all admissible vector fields if and only if the colouring associated with the polydiagonal is balanced. Assume $Z_0(t)$ is a hyperbolic periodic solution to the admissible system $\dot{Z} = F(Z)$; the associated polydiagonal $\Delta(Z_0(t))$ has a balanced colouring; and $\dot{Z} = G(Z, \varepsilon)$, where $G(Z, 0) = F(Z)$, is a perturbed admissible system. For small ε it follows from uniqueness that the perturbed periodic solution $Z_\varepsilon(t)$ to the admissible perturbed system must lie in $\Delta(Z_0(t))$. Moreover, $\Delta(Z_\varepsilon(t)) = \Delta(Z_0(t))$ and the colouring associated with Z_ε is identical to the colouring associated with $Z_0(t)$. In this situation, we say that the colouring associated with the hyperbolic periodic solution $Z_0(t)$ is *rigid*.

As a special case, colourings associated with hyperbolic equilibria can be rigid. It was proved in [2] that a colouring associated with a hyperbolic equilibrium is rigid if and only if it is balanced. In theorem 6.1 we prove that a colouring associated with a hyperbolic periodic solution is rigid if and only if it is balanced. This theorem can also be thought of as a perturbation result—at least when one argues by contradiction. Suppose that the colouring is not balanced, then there must exist a pair of nodes i and j with the same colour (that is, $z_i^0(t) - z_j^0(t) = 0$ for all t) whose input sets are not colour isomorphic. In this case we must construct an admissible perturbation with enough control of the perturbed periodic solutions $Z_\varepsilon = (z_1^\varepsilon, \ldots, z_n^\varepsilon)$ such that $z_i^\varepsilon(t) - z_j^\varepsilon(t) \neq 0$ for all small ε. It turns out that the class of perturbations that worked for the fully oscillatory results also works for the rigidity results.

Stewart and Parker [5] discuss the fact that the phase shifts in periodic solutions can be rigid (unperturbed by small admissible vector field perturbations) only if symmetry (in a certain sense) exists. More precisely, every balanced colouring (such as the balanced colouring associated with the synchronous nodes of a hyperbolic periodic solution, as follows from theorem 6.1) leads to a quotient network (see [2]). Stewart and Parker prove that if this quotient network is all-to-all coupled (with perhaps many different arrow types), then there is a cyclic symmetry of the quotient network that is responsible for the rigid phase shift synchrony in the original solution. It is likely that this result is valid if the quotient network is transitive. It is also likely that morally this result is valid for feed-forward networks as well—but the exact statement will be more complicated.

Structure of the paper. This paper is constructed as follows. The main results on fully oscillatory solutions are discussed in section 2. The use of admissible perturbations of the form $\Psi_1 A \Psi_2$ is discussed in section 4 and the use of admissible perturbations of the form Φ is shown in section 5. The basic mode of proof is that the number of admissible perturbations that permit constant cells to stay constant is in a sense finite-dimensional, whereas the number of admissible perturbations is infinite-dimensional. This point is discussed in section 3. Finally, the main results on rigidity are discussed in section 6. The structure of the proof is similar to the structure of the proofs of the fully oscillatory results.

2. Results on fully oscillatory periodic solutions

In this section, we present the main results concerning fully oscillatory periodic solutions. Let $\mathcal{G}$ be a coupled cell network and let $F : \mathcal{P}_\mathcal{G} \to \mathcal{P}_\mathcal{G}$ be an admissible vector field.

Theorem 2.1. *Assume that the system*

$$\dot{Z} = F(Z) \tag{2.1}$$

has a hyperbolic periodic solution $Z_0(t)$. Suppose $Z_0(t)$ is constant in cell c and cell c receives an input from a cell in which $Z_0(t)$ is time-varying. Then there is an arbitrarily small admissible perturbation of (2.1) whose perturbed periodic solution is time-varying in cell c.

Theorem 2.2 is a corollary of theorem 2.1.

Theorem 2.2. *Suppose the network $\mathcal{G}$ is transitive and the admissible system (2.1) has a hyperbolic periodic solution Z_0 that is not fully oscillatory. Then there is an arbitrarily small admissible perturbation of (2.1) whose perturbed periodic solution is fully oscillatory.*

Proof. Since Z_0 is not fully oscillatory, the transitivity of the network implies that there exists a constant cell c that receives input from a time-varying cell. By theorem 2.1, there exists an arbitrarily small admissible perturbation of (2.1) whose perturbed periodic solution is time-varying in cell c. Continuity implies that time-varying cells stay time-varying under small perturbation. So for small enough perturbations, the perturbed periodic solution has more oscillating cells than Z_0 does. Since the sum of a finite number of small perturbations is a small perturbation, induction implies there exists an admissible perturbation such that the perturbed periodic solution is fully oscillatory. □

We prove theorem 2.1 locally; that is, we prove the theorem on a small interval J in time t whose choice is made using lemmas 2.3 and 2.6.

Lemma 2.3. *Let*

$$Z_0(t) = (z_1^0(t), \ldots, z_n^0(t))$$

be a nonconstant periodic solution to (2.1), let $J_0 \subset \mathbf{R}$ be an open interval, and suppose that cell c receives input from a cell that is time-varying on J_0. Then there exists an open interval $J_1 \subset J_0$ such that

(a) for each cell i, either z_i^0 is constant on J_1 or $\dot{z}_i^0$ is nowhere zero on J_1,
(b) for each pair of cells i, j, either $z_i^0(t) = z_j^0(t)$ on J_1 or $z_i^0(J_1)$ and $z_j^0(J_1)$ are disjoint and
(c) cell c receives input from a cell that is time-varying on J_1.

Proof. By hypothesis, cell c receives an input from a cell d such that z_d^0 is time-varying on J_0.

(c) Then there exists a point $t_0 \in J_0$ where $\dot{z}_d(t_0) \neq 0$. By continuity, there exists an open interval J (containing t_0) such that $\dot{z}_d(t) \neq 0$ for $t \in J$.
(a) For each j we can shrink J such that either $z_j^0(t)$ is constant on J or $\dot{z}_j^0(t)$ is nowhere zero on J.
(b) We can further shrink J to an open interval $J_1 \subset J \subset J_0$ such that for each pair i, j either $z_i^0(t) = z_j^0(t)$ on J_1 or $z_i^0(J_1)$ and $z_j^0(J_1)$ are disjoint. □

Remark 2.4. Since Z_0 must be nonconstant on every interval, there always exists an interval $J_1 \subset J_0$ satisfying conditions (a) and (b); we require condition (c) as well to ensure the conclusion of theorem 2.1 holds.

Next we set notation. Let $Z(t)$ be periodic and define the sets

$$\mathcal{C}(Z, J_1) = \{i : \dot{z}_i(t) = 0 \text{ for all } t \in J_1\}$$

$$\mathcal{O}(Z, J_1) = \{i : \dot{z}_i(t) \neq 0 \text{ for all } t \in J_1\}$$

and the polydiagonal subspace

$$\Delta(Z, J_1) = \{x \in \mathcal{P}_{\mathcal{G}} : x_i = x_j \text{ whenever } z_i^0(t) = z_j^0(t) \text{ for every } t \in J_1\}.$$

Let J_1 be an open interval whose existence is guaranteed by lemma 2.3. Then it follows from the choice that

$$\mathcal{C}(Z_0, J_1) \cup \mathcal{O}(Z_0, J_1) = \{1, \ldots, n\}.$$

We claim that without loss of generality, we can find an open subinterval $J \subset J_1 \subset J_0$ such that the three sets $\mathcal{C}(Z_0, J)$, $\mathcal{O}(Z_0, J)$, $\Delta(Z_0, J)$ are *rigid* in a way we now define.

Definition 2.5. A property is *rigid* if and only if that property remains unchanged under all sufficiently small admissible perturbations.

For example, we can consider the set $\mathcal{C}(Z_0, J_1)$ to be a *property* of the periodic solution Z_0. That property is rigid if the set does not change on perturbation of the periodic solution by an admissible perturbation of the vector field.

Lemma 2.6. *The periodic solution* Z_0 *can be perturbed by an arbitrarily small admissible perturbation so that the sets* $\mathcal{C}$, $\mathcal{O}$ *and* Δ *are rigid on an open subinterval* $J \subset J_1$.

Proof. Let

$$\dot{Z} = \hat{F}(Z)$$

be a small admissible perturbation of (2.1). By hyperbolicity of Z_0 there exists a unique hyperbolic periodic solution $\hat{Z}_0$ to the perturbed equation. By continuity, time-varying cells in Z_0 remain time-varying in $\hat{Z}_0$ under a small enough perturbation, but constant cells may become time-varying under perturbation. Therefore,

$$\mathcal{C}(\hat{Z}_0, J_1) \subset \mathcal{C}(Z_0, J_1),$$

$$\mathcal{O}(Z_0, J_1) \subset \mathcal{O}(\hat{Z}_0, J_1).$$

Let $\hat{Z}_0 = (\hat{z}_1^0, \ldots, \hat{z}_n^0)$. By lemma 2.3(a), there exists an open subinterval $J_2 \subset J_1$ such that the periodic solution $\hat{Z}_0$ satisfies $\dot{\hat{z}}_i(t) = 0$ for every $t \in J_2$ or $\dot{\hat{z}}_i(t) \neq 0$ for every $t \in J_2$. This implies $\mathcal{C}(\hat{Z}_0, J_2) \cup \mathcal{O}(\hat{Z}_0, J_2) = \{1, \ldots, n\}$. Since small perturbations can only decrease the number of constant cells and there are only a finite number of cells, we only need to make a finite number of small perturbations to reach a state where these two sets are rigid. Since the sum of a finite number of small perturbations is again a small perturbation, it follows that, after shrinking J_2 if necessary, there exists an admissible perturbation with the perturbed periodic solution $\hat{Z}_0$, such that $\mathcal{C}(\hat{Z}_0, J_2)$ and $\mathcal{O}(\hat{Z}_0, J_2)$ are rigid.

Also note that under small perturbation $\hat{z}_i^0(t) \neq \hat{z}_j^0(t)$ for every $t \in J_2$ if $z_i^0(t) \neq z_j^0(t)$ for every $t \in J_1$. That is, asynchronous cells stay asynchronous. However, synchronous cells may be desynchronized under small perturbation. Therefore

$$\Delta(Z_0, J_1) \subset \Delta(\hat{Z}_0, J_2).$$

As above, we can shrink J_2 to J by successive small perturbations until $\Delta(\hat{Z}_0, J_2)$ can no longer grow under perturbation, and then Δ is rigid. □

It follows from lemma 2.6 that we can assume

$$\begin{aligned} \mathcal{C}(Z_0, J) &= \mathcal{C}(\hat{Z}_0, J), \\ \mathcal{O}(Z_0, J) &= \mathcal{O}(\hat{Z}_0, J), \\ \Delta(Z_0, J) &= \Delta(\hat{Z}_0, J), \end{aligned} \tag{2.2}$$

where $\hat{Z}_0$ is the perturbed hyperbolic periodic solution to any sufficiently small perturbation. We will often denote these sets simply by $\mathcal{C}$, $\mathcal{O}$ and Δ if there is no danger of confusion, suppressing their dependence on Z_0 and J. In addition, we will refer to the elements of $\mathcal{C}$ as *$\mathcal{C}$-cells* and the elements of $\mathcal{O}$ as *$\mathcal{O}$-cells*.

Remark 2.7. We can associate a *colouring* of the cells with $\Delta(Z_0, J)$ by assigning the same colour to cells i and j if and only if $z_i^0(t) = z_j^0(t)$ for all $t \in J$; we may thus identify a colour with the set L of all cells of that colour. We will call a colour L an *$\mathcal{O}$-colour* if the cells in L oscillate for Z_0, and call L a *$\mathcal{C}$-colour* otherwise. We say that the colouring of cells associated with the polydiagonal $\Delta(Z_0, J)$ is *rigid* if the colouring does not change on small admissible perturbation of the admissible system.

Now we discuss a special case of theorem 5.1 in [5] (rigid input theorem).

Lemma 2.8. *Suppose the colouring associated with $\Delta(Z_0, J)$ is rigid. If cells i and j have the same colour, then they are input equivalent.*

Proof. We argue by contradiction. Suppose cells i and j have the same colour, but are not input equivalent. Let $\Phi = (\varphi_1, \ldots, \varphi_n)$ be a strongly admissible change of coordinates. Since cells i and j are not input equivalent, φ_i and φ_j are independently defined maps. For example, we can choose φ_j to be the identity and φ_i to be any diffeomorphism. Hence, we can choose a strongly admissible, near identity, change of coordinates Φ, such that $\varphi_i(z_i^0(t)) \neq z_j^0(t)$ for some $t \in J$. Hence, the colouring is not rigid. This contradicts the assumption that the colouring is rigid. $\square$

Definition 2.9. Let $Z_0(t)$ be a periodic state and $J \subset \boldsymbol{R}$ an open interval. We say that Z_0 is *nondegenerately rigid on J* if

(a) $\mathcal{C}(Z_0, J)$, $\mathcal{O}(Z_0, J)$ and $\Delta(Z_0, J)$ are rigid.
(b) For each cell i, either z_i^0 is constant on J or $\dot{z}_i^0$ is never zero on J.
(c) For each pair of cells i and j, either $z_i^0(t) = z_j^0(t)$ on J or $z_i^0(J)$ and $z_j^0(J)$ are disjoint.

Remark 2.10. Henceforth we make two *standard assumptions*. We assume that Z_0 is a hyperbolic periodic solution to (2.1) that is nondegenerately rigid on an open interval $J \subset \boldsymbol{R}$. We also assume that cell c receives input from a cell that is oscillating on J.

Given a hyperbolic periodic solution to (2.1) and an open interval $J_0 \subset \boldsymbol{R}$, lemmas 2.3 and 2.6 prove that there is an arbitrarily small admissible perturbation of (2.1) and an open subinterval $J \subset J_0$ on which the associated perturbed periodic solution satisfies the standard assumptions. Proposition 2.11 shows that this is the perturbation whose existence is claimed in theorem 2.1.

Proposition 2.11. *Suppose that the periodic solution $Z_0(t) = (z_1^0(t), \ldots, z_n^0(t))$ satisfies the standard assumptions on an open interval J. Then cell c oscillates in Z_0.*

3. Overview of proof of proposition 2.11

Let P be an admissible map and let $p(Z)$ be the c-component of $P(Z)$. We shall also call p *admissible*. Consider the perturbed admissible system

$$\dot{Z} = F(Z) + \varepsilon P(Z) \tag{3.1}$$

for small ε. Let

$$Z_\varepsilon(t) = (z_1^\varepsilon(t), \ldots, z_n^\varepsilon(t))$$

be the periodic solution of (3.1) that is a small perturbation of Z_0. Of course, Z_ε depends on P. So we define the function

$$\alpha(P) = \frac{\partial}{\partial\varepsilon} Z_\varepsilon \bigg|_{\varepsilon=0}.$$

Let $f(Z)$ be the c-component of F. Then the differential equation for z_c^ε is

$$\dot{z}_c^\varepsilon(t) = f(Z_\varepsilon(t)) + \varepsilon p(Z_\varepsilon(t)). \tag{3.2}$$

On differentiating both sides of (3.2) with respect to ε and evaluating at $\varepsilon = 0$, we obtain

$$\frac{\partial}{\partial\varepsilon}\dot{z}_c^\varepsilon(t)\bigg|_{\varepsilon=0} = f_Z(Z_0(t))\,\frac{\partial}{\partial\varepsilon}Z_\varepsilon(t)\bigg|_{\varepsilon=0} + p(Z_0(t)) = f_Z(Z_0(t))\alpha(P)(t) + p(Z_0(t)). \tag{3.3}$$

We prove proposition 2.11 by contradiction. Suppose that cell c is constant under the standard assumptions. Since $\mathcal{C}$ is rigid, cell c remains constant for all sufficiently small admissible perturbations P. It follows that the left-hand side of (3.3) is 0 for all admissible p. So (3.3) becomes

$$0 = f_Z(Z_0)\alpha(P) + p(Z_0), \tag{3.4}$$

which must be valid for all admissible P. Let $\mathcal{F}_J$ denote the space of functions from J to $\boldsymbol{R}^N$, where the value of N will depend on the context; here we take $N = k_c$. We establish the contradiction by showing that the right-hand side of (3.4) contains an infinite-dimensional subspace of $\mathcal{F}_J$ and hence cannot be 0 for all admissible maps P.

We state this approach more abstractly. Let $\mathcal{A}$ be the space of all admissible maps. Let $\pi : \mathcal{A} \to \mathcal{F}_J$ be the map given by

$$P \mapsto f_Z(Z_0(t))\alpha(P)(t) + p(Z_0(t)).$$

We will show that the image of π spans an infinite-dimensional subspace of $\mathcal{F}_J$ and to do this we need to consider two cases.

An *$\mathcal{O}$-coloured sum associated with f* is a function on J of the form

$$\sum_{i\in L} f_{z_i}(Z_0(t)), \tag{3.5}$$

where L is an $\mathcal{O}$-colour. Then there are two cases, depending on the values of $\mathcal{O}$-coloured sums. The first case is when all $\mathcal{O}$-coloured sums (3.5) are zero and the second is when some $\mathcal{O}$-coloured sum is nonzero. In each case, we will exhibit sufficient P to show that the image of π is infinite-dimensional. In the first case, we choose $P = \Psi A \Phi$, where Ψ and Φ are strongly admissible maps on $\mathcal{P}_\mathcal{G}$ and A is an admissible map on $\mathcal{P}_\mathcal{G}$ derived from the adjacency matrix of a certain coupling type. In this case we say that P is of *type I*. In the second case we choose $P = \Phi$, where Φ is again a strongly admissible map on $\mathcal{P}_\mathcal{G}$, and in this case we say that P is of *type II*. These two cases are discussed in sections 4 and 5.

4. Case 1: $\mathcal{O}$-coloured sums are zero

We begin by defining the admissible matrix A. By our choice of J, cell c receives input from an $\mathcal{O}$-cell. Let d be an $\mathcal{O}$-cell that is coupled to c, let the coupling occur through an edge of type e, and let $\mathfrak{L}$ be the colour of d. Let $A_0 = (a_{ij})$, where a_{ij} are nonnegative integers, be the adjacency matrix of the subnetwork that consists of all nodes and all edges of type e in the network $\mathcal{G}$. For each pair of phase spaces $\boldsymbol{R}^{k_i}$ and $\boldsymbol{R}^{k_j}$, we arbitrarily choose two positive integers $s \leqslant k_i$ and $r \leqslant k_j$. Let E_{ij} be the $k_i \times k_j$ matrix whose entry at position (s, r) is 1 and whose other entries are zero. If the pairs of cells i, $\hat{i}$ and j, $\hat{j}$ are each of the same cell type, then we further require that $E_{ij} = E_{\hat{i}\hat{j}}$. In block form we define a linear admissible map on $\mathcal{P}_G$ (many other choices would work later) by

$$A \equiv (a_{ij} E_{ij}). \tag{4.1}$$

Recall that a *strongly admissible* map of a general network has the form

$$\Phi(Z) = (\varphi_1(z_1), \ldots, \varphi_n(z_n)), \tag{4.2}$$

where $\varphi_i : \boldsymbol{R}^{k_i} \to \boldsymbol{R}^{k_i}$. Moreover, if cells i and j are input equivalent, then $k_i = k_j$ and $\varphi_i = \varphi_j$. Recall [2] also that the composition of a strongly admissible map with an admissible map is always admissible, so that $\Psi A \Phi$ is admissible if Ψ and Φ are strongly admissible. In the following, we denote the c-component of a vector V by V_c.

Type I admissibles $(\Psi A \Phi Z_0)_c$ *are infinite-dimensional*

Lemma 4.1. *Let A be the matrix defined in (4.1). Assume that $Z_0(t)$ satisfies the standard assumptions on the open interval J. Then the set*

$$\mathcal{S} = \{(\Psi A \Phi Z_0(t))_c : \ \Phi, \Psi \text{are strongly admissible maps of } \mathcal{G} \text{ and } t \in J\}$$

contains an infinite-dimensional subspace of $\mathcal{F}_J$.

Proof. Let $\Phi = (\varphi_1(z_1), \ldots, \varphi_n(z_n))$ be a strongly admissible map of $\mathcal{G}$. Then

$$(A\Phi(Z_0(t)))_c = \sum_{j=1}^{n} a_{cj} E_{cj} \varphi_j(z_j^0). \tag{4.3}$$

Since the colouring associated with $\Delta(Z_0, J)$ is rigid by hypothesis, it follows from lemma 2.8 that cells of the same colour L must be input equivalent. Thus $\varphi_i = \varphi_j$ for all i and j in L, and we may denote their common value by φ^L. We may similarly denote by z^L the common value of z_i^0 for i in L. Then (4.3) can be rewritten as

$$(A\Phi(Z_0(t)))_c = \sum_{\text{colours } L} \left(\sum_{j \in L} a_{cj} E_{cj} \right) \varphi^L(z^L(t)) \tag{4.4}$$

for all $t \in J$.

Recall that d is an $\mathcal{O}$-cell of colour $\mathfrak{L}$. Since $\dot{z}_d^0(t) \neq 0$ for $t \in J$, we can choose φ_d so that

$$\varphi_d(z_d^0(t)) = \varphi^{\mathfrak{L}}(z^{\mathfrak{L}}(t)) \text{ is time-varying on } J.$$

Moreover, since E_{cd} is a nonzero linear map from $\boldsymbol{R}^{k_d}$ to $\boldsymbol{R}^{k_c}$ and φ_d can be any map on $\boldsymbol{R}^{k_d}$, we can always choose φ_d so that

$$E_{cd}\varphi_d(z_d^0(t)) \text{ is time-varying on } J.$$

In addition, since $z_i^0(J)$ and $z_d^0(J)$ are disjoint for every $i \notin \mathfrak{L}$, we can choose a map φ_i such that

$$\varphi_i(z_i^0(t)) = 0 \text{ for all } i \notin \mathfrak{L}.$$

Thus $\varphi^L(z^L(t)) = 0$ for all $L \neq \mathfrak{L}$. T These choices define an admissible map Φ, such that (4.3) becomes

$$(A\Phi Z_0(t))_c = \left(\sum_{j\in\mathfrak{L}} a_{cj}\right) E_{cd}\varphi_d(z_d^0(t)). \tag{4.5}$$

Since cell d is coupled to cell c through an edge of type e,

$$\sum_{j\in\mathfrak{L}} a_{cj} > 0.$$

It follows that there exists a strongly admissible Φ such that

$$(A\Phi Z_0(t))_c \text{ is time-varying.}$$

Since Ψ is arbitrary, we see that $\mathcal{S}$ is an infinite-dimensional subspace of $\mathcal{F}_J$. □

Image of π using type I admissibles is infinite-dimensional

Recall (3.4) that

$$0 = f_Z(Z_0)\left.\frac{\partial Z_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} + p(Z_0)$$

for all admissible P. For convenience let

$$\alpha_i(t) = \left.\frac{\partial z_i^\varepsilon}{\partial \varepsilon}(t)\right|_{\varepsilon=0}. \tag{4.6}$$

Then we can rewrite (3.4) as

$$0 = \sum_{\text{colours } L}\ \sum_{i\in L} f_{z_i}(Z_0(t))\alpha_i(t) + p(Z_0(t)). \tag{4.7}$$

The rigidity of Δ implies that $z_i^\varepsilon(t) = z_j^\varepsilon(t)$ for all $t \in J$ whenever $i, j \in L$. Hence, $\alpha_i(t) = \alpha_j(t)$ for all $t \in J$. Let $\alpha^L(t)$ be the common value of $\alpha_i(t)$ for $i \in L$. Then (4.7) can be rewritten as

$$0 = \sum_{\text{colours } L}\left(\sum_{i\in L} f_{z_i}(Z_0(t))\right)\alpha^L(t) + p(Z_0(t)). \tag{4.8}$$

By hypothesis, all $\mathcal{O}$-coloured sums are zero, so that (4.8) reduces to

$$0 = \sum_{\mathcal{C}\text{-colours } L}\left(\sum_{i\in L} f_{z_i}(Z_0(t))\right)\alpha^L(t) + p(Z_0(t)). \tag{4.9}$$

Since for $i \in \mathcal{C}$, we have $\alpha_i(t) = \alpha_i \in \boldsymbol{R}^{k_i}$ is constant on J, and since $f_{z_i}(Z_0(t))$ is independent of the perturbation P, the first term on the right-hand side of (4.9) lies in the finite-dimensional subspace of $\mathcal{F}_J$ spanned by $f_{z_i}(Z_0(t))$. However, as we have shown in lemma 4.1, the function $p(Z_0(t)) = (\Psi A\Phi Z_0(t))_c$ can be chosen from an infinite-dimensional subspace of $\mathcal{F}_J$, which contradicts (3.4).

5. Case 2: some $\mathcal{O}$-coloured sum is nonzero

We now consider perturbations of the form $P = \Phi$, where $\Phi = (\varphi_1, \ldots, \varphi_n)$ is a strongly admissible map. Let L be a colour and let

$$\mathcal{Q}_L = \{\Phi : \ \Phi \text{ is strongly admissible and } \varphi_i(z_i^0(J)) = 0 \text{ for } i \notin L\}. \tag{5.1}$$

Remark 5.1. Note that lemma 2.8 is not required for the space $\mathcal{Q}_L$ to be well defined.

Type II admissibles $(\Phi Z_0(t))_c$ *are infinite-dimensional*

Lemma 5.2. *Assume that the periodic solution* $Z_0(t)$ *satisfies the standard assumptions on an open interval* J*, and suppose that* Z_0 *oscillates in a cell* i *of colour* L*. Let* $M : J \to$ $k_i \times k_i$*-matrices be a fixed nonzero matrix-valued function. Then the set*

$$\left\{ M(t) \int_{t_0}^{t} \varphi_i(z_i^0(s))\,\mathrm{d}s : \ \Phi \in \mathcal{Q}_L \right\}$$

spans an infinite-dimensional subspace of $\mathcal{F}_J$*.*

Proof. Let $\boldsymbol{R}^{k_i}$ be the phase space of cell i. We assert that there is a vector $V \in \boldsymbol{R}^{k_i}$ and an interval $J_0 \subset J$ on which

$$M(t)V \neq 0 \text{ for } t \in J_0.$$

This follows from the fact that $M(t)$ is nonzero on J. We can then choose Φ so that φ_i is a scalar multiple of V; that is, $\varphi_i(z_i) = b(z_i)V$ where $b \in \boldsymbol{R}$. Note that since z_i^0 is time-varying on J_0, $z_i^0(J_0)$ is an embedded curve in $\boldsymbol{R}^{k_i}$. Also note that b can be any scalar function defined on $z_i^0(J_0)$. Hence the collection of functions of the form

$$\int_{t_0}^{t} \varphi_i(z_i^0(s))\mathrm{d}s = \left(\int_{t_0}^{t} b(z_i^0(s))\,\mathrm{d}s \right) V \tag{5.2}$$

spans an infinite-dimensional subspace. □

Image of π *using type II admissibles is infinite-dimensional*
We denote $Z(t) = (X(t), Y(t))$ where $X(t)$ represents the variables of the cells constant in $Z_0(t)$ on J, $Y(t)$ represents the variables of the cells oscillating in $Z_0(t)$ on J. Similarly, we denote the perturbed periodic solution $Z_\varepsilon(t) = (X_\varepsilon(t), Y_\varepsilon(t))$. Define

$$\gamma(P) = \left.\frac{\partial X_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \qquad \text{and} \qquad \beta(P) = \left.\frac{\partial Y_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}, \tag{5.3}$$

where the dependence of β and γ on the perturbation P in (3.1) is indicated explicitly. Note that $\gamma(P)$ must be constant since the standard assumptions assert that $\mathcal{O}$ is rigid.

With this notation, we can rewrite (3.4) as

$$0 = f_X(Z_0(t))\gamma(P) + f_Y(Z_0(t))\beta(P) + p(Z_0(t)). \tag{5.4}$$

Now let $\mathfrak{M}$ be an $\mathcal{O}$-colour whose corresponding $\mathcal{O}$-coloured sum is nonzero. Since c is by assumption a $\mathcal{C}$-colour, and thus not in $\mathfrak{M}$, we have that

$$p(Z_0(t)) = \varphi_c(z_c(t)) = 0$$

for $P = \Phi \in \mathcal{Q}_{\mathfrak{M}}$. Thus, we will have arrived at a contradiction if we can show that the set

$$\mathcal{T} = \{f_Y(Z_0(t))\beta(\Phi)(t) \in \mathcal{F}_J : \ \Phi \in \mathcal{Q}_{\mathfrak{M}}\} \tag{5.5}$$

spans an infinite-dimensional subspace of $\mathcal{F}_J$. Note that in our proof of this fact, we will not rely on any properties of f other than the fact that one of the $\mathcal{O}$-coloured sums associated with f is nonzero; this observation will be needed later for the statement of lemma 5.3.

Let G be the part of the differential equation of F corresponding to the oscillating cells, that is,

$$\dot{Y}(t) = G(X(t), Y(t))$$

and let Φ^Y be the coordinates of Φ corresponding to the oscillating cells. Then the oscillating cells Y in the perturbed periodic solution satisfy

$$\dot{Y}_\varepsilon = G(Z_\varepsilon) + \varepsilon\Phi^Y(Y_\varepsilon). \tag{5.6}$$

On differentiating both sides of (5.6) with respect to ε and evaluating at $\varepsilon = 0$, we obtain

$$\dot{\beta} = G_X(Z_0(t))\gamma + G_Y(Z_0(t))\beta + \Phi^Y(Y_0) \tag{5.7}$$

where γ and β are defined in (5.3). To simplify notation, we dropped the explicit dependence of γ and β on Φ.

To arrive at our contradiction, we need to determine how β depends on Φ. Choose $t_0 \in J$ and let $W(t)$ be the fundamental solution to the homogeneous linear ODE system

$$\dot{\beta}(t) = G_Y\big(Z_0(t)\big)\beta(t) \tag{5.8}$$

with $W(t_0) = I$, the identity matrix. Then the general solution to the inhomogeneous equation (5.7) is

$$\begin{aligned}\beta(t) &= W(t)\bigg(\int_{t_0}^{t} W^{-1}(s)(G_X(Z_0(s))\gamma + \Phi^Y(Y_0(s)))\,\mathrm{d}s + K\bigg)\\ &= W(t)\bigg(\int_{t_0}^{t}(W^{-1}(s)G_X(Z_0(s)))\,\mathrm{d}s\bigg)\gamma\\ &\quad + W(t)K + W(t)\int_{t_0}^{t}(W^{-1}(s)\Phi^Y(Y_0(s)))\,\mathrm{d}s\end{aligned} \tag{5.9}$$

where K is the initial condition. Since $W(t)$ is independent of Φ we see that the first two terms in the computation of $\beta(t)$ on the right-hand side of (5.9) stay in a finite-dimensional subspace of $\mathcal{F}_J$. Therefore, if there is an infinite-dimensional subspace of possible $\beta(t)$, it must come from the last term in (5.9); namely

$$W(t)\int_{t_0}^{t} W^{-1}(s)\Phi^Y(Y_0(s))\,\mathrm{d}s. \tag{5.10}$$

In particular, we will have our contradiction if we can show that

$$f_Y(Z_0(t))W(t)\int_{t_0}^{t} W^{-1}(s)\Phi^Y(Y_0(s))\,\mathrm{d}s \tag{5.11}$$

spans an infinite-dimensional subspace of $\mathcal{F}_J$ when $\Phi \in \mathcal{Q}_{\mathfrak{M}}$.

Since $W(t)$ is unknown, calculations are difficult. However, we can gain approximate control of (5.10), and hence of (5.11), on a small interval of time by recalling that $W(t_0) = I$. Indeed, choose a small interval $J_1 \subset J$ containing t_0. Then, in this interval, let

$$W(t) = I + \tau\hat{W}^{\tau}(t). \tag{5.12}$$

with $\tau \ll 1$. Then

$$f_Y(Z_0(t))W(t)\int_{t_0}^{t} W^{-1}(s)\Phi^Y(Y_0(s))\,\mathrm{d}s = f_Y(Z_0(t))\int_{t_0}^{t}\Phi^Y(Y_0(s))\,\mathrm{d}s + O(\tau). \tag{5.13}$$

Recall that $z^L(t)$ and φ^L denote the common values of $z_i^0(t)$ and φ_i, respectively, for all cells i of colour L. Thus (5.13) becomes

$$\begin{aligned}&f_Y(Z_0(t))W(t)\int_{t_0}^{t} W^{-1}(s)\Phi^Y(Y_0(s))\,\mathrm{d}s\\ &\qquad = \sum_{\mathcal{O}\text{-colours } L}\ \sum_{i\in L} f_{z_i}(Z_0(t))\int_{t_0}^{t}\varphi^L(z^L(s))\,\mathrm{d}s + O(\tau).\end{aligned} \tag{5.14}$$

It follows that if

$$\sum_{\mathcal{O}\text{-colours } L}\ \sum_{i\in L} f_{z_i}(Z_0(t))\int_{t_0}^{t}\varphi^L(z^L(s))\,\mathrm{d}s \tag{5.15}$$

spans an infinite-dimensional subspace of $\mathcal{F}_J$, then we have our contradiction.

We can choose J containing t_0 small enough to guarantee that the approximation of $W(t)$ by (5.12) is valid. Since $\Phi \in \mathcal{Q}_{\mathfrak{M}}$, we have $\varphi^L(z^L(J)) = 0$ for $L \neq \mathfrak{M}$. Thus

$$\sum_{\mathcal{O}\text{-colours } L} \left(\sum_{i \in L} f_{z_i}(Z_0(t)) \right) \int_{t_0}^{t} \varphi^L(z^L(s))\,\mathrm{d}s = \sum_{i \in \mathfrak{M}} f_{z_i}(Z_0(t)) \int_{t_0}^{t} \varphi^{\mathfrak{M}}(z^{\mathfrak{M}}(s))\,\mathrm{d}s. \tag{5.16}$$

Let $M(t)$ be the nonzero $\mathcal{O}$-coloured sum $\sum_{i\in\mathfrak{M}} f_{z_i}(Z_0(t))$. It suffices to show that

$$\sum_{i \in \mathfrak{M}} f_{z_i}(Z_0(t)) \int_{t_0}^{t} \varphi^{\mathfrak{M}}(z^{\mathfrak{M}}(s))\,\mathrm{d}s = M(t) \int_{t_0}^{t} \varphi^{\mathfrak{M}}(z^{\mathfrak{M}}(s))\,\mathrm{d}s \tag{5.17}$$

spans an infinite-dimensional subspace of $\mathcal{F}_J$ while Φ varies in $\mathcal{Q}_{\mathfrak{M}}$. But this follows from lemma 5.2, so that we have our contradiction. □

This completes the proof of proposition 2.11. Recall from the observation following the definition of $\mathcal{T}$ in (5.5) that, aside from the condition that one of the $\mathcal{O}$-coloured sums associated with f be nonzero, no particular properties of f were used in the proof that $\mathcal{T}$ spans an infinite-dimensional subspace of $\mathcal{F}_J$. Thus, we may replace the coordinate function f in this proof by any of a more general class of functions that are independent of the vector field F. We incorporate this observation in the following lemma, which summarizes the results of the proof, and then state a corollary that will be useful in the next section.

Let $g : J \to \boldsymbol{R}^k$ be a smooth function. Analogous to the case for f, we define an *$\mathcal{O}$-coloured sum associated with g* to be a function of the form

$$\sum_{i \in L} g_{z_i}(Z_0(t)),$$

where L is an $\mathcal{O}$-colour. As before, we let g_Y represent the partial derivative of g about the oscillating cells.

Lemma 5.3. *Assume that the hyperbolic periodic solution Z_0 to (2.1) is nondegenerately rigid on J, and let $g : \mathcal{P}_{\mathcal{G}} \to \boldsymbol{R}^k$ be a smooth function into the phase space of one of the cells of $\mathcal{G}$. Suppose there exists an $\mathcal{O}$-colour L such that the corresponding $\mathcal{O}$-coloured sum associated with g is nonzero. Then the set*

$$\{g_Y(Z_0(t))\beta(\Phi)(t) \in \mathcal{F}_J : \Phi \in \mathcal{Q}_L\}$$

spans an infinite-dimensional subspace of $\mathcal{F}_J$, where $\beta(\Phi)$ and $\mathcal{Q}_L$ are as defined in (5.3) and (5.1), respectively.

Corollary 5.4. *Assume that the hyperbolic periodic solution Z_0 to (2.1) is nondegenerately rigid on J, and let $g : \mathcal{P}_{\mathcal{G}} \to \boldsymbol{R}^k$ be a smooth function into the phase space of one of the cells of $\mathcal{G}$. Suppose that one of the $\mathcal{O}$-coloured sums associated with g is nonzero on J. Then there exists a strongly admissible map Φ such that the perturbed periodic solution Z_ε to (3.1), where $P = \Phi$, satisfies*

$$g(Z_\varepsilon) \neq 0 \text{ on } J \tag{5.18}$$

for any small positive ε.

Proof. Let Φ be strongly admissible, and let Z_ε be the corresponding perturbed periodic solution to (3.1). On differentiating $g(Z_\varepsilon)$ with respect to ε and evaluating at $\varepsilon = 0$, we obtain

$$\left.\frac{\partial}{\partial\varepsilon} g(Z_\varepsilon)\right|_{\varepsilon=0} = g_X(Z_0(t))\gamma(\Phi) + g_Y(Z_0(t))\beta(\Phi). \tag{5.19}$$

where $\gamma(\Phi)$ and $\beta(\Phi)$ are as defined in (5.3). It now follows from lemma 5.3 that the set

$$\{g_Y(Z_0(t))\beta(\Phi)(t) \in \mathcal{F}_J : \Phi \text{ is strongly admissible}\} \tag{5.20}$$

spans an infinite-dimensional subspace of $\mathcal{F}_J$. Thus there must exist a strongly admissible Φ such that

$$\frac{\partial}{\partial \varepsilon} g(Z_\varepsilon)\bigg|_{\varepsilon=0} \neq 0 \text{ on } J, \tag{5.21}$$

so that $g(Z_\varepsilon) \neq 0$ on J for all small positive ε. ☐

6. Rigidity of periodic solutions

In this section, we study the relation between rigidity and balanced colorings. Let $Q_0 = (q_1^0, \ldots, q_n^0)$ be a point in the phase space $\mathcal{P}_\mathcal{G}$. Define the polydiagonal

$$\Delta(Q_0) = \{q \in \mathcal{P}_\mathcal{G} : \; q_i = q_j \text{ if } q_i^0 = q_j^0\}.$$

Suppose that Q_0 is a hyperbolic equilibrium of (2.1). It is shown in [2] that $\Delta(Q_0)$ is rigid if and only if the associated colouring is balanced (or that $\Delta(Q_0)$ is flow-invariant). Here we discuss the analogue for hyperbolic periodic solutions Z_0. Let $\Delta(Z_0) \equiv \Delta(Z_0, \boldsymbol{R})$. Our main theorem is as follows.

Theorem 6.1. *Suppose $Z_0(t)$ is a hyperbolic periodic solution of (2.1). Then the colouring associated with $\Delta(Z_0)$ is rigid if and only if it is balanced.*

Our proof of theorem 6.1 will use lemmas 6.2–6.4, which we now state and prove.

Lemma 6.2. *Assume the hyperbolic periodic solution Z_0 to (2.1) is nondegenerately rigid on J. Suppose cells c and d are in the same colour class, let g and h be the corresponding components of F, and let $f = g - h$. Then every $\mathcal{O}$-coloured sum associated with f is zero on J.*

Proof. Consider a perturbation of (2.1) of type II,

$$\dot{Z} = F(Z) + \varepsilon\Phi(Z),$$

where $\Phi = (\varphi_1, \ldots, \varphi_n)$ is an arbitrary strongly admissible map, and let $Z_\varepsilon = (z_1^\varepsilon, \ldots, z_n^\varepsilon)$ be the corresponding perturbed periodic solution. Then z_c^ε and z_d^ε satisfy

$$\begin{aligned} \dot{z}_c^\varepsilon &= g(Z_\varepsilon) + \varepsilon\varphi_c(z_c^\varepsilon), \\ \dot{z}_d^\varepsilon &= h(Z_\varepsilon) + \varepsilon\varphi_d(z_d^\varepsilon). \end{aligned} \tag{6.1}$$

By lemma 2.8, cells c and d are input equivalent, so that $\varphi_c = \varphi_d$. Now, since Z_0 is nondegenerately rigid on J by hypothesis, the colouring associated with Z_0 is rigid, so that $z_c^\varepsilon(t) = z_d^\varepsilon(t)$ for $t \in J$. Thus, subtracting the second equation in (6.1) from the first, we find that

$$0 = f(Z_\varepsilon) \tag{6.2}$$

for all small ε. Since Φ is arbitrary, it now follows from corollary 5.4 that every $\mathcal{O}$-coloured sum associated with f is zero on J. ☐

Lemma 6.3. *Assume the hyperbolic solution Z_0 is nondegenerately rigid on J, and let X and Y be the corresponding $\mathcal{C}$- and $\mathcal{O}$-cells of $\mathcal{G}$, respectively. Then (2.1) is a skew product*

$$\begin{aligned} \dot{X} &= H(X), \\ \dot{Y} &= G(X, Y). \end{aligned} \tag{6.3}$$

Moreover, if we write $Z_0 = (X_0, Y_0)$, then the polydiagonal on the X cells $\Delta(X_0)$ is balanced with respect to the network of X cells.

Proof. Theorem 2.1 implies that $\mathcal{C}$-cells only receive signals from $\mathcal{C}$-cells. Therefore, (2.1) can be put in skew-product form (6.3). Since $X_0(t)$ is constant on the open interval J, X_0 is an equilibrium of $\dot{X} = H(X)$. Also, because $Z_0(t)$ is nondegenerately rigid on J, $\Delta(X_0)$ is rigid. By theorem 7.6 in [2], $\Delta(X_0)$ is flow-invariant and therefore balanced. $\square$

We motivate the next lemma, lemma 6.4, by recalling the outline of the proof of theorem 6.1. This theorem is proved locally. Assume Z_0 is nondegenerately rigid on J. We prove that $\Delta(Z_0, J)$ is flow-invariant by contradiction. Suppose there exists a point $Q_0 \in \Delta(Z_0, J)$ and an admissible map B such that $B(Q_0) \notin \Delta(Z_0, J)$. Then we show that under one of the perturbations $\varepsilon\Psi B\Phi$ or $\varepsilon(\Psi B\Phi + \Phi)$, where Ψ and Φ are strongly admissible maps, $\Delta(Z_0, J)$ is not rigid, which contradicts the fact that Z_0 is nondegenerately rigid on J. The choice of the perturbation that forces $\Delta(Z_0, J)$ out of rigidity is based on the derivatives of $B\Phi$ and $B\Phi + \Phi$ at $s \in J$, where $\Phi(Z_0(s)) = Q_0$. Lemma 6.4 discusses these derivatives.

Lemma 6.4. *Assume the hyperbolic solution Z_0 is nondegenerately rigid on J. Let c be an $\mathcal{O}$-cell, let B be an admissible map, let $s \in J$, and let $Q_0 \in \Delta(Z_0, J)$. There exists a strongly admissible map Φ with $Q_0 = \Phi(Z_0(s))$ such that either*

$$\frac{\mathrm{d}}{\mathrm{d}t}(B\Phi Z_0(t))_c\bigg|_{t=s} \neq 0 \tag{6.4}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(B\Phi Z_0(t) + \Phi Z_0(t))_c\bigg|_{t=s} \neq 0. \tag{6.5}$$

Proof. Since Z_0 is nondegenerately rigid on J, $\Delta(Z_0(s)) = \Delta(Z_0, J)$. $Z_0(s)$ is called a generic point of $\Delta(Z_0, J)$ in [2] and by lemma 7.5 in [2], there exists a strongly admissible map Φ such that $Q_0 = \Phi(Z(s))$. Suppose for all strongly admissible maps Φ with $Q_0 = \Phi(Z_0(s))$, (6.4) fails; that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}(B\Phi Z_0(t))_c\bigg|_{t=s} = 0. \tag{6.6}$$

Since $\Phi = (\varphi_1, \ldots, \varphi_n)$ and φ_c can be any map on $\boldsymbol{R}^{k_c}$, we can choose φ_c such that $D\varphi_c(z_c^0(s)) \neq 0$. Also since cell c is an $\mathcal{O}$-cell, $\dot{z}_c^0 \neq 0$ on J. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(B\Phi Z_0(t) + \Phi Z_0(t))_c\bigg|_{t=s} = \frac{\mathrm{d}}{\mathrm{d}t}(\Phi Z_0(t))_c\bigg|_{t=s} = D\varphi_c(z_c^0(s))\dot{z}_c^0(s) \neq 0. \qquad \square$$

Proof of theorem 6.1. When $Z_0(t)$ is hyperbolic standard results show that balanced implies rigid. We prove that rigid implies flow-invariance and hence balanced. We prove the theorem locally. By lemma 2.6 we may assume there is an open interval $J \subset R$ such that Z_0 is nondegenerately rigid on J.

Suppose $\Delta(Z_0, J)$ is flow-invariant. We claim that $\Delta(Z_0) = \Delta(Z_0, J)$ and hence that $\Delta(Z_0)$ is flow-invariant. By definition $\Delta(Z_0, J) \subset \Delta(Z_0, \boldsymbol{R}) = \Delta(Z_0)$. The flow-invariance of $\Delta(Z_0, J)$ implies that $Z_0(t) \in \Delta(Z_0, J)$ for all $t \in \boldsymbol{R}$, since $Z_0(t_0) \in \Delta(Z_0, J)$ for any $t_0 \in J$. Thus $\Delta(Z_0, J)$ is a polydiagonal that contains the entire trajectory $Z_0(\boldsymbol{R})$. However, by definition, $\Delta(Z_0)$ is the smallest polydiagonal that contains this trajectory. Thus, $\Delta(Z_0) \subset \Delta(Z_0, J)$, which verifies the claim.

We next show that $\Delta(Z_0, J)$ is flow-invariant. That is, for every point $Q \in \Delta(Z_0, J)$ and every admissible map $B = (b_1, \ldots, b_n)$, $B(Q) \in \Delta(Z_0, J)$. That is, $z_i(t) = z_j(t)$ on J implies $b_i(Q) = b_j(Q)$. The proof proceeds by contradiction. Suppose there exists

$Q_0 \in \Delta(Z_0, J)$ and one admissible map B such that $B(Q_0) \notin \Delta(Z_0, J)$. That is, there exist two cells c and d of the same colour, whose corresponding components $b_c(Q_0)$ and $b_d(Q_0)$ are not equal. If cells c and d were $\mathcal{C}$-cells, then by lemma 6.3 we would have $b_c(Q_0) = b_d(Q_0)$. Therefore, cells c and d must be $\mathcal{O}$-cells. Since cell c is an $\mathcal{O}$-cell and Z_0 is nondegenerately rigid on J, lemma 6.4 implies that we can choose $s \in J$ and a strongly admissible map Φ where $Q_0 = \Phi(Z_0(s))$ such that either (6.4) or (6.5) is valid for c.

Suppose Φ satisfies (6.4), and consider the system

$$\dot{Z} = F(Z) + \varepsilon \Psi B \Phi(Z) \tag{6.7}$$

obtained by perturbing (2.1), where Ψ is an arbitrary strongly admissible map. Let $Z_\varepsilon = (z_1^\varepsilon, \ldots, z_n^\varepsilon)$ be the perturbed periodic solution, and let g and h be the components of F corresponding to cells c and d, respectively. Then z_c^ε and z_d^ε satisfy

$$\begin{aligned} \dot{z}_c^\varepsilon &= g(Z_\varepsilon) + \varepsilon((\Psi B \Phi)_c(Z_\varepsilon)), \\ \dot{z}_d^\varepsilon &= h(Z_\varepsilon) + \varepsilon((\Psi B \Phi)_d(Z_\varepsilon)). \end{aligned} \tag{6.8}$$

Letting $f = g - h$ and $u = (\Psi B \Phi)_c - (\Psi B \Phi)_d$, it follows that

$$0 = \dot{z}_c^\varepsilon - \dot{z}_d^\varepsilon = f(Z_\varepsilon) + \varepsilon u(Z_\varepsilon). \tag{6.9}$$

Now, if we define

$$\alpha_i(t) = \left.\frac{\partial z_i^\varepsilon(t)}{\partial \varepsilon}\right|_{\varepsilon=0}$$

as in (4.6), then on differentiating (6.9) with respect to ε and evaluating at $\varepsilon = 0$, we obtain

$$0 = \sum_{\text{colours } L} \sum_{i \in L} f_{z_i}(Z_0(t)) \alpha_i(t) + u(Z_0(t)) \tag{6.10}$$

$$= \sum_{\text{colours } L} \left(\sum_{i \in L} f_{z_i}(Z_0(t)) \right) \alpha^L(t) + u(Z_0(t)), \tag{6.11}$$

where α^L denotes the common value of α_i for $i \in L$. By lemma 6.2, all the $\mathcal{O}$-coloured sums associated with f must be zero, so that (6.11) becomes

$$0 = \sum_{\mathcal{C}\text{-colours } L} \left(\sum_{i \in L} f_{z_i}(Z_0(t)) \right) \alpha^L(t) + u(Z_0(t)). \tag{6.12}$$

Note that for any $\mathcal{C}$-colour L, the function $\alpha^L(t)$ is constant, so that as Ψ varies, the function

$$\sum_{\mathcal{C}\text{-colours } L} \left(\sum_{i \in L} f_{z_i}(Z_0(t)) \right) \alpha^L(t)$$

is constrained to lie in a finite-dimensional function space. However, recalling that $u = (\Psi B \Phi)_c - (\Psi B \Phi)_d$, we claim that having fixed B and Φ,

$$\mathcal{B} = \{(\Psi B \Phi)_c(Z_0(t)) - (\Psi B \Phi)_d(Z_0(t)) : \ \Psi \text{ is strongly admissible}, \ t \in J\}$$

contains an infinite-dimensional function space on J. Recall $B(\Phi(Z_0(s))) = B(Q_0)$. Since we have assumed that $b_c(x_0) \neq b_d(x_0)$, we have

$$B\Phi(Z_0(s))_c \neq B\Phi(Z_0(s))_d.$$

By continuity, there exists an open neighbourhood $J_s \subset J$ of s, such that

$$B\Phi(Z_0(J_s))_c \cap B\Phi(Z_0(J_s))_d = \emptyset.$$

Note that (6.4) implies $B\Phi(Z_0(t))_c$ is time-varying on J_s and Ψ can be any strongly admissible map. It follows that

$$\mathcal{B}_c = \{(\Psi B\Phi)_c(Z_0(t)) : \ \psi_c((B\Phi(Z_0(J_s)))_d) = 0, t \in J.\}$$

contains an infinite-dimensional function space on J. Since $\mathcal{B}_c \subset \mathcal{B}$, we always can find strongly admissible maps Ψ such that (6.12) is invalid.

Suppose Φ satisfies (6.5). Then we consider the perturbed system

$$\dot{Z} = F(Z) + \varepsilon\Psi(B\Phi(Z) + \Phi(Z)). \tag{6.13}$$

The rest of the argument follows exactly as the previous case. □

Remark 6.5. Note that if an admissible map B satisfies $B(Q_0) \not\subseteq \Delta(Z_0, J)$, then in the proof we could have chosen B to be linear. This follows since we did show that $\Delta(Z_0, J)$ is flow-invariant, and it was proved in [2] that a polydiagonal is flow-invariant if and only if it is flow-invariant under all linear admissible maps.

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