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THE USE OF SYMMETRY AND BIFURCATION TECHNIQUES IN STUDYING FLAME STABILITY

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ABSTRACT. We use group theory techniques to study the mode interaction of Hopf bifurcations with O(2) symmetry. The particular mathematical problems we study are motivated by experimental results on mesh stabilized porous plug burner flames.

1. INTRODUCTION

In this paper we explore the use of singularity theory and group theory to analyse the dynamic behavior of mesh-stabilized flames on a porous plug burner. The mathematical analysis takes place on the "normal form equations": a reduced set of equations that might be obtained by performing a center-manifold reduction on a set of partial differential equations and boundary conditions that model the experiment, and following this by a change of coordinates. These normal form equations are not, superficially, much different from what one finds by using a classical mode analysis/perturbation theory approach, as has been done frequently in similar problems. A good example can be found in Erneux and Matkowsky [1984] which examines a degenerate bifurcation with O(2) symmetry, closely related to the one we will present.

There are, however, two important differences. The first

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is that we do not start from a specific model of the physics and chemistry in the problem. This means, for example, that we are not assuming a separation of fluid dynamic from chemical effects, nor are we invoking large activation energy asymptotics. Rather, we use some of the experimental observations directly to indicate an appropriate normal form. In this case, the experiments suggest the possible interaction of two (and perhaps even three) modes, as we shall spell out later. They further indicate particular symmetry properties of the modes. Recent experience in unfolding of dynamical systems suggests considering codimension two bifurcations in order to understand global behavior. We remark that there are several free parameters in the experiment to justify this. The dynamic unfolding of this mode interaction when symmetry does not play a role has been worked out and is presented in Guckenheimer and Holmes [1983], but the effects of symmetry have not previously been considered. The mathematical details of the present analysis will be presented elsewhere. (See Chossat, Golubitsky and Keyfitz [1985] and Chossat [1985].)

The other contribution of the singularity theory/group theory normal form approach is that it provides a systematic understanding of the structure of the problem, either as a bifurcation problem or as a way of establishing the global dynamics. In particular, unfolding theory describes (in a sense to be explained later) the perturbations that can occur in the neighborhood of a degenerate (or singular) point. One can use this theory to make a refined guess that quasiperiodic solutions or chaos, say, may or may not be present, and in what sort of region. The theory also establishes which higher order terms will be important in determining this structure, and which can safely be neglected. This is a nontrivial contribution in problems where symmetry is present, because the symmetry may affect the intuitive ordering of terms by polynomial degree.

Let us contrast briefly this new approach with established methods in order to understand why an applied mathematician might want to learn a theory which, at first, might seem a bit overwhelming. Let us suppose that one thing the applied mathematician would like to do in a problem like this is to uncover which aspects of the physics are responsible for each phenomenon observed in the experiments. The physics, as heretofore studied, enters the models through the structure of the underlying equations, and through the parameters (Lewis number, Reynolds number and so on) that reflect the relative importance of different physical effects. Classical PDE and singular perturbation theory have been effective tools for studying these aspects of the models. Symmetries, whether due to geometry or to frame indifference, have certainly been recognized as playing a role in the analysis, but have generally been regarded as less deep, or less important. Often they have been dismissed with an initial statement about how the variables are to be chosen. The new techniques allow a more accurate assessment of the importance of symmetry in physical problems.

II. A SUMMARY OF THE EXPERIMENTAL RESULTS

The experiments of Blackshear, Mapp and Gorman [1984] and Mapp, Blackshear and Gorman [1985] were performed in a low-pressure chamber, using a methane-air flame over a circular porous plug burner. The equivalence ratio, the total flow rate and the pressure in the chamber were all varied. In addition, a mesh screen whose position could be adjusted was suspended above the flame. While the parameter region has not yet been completely mapped out, some observations can be reported.

The "steady-state" or "trivial" solution is a steady, planar flame over the entire burner. Owing to edge effects, this "plane" flame is actually slightly nonplanar at the edges. However, it has the complete circular symmetry of the burner. We note that the symmetry group is O(2), the group of rotations and reflections in the plane. It is clear that both the burner and the trivial solution would appear exactly the same if the observer's frame were rotated by any angle, or if the experiment were observed in a vertical mirror. (The reflectional symmetry would not hold were the flame itself rotating, since then the mirror image would appear to be rotating in the opposite direction; in that case the group of symmetries would only be SO(2), the group of rotations. We shall see this group shortly.) This trivial solution is observed (and hence may be presumed to be stable) for given values of the experimental parameters when the mesh screen is very close to the burner.

When this solution becomes unstable, it is typically to a pulsating mode: a time-periodic solution. The simplest solutions of this form are planar pulsating modes: a snapshot of such a flame would show it to be planar (except for the edge effects). It oscillates with a well-defined frequency, either axially (by changing its distance from the burner periodically) or radially (by periodically changing its extent over the burner). These flames are called radially or axially pulsating respectively. Each oscillates with a single well-defined frequency: typically, the radial pulsations have lower frequency. Because of edge effects it may appear that the axial pulsations have a component of radial motion, and conversely, but there is only one period present, and the two types are distinguished by the magnitude of the frequency. It may even be the case that they are dependent effects, instances of the same mode. This is currently being tested by further experiments. The important point here, for our approach, is that the pulsating solution has the same geometrical symmetry (that is, O(2) invariance) as the steady-state solution; however, the steady-state is time-invariant, while the pulsating solution has broken this temporal symmetry.

It will prove critically important, in what follows, to recognize that the time variable, as well as the spatial ones, admits symmetry, and, in particular, that there can be solutions that exhibit <u>mixed spatial-temporal symmetries</u>. For example, a steady, nonplanar flame (which is not observed in this set of experiments) would break the spatial symmetry, but not the temporal symmetry, while a nonplanar flame that rotated, in time, around the burner but did not change its shape would exhibit a mixed symmetry. (We conjecture that a flame of this sort plays a role in the analysis of the more complicated patterns that are found.)

Nonplanar flames are observed, and details of their structure are being investigated. In recent experiments, a nonplanar flame has been observed bifurcating from the trivial solution. It is definitely oscillatory; either one frequency or two may be present. (In what follows we shall suppose one frequency is present, and call this state a standing wave.)

A third type of flame, which definitely exhibits a spatial rotation, has been given the name spiral combustion. In a definite region of parameter space, what is observed is a hot spot which rotates partway around the burner and then is extinguished. This phenomenon is then repeated periodically. Provisionally, we identify this as a rotating wave state.

The explanation of the phenomena observed that we shall pursue here is that there is an interaction, or competition, between two modes: a symmetry preserving, O(2)-invariant, time-dependent mode represented by the (radially or axially) pulsating flame, and a symmetry breaking, time-dependent, O(2)-invariant mode represented by the nonplanar and the spiral flame. The mathematical framework for bifurcation to time-periodic solutions is Hopf bifurcation: a pair of complex conjugate eigenvales crosses the imaginary axis away from the origin. Mode interaction will be studied by considering a degenerate case in which more than one eigenvalue pair crosses the imaginary axis for a particular value of the bifurcation parameter. This situation would not be observed experimentally unless very special choices of the experimental conditions were made, but the unfolding of it will include the observed effects of this mode interaction. We should comment here that the experimental evidence supports the existence of two independent frequencies for the two types of modes. It is characteristic of the presence of the O(2) symmetry that the symmetry breaking mode corresponds to a double purely imaginary eigenvalue and gives rise to two types of solutions: rotating waves and standing waves.

Thus we are motivated to consider the interaction of two Hopf bifurcations, with different symmetry properties. We now set up the background for this.

III. SYMMETRY AND BIRKHOFF NORMAL FORMS

Under the assumption that the qualitative behavior observed in the experiment is governed by the interaction of a finite number of modes, we may treat the entire problem as a finite dimensional one: that is, as a dynamical system, or ODE, rather than a PDE. If we had modelled the experiment by a system of PDE with boundary conditions, we would have had to start by linearizing about the trivial solution, looking for parameter values where a pair of eigenvalues crossed the imaginary axis, and then throwing away all the modes whose eigenvalues had negative real part, since they decay to zero. The rigorous justification of this last step is the center manifold theorem. (There are some technical restrictions which must be observed in interpreting any analysis based on normal forms, but they do not concern us at this stage.)

The center manifold reduction respects the symmetry: that is, the reduced system of ODE also commutes with the group action.

The act of putting our system of ODE in normal form by change of coordinates also introduces new symmetries in addition to the O(2) symmetries we have just discussed, and these symmetries are crucial to our analysis. As is well-known the periodicity in Hopf bifurcation induces on the original problem a circle (or S^1 or SO(2)) action given by phase shifts. This purely temporal symmetry exists independent of the spatial O(2) symmetry. Suppose now that we consider the double degeneracy of two Hopf bifurcations occurring at the same parameter value with eigenvalues $\pm\omega_0 i$ and $\pm\omega_1 i$. It can be proved that if ω_0 and ω_1 are rationally independent, or nonresonant, that is, if $m_1\omega_0 + m_2\omega_1 \neq 0$ for any integers m_1 and m_2, then there is, in fact, a torus action of $T^2 = S^1 \times S^1$ induced on the reduced equations in normal form. If the eigenvalues are resonant, then the induced action is S^1 instead. If they are weakly resonant, then the torus action will be valid only up to some order. The consequence of this group action is that a change of variables can be found in which the reduced equations are invariant under T^2. This form of the equations is called the Birkhoff normal form. It was derived by Takens [1974] when no O(2) symmetry is present. The consequences of the phase shift symmetries for unfolding the singularity can be found in Guckenheimer and Holmes [1983]. A group theoretic proof of the existence of the torus symmetry can be found in Chossat, Golubitsky and Keyfitz [1985]. We shall assume nonresonance from now on.

What then can we say about the center manifold reduction at this point? Group theoretic considerations using O(2) symmetry, as spelled out in Golubitsky and Stewart [1985], imply that generically each O(2)-invariant eigenspace is either two- or four-dimensional. Furthermore, O(2) acts trivially on the two-dimensional and nontrivially on the four-dimensional space. Letting ω_0 denote the frequency associated with the symmetry preserving solution (planar pulsations) and ω_1 denote the frequency of the symmetry breaking solution (nonplanar flames and spiral combustion), then we see that the case we want to consider yields a six-dimensional center manifold on which the linearized part of the vector field, L, is

$$\begin{bmatrix} 0 & \omega_0 & 0 & 0 \\ -\omega_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_1 I_2 \\ 0 & 0 & -\omega_1 I_2 & 0 \end{bmatrix}.$$

(In a slightly wider context, Chossat, Golubitsky and Keyfitz [1985] consider mode interaction of two nonresonant Hopf bifurcations where each mode may be either symmetry preserving or symmetry breaking. (See also Chossat [1985].) The three cases have center manifolds of dimension four, six or eight depending on whether each mode is symmetry preserving (two-dimensional) or symmetry breaking (four-dimensional). Each of these mode interactions has codimension two (in the world of O(2) symmetry) if vectorfield unfoldings are being considered; as bifurcation theory problems they are all codimension one: one splitting parameter is required to separate the two bifurcation points. Because of the symmetry, however, there are many possibilities for secondary and tertiary bifurcation. The analysis of these possibilities is the heart of our paper.

We now show how these general remarks on symmetry and reduction imply the existence of satisfactory coordinates for a simple normal form.

Write the six-dimensional center manifold as $\mathbf{R}^2 \oplus \mathbf{C}^2$, and further identify the first, symmetry preserving, summand with $\mathbf{C}$. It is shown in Chossat, Golubitsky and Keyfitz [1985] that the normal form for the vectorfield is

$$(p_0+iq_0)\begin{pmatrix} z_0 \\ 0 \\ 0 \end{pmatrix} + (p_1+iq_1)\begin{pmatrix} 0 \\ z_1 \\ z_2 \end{pmatrix} + (p_2+iq_2)\delta\begin{pmatrix} 0 \\ z_1 \\ -z_2 \end{pmatrix},$$

where p_0, p_1, p_2, q_0, q_1 and q_2 are functions of ρ, N and Δ; $\rho = \|z_0\|^2$, $N = \|z_1\|^2 + \|z_2\|^2$, $\delta = \|z_2\|^2 - \|z_1\|^2$ and $\Delta = \delta^2$. Moreover,

$$p_0(0,0,0) = p_1(0,0,0) = 0$$
$$q_0(0,0,0) = \omega_0,\ q_1(0,0,0) = \omega_1.$$

When the complex z_j's are written in polar coordinates, the normal form actually separates into amplitude and phase equations:

$$\begin{aligned} \dot{r}_0 &= -p_0 r_0 & \dot{\psi}_0 &= -q_0 \\ \dot{r}_1 &= -(p_1+\delta p_2)r_1 & \dot{\psi}_1 &= -(q_1+\delta q_2) \\ \dot{r}_2 &= -(p_1-\delta p_2)r_2 & \dot{\psi}_2 &= -(q_1-\delta q_2) \ . \end{aligned} \tag{3.1}$$

(Here and throughout we write the equations as $\dot{u} + X(u) = 0$ so that positive eigenvalues denote stability.) Since each phase angle just rotates with a nonzero speed, we have actually reduced the problem to consideration of the three amplitude variables r_0, r_1, r_2. This is analogous to the reduction performed by Takens in the four-dimensional case, without a spatial group acting. (See Guckenheimer and Holmes [1983].) In the eight dimensional case of two symmetry breaking Hopf bifurcations, the phase variables cannot be eliminated at this stage, and the analysis is more complicated.

Most of the symmetry has disappeared explicitly from the three amplitude equations on which we will perform the remaining analysis. In a sense, one could say that the three circle group actions, O(2) and the two S^1's, have removed three dimensions by eliminating the dependence of the vector field on the phase variables. (This is why one phase variable will remain in the eight-dimensional case.) Note however that the amplitude equations still commute with the action of $\mathbf{Z}_2 \times \mathbf{D}_4$, where $\mathbf{Z}_2$ acts by ± 1 on r_0 and $\mathbf{D}_4$ acts on (r_1, r_2) as the symmetries of the square.

IV. CLASSIFICATION OF SOLUTIONS BY ISOTROPY

The background for the results in this section was established by Golubitsky and Stewart [1985], where the notion of distinguishing periodic solutions by their isotropy subgroups was developed. In the current discussion, we actually work with steady-state bifurcation in the amplitude equations, and thus avoid some of the difficulties associated with the dynamics.

This also means that we cannot rigorously answer the question of how much of the dynamics we observe translates from the normal form to the original system. This question is addressed in Chossat, Golubitsky and Keyfitz [1985].

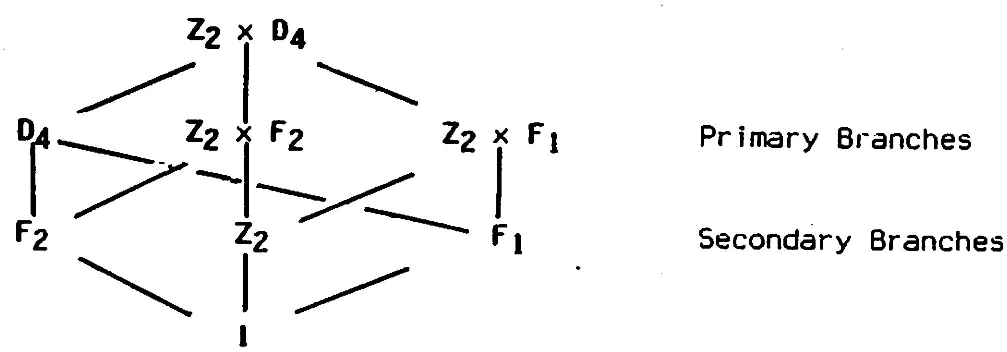
Our equations are equivariant under the action of $G = \mathbf{Z}_2 \times \mathbf{D}_4$. For a compact Lie group G acting on a vector space V, the <u>isotropy subgroup</u> of a vector v in V is the subgroup of G which leaves v fixed:

$$\Sigma_v = \{ \sigma \in G : \sigma v = v \}.$$

The <u>fixed-point subspace</u> of a subgroup Σ of G is the subspace of V consisting of vectors v left fixed by each $\sigma \in \Sigma$:

$$V^{\Sigma} = \{ v \in V : \sigma v = v \ \text{for all} \ \sigma \in \Sigma \}.$$

In the case we are interested in here, the lattice of (conjugacy classes of) isotropy subgroups is given by (with the notation explained below):



Here we use the notation $G = \mathbf{Z}_2 \times \mathbf{D}_4$ acts on (r_0, r_1, r_2) as follows. The action of $\mathbf{Z}_2$ is by ± 1 on r_0. The flip F_1 sends (r_1, r_2) to (r_2, r_1) and the flip F_2 sends r_2 to $-r_2$. F_1 and F_2 generate the two element groups $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively. The group $\mathbf{D}_4$ is generated by F_1 and F_2. In the following table we list the isotropy subgroups along with their fixed-point subspaces and the type of solution in the original problem. From now on we shall refer to the isotropy subgroups by their number in this table.

Type	Σ	V^Σ	Solution
0	$Z_2 \times D_4$	$\{(0,0,0)\}$	Trivial Solution
1	D_4	$\{(r_0,0,0)\}$	Invariant Hopf
2	$Z_2 \times F_2$	$\{(0,r_1,0)\}$	Rotating Wave
3	$Z_2 \times F_1$	$\{(0,r_1,r_1)\}$	Standing Wave
4	F_2	$\{(r_0,r_1,0)\}$	
5	Z_2	$\{(0,r_1,r_2)\}$	
6	F_1	$\{(r_0,r_1,r_1)\}$	
7	1	$\{(r_0,r_1,r_2)\}$	

Table 4.1

Recall that there is a phase associated with each nonzero value of the amplitude variable r_j. Thus solutions of types 1, 2 and 3 are actually periodic, while 4, 5 and 6 are doubly periodic. Solution 3, the standing wave, corresponds to an invariant torus with each trajectory on the torus being periodic. We find that cases 5 and 7 do not occur in the least degenerate (codimension two) case we are considering here.

In our analysis we identify a branch of solutions with each of the remaining isotropy subgroups. We also determine the stability of solutions on each branch, at least for the normal form equations, by finding the signs of the eigenvalues of the linearized solutions.

In these reduced amplitude equations, the primary and secondary bifurcations are both steady-state, although these correspond to periodic and quasiperiodic solutions in the normal form equations, as mentioned above. We shall see that there may also be a Hopf bifurcation in the reduced equations: it will be a tertiary bifurcation and will introduce a third, independent frequency into the dynamics. Characteristic of this analysis is the fact that this tertiary bifurcation can be inferred directly from an exchange of stability along a secondary branch of solutions in the normal form equations. Furthermore this branch should terminate via a homoclinic bifurcation, in the manner discussed in Guckenheimer and Holmes [1983].

V. DETAILS OF THE NORMAL FORM ANALYSIS

We have chosen a distinguished parameter in the analysis, because the number of possibilities is so large that a bifurcation diagram is a convenient way to summarize them. Also, in the experiments, we will be looking for transitions as some parameter (total flow rate or screen height, for example) is varied. In the mathematical analysis, this parameter will be denoted by λ. From the reduction to the amplitude equations, and their symmetries, it follows that one can write

$$\begin{aligned} p_0 &= \alpha_0\lambda + a_0\rho + b_0N + c_0\Delta + \beta + \ldots \\ p_1 &= \alpha_1\lambda + a_1\rho + b_1N + \ldots \\ p_2 &= p_2 + \ldots \end{aligned} \tag{5.1}$$

where $\rho = r_0^2$, $N = r_1^2 + r_2^2$, $\delta = r_2^2 - r_1^2$, $\Delta = \delta^2$ and the dots represent higher order terms. Using techniques of singularity theory, one can show that if certain nondegeneracy conditions are satisfied, then the higher order terms can be omitted without changing the nature of the bifurcation diagrams or, in some context, changing the dynamics. These conditions are that all coefficients in (5.1) be nonzero and that $a_1b_0-a_0b_1$, b_1-p_2, $a_0\alpha_1-a_1\alpha_0$, $a_1b_0-a_0(b_1-p_2)$ be nonzero. Note that $\beta = 0$ corresponds to the multiple eigenvalue: β is the splitting or unfolding parameter.

Further, we can scale some of the parameters to one. We choose to take α_0, α_1, c_0, and a_0 to be ± 1.

In fact, we may take $\alpha_0 = -1$ and $\alpha_1 = -1$, which will make the trivial solution stable subcritically since positive eigenvalues imply stability.

Thus we must analyse the bifurcating branches and stability of

$$\dot{r} + f(r) = 0$$

or

$$\dot{r}_0 = -p_0 r_0 = -\{-\lambda + a_0 r_0^2 + b_0(r_1^2+r_2^2) \pm \Delta + \beta\}r_0$$

$$\dot{r}_1 = -\{-\lambda + a_1 r_0^2 + b_1(r_1^2+r_2^2) + p_2(r_2^2-r_1^2)\}r_1 \quad (5.2)$$

$$\dot{r}_2 = -\{-\lambda + a_1 r_0^2 + b_1(r_1^2+r_2^2) - p_2(r_2^2-r_1^2)\}r_2$$

for all choices $a_0 = \pm 1$ and all values of the remaining parameters a_1, b_0, b_1, p_2 and β. The term $\pm\Delta$ is necessary to avoid degeneracy but does not affect the branches or stability to lowest order, so we will ignore it for now.

Notice that the symmetry preserving (r_0) branch bifurcates at $\lambda = \beta$, while the symmetry breaking bifurcation is double (in the sense that two branches always emerge) and remains at $\lambda = 0$. The stability of the bifurcating branches at the bifurcation point is determined by the eigenvalues of $df = \partial f/\partial r$, a 3x3 matrix. These eigenvalues are not hard to calculate, especially if the $Z_2 \times D_4$ symmetry of f is used. (Results of the calculation are given in Table 5.1.) Details are in Chossat, Golubitsky and Keyfitz [1985]. Clearly, branch 1 emerges first (that is, for a lower value of λ) precisely when β is negative. Branches 1, 2 and 3 have the properties: 1 is supercritical if $a_0 > 0$; 2 is supercritical if $b_1 - p_2 > 0$; 3 is supercritical if $b_1 > 0$. The signs of the four quantities β_0, a_0, $b_1 - p_2$ and b_1 are all independent; thus there are sixteen possibilities for the order of bifurcation and the super- or sub-criticality of the primary branches.

Which branches can be stable? First note that since one eigenvalue along the trivial solution becomes negative at the first branch and remains negative along the second, the second branch can never be stable at the bifurcation point (though it can gain stability by a secondary bifurcation). If the first branch is the symmetry-preserving branch 1, stability is exactly the same as in the nondegenerate Hopf bifurcation: this branch

Branch	Equation of Branch	Eigenvalues of df	Bifurcation Point	Conditions for (>0) Supercritical Bifurcation	Conditions for (>0) Stability at Bifurcation Point	Secondary Bifurcation occurs at	Condition for Secondary Bifurcation (>0)
1	$\lambda=a_0r_0^2+\beta$	a_0, $a_1(\lambda-\beta)-a_0\lambda$ (2)	β	a_0	$a_0, -\beta$	$\lambda_1=\frac{a_1\beta}{a_1-a_0}$	$\frac{\beta}{a_1-a_0}$ (1)
2	$\lambda=(b_1-p_2)r_1^2$	b_1-p_2, p_2, $\beta-(\frac{b_1-p_2-b_0}{b_1-p_2})\lambda$	0	b_1-p_2	β, b_1-p_2, p_2	$\lambda_2=\frac{(b_1-p_2)\beta}{b_1-p_2-b_0}$	$\frac{\beta}{b_1-p_2-b_0}$ (2)
3	$\lambda=2b_1r_1^2$	$b_1, -p_2$, $\beta-\frac{(b_1-b_0)}{b_1}\lambda$	0	b_1	$\beta, -p_2, b_1$	$\lambda_3=\frac{b_1\beta}{b_1-b_0}$	$\frac{\beta}{b_1-b_0}$ (3)
4	$\lambda=\beta+a_0r_0^2+b_0r_1^2$ $(a_1-a_0)r_0^2+(b_1-b_0-p_2)r_1^2=\beta$	p_2 $tr=a_0r_0^2+(b_1-p_2)r_1^2$ $det=a_0(b_1-p_2)-a_1b_0$	λ_1	$\frac{a_1b_0-a_0(b_1-p_2)}{a_1-a_0}$	p_2, a_0, $a_0(b_1-p_2)-a_1b_0$	Hopf	(1), (2) and $a_0(b_1-p_2)<0$
			λ_2	$\frac{a_0(b_1-p_2)-a_1b_0}{b_1-p_2-b_0}$	p_2, b_1-p_2, $a_0(b_1-p_2)-a_1b_0$		
6	$\lambda=\beta+a_0r_0^2+2b_0r_1^2$ $(a_1-a_0)r_0^2+2(b_1-b_0)r_1^2=\beta$	$-p_2$ $tr=a_0r_0^2+2b_1r_1^2$ $det=a_0b_1-a_1b_0$	λ_1	$\frac{a_1b_0-a_0b_1}{a_1-a_0}$	$-p_2, a_0$, $a_0b_1-a_1b_0$	Hopf	(1), (3) and $a_0b_1<0$
			λ_3	$\frac{a_0b_1-a_1b_0}{b_1-b_0}$	$-p_2, b_1$, $a_0b_1-a_1b_0$		

Table 5.1

is stable when it is supercritical. This happens in four of the sixteen cases: $\beta < 0$ and $a_0 > 0$.

If $\beta > 0$ so 2 and 3 bifurcate first, the following occurs: there are no stable branches unless both 2 and 3 are supercritical; then precisely one is stable. That is, we must have both $b_1 - p_2 > 0$ and $b_1 > 0$. Then 2 is stable if $p_2 > 0$ and 3 is stable if $p_2 < 0$. This is the circumstance of the simpler case of O(2)-equivariant Hopf bifurcation (without the presence of a second mode). This situation was first discovered by Ruelle [1973] and a number of authors since. (See Golubitsky and Stewart [1985] and van Gils [1984].)

In all other cases, all three primary branches are unstable.

Of the three secondary branches, 5 requires $p_2 = 0$, and so does not occur in the least degenerate case. Branch 4 is characterized by $r_2 = 0$ and satisfies $p_0 = p_1 - r_1{}^2 p_2 = 0$. Hence it is given by

$$(a_1 - a_0) r_0{}^2 + (b_1 - b_0 - p_2) r_1{}^2 = \beta \tag{5.3}$$

$$\lambda = \beta + a_0 r_0{}^2 + b_0 r_1{}^2 \tag{5.4}$$

From these equations the bifurcation point(s) may be calculated explicitly. The branch exists unless the coefficients in (5.3) satisfy $\operatorname{sgn}(a_1 - a_0) = \operatorname{sgn}(b_1 - b_0 - p_2) = -\operatorname{sgn}(\beta)$. This branch is finite (an ellipse), transiting from 1 to 2 or infinite (a hyperbola), transiting from 1 ($r_1 = 0$) or 2 ($r_0 = 0$) to infinity depending on whether or not $\operatorname{sgn}(a_1 - a_0) = \operatorname{sgn}(b_1 - b_0 - p_2)$. Furthermore, stability for any solution on the branch holds if p_2, $a_0(b_1 - p_2) - a_1 b_0$ and $a_0 r_0{}^2 + (b_1 - p_2) r_1{}^2$ are all positive. The third quantity will change sign along a finite branch if a_0 and $b_1 - p_2$ have opposite signs. In this case we know that in the amplitude equations there will be a Hopf bifurcation. (Note: the second and third quantities just listed are not eigenvalues but determinant and trace of a 2x2 matrix for the eigenvalues.) The Hopf bifurcation will be super- or sub-critical according to the sign of a certain fifth-order expression (for example, the $\pm c_0$ in the normal form), and will give a branch of stable periodic solutions in the amplitude equations (determined by appropriate exchange of stabilities), in some cases terminating in a homoclinic bifurcation (which could be calculated following the recipe in Guckenheimer and Holmes [1983]).

By a similar calculation, type 6 solutions, which satisfy (5.2) with $r_1 = r_2$ and $p_0 = p_1 = 0$, are given by

$$(a_1-a_0)r_0^2 + 2(b_1-b_0)r_1^2 = \beta \tag{5.5}$$

$$\lambda = \beta + a_0r_0^2 + 2b_0r_1^2 \tag{5.6}$$

and join branches 1 and 3 according to the signs of coefficients in (5.5). Stability here requires $p_2 < 0$, $a_0b_1-a_1b_0 > 0$ and $a_0r_0^2+2b_1r_1^2 > 0$. Again a Hopf bifurcation and a homoclinic bifurcation are possible. Note that (5.3) becomes the same as (5.5) and (5.4) becomes (5.6) if $r_1 = 0$; hence 4 and 6 bifurcate from 1 as a double branch (but many combinations of sub- and super-critical and finite and infinite branches are possible). Note that (because p_2 enters with opposite sign in each set of conditions) at most one branch can be stable anywhere. This double bifurcation is again a consequence of O(2)-equivariant Hopf bifurcation mentioned above for the primary branches. Thus only if both branches are super- or sub-critical can even one be stable at inception, and then stability happens precisely when 1 itself is supercritical, for then it gains or loses stability at the secondary bifurcation.

If 4 is stable near its bifurcation from 2, then p_2, b_1-p_2 and $a_0(b_1-p_2)-a_1b_0$ are all positive, which implies that 2 is supercritical and gains or loses stability at the secondary bifurcation. The same is true of 6 from 3.

It is clear that there are too many possibilities to document. Further study will undoubtedly reveal patterns and simplifications. It is worth noting that for any choices of the coefficients, the bifurcation diagrams could easily be drawn - it is only the multitude of them which makes the exercise unattractive. We mention some general principles that may be useful in analysing the possibilities and then draw some sample bifurcation diagrams for interesting cases.

(a) If all three primary branches are supercitical, exactly one is stable near the trivial solution.

(b) Only if a primary branch is supercritical can it ever be stable.

(c) In order for a secondary bifurcation to be stable at inception, the primary must be supercritical and stable on one side of the bifurcation point.

It is possible for finite secondary branches to be stable over their entire length when all three branches are supercritical (and not otherwise). For example, Figure 5.1 shows how a secondary (mixed mode) branch may connect the symmetry preserving (pulsating flame) branch with a symmetry breaking (rotating wave) branch. The mixed mode in this case would exhibit two distinct frequencies in its spectrum. Other choices of the coefficients allow the transition to proceed the other way (from rotating wave to pulsating flame as λ is increased), and also the transitions to the other equivariant branch, the standing waves (nonplanar mode). The mixed mode solution in this case will have frequencies characteristic of pulsation and of the nonplanar mode.

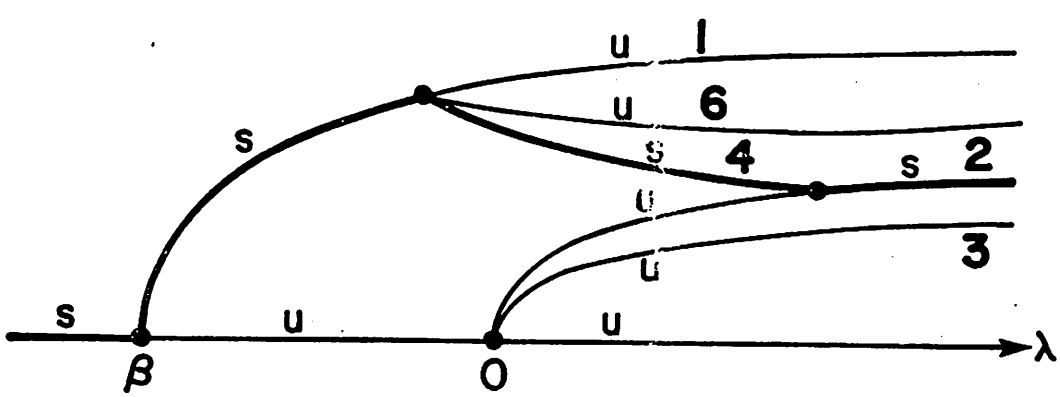


Figure 5.1

Some caution should be observed in interpreting these diagrams. Strictly speaking, we have analysed only the normal form equations, and, at that, only the amplitude equations. The mixed mode branch corresponds to an invariant torus in the Birkhoff normal form, and this torus, when mapped back to the original equations, may break up because the T^2 symmetry that was invoked to reduce to the Birkhoff normal form does not exist in the original coordinates. Even in the somewhat simpler case of S^1 symmetry due to a periodic solution there is no theorem that says that all of the dynamics in the normal form equations will be valid in the original system. The present case is even more complicated since, for example, the rational dependence of the two frequencies in the torus will be extremely sensitive to perturbation. In fact, it is likely that all of the dynamics in the unfolding will be very complicated indeed. However, an experiment will not distinguish between rational and irrational dependence either, and a reasonable but difficult problem is to give a mathematical framework for the entire reduction we have used here. We emphasize that this difficulty exists in any formal "mode analysis" calculation, and is not caused by the explicit use of group theoretic techniques. In fact, this approach may be useful for sharpening the focus on the problem.

Returning to the possibilities for qualitatively different bifurcations in the normal form equations, one can see that there are many possible ways for mode-jumping to occur when a finite branch 4 or 6 is unstable. The case where the invariant solution jumps to the rotating wave as λ increases past a critical value (λ_1) and jumps back as λ decreases beyond a different value (λ_2) is illustrated in Figure 5.2. For different signs of the parameters, other possibilities occur.

One case that will not occur in this codimension two problem is a transition between the two symmetry breaking modes 2 and 3. Such branches occur only in unfoldings of the degeneracy $p_2 = 0$. This is the case studied by Erneux and Matkowsky, and further analysed in Golubitsky and Roberts [1985].

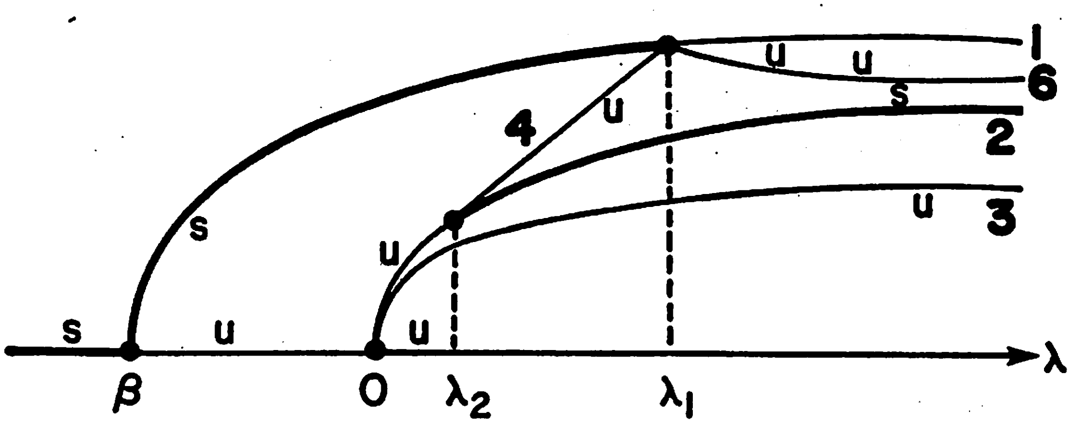


Figure 5.2

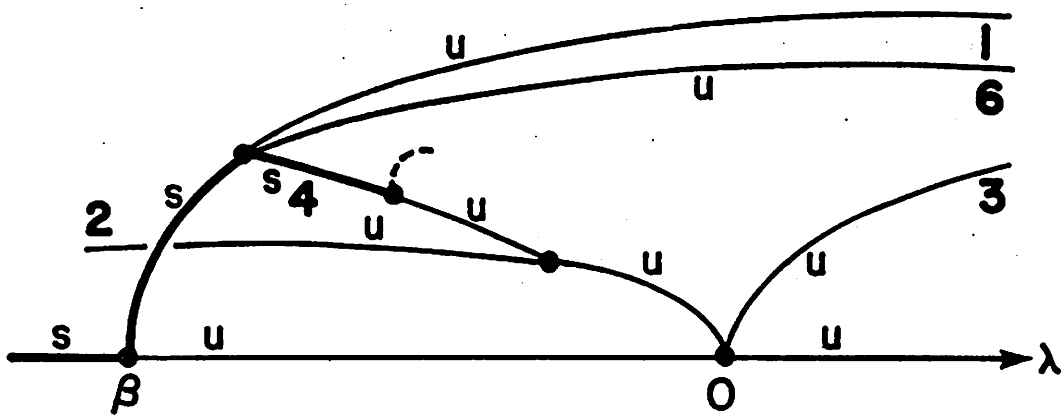


Figure 5.3

Finally, Hopf and homoclinic bifurcation to stable tertiary states will occur for certain choices of the fifth-order terms, which have been omitted here. A Hopf bifurcation to a stable tertiary mode may occur in 4 if 4 is a finite branch which is stable at one end and not at the other. That is accomplished by choosing $p_2 > 0$, $a_1b_0 < a_0(b_1-p_2) < 0$ and the usual condition that (5.3) be an ellipse. One case is illustrated in Figure 5.3, but there are many others. An example of a case involving branch 6, and where also the stable Hopf bifurcation is subcritical, is shown in Figure 5.4. The tertiary branches in these two diagrams represent periodic orbits in the amplitude equations, and hence 3-tori, or quasiperiodic solutions with three independent frequencies, in the original physical model.

Two of the three frequencies are close to ω_0 and ω_1, but the third varies from zero at the homoclinic point (the infinite period bifurcation) to a value determined by the third-order terms at the bifurcation point.

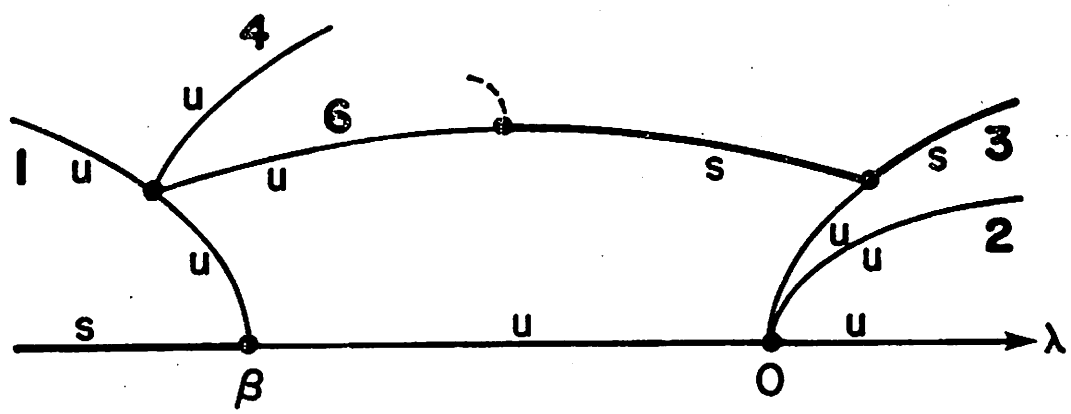


Figure 5.4

VI. CONCLUSIONS AND DIRECTIONS FOR FURTHER STUDY

We have presented a sketch of the normal form calculation for mode interaction of an O(2) symmetry preserving Hopf bifurcation with an O(2) symmetry breaking Hopf bifurcation. Our motivation for this study was the thought that such a mode interaction might be the basis for observed phenomena in a series of flame stability experiments. In particular, the existence of transitions from steady-states to radially symmetric pulsating solutions (identified with type 1) and to nonsymmetric, time dependent solutions (tentatively identified with type 2 or 3 or both) under different experimental conditions motivates studying what happens when these two transitions are brought together in the least degenerate way. That gives rise to the codimension two problem studied here.

Let us suppose for a moment that under all experimental conditions the parameters a_0, a_1, etc. in equation (5.2) and all nondegeneracy conditions retain the same signs. Then the qualitatively different bifurcation sequences which are observed will be the pairs obtained by considering both $\beta < 0$ and β

> 0 for a given set of the other parameters. A choice consistent with the hypothesis that spiral combustion is a rotating wave is, for example, $a_0 > 0$, $b_1 - p_2 > 0$, $b_1 > 0$ and $p_2 > 0$, which means that all primary branches are supercritical and that 2 rather than 3 is stable. As we saw branch 1 will be stable, at the bifurcation point, when $\beta < 0$, and branch 2 will be stable when $\beta > 0$. In addition, secondary bifurcation from 1 will occur if $\beta(a_1 - a_0) > 0$; since the sign of $a_1 - a_0$ is assumed fixed this means that if this term is negative the primary branch will become unstable and if, in addition, $b_0 a_1 - a_0(b_1 - p_2) > 0$ a stable mixed mode will occur, as in Figure 5.1. In fact, for definiteness, let us assume the other conditions of Figure 5.1: that this mixed mode branch is finite and stable over its entire length. Then under other experimental conditions where everything is the same except the sign of β we find that there is no secondary bifurcation at all from the stable primary branch 2: the situation is as pictured in Figure 6.1. This pair of predictions can be tested by further experiments; one of our immediate goals is to test the identification of the branches and the usefulness of the model by examining transitions.

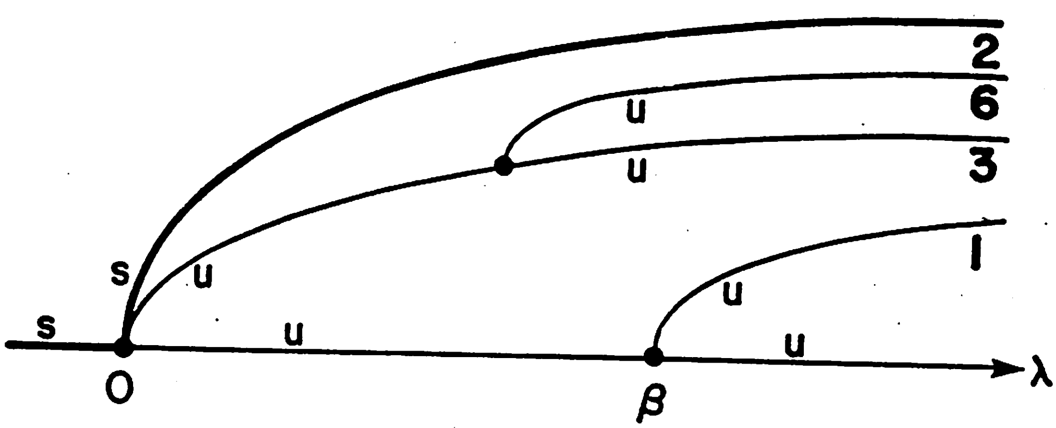


Figure 6.1

It is also possible that varying experimental conditions changes the signs of some of the parameters. Then further degeneracies, and a higher codimension problem, may be present. This will necessarily be the case if both rotating and standing waves can be observed, since these correspond to different signs of p_2. At the moment the observations are not precise enough to answer this. Normal form analysis could be useful in such a case, since any model will be very complicated.

If further experiments confirm that mode interaction is present, then it will be time to look at PDE models for the porous plug burner, for example the model considered by Buckmaster [1983]. A necessary feature of a successful model will be that the linearized problem will have purely imaginary eigenvalues as described in the normal form model.

Finally, more mathematical analysis is necessary to put the normal form analysis in a rigorous setting. It is not yet understood how the invariant tori (quasiperiodic solutions corresponding to branches 4 or 6) behave in even a finite dimensional system of ODE which is not in normal form, let alone in a PDE model. But even the small amount of work done so far suggests that this approach may give interesting results and that the use of symmetry techniques may show a way in which some quite diverse physical problems have many features in common.

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