SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 10, No. 4, pp. 1404–1442



# Synchrony-Breaking Bifurcation at a Simple Real Eigenvalue for Regular Networks 1: 1-Dimensional Cells*

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**Abstract.** We study synchrony-breaking local steady-state bifurcation in networks of dynamical systems when the critical eigenvalue is real and simple, using singularity theory to transform the bifurcation into normal form. In a general dynamical system, a generic steady-state local bifurcation from a trivial state is transcritical. In the presence of symmetry, a pitchfork is also possible generically. Network structure introduces constraints that may change the generic behavior. We consider regular networks, in which all cells have the same type and all arrows have the same type, and every cell receives inputs from the same number of arrows. A characterization of all smooth admissible maps permits a singularity-theoretic analysis based on Liapunov–Schmidt reduction. Assuming that the cells have 1-dimensional internal dynamics, we give conditions on the critical eigenvectors of the linearization and its transpose that determine when a generic bifurcation is transcritical, pitchfork, or more degenerate. We prove that for all regular $n$-cell networks, such bifurcations are generically $n$-determined. In the path-connected case, this is improved to $(n-1)$-determined. In bidirectional networks, generic bifurcation is transcritical or pitchfork, but the role of symmetry is minor. In the general case, degenerate cases can occur: the network must have at least 4 cells (5 in the path-connected case). We give examples of networks for which generic bifurcations are degenerate, including a 6-cell network with a normal form that is determined only at degree 6 and a path-connected 5-cell network with a normal form that is determined only at degree 4.

**Key words.** network, bifurcation, singularity, normal form, synchrony-breaking

**AMS subject classifications.** 37C20, 37C80, 34C23, 37G05, 37G10, 37G40

**DOI.** 10.1137/110825418

## 1. Introduction.

In a 1-parameter family of dynamical systems, generic steady-state bifurcation from a trivial branch is transcritical and occurs at a simple real eigenvalue. However, what is generic can change if the system has special features. For instance, in the presence of $\mathbb{Z}_2$ symmetry acting nontrivially on the critical eigenspace, generic symmetry-breaking bifurcation from a trivial branch is a pitchfork. We investigate whether these results remain valid for a network of coupled dynamical systems [7, 9, 21]. The dynamics and bifurcations of networks are known to be constrained by the network architecture [4, 15]. Even the linear structure may be degenerate: multiple eigenvalues and nontrivial Jordan blocks can occur generically for particular network architectures [15], and any Jordan normal form for a fixed imaginary or zero eigenvalue can be generic in a suitable network [3].

*Received by the editors February 22, 2011; accepted for publication (in revised form) by T. Kaper September 1, 2011; published electronically December 6, 2011.
http://www.siam.org/journals/siads/10-4/82541.html

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This author was supported by the University of Houston, the Turing Institute, the University of Manchester, and the Mathematical Biosciences Institute, who partially funded his visits during which part of this research was carried out.

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Here we make a systematic study of local steady-state bifurcation in network dynamics at a simple real eigenvalue. We impose a strong constraint on the network topology by assuming it to be *regular*: that is, it has one type of cell and one type of coupling, and all cells receive the same number of inputs, called the *valency*. We also assume that the cells of the network have 1-dimensional internal dynamics, a restriction that simplifies the calculations considerably. In a second paper [20] we will discuss higher-dimensional cells, showing that the main results of this paper extend to higher-dimensional cells for some networks but not for others. This extension to higher dimensions relies on the 1-dimensional case, so this paper is a necessary step towards the general case.

We obtain necessary and sufficient conditions for generic synchrony-breaking steady-state bifurcation at a simple real eigenvalue to be transcritical or pitchfork, expressed as algebraic properties of the critical eigenvectors of the adjacency matrix of the network and its transpose. We also exhibit network architectures for which these conditions are not satisfied for any ODE compatible with the network structure, so generic bifurcations for these networks are always more degenerate. These exceptional networks appear to be rare, but their existence is surprising and shows that the network architecture can cause generic bifurcations to be more degenerate than they are for general or equivariant dynamical systems.

Throughout the paper we adopt the formalism of [7, 9], which permits multiple arrows and self-loops. In particular, an ODE is *admissible* if it respects the network architecture (Definition 2.1). Define a network to be *path-connected* if any pair of distinct cells can be connected, in either direction, by a sequence of directed edges (other terms for this property are *strongly connected* and *transitive*). A network that is not path-connected is *feed-forward*.

The examples of degenerate bifurcation that we construct arise in networks with few cells but having arrows of high multiplicity. We briefly explain why such networks, which may appear artificial, are significant. First, our main aim is to discover the possible phenomena in this area and to prevent fruitless attempts to prove that transcritical and pitchfork bifurcations exhaust the generic possibilities. Second, multiple arrows can be removed by appealing to the lifting theorem [18, 19], which proves that any network with self-loops and multiple arrows is a quotient (see below or [9]) of a network with no self-loops and no multiple arrows. This construction, applied to any of our examples, leads to a conventional single-arrow network with a number of cells that is roughly comparable to the original arrow multiplicities; it remains regular. The precise relationship is derived in [19]. The corresponding bifurcation for the lifted network remains degenerate. Even if that eigenvalue is not simple in the lifted network, the lift will have a degenerate branch of equilibria because the eigenvalue in the original network is simple. For applications, the most useful results are Theorem 5.2 and Corollary 5.5, which give necessary and sufficient conditions for transcritical and pitchfork bifurcations to occur.

This paper was motivated by the systematic study of regular 3-cell networks in Leite and Golubitsky [15]. Their Theorem 2.4 classifies all connected regular 3-cell networks of valency 2, finding 38 distinct topological types, in agreement with the enumeration of Aldosray and Stewart [1], 34 of which have distinct spaces of admissible vector fields. Assuming 1-dimensional cell phase spaces, which causes no loss of generality in the linear theory, these authors analyze all possible generic synchrony-breaking steady-state and Hopf bifurcations in these systems. Their Table 3 lists the possibilities for simple critical eigenvalues, case S1 of the analysis. For steady-state bifurcation, 21 networks have adjacency matrices with real simple eigenvalues, and all of the associated synchrony-breaking bifurcations are either transcritical or pitchfork. Pitchforks occur in seven cases, and can be explained by $\mathbb{Z}_2$ symmetry, either on the full network, a quotient network, or a subnetwork that is "decoupled" from the other cells. In particular, regular 3-cell networks can be classified into a small number of types, each with "the same" generic bifurcation behavior. These results raise some general questions:

(1) Are all generic simple-eigenvalue synchrony-breaking local steady-state bifurcations in networks transcritical or pitchfork?
(2) Is $\mathbb{Z}_2$ symmetry on some associated network, such as a quotient, necessary for a pitchfork bifurcation to occur generically?
(3) Is the range of possible generic bifurcations more limited than that of the possible topological types of networks?
(4) Are there systematic structural features that explain how the bifurcations relate to the topology?

In equivariant dynamics, the answer to the analogue of question (1) is "yes," because the equivariant mapping $||x||^2x$ has degree 3 and is generically nonzero. For a network, however, this map need not be admissible, and we show that the answer to (1) is "no": generic simple-eigenvalue steady-state bifurcations can be more degenerate than transcritical or pitchfork; see Examples 7.1, 8.2, and 8.4 and sections 9 and 10. The answer to (2) is also negative: in networks, symmetry is *not* necessary for generic pitchfork bifurcation; see Example 5.7. These degenerate cases seem to be rare, so (3) has a positive but currently not systematic answer. As regards (4), we find a number of connections between network topology and bifurcation type. However, the main features that determine the type of bifurcation are combinatorial and number-theoretic properties of the critical eigenvectors of the adjacency matrix of the network and its transpose, rather than overt geometric features of network topology.

**1.1. Outline of the paper.** In order to summarize the main results of the paper, we set up some terminology and notation.

Let $\mathcal{G}$ be a regular network with $n$ cells, and consider a 1-parameter family of admissible ODEs for $\mathcal{G}$:

$$\dot{X} = \Phi(X, \lambda), \qquad X \in \mathbb{R}^n, \lambda \in \mathbb{R}. \tag{1.1}$$

Steady states of (1.1) are solutions $x$ of the *bifurcation problem*

$$\Phi(X, \lambda) = 0. \tag{1.2}$$

Steady-state bifurcation theory describes how the solutions $x$ of (1.2) vary with $\lambda$. A bifurcation occurs when the topology of the solution set (and in particular the number of solutions) changes near some point $\lambda_0$, known as a *bifurcation point*.

Let $\Delta = \{(y, y, \ldots, y) \in \mathbb{R}^n\}$ be the diagonal subspace of *fully synchronous* states. The subspace $\Delta$ is flow-invariant for regular networks, so the system (1.1) can have a fully synchronous equilibrium $(u, u, \ldots, u)$ for suitable $u \in \mathbb{R}$. Without loss of generality we may translate $u$ to 0 in each cell, because this is a strongly admissible diffeomorphism [9].

Suppose that (1.1) undergoes a steady-state bifurcation at $\lambda = 0$ from this synchronous equilibrium. Then the Jacobian $L = \mathrm{D}\Phi|_{(0,0)}$ must be singular by the implicit function theorem, so it has a zero eigenvalue. Further, suppose that the eigenvalue is simple, which implies that

$$\dim \ker L = 1.$$

The eigenvector $v$ of $L$ is either in $\Delta$ or transverse to $\Delta$. The first case leads to a *synchrony-preserving* bifurcation; generically such bifurcations are saddle-node bifurcations because $\Phi|\Delta$ is arbitrary. We call the second case a *synchrony-breaking* bifurcation. The simple eigenvalue assumption implies that $L|\Delta$ is nonsingular; so there is a unique curve of equilibria $(u(\lambda), u(\lambda), \ldots, u(\lambda))$ in $\Delta$. Again, without loss of generality, we may assume that this curve of equilibria is at the origin, so

$$\Phi(0, \lambda) \equiv 0.$$

We assume this property of $\Phi$ throughout and say that $\Phi$ has a *trivial branch*.

To compute the low-degree terms in the reduced mapping, it is convenient to introduce some nonstandard notation. Let $\langle , \rangle$ denote the usual inner product on $\mathbb{R}^n$. If $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, define the *componentwise product*

$$v \star w = (v_1 w_1, \ldots, v_n w_n)$$

and write

$$v^{[p]} = (v_1^p, \ldots, v_n^p) = \underbrace{v \star \cdots \star v}_{p}.$$

The operation $\star$ is bilinear, and gives $\mathbb{R}^n$ the structure of a commutative associative algebra, with an identity element $e = (1, 1, \ldots, 1)$. This algebra provides a natural description of admissible polynomial maps in Theorem 4.3.

We now summarize the main results of this paper, assuming the following standard hypotheses throughout.

*Standard hypotheses.* Let $\mathcal{G}$ be a regular network with 1-dimensional cell phase spaces, having adjacency matrix $A$. Suppose that $\mu$ is a simple real eigenvalue of $A$ associated with a steady-state synchrony-breaking bifurcation from the fully synchronous state. The theorems refer to this bifurcation. ■

Let $v, u$ be eigenvectors for eigenvalue $\mu$ of $A$ and $A^{\mathrm{T}}$, respectively. Let $u^{\perp}$ be the orthogonal complement of $u$, so that $\mathbb{R}^n = \mathbb{R}\{v\} \oplus u^{\perp}$; see (2.2), (2.3) below. Then

$$(A - \mu I)|_{u^{\perp}}$$

is nonsingular on $u^{\perp}$ because $\mu$ is simple. We also denote this restriction by $L$, so $L^{-1}$ is a well-defined map on $u^{\perp}$. We will apply $L^{-1}$ only to vectors known to lie in $u^{\perp}$.

The outline of the paper is as follows. We begin by setting up necessary background from linear algebra (especially the Perron–Frobenius theorem) and networks in section 2. Liapunov–Schmidt reduction and singularity theory are briefly recalled in section 3. We characterize smooth and polynomial admissible maps in section 4.

This characterization is specialized to quadratic and cubic maps in section 5, and we compute the corresponding terms in the reduced map $g$. In particular, we prove (Theorem 5.2) that generically (in the admissible map $\Phi$) the bifurcation is transcritical if and only if

$$\langle u, v^{[2]} \rangle \neq 0.$$

If this term vanishes, it is generically a pitchfork (Corollary 5.5) if and only if at least one of the conditions

$$\langle u, v^{[3]}\rangle \neq 0, \qquad \langle u, v\star Av^{[2]}\rangle \neq 0, \qquad \langle u, v\star L^{-1}v^{[2]}\rangle \neq 0$$

is valid. (In general these conditions are independent.) We describe connections between pitchforks and symmetry (which is *not* a necessary condition for a pitchfork). In Theorem 5.8 we prove that in bidirectional networks, generic bifurcations are either transcritical or pitchfork.

Section 6 proves three determinacy theorems, which state that the normal form of the bifurcation is generically determined by its Taylor series truncated at finite degree. If the degree required is $k$, we say that the bifurcation is $k$-determined. Theorem 6.1 states that in an $n$-cell network the bifurcation is generically $(n+1)$-determined. Theorem 6.13 uses this result to improve the conclusion to "generically $n$-determined." Theorem 6.14 builds on the previous theorems to show that in a path-connected $n$-cell network with $n \geq 4$ cells, the bifurcation is generically $(n-1)$-determined. (These theorems place constraints on possible examples of degeneracy and were used to find the examples that follow.)

Section 7 constructs a regular 4-cell network of valency 736 in which generic bifurcation at a simple real eigenvalue is 3-degenerate but 4-determined. This construction uses a method we call "bordering" and leads to a feed-forward network, that is, one that is not path-connected. Section 8 considers higher degeneracies, leading to a regular 5-cell network of valency 390 in which generic bifurcation at a simple real eigenvalue is 4-degenerate but 5-determined. We exhibit a simpler example with valency 84, which has a $\mathbb{Z}_2$ symmetry. Section 9 constructs a regular 6-cell network of valency 885920 in which generic bifurcation at a simple real eigenvalue is 5-degenerate but 6-determined. Finally, section 10 constructs a regular path-connected 5-cell network of valency 6273504 in which generic bifurcation at a simple real eigenvalue is 3-degenerate but 4-determined.

Appendix A derives conditions for 4-degeneracy, and Appendix B contains a computation of terms of degree 4 in the reduced map.

## 2. Background from linear algebra and networks.

Throughout the paper we make repeated use of some simple results from linear algebra, which we recall here along with the notation we use. We also introduce some basic network notation.

If $\mu$ is an eigenvalue of an $n\times n$ matrix $A$, then the corresponding *real generalized eigenspace* is

$$E_\mu = \{x \in \mathbb{R}^n : (A-\mu I)^n x = 0\}$$

if $\mu \in \mathbb{R}$, and

$$E_\mu = \{x \in \mathbb{R}^n : [(A-\mu I)(A-\overline{\mu} I)]^n x = 0\}$$

if $\mu \notin \mathbb{R}$. The space $\mathbb{R}^n$ is the direct sum of all generalized eigenspaces of $A$.

We write column vectors in the form $[v_1, \ldots, v_n]^{\mathrm{T}}$, where T denotes transpose. The vector $e = [1, \ldots, 1]^{\mathrm{T}}$ plays a crucial role; in the context of networks we call it the *synchronous eigenvector*.

Suppose that a matrix $A$ has a real simple eigenvalue $\mu$ with eigenvector $v$. Let $u$ be the corresponding eigenvector of $A^{\mathrm{T}}$. Then it is well known that the following hold:

(2.1) For all $x, y \in \mathbb{R}^n$, $\langle x, Ay\rangle = \langle A^{\mathrm{T}}x, y\rangle$.

(2.2) The vector $u$ is orthogonal to all generalized eigenspaces $E_\nu$ of $A$ with $\nu \neq \mu$.

(2.3) The vectors $v$ and $u$ are not orthogonal. That is, $\langle u, v\rangle \neq 0$.

(2.4) If $A$ has constant row-sums $k$ and $\mu \neq k$, then $u_1 + \cdots + u_n = 0$.

Another useful result is the following:

(2.5) Suppose that $v = [v_1, \ldots, v_n]^{\mathrm{T}}$, where all $v_j$ are distinct. Then the vectors $e, v, v^{[2]}, \ldots, v^{[n-1]}$ are linearly independent.

The entries of any adjacency matrix are nonnegative integers, and this restriction on their sign has crucial implications for the theory. A fundamental result here is the Perron–Frobenius theorem. Recall that a square matrix is *irreducible* if it cannot be put into nontrivial block-triangular form by a permutation of the coordinates; see [14, section 10.7]. The adjacency matrix of a network is irreducible if and only if the network is path-connected. The Perron–Frobenius theorem states that if $A$ is an irreducible real matrix, all of whose entries are nonnegative, then there exists a real eigenvalue $\sigma > 0$ of $A$ such that the following hold:

(2.6) Every other eigenvalue $\lambda$ satisfies $|\lambda| < \sigma$. In particular, $\mathrm{Re}(\lambda) < \sigma$.

(2.7) The eigenvalue $\sigma$ is simple.

(2.8) There is an eigenvector $X$ with eigenvalue $\sigma$ with all components positive ($X_j > 0$).

(2.9) If an eigenvector of $A$ has all components positive, then it is a scalar multiple of the eigenvector $X$ in (2.8).

A proof of (2.6), (2.7), (2.8) is given in Lancaster and Tismenetsky [14, section 15.4]. Statement (2.9) is Exercise 15.3.11 of that reference. It implies that in our case, the maximal eigenvalue $\sigma$ is the valency $k$ of the network, because the corresponding eigenvector is $e = [1, 1, \ldots, 1]^{\mathrm{T}}$ with all entries positive.

We also recall a few basic concepts about coupled cell networks [7, 9, 21], that will be required below, specialized to the regular case.

A *coupled cell network* $\mathcal{G}$ is a directed graph. The nodes ("cells") represent dynamical systems and the edges ("arrows") represent couplings. That is, an arrow $e$ from cell $i$ to cell $c$ indicates that the dynamics of $i$ influences the dynamics of $c$. We denote the set of cells by $\mathcal{C}$ and the set of arrows by $\mathcal{E}$. In a *regular* network all cells have the same type (that is, the same phase space and the same internal dynamics), and all arrows have the same type (all couplings have the same form apart from the choice of variables).

If $a \in \mathcal{E}$ connects cell $i$ to cell $c$, then we write

$$c = \mathcal{H}(a), \qquad i = \mathcal{T}(a)$$

and call these cells the *head* and *tail* of $a$, respectively. The *input set* $I(c)$ of $c \in \mathcal{C}$ is the set of all arrows whose head is $c$, so that

$$I(c) = \{a \in \mathcal{E} : \mathcal{H}(a) = c\}.$$

For a regular network, all input sets have the same cardinality, so $I(c) = k$ for all $c \in \mathcal{C}$. We call $k$ the *valency* of the network.

Associated with each regular coupled cell network is a class of admissible vector fields, determining admissible ODEs. This concept is a generalization of group equivariance [6, 8] to the network case. For regular networks, the admissible vector fields are constructed as follows. Choose a *cell phase space*, which for the present purposes (local bifurcation theory) we assume to be a real vector space $\mathbb{R}^r$ of dimension $r \geq 1$. Write

$$\mathbb{R}^{kr} = \mathbb{R}^r \oplus \cdots \oplus \mathbb{R}^r$$

with $k$ summands. Then the phase space of the coupled system is $P = \mathbb{R}^{nr}$. The symmetric group $\mathbb{S}_k$ acts on $\mathbb{R}^{kr}$ by permuting the $k$ entries:

$$\sigma(x_1, \ldots, x_k) = (x_{\sigma.1}, \ldots, x_{\sigma.k}),$$

where each $x_j \in \mathbb{R}^r$ and the action $\sigma.j = \sigma^{-1}(j)$. We call $\mathbb{S}_k$ the *vertex group* of the network.

Definition 2.1. *Let $f$ be any smooth function*

$$f : \mathbb{R}^r \oplus \mathbb{R}^{kr} \to \mathbb{R}^r$$

*that is invariant under the permutation action of $\mathbb{S}_k$ on $\mathbb{R}^{kr}$. The function $f$ determines a vector field $F = (f_1, \ldots, f_n)$ on $P$ as follows:*

$$f_c(x) = f(x_c, x_{T(c)}), \tag{2.10}$$

*where $x_{T(c)}$ denotes the $k$-tuple of tail cells of the input arrows $a \in I(c)$ of cell $c$:*

$$x_{T(c)} = x_{\mathcal{T}(I(c))}.$$

*Any such vector field is said to be* admissible. *The same term is applied to the corresponding ODE*

$$\frac{\mathrm{d}}{\mathrm{d}t} x_c = f_c(x). \qquad \blacksquare$$

Invariance under the action of $\mathbb{S}_k$ is a consequence of the general coupled cell formalism. It permits the classification of admissible smooth and polynomial maps; see section 4.

## 3. Normal forms for local bifurcation.

Let $A = (a_{ij})$ be the *adjacency matrix* of the network, so that $a_{ij}$ is the number of arrows leading from cell $j$ to cell $i$. Since $\mathcal{G}$ is regular and cells are 1-dimensional, the linear admissible vector fields have the form $LX$, where

$$L = \alpha I + \beta A \tag{3.1}$$

for $\alpha, \beta \in \mathbb{R}$; see Leite and Golubitsky [15, section 3.1]. The entries of $A$ are nonnegative integers, and regularity implies that all rows of $A$ have the same sum, equal to the valency.

Note that $L$ has a simple zero eigenvalue with eigenvector $v$ if and only if $A$ has a simple real eigenvalue $\mu = -\frac{\alpha}{\beta}$ with eigenvector $v$, and $\beta$ is nonzero.

Let D denote the derivative with respect to the state variables $x$. Then

$$L(\lambda) = \mathrm{D}\Phi|_{(0,\lambda)} = \alpha(\lambda)I + \beta(\lambda)A$$

is the linearization of $\Phi$ along the trivial solution. We can normalize $\Phi(x, \lambda)$ by dividing by $\beta(\lambda)$ (which up to sign is the same as rescaling time) and thus assume that $\beta(\lambda) = 1$. Moreover, generically we can assume that the *eigenvalue crossing condition* is valid; that is, the critical eigenvalue crosses 0 with nonzero speed, so $\alpha'(0) \neq 0$. Changing coordinates in $\lambda$ we may set $\alpha(\lambda) = \lambda - \mu$. Now

$$L(\lambda) = (\lambda - \mu)I + A, \tag{3.2}$$

where $\mu$ is a simple real eigenvalue of $A$. Throughout the paper we assume that $L$ has been normalized in this way.

The technique of Liapunov–Schmidt reduction (see, for example, [5]) transforms the bifurcation problem (1.1) into a *reduced equation*

$$g(x, \lambda) = 0, \qquad x \in \mathbb{R}, \lambda \in \mathbb{R},$$

where $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth. Here $\mathbb{R}$ is identified with $\ker L$. Singularity-theoretic methods [5] classify the resulting types of bifurcation according to certain features of the Taylor series of $g$. We introduce some useful terminology. A bifurcation problem is *finitely determined* if it is equivalent to its Taylor series truncated at some finite order. It is $r$-*determined* if it is equivalent to its Taylor series truncated at order $r$, and *strictly* $r$-*determined* if it is $r$-determined but not $(r-1)$-determined. If for some $r$ the terms in the Taylor series vanish for all degrees $l$ with $2 \leq l \leq r$, we say that the bifurcation is $r$-*degenerate*.

Theorem 3.1. *Assuming* (3.2), *suppose that* $r \geq 2$ *and*

$$g = g_x = \cdots = g_{x^{r-1}} = 0 \quad \text{and} \quad g_{x^r} \neq 0$$

*at* $x = \lambda = 0$. *Then the bifurcation problem* $g$ *is strictly* $r$-*determined and equivalent to the normal form*

$$g_r^{\pm}(x, \lambda) \equiv \lambda x \pm x^r. \qquad \blacksquare$$

In applications, one important feature of the normal form is the growth rate of the branch(es). For the normal form $g_r^{\pm}$, the synchrony-breaking branch (branches when $r$ is odd) grows at a rate $|x| \sim |\lambda|^{1/(r-1)}$ near 0. Because the equivalence relation employed in singularity theory is a diffeomorphism, the same asymptotic growth rate occurs for the corresponding branch of the original bifurcation problem (1.1); see Corollary 6.12.

Using singularity theory we can put almost all bifurcation problems (1.2) into normal form by applying suitable changes of coordinates. The main step is to apply Liapunov–Schmidt reduction [5, 8]. This leads to a *reduced bifurcation equation* $g(x, \lambda) = 0$, and $x \in \mathbb{R}$ when the critical eigenvalue is simple. We use the notation of Golubitsky and Schaeffer [5, pages 25–35], specialized to one bifurcation parameter $\lambda$.

Translating coordinates if necessary, we may assume that the bifurcation occurs at the origin, so $X, \lambda$ are small and the linearization $L = \mathrm{D}\Phi|_{0,0}$ is singular. Assume that 0 is a simple eigenvalue of $L$, so that $\ker L$ is 1-dimensional.

The reduction method requires a choice of a complement $N$ to $\ker L$ in $\mathbb{R}^n$. Since $(\ker L) = E_\mu$, we know that

$$M = \operatorname{range} L = \bigoplus_{\nu \neq \mu} E_\nu \tag{3.3}$$

is a complement, so we take $N = M = u^\perp$. This space is $A$-invariant. By (2.2),

$$M = u^\perp = \{x : \langle u, x\rangle = 0\}.$$

Let $E$ be projection onto $u^\perp$ with kernel $\mathbb{R}\{v\}$. This is the projection employed in the Liapunov–Schmidt process, and there exist formulas for the Taylor coefficients of the reduced map $g$ in terms of $E$.

Throughout the paper we consider the $m$th derivative of $\Phi$, relative to the state variables $x$ and evaluated at the origin, as a symmetric $m$-linear form. We denote this form by $\mathrm{D}^m\Phi$.

We now state explicit formulas for the first few Taylor coefficients of the reduced map $g$ at $(0, 0)$ in terms of $\Phi$. Equations (3.4), (3.5), (3.6) are proved in [5], and (3.7) can be derived using similar methods:

$$g_x = 0, \tag{3.4}$$

$$g_{xx} = \langle u, \mathrm{D}^2\Phi(v, v)\rangle, \tag{3.5}$$

$$g_{xxx} = \langle u, \mathrm{D}^3\Phi(v, v, v)\rangle - 3\langle u, \mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v, v))\rangle, \tag{3.6}$$

$$\begin{aligned} g_{xxxx} = \langle u, \big[&\mathrm{D}^4\Phi(v, v, v, v) - 6\mathrm{D}^3\Phi(v, v, L^{-1}E\mathrm{D}^2\Phi(v, v)) \\ &+ 3\mathrm{D}^2\Phi(L^{-1}E\mathrm{D}^2\Phi(v, v), L^{-1}E\mathrm{D}^2\Phi(v, v)) \\ &- 4\mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^3\Phi(v, v, v)) \\ &+ 12\mathrm{D}^2\Phi(v, L^{-1}ED^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v, v)))\big]\rangle. \end{aligned} \tag{3.7}$$

**4. Characterization of admissible maps.** In order to consider higher degeneracies, we characterize smooth admissible maps. The terms of given degree in the Taylor series of an admissible map constitute an admissible map, so this also leads to a classification of polynomial admissible maps of any given degree.

We begin by characterizing the smooth admissible maps. By invariant theory (see Macdonald [16]), the polynomial $\mathbb{S}_k$-invariant $\mathbb{R}$-valued functions on $\mathbb{R}^k$ are generated by a finite set of invariants. Since $\mathbb{S}_k$ acts trivially on $\mathbb{R}$, the same holds for the action on $\mathbb{R} \times \mathbb{R}^k$, with $\mathbb{R}$ acting as a parameter. We can then prove, for any given network $\mathcal{G}$, the following theorem.

Theorem 4.1. *Suppose that $\gamma_0, \dots, \gamma_p$ is any finite set of generators for the $\mathbb{S}_k$-invariant polynomial functions $\mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. Then a map $\Phi : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is admissible if and only if there exists a smooth map $\zeta : \mathbb{R}^p \to \mathbb{R}$ such that*

$$\Phi_c(x) = \zeta(\gamma_0(x_c, x_{T(c)}), \dots, \gamma_p(x_c, x_{T(c)}))$$

*for all cells $c$.*

*Proof.* "If" is clear. We prove "only if." Let $\Phi$ be admissible, and write

$$\Phi(x) = [\Phi_1(x), \dots, \Phi_n(x)]^{\mathrm{T}}.$$

Then for all cells $c$ there exists $\hat{\Phi}_c : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ such that

$$\Phi_c(x) = \hat{\Phi}_c(x_c, x_{T(c)}),$$

where $T(c) = \mathcal{T}(I(c)) \in \mathbb{R}^k$. The vertex group $\mathbb{S}_k$ acts on $\mathbb{R}^k$, and by regularity of $\mathcal{G}$ there is a smooth map $\phi : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ such that $\hat{\Phi}_c = \phi$ for all $c$, and $\phi$ is $\mathbb{S}_k$-invariant. In a regular network this condition, together with smoothness of $\phi$, is also sufficient for $\Phi$ to be admissible [21].

There exists a finite set of polynomial generators $\gamma_0, \dots, \gamma_p$ for the $\mathbb{S}_k$-invariant polynomial functions $\mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. By Schwarz [17], there exists a smooth map $\zeta : \mathbb{R}^{p+1} \to \mathbb{R}$ such that

$$\phi(y_0, y_1, \dots, y_k) = \zeta(\gamma_1(y), \dots, \gamma_p(y))$$

as claimed. ∎

We can now make a specific choice of the generators $\gamma_j$ that is particularly convenient for networks. If $y = (y_1, \dots, y_k)$, then it is well known (see, for example, Macdonald [16]) that the $\mathbb{S}_k$-invariant polynomial functions are generated as an $\mathbb{R}$-algebra by the power-sums

$$\pi_p(y) = y_1^p + \cdots + y_k^p.$$

To account for the trivial component $\mathbb{R}$ we take $y_0 \in \mathbb{R}$, which acts as a free parameter. So we can take

$$\begin{aligned} \gamma_0(y) &= y_0, \\ \gamma^j(y) &= y_1^j + \cdots + y_k^j, \quad 1 \le j \le k. \end{aligned}$$

Now we immediately have the following theorem.

Theorem 4.2. *With the above choice of the $\gamma_j$,*

$$\phi(x_c, x_{T(c)}) = \zeta(\gamma_0(x), \gamma_1(x), \dots, \gamma_k(x)),$$

*where*

$$\begin{aligned} \gamma_0(x) &= x_c, \\ \gamma^j(x) &= (Ax^{[j]})_c, \end{aligned}$$

*where $(Ax^{[j]})_c$ is the cth row of $Ax^{[j]}$.*

*Proof.* Observe that $\gamma^j(x) = \sum_{t \in T(c)} x_t^j = \sum_{i=1}^n a_{ci} x_i^j = (Ax^{[j]})_c$. ∎

We now specialize to polynomial admissible maps, which arise as truncated Taylor series of smooth admissible maps.

Theorem 4.3. *Suppose that the network is regular, with 1-dimensional cell phase spaces. Then the admissible polynomial maps of degree $m$ are linear combinations over $\mathbb{R}$ of maps of the form*

$$x^{[p_0]} \star Ax^{[p_1]} \star \cdots \star Ax^{[p_s]}, \tag{4.1}$$

*where $s \ge 0$, $p_0 \ge 0$, $p_j > 0$ for $1 \le j \le s$, and $p_0 + \cdots + p_s = m$. (When $s = 0$, no factors $Ax^{[p_j]}$ occur.)*

*Proof.* This is a direct consequence of Theorem 4.2. ∎

**5. Quadratic and cubic terms of the reduced map.** We now specialize the classification of Theorem 4.3 to the quadratic and cubic terms in the reduced map, assuming the standard hypotheses on the bifurcation and the standard choice of spaces for the Liapunov–Schmidt procedure.

The linearization $L$ in the Liapunov–Schmidt procedure is

$$L = A - \mu I,$$

so the kernel of $L$ is spanned by $v$. The vector $u$ lies in $(\text{range } L)^\perp$, so we may use these choices of $v, u$ in the formulas (3.4), (3.5), (3.6), (3.7) for Taylor series coefficients.

Statements (2.2), (2.3) imply that $\mathbb{R}^n = \mathbb{R}\{v\} \oplus u^\perp$. Now the restriction

$$L = (A - \mu I)|_{u^\perp}$$

(we continue to call it $L$) is nonsingular on $u^\perp$ because $\mu$ is simple. So $L^{-1}$ is a well-defined map on $u^\perp$. We will apply $L^{-1}$ only to vectors known to lie in $u^\perp$.

By Theorem 4.3 any quadratic admissible map is a linear combination of

$$x^{[2]}, \qquad x\star Ax, \qquad Ax^{[2]}, \qquad (Ax)^{[2]}.$$

More explicitly,

$$\begin{array}{rcl} q_1 &=& x_c^2, \\ q_2 &=& x_c(x_{i_1} + \cdots + x_{i_k}), \\ q_3 &=& x_{i_1}^2 + \cdots + x_{i_k}^2, \\ q_4 &=& (x_{i_1} + \cdots + x_{i_k})^2, \end{array} \tag{5.1}$$

where the sums are taken over the tail cells of all input arrows to cell $c$.

The cubic admissible maps are linear combinations of

$$x^{[3]}, \qquad x^{[2]}\star Ax, \qquad x\star Ax^{[2]}, \qquad x\star(Ax)^{[2]}, \qquad Ax^{[3]}, \qquad (Ax)\star(Ax^{[2]}), \qquad (Ax)^{[3]}$$

or, more explicitly,

$$\begin{aligned} Q_1 &= x_c^3, \\ Q_2 &= x_c^2(x_{i_1} + \cdots + x_{i_k}), \\ Q_3 &= x_c(x_{i_1}^2 + \cdots + x_{i_k}^2), \\ Q_4 &= x_c(x_{i_1} + \cdots + x_{i_k})^2, \\ Q_5 &= x_{i_1}^3 + \cdots + x_{i_k}^3, \\ Q_6 &= (x_{i_1} + \cdots + x_{i_k})(x_{i_1}^2 + \cdots + x_{i_k}^2), \\ Q_7 &= (x_{i_1} + \cdots + x_{i_k})^3. \end{aligned}$$

Now we compute the quadratic terms in the reduced equation.

Theorem 5.1. *Let the quadratic terms in* $\Phi$ *be*

$$\Phi^2(x) = ax^{[2]} + bx\star Ax + cAx^{[2]} + d(Ax)^{[2]}, \qquad a, b, c, d \in \mathbb{R}. \tag{5.2}$$

*Then the reduced map, to quadratic order, is*

$$g(x) = (\lambda - \mu)x + \sigma\langle u, v^{[2]}\rangle x^2,$$

*where*

$$\sigma = a + \mu b + \mu c + \mu^2 d. \tag{5.3}$$

*Proof.* The linear term is $(\lambda - \mu)x$.

By Taylor's theorem, $\Phi^2(x) = \frac{1}{2}\mathrm{D}^2\Phi(x, x)$, and the quadratic term in the Liapunov–Schmidt reduced map is $\frac{1}{2}g_{xx}x^2$. By (3.5),

$$\frac{1}{2}g_{xx} = \frac{1}{2}\langle u, \mathrm{D}^2\Phi(v, v)\rangle = \langle u, \Phi^2(v)\rangle. \tag{5.4}$$

Since $Av = \mu v$,

$$\begin{aligned}
\Phi^2(v) &= av^{[2]} + bv{\star}Av + cAv^{[2]} + d(Av)^{[2]} \\
&= av^{[2]} + bv{\star}\mu v + cAv^{[2]} + d(\mu v)^{[2]} \\
&= av^{[2]} + b\mu v^{[2]} + cAv^{[2]} + d\mu^2 v^{[2]} \\
&= (a + \mu b + \mu^2 d)v^{[2]} + cAv^{[2]}. \qquad (5.5)
\end{aligned}$$

The coefficient of $x^2$ is therefore

$$\begin{aligned}
\langle u, \Phi^2(v)\rangle &= \langle u, (a + \mu b + \mu^2 d)v^{[2]} + cAv^{[2]}\rangle \\
&= (a + \mu b + \mu^2 d)\langle u, v^{[2]}\rangle + c\langle u, Av^{[2]}\rangle \\
&= (a + \mu b + \mu^2 d)\langle u, v^{[2]}\rangle + c\langle A^{\mathrm{T}}u, v^{[2]}\rangle \\
&= (a + \mu b + \mu^2 d)\langle u, v^{[2]}\rangle + c\langle \mu u, v^{[2]}\rangle \\
&= (a + \mu b + \mu c + \mu^2 d)\langle u, v^{[2]}\rangle
\end{aligned}$$

as stated. ∎

The characterization of transcritical bifurcations follows immediately.

Theorem 5.2. *The bifurcation is generically transcritical if and only if $\langle u, v^{[2]}\rangle \neq 0$. In this case the genericity condition is $\sigma \neq 0$, where $\sigma$ is defined in* (5.3). ∎

The same type of calculation applies to cubic terms. We assume that the quadratic term in the reduced map vanishes and compute $\Phi^3(x) = \frac{1}{6}\mathrm{D}^3\Phi(x, x, x)$. If this term is nonzero, then the bifurcation is a pitchfork by Theorem 3.1.

By (3.7), the cubic term in the reduced map is $\frac{1}{6}g_{xxx}x^3$, where

$$g_{xxx} = \langle u, \mathrm{D}^3\Phi(v, v, v)\rangle - 3\langle u, \mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v, v))\rangle. \tag{5.6}$$

The first term $\langle u, \mathrm{D}^3\Phi(v, v, v)\rangle$ arises from the cubic terms in the admissible vector field $\Phi$. The second term $-3\langle u, \mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v, v))\rangle$ arises from the quadratic terms in $\Phi$ by way of the implicit function theorem.

Theorem 5.3. *Suppose that $\langle u, v^{[2]}\rangle = 0$, the quadratic terms in $\Phi$ are*

$$\Phi^2(x) = ax^{[2]} + bx{\star}Ax + cAx^{[2]} + d(Ax)^2$$

*for $a, b, c, d \in \mathbb{R}$, and the cubic terms are*

$$\Phi^3(x) = Px^{[3]} + Qx^{[2]}{\star}Ax + Rx{\star}Ax^{[2]} + Sx{\star}(Ax)^{[2]} \\ + TAx^{[3]} + U(Ax){\star}Ax^{[2]} + V(Ax)^{[3]},$$

*where $P, Q, R, S, T, U, V \in \mathbb{R}$. Then the cubic coefficient of the reduced map is*

$$\frac{1}{6}g_{xxx} = \sigma_1\langle u, v^{[3]}\rangle + \sigma_2\langle u, v{\star}Av^{[2]}\rangle + \sigma_3\langle u, v{\star}L^{-1}v^{[2]}\rangle, \tag{5.7}$$

*where the $\sigma_j$ are polynomial functions of $a, b, c, d, P, Q, R, S, T, U, V, \mu$.*

*Proof.* The proof is similar to that of Theorem 5.1 and is given in Appendix A. ∎

*Example* 5.4. The condition $\langle u, v{\star}L^{-1}v^{[2]}\rangle = 0$ is not a consequence of $\langle u, v^{[2]}\rangle = \langle u, v^{[3]}\rangle = \langle v{\star}Av^{[2]}\rangle = 0$. Consider the 5-cell network for which

$$A = \begin{bmatrix} 0 & 2 & 2 & 1 & 22 \\ 6 & 8 & 11 & 2 & 0 \\ 6 & 1 & 2 & 2 & 16 \\ 1 & 1 & 1 & 0 & 24 \\ 0 & 0 & 0 & 0 & 27 \end{bmatrix},$$

which is regular of valency 27. The eigenvectors for eigenvalue $-1$ are

$$v = [-1, -2, 2, 1, 0]^{\mathrm{T}}, \qquad u = [-8, 1, -1, 8, 0]^{\mathrm{T}}.$$

The other eigenvalues are 27 and the three roots of an irreducible cubic, so $-1$ is a simple eigenvalue. Direct calculation shows that $\langle u, v^{[2]}\rangle = \langle u, v^{[3]}\rangle = \langle v{\star}Av^{[2]}\rangle = 0$. However, $(L+I)^{-1}v^{[2]} = [\frac{1}{2}, 0, 0, \frac{1}{2}, 0]^{\mathrm{T}}$ so that $\langle u, v{\star}L^{-1}v^{[2]}\rangle = 8 \neq 0$. ∎

We immediately deduce a characterization of pitchfork bifurcations.

Corollary 5.5. *With the usual hypotheses, suppose that $\langle u, v^{[2]}\rangle = 0$. Then generically the bifurcation is a pitchfork if and only if at least one of $\langle u, v^{[3]}\rangle$, $\langle u, v{\star}Av^{[2]}\rangle$, or $\langle u, v{\star}L^{-1}v^{[2]}\rangle$ is nonzero.* ∎

The most obvious context in which the quadratic term in the reduced map must vanish for all admissible vector fields is symmetry. The spaces involved in the Liapunov–Schmidt reduction can be chosen to be invariant under the symmetry group, so the reduced map is also symmetric [5]. If the network $\mathcal{G}$ has a global symmetry group $\Gamma$, and $A$ has a simple eigenvalue $\mu$, then $\Gamma$ leaves the eigenspace $E_\mu = \mathbb{R}\{v\}$ invariant. Either $\Gamma$ acts trivially on $E_\mu$, in which case the bifurcation preserves the symmetry, or $\Gamma$ acts nontrivially on $E_\mu$, in which case the action factors through $\mathbb{Z}_2$ and the least degenerate bifurcation is a pitchfork. More generally, the same applies on a quotient network (we omit the proof, which is straightforward).

Proposition 5.6. *Suppose that $\mathcal{G}$ has a quotient network with a symmetry group that changes the sign of the eigenvector associated with the bifurcation. Then the Liapunov–Schmidt reduced bifurcation equation has only odd degree terms in $x$, and thus is pitchfork or more degenerate.* ∎

However, symmetry—even on a quotient network—is *not* necessary for pitchforks to be generic. The next example shows that they can occur for combinatorial reasons.

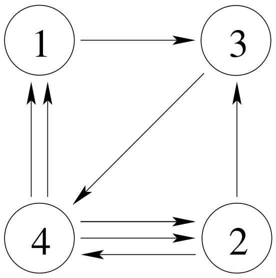


**Figure 1.** *4-cell path-connected regular network with trivial symmetry.*

*Example* 5.7. Consider network $\mathcal{G}_{49}$ of Figure 1, which is #49 on the list of all 416 connected 4-cell valency-2 networks in Kamei [12, Chapter 3, Figure 3.3] and Kamei [13]. The adjacency matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A$ are $2, 0, -1 \pm i$, so they are all simple. The eigenvectors of $A$ and $A^{\mathrm{T}}$ for eigenvalue 0 are

$$v = [1, -1, 1, 0]^{\mathrm{T}}, \qquad u = [-1, 1, 0, 0]^{\mathrm{T}}.$$

Now $\langle u, v^{[2]} \rangle = 0$, so there is a degeneracy at the quadratic level. The cubic term $\langle u, v^{[3]} \rangle = -2 \neq 0$, so generically the bifurcation is a pitchfork.

By inspection of Figure 1, the symmetry group of $\mathcal{G}_{49}$ is trivial. The only nontrivial polydiagonal (see [9, 21]) containing $v$ is $\{(x, y, x, z)\}$, corresponding to the coloring $\bowtie$ with classes $\{1, 3\}, \{2\}, \{4\}$. However, this coloring is not balanced, so there is no nontrivial quotient network. ■

Corollary 5.5 has a strong implication for bidirectional networks. Recall that a network is *bidirectional* if every arrow from cell $c$ to cell $d$ corresponds bijectively to an arrow of the same type from $d$ to $c$. Clearly a regular network is bidirectional if and only if $A$ is a symmetric matrix. All regular bidirectional networks are 3-determined.

Theorem 5.8. *With the usual hypotheses, suppose that the network is bidirectional. Then generically the bifurcation problem is 3-determined. Thus real simple-eigenvalue bifurcation in a bidirectional network is generically transcritical or pitchfork. If $v_1^3 + \cdots + v_n^3 \neq 0$, then the bifurcation is transcritical. Otherwise, it is a pitchfork.*

*Proof.* Since

$$A = A^{\mathrm{T}},$$

we may take $u = v$. Therefore

$$\langle u, v^{[3]} \rangle = v_1^4 + \cdots + v_n^4 \neq 0,$$

so at least one cubic term in the Liapunov–Schmidt reduction is generically nonzero. If $\langle u, v^{[2]} \rangle = v_1^3 + \cdots + v_n^3 \neq 0$, then generically the bifurcation is transcritical. Otherwise, it is generically a pitchfork. ■

The symmetric matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

has eigenvalues $3, \sqrt{3}, -\sqrt{3}$. The eigenvector for $\sqrt{3}$ is $[1-\sqrt{3}, -2+\sqrt{3}, 1]^{\mathrm{T}}$. Taking this for $v$ we get $v_1^3 + v_2^3 + v_3^3 = -15 + 9\sqrt{3} \neq 0$, so the transcritical case can occur. The next example shows that symmetry is not necessary for generic bifurcation to be pitchfork in the bidirectional case.

*Example* 5.9. Let

$$A = \begin{bmatrix} 0 & 47 & 54 & 43 & 36 \\ 47 & 129 & 0 & 0 & 4 \\ 54 & 0 & 126 & 0 & 0 \\ 43 & 0 & 0 & 129 & 8 \\ 36 & 4 & 0 & 8 & 132 \end{bmatrix}.$$

This has valency 180, and

$$v = [1, 5, 9, -7, -8]^{\mathrm{T}}$$

is a simple eigenvector with eigenvalue 132. Now $u = v$, and the choice of $v$ makes the associated bifurcation 2-degenerate. Clearly no symmetry can invert $v$; in fact, the symmetry group is trivial. The corresponding network is path-connected. All entries of $A$ can be made nonzero with slightly different choices, at the expense of making the integers larger when scaling away fractions. ■

**6. Finite determinacy.** Theorem 5.8 is a simple example of a *determinacy theorem*, stating that under suitable hypotheses the Liapunov–Schmidt reduced bifurcation problem is finitely determined; that is, its normal form can be obtained from a finite truncation of its Taylor series. In this section we discuss determinacy theorems for general networks. These provide limits on the degree of degeneracy of network bifurcations at a simple real eigenvalue. We prove three determinacy results of increasing strength: the proof of each relies on the previous one.

The first such result is Theorem 6.1. This implies Corollary 6.2: generically (in the admissible vector field) all $n$-cell networks have $(n+1)$-determined bifurcations at simple eigenvalues for $n \geq 3$. Next, Theorem 6.13 improves this to $n$-determined bifurcations. Finally, Theorem 6.14 shows that in the path-connected case $n$-determinacy can be improved to $(n-1)$-determinacy when $n \geq 4$.

To state the first theorem in its strongest form, say that a bifurcation problem is *strongly* $l$*-determined* if $\langle u, \Phi^l \rangle \neq 0$ for some admissible vector field $\Phi^l$ of degree $l$. In the Liapunov–Schmidt procedure, the most straightforward way to get a nonzero contribution to the normal form $g_{x^{[l]}}$ is to ensure that

$$\langle u, v^{[l]} \rangle \neq 0.$$

We can now state and prove the following theorem.

Theorem 6.1. *Suppose that $v$ has $l$ distinct nonzero entries. Then generically $\langle u, v^{[r]} \rangle \neq 0$ for some $r$ with $2 \leq r \leq l+1$, so generically the bifurcation is strongly $(l+1)$-determined.*

*Proof.* We claim that there is a polynomial $p(x)$ of degree $l+1$ with $p(0) = 0$ and the coefficient of $x$ in $p$ nonzero, such that

$$p(v_j) = 0$$

for all entries $v_j$ of $v$. To construct $p$, let $w_1, \dots, w_l$ be the distinct nonzero entries $v_j$, and define

$$p(x) = x(x - w_1) \cdots (x - w_l).$$

Then $p(0) = 0$ and $p(w_j) = 0$. The coefficient of $x$ in $p$ is $(-1)^l w_1 \cdots w_l \neq 0$. Write

$$p(x) = a_1 x + a_2 x^2 + \cdots + a_{l+1} x^{l+1}$$

so that

$$v_j = \frac{-1}{a_1}(a_2 v_j^2 + \cdots + a_{l+1} v_j^{l+1}). \tag{6.1}$$

Now suppose for a contradiction that $\langle u, v^{[r]} \rangle = 0$ for $2 \leq r \leq l+1$. Then

$$\langle u, a_2 v^{[2]} + \cdots + a_{l+1} v^{[l+1]} \rangle = 0.$$

But now (6.1) implies that $\langle u, v \rangle = 0$, contrary to (2.3). ■

Corollary 6.2. *In an $n$-cell network, generically the bifurcation is $(n+1)$-determined.* ■

Corollary 6.3. *If $v$ has at most two distinct nonzero entries, then generically the bifurcation is 3-determined.* ■

This corollary is interesting because balanced polydiagonals with only two distinct entries (patterns of synchrony with two clusters) are analogous to the fixed-point subspaces of "axial" isotropy subgroups in the group-equivariant case [8]. Call such a polydiagonal *axial*. Then we deduce the following corollary.

Corollary 6.4. *If $v$ lies in an axial balanced polydiagonal, then generically the bifurcation is 3-determined.* ■

**6.1. Eigenstructure of $v^{[2]}$.** The vector $v^{[2]}$ plays a key role in the theory and has arisen naturally in several previous examples as an eigenvector of $A$. To improve Corollary 6.2 we examine this vector in more detail. First, note a simple property of the componentwise product:

$$\langle x \star y, z \rangle = \langle x, y \star z \rangle. \tag{6.2}$$

This identity is clear because both sides reduce to

$$\sum_i x_i y_i z_i.$$

Using this identity we prove the following proposition.

Proposition 6.5. *For a path-connected network at a 3-degenerate bifurcation, $v^{[2]}$ cannot be an eigenvector of $A$.*

*Proof.* All entries of $v^{[2]}$ are nonnegative. Since the network is path-connected, the adjacency matrix is irreducible. Now (2.9) implies that $v^{[2]} = \phi e$ for some $\phi \in \mathbb{R}$. Clearly $\phi \neq 0$. By (6.2),

$$0 = \langle u, v^{[3]} \rangle = \langle u \star v, v^{[2]} \rangle = \langle u \star v, \phi e \rangle = \phi \langle u, v \rangle \neq 0,$$

which is a contradiction. ■

**6.2. $n$-cell networks are generically $n$-determined.** In this section we improve Theorem 6.1, replacing the determinacy bound $n+1$ by $n$. Example 7.1 shows that $n$ cannot be replaced by $n-1$ in general. Section 10 shows that in the path-connected case it cannot be replaced by $n-2$.

We first dispose of 2- and 3-cell networks, which have special features that do not apply in the general case.

It is easy to analyze the 2-cell case, and there are no surprises: all regular 2-cell networks are 3-determined, and pitchforks occur only when the network has $\mathbb{Z}_2$ symmetry. We therefore consider 3-cell networks. Again, all regular 3-cell networks have 3-determined bifurcations at any simple real eigenvalue. The proof is not as straightforward as in the 2-cell case. It contains some ideas that generalize to more cells, but it also relies on special features of 3-cell networks, and it is needed below to deal with this case in Theorem 6.1.

We establish a slightly stronger result, not involving $\langle u, v_\star L^{-1} v^{[2]}\rangle$.

Lemma 6.6. *Let $n = 3$. If*

$$\langle u, v^{[2]}\rangle = \langle u, v^{[3]}\rangle = \langle u, v_\star A v^{[2]}\rangle = 0, \tag{6.3}$$

*then $v^{[2]}$ is an eigenvector of $A$ and $v^{[3]} = \alpha v^{[2]} + \beta e$, where $\alpha, \beta \in \mathbb{R}$ and $e = [1, 1, 1]^{\mathrm{T}}$.*

*Proof.* Recall that $e$ is an eigenvector of $A$ with eigenvalue $k$. Assume (6.3). If any entry of $v$ is zero, or two entries are equal, then Theorem 6.1 implies that the bifurcation is strongly 3-determined, contradicting (6.3). So

$$v = [v_1, v_2, v_3]^{\mathrm{T}},$$

where all $v_j$ are distinct nonzero.

Statement (2.5) of section 2 implies that $\{e, v, v^{[2]}\}$ is a basis for $\mathbb{R}^3$, so

$$Av^{[2]} = \alpha v^{[2]} + \beta v + \gamma e \tag{6.4}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$. Therefore

$$v_\star A v^{[2]} = \alpha v^{[3]} + \beta v^{[2]} + \gamma v.$$

By (6.3), the vectors $v_\star A v^{[2]}, v^{[3]}, v^{[2]}$ are orthogonal to $u$. But $v$ is not orthogonal to $u$, so $\gamma = 0$. Now (6.4) becomes

$$Av^{[2]} = \alpha v^{[2]} + \beta v.$$

However, both $v^{[2]}$ and $Av^{[2]}$ are orthogonal to $u$ (the latter using $\langle u, Av^{[2]}\rangle = \langle A^{\mathrm{T}} u, v^{[2]}\rangle$), but $\langle u, v\rangle \neq 0$ by (2.3), so $\beta = 0$. Therefore

$$Av^{[2]} = \alpha v^{[2]} \tag{6.5}$$

and $v^{[2]}$ is an eigenvector with eigenvalue $\alpha$. ■

We now prove that (6.3) cannot hold when $n = 3$.

Theorem 6.7. *No regular 3-cell network can satisfy (6.3). In particular, every generic synchrony-breaking simple-eigenvalue steady-state bifurcation of a regular 3-cell network is 3-determined.*

*Proof.* First, assume that the network is path-connected. By Theorem 6.1, all entries of $v$ are nonzero. Therefore all entries of the eigenvector $v^{[2]}$ are positive. By the Perron–Frobenius theorem (2.9), $v^{[2]} = \alpha e$ for some $\alpha \in \mathbb{R}$, and clearly $\alpha > 0$. Therefore $v = [\pm\sqrt{\alpha}, \ldots, \pm\sqrt{\alpha}]^{\mathrm{T}}$, with at most two distinct entries, so Theorem 6.1 implies that (6.3) cannot be valid.

Next, suppose that $\mathcal{G}$ is not path-connected. By Lemma 6.6, equation (6.5) holds. By Theorem 6.1, the entries of $v$ are all distinct and nonzero. There are two cases: either $\mathcal{G}$ can be decomposed into a 2-cell network that forces a 1-cell network, or $\mathcal{G}$ can be decomposed into a 1-cell network that forces a 2-cell network. In the first case,

$$A = \begin{bmatrix} a & k-a & 0 \\ b & k-b & 0 \\ c & d & k-c-d \end{bmatrix} \tag{6.6}$$

for nonnegative integers $a, b, c, d$. In the second case,

$$A = \begin{bmatrix} k & 0 & 0 \\ a & b & k-a-b \\ c & d & k-c-d \end{bmatrix} \tag{6.7}$$

for nonnegative integers $a, b, c, d$.

Assume (6.6). Then $[v_1, v_2]^{\mathrm{T}}$ is an eigenvector for eigenvalue $\mu$ of the $2 \times 2$ block

$$\begin{bmatrix} a & k-a \\ b & k-b \end{bmatrix}$$

whose eigenvalues are $k, a-b$. Since $\mu \neq k$, we must have $\mu = a - b$. Now the corresponding eigenvector of $\mathbf{A}^{\mathrm{T}}$ is $[1, -1, 0]^{\mathrm{T}}$, and since $u$ is orthogonal to both $v^{[2]}$ and $v^{[3]}$, we have $v_1^2 = v_2^2$ and $v_1^3 = v_2^3$. Therefore $v_1 = v_2$ (the $v_j$ are nonzero), a contradiction since the $v_j$ are also distinct.

It remains to consider the possibility (6.7). Now $Av = \mu v$ implies that $\mu = k$ by considering the coefficient of $v_1$, a contradiction. ■

This result generalizes to the $n$-cell case (see Theorem 6.13), but the proof is different for $n \geq 4$, as we now explain. First we need a general fact about Liapunov–Schmidt reduction.

Lemma 6.8. *Liapunov–Schmidt reduction preserves the branch of trivial solutions.*

*Proof.* Suppose that $\Phi(0, \lambda) = 0$. We need to prove that the reduced map satisfies $g(0, \lambda) = 0$. In the notation of [5] it suffices to show that the implicitly defined map $W(0, \lambda) = 0$.

For each fixed $\lambda$ near 0 the set $\Phi(M, \lambda)$ contains the origin, since $\Phi(0, \lambda) = 0$. It is a codimension-1 submanifold of $\mathbb{R}^n$ that is transverse to $(\text{range } L)^{\perp}$ at the origin. Therefore the only $w \in M$ for which $\Phi(w, \lambda) \in (\text{range } L)^{\perp}$ is $w = 0$.

Now $W(v, \lambda)$ is the unique function that satisfies $W(0, 0) = 0$ and the equation

$$E\Phi(v + W(v, \lambda), \lambda) = 0.$$

Therefore $\Phi(W(0, \lambda), \lambda) \subset (\text{range } L)^{\perp}$. Since $W(0, \lambda) \in M$, we have $W(0, \lambda) = 0$. ■

Next, we need a property of the growth rates of bifurcating branches.

Theorem 6.9. *Assuming* (3.2), *suppose that* $r \geq 2$ *and*

$$g = g_x = \cdots = g_{x^{r-1}} = 0 \quad \textit{and} \quad g_{x^r} \neq 0$$

*at* $x = \lambda = 0$. *Then the bifurcation problem* $g$ *is strictly* $r$*-determined and equivalent to the normal form*

$$g_r^{\pm}(x, \lambda) \equiv \lambda x \pm x^r.$$

*Proof.* By Lemma 6.8, Liapunov–Schmidt reduction preserves a trivial branch of solutions, so $g(0, \lambda) = 0$ and $g = g_\lambda = 0$. Also, the eigenvalue crossing condition guarantees that $g_{x\lambda}(0, 0) \neq 0$. Proposition II, 9.2 of [5] proves the theorem when $r \geq 3$. When $r = 2$ we use the fact that $g_{\lambda\lambda} = 0$ and Proposition II, 9.3 of [5]. ■

We now discuss the growth rate of a branch in more detail. For this purpose a (steady-state) *branch* of a bifurcation problem $\Phi$ is a connected component of the zero-set of $\Phi$ minus the origin. Near the origin, each branch exists either for $\lambda > 0$ or $\lambda < 0$.

Definition 6.10. *A branch has* growth rate $|\lambda|^a$ *near the origin, for* $a \in \mathbb{R}$, *if along that branch*

$$\frac{||X||}{K|\lambda|^a} \to 1 \qquad \textit{as } \lambda \to 0$$

*for a constant* $K > 0$. ■

In the normal form $g_r^{\pm}$ the nontrivial branches have growth rate $|\lambda|^{1/(r-1)}$ near the origin, and this growth rate characterizes the normal form because $1/(r-1)$ determines $r$ uniquely. The same growth rate occurs on the corresponding branch of $\Phi$ by Proposition 6.11.

In general, if a bifurcating branch is observed in terms of the dynamics of a single cell, the growth rate may differ from that of the overall branch: examples can be found in Leite and Golubitsky [15, Table 4]. This possibility arises because projections need not preserve growth rates. However, there is one simple condition that avoids this issue: this applies when the critical eigenvector has no zero entries. We derive the appropriate result as a corollary of the next proposition, which is well known.

Proposition 6.11. *Assume the standard hypotheses, so that in particular* $\mu$ *denotes a simple real eigenvalue and* $v$ *is the corresponding eigenvector. Suppose that the corresponding steady-state bifurcation has normal form* $g_r^{\pm}$. *Then each nontrivial bifurcating branch is asymptotic to a curve of the form*

$$X = \pm K|\lambda|^{1/(r-1)}v, \quad X \in \mathbb{R}^n,$$

*with an appropriate sign depending on the direction of the branch, as* $\lambda \to 0$. *Here* $K > 0$ *is a constant.*

*Proof.* The result follows from the Liapunov–Schmidt reduction procedure and the use of diffeomorphisms in contact equivalence for singularities. ■

Corollary 6.12. *For any cell* $c$ *such that* $v_c \neq 0$, *the growth rate of the projection of a bifurcating branch onto cell* $c$, *that is, the component* $X_c$ *of the branch* $X$, *is the same as that of the branch.*

*Proof.* With the notation of the previous proposition,

$$X_c = \pm K|\lambda|^{1/(r-1)}v_c,$$

which yields the same growth rate provided $v_c \neq 0$. ∎

We can now state and prove the following theorem.

Theorem 6.13. *In any $n$-cell network, $n \geq 3$, simple eigenvalue bifurcations are generically $n$-determined.*

*Proof.* We may assume $n \geq 4$ by Theorem 6.7. For a contradiction, assume the bifurcation is $n$-degenerate. In particular, it is 4-degenerate.

Let $v$ be the eigenvector of $A$ with eigenvalue $\mu$, and let $u$ be the corresponding eigenvector of $A^{\mathrm{T}}$. Unless all entries of $v$ are distinct and nonzero, the result follows from Theorem 6.1. So the $v_j$ are distinct and nonzero. Therefore the vectors $e, v, v^{[2]}, \dots, v^{[n-1]}$ form a basis for phase space $P = \mathbb{R}^n$ by section 2. So there exist scalars $c_0, \dots, c_{n-1}$ such that

$$Av^{[2]} = c_0 e + c_1 v + c_2 v^{[2]} + \cdots + c_{n-1} v^{[n-1]}.$$

Taking the inner product with $u$ shows that $c_1 = 0$.

The quartic term $v^{[2]} \star A v^{[2]}$ is orthogonal to $u$ since the bifurcation is 4-degenerate. Now

$$v^{[2]} \star A v^{[2]} = c_0 v^{[2]} + c_2 v^{[4]} + \cdots + c_{n-2} v^{[n]} + c_{n-1} v^{[n+1]}. \tag{6.8}$$

Consider the polynomial

$$p(x) = (x - v_1) \dots (x - v_n) = x^n - \sigma_1 x^{n-1} + \cdots + \sigma_{n-2} x^2 - \sigma_{n-1} x + \sigma_n,$$

where the $\sigma_i$ are elementary symmetric polynomials in the $v_j$. Clearly

$$v^{[n]} - \sigma_1 v^{[n-1]} + \cdots + \sigma_{n-2} v^{[2]} - \sigma_{n-1} v + \sigma_n e = 0$$

since each $v_j$ is a zero of $p$. Therefore

$$v^{[n+1]} = \sigma_1 v^{[n]} - \cdots \pm \sigma_{n-2} v^{[3]} \mp \sigma_{n-1} v^{[2]} \pm \sigma_n v.$$

By (6.8),

$$v^{[2]} \star A v^{[2]} = c_0 v^{[2]} + c_2 v^{[4]} + \cdots + c_{n-2} v^{[n]} + c_{n-1}(\sigma_1 v^{[n]} - \cdots \pm \sigma_{n-2} v^{[3]} \mp \sigma_{n-1} v^{[2]} \pm \sigma_n v).$$

Taking the inner product with $u$ and using 4-degeneracy we get $c_{n-1}\sigma_n = 0$. But $\sigma_n = v_1 v_2 \dots v_n \neq 0$, so $c_{n-1} = 0$.

Similar calculations for the terms

$$v^{[l]} \star A v^{[2]},$$

where $l = 3, \dots, n-2$, show inductively that $c_{n-2} = \cdots = c_3 = 0$. Now

$$Av^{[2]} = c_0 e + c_2 v^{[2]}$$

so that

$$v \star A v^{[2]} = c_0 v + c_2 v^{[3]}.$$

Taking the inner product with $u$ and using 3-degeneracy we get $c_0 = 0$. Therefore

$$Av^{[2]} = c_2 v^{[2]}$$

and $v^{[2]}$ is an eigenvector of $A$.

Proposition 6.5 now implies that the network is not path-connected. Denote this network by $\mathcal{G}$, and decompose $\mathcal{G}$ into its path-connected components, which have a natural partial ordering defined by the existence of directed paths; see Josić and Török [11, Proposition 15]. By finiteness, some path-connected component $\mathcal{H}$ is maximal with respect to this ordering. This component is a path-connected subnetwork that receives no inputs from the remainder of the network. Therefore, with a suitable ordering of the cells, there is a block decomposition

$$A = \begin{bmatrix} P & 0 \\ Q & R \end{bmatrix}$$

in which $P$ is an $m \times m$ matrix, $Q$ is an $(n-m) \times m$ matrix, and $R$ is an $(n-m) \times (n-m)$ matrix, where $1 \leq m \leq n-1$.

Write the eigenvector $v$ in the corresponding block form

$$v = \begin{bmatrix} y \\ z \end{bmatrix}.$$

Then $Av = \mu v$ implies that $Py = \mu y$. Since $v$ has no zero entries, $y$ is an eigenvector of $P$ with eigenvalue $\mu$. Moreover, $\mathcal{H}$ is a regular subnetwork of valency $k$ with adjacency matrix $P$, and it is path-connected.

Because $\mathcal{H}$ receives no inputs from the remainder of the network $\mathcal{G}$, any admissible ODE has a corresponding block structure. So there is a natural projection of the dynamics on $\mathcal{G}$ onto the dynamics of $\mathcal{H}$. The vector field $\Psi$ for $\mathcal{H}$ is given by the first $m$ components of the vector field $\Phi$ for $\mathcal{G}$. Trajectories of $\Phi$ project to give trajectories of $\Psi$. Therefore the bifurcation diagram for $\Psi$ on $\mathcal{H}$, near a nontrivial branch determined by the critical eigenvalue $\mu$, is the projection of the bifurcation diagram for $\Phi$ on $\mathcal{G}$ near the corresponding branch determined by the same critical eigenvalue $\mu$.

Because $v$ has no zero entries, Corollary 6.12 implies that the original and projected branches have the same growth rate, as a function of $|\lambda|$, when observed on any cell. By Theorem 6.1, $\Psi$ is generically $(m+1)$-determined and hence generically $n$-determined since $m \leq n-1$. Therefore (generically in $\Psi$) the branch for $\Psi$ has growth rate $|\lambda|^{1/(r-1)}$ for some $r$ such that $2 \leq r \leq n$. But we have just shown that this is also the growth rate (generically in $\Phi$) for the corresponding branch for $\Phi$. Since the growth rate characterizes the normal form, and the bifurcation problem $\Phi$ is generically finitely determined, it follows that $\Phi$ is generically $n$-determined.

(Since the network is regular, any $\mathcal{H}$-admissible perturbation of $\Psi$ extends naturally to a $\mathcal{G}$-admissible perturbation of $\Phi$. So the use of genericity is unambiguous here.) ■

**6.3. $n$-cell path-connected networks are generically $(n-1)$-determined.** Below, we construct degenerate bifurcations in several feed-forward networks, where a special method ("bordering") makes the construction simpler. The construction of path-connected examples is constrained by a further improvement on the determinacy theorem. We now prove that when $n \geq 4$ all $n$-cell path-connected networks have $(n-1)$-determined bifurcations.

Theorem 6.14. *If $n \geq 4$, then every simple-eigenvalue steady-state bifurcation of an $n$-cell path-connected regular network is generically $(n-1)$-determined.*

The proof, which we postpone to section 6.4 in order to set up the ideas involved, makes repeated use of two simple properties of the componentwise product. The first is (6.2), and the second is

$$x{\star}y^{[p]} = 0 \Longrightarrow x{\star}y^{[q]} = 0, \qquad 1 \le q < p. \tag{6.9}$$

This implication is trivial, but it plays a key role in the proof. Since $y$ may have some components equal to 0, we cannot cancel $y$ completely to get $x = 0$. The proof of (6.9) is straightforward: if $x_i y_i^p = 0$, then $x_i = 0$ or $y_i = 0$, so $x_i y_i^q = 0$. We call (6.9) the *semicancellation law.*

Define a sum of generalized eigenspaces

$$\mathcal{X} = \bigoplus_{\nu \neq k,\mu} E_\nu.$$

Then the phase space $P$ decomposes as

$$P = E_k \oplus E_\mu \oplus \mathcal{X}$$

and all three summands are $A$-invariant. In the path-connected case, $\dim E_k = 1$ by the Perron–Frobenius theorem (2.7). Since $\mu$ is simple, $\dim E_\mu = 1$. Denote the eigenvector of $A^{\mathrm{T}}$ for eigenvalue $k$ by $w$. Since the network is path-connected, the Perron–Frobenius theorem (2.8) implies that we can choose $w$ so that $w_i > 0$ for $1 \le i \le n$. By (2.2), we have

$$u^\perp = E_k \oplus \mathcal{X}, \tag{6.10}$$

$$w^\perp = E_\mu \oplus \mathcal{X}. \tag{6.11}$$

Lemma 6.15. *Assume that $A$ is $m$-degenerate. Then the vectors*

$$u \quad u{\star}v \quad u{\star}v^{[2]} \quad \dots \quad u{\star}v^{[m-2]} \tag{6.12}$$

*are linearly independent.*

*Proof.* We may assume $m \ge 4$ since the statement is obvious for $m \le 3$. The proof makes repeated use of (2.2), which states that $\langle u, v\rangle \neq 0$.

Suppose that

$$\alpha_0 u + \alpha_1 u{\star}v + \alpha_2 u{\star}v^{[2]} + \cdots + \alpha_{m-2} u{\star}v^{[m-2]} = 0.$$

Take the inner product with $v$. Equation (6.2) implies that $\langle v, u{\star}v^{[p]}\rangle = \langle u, v^{[p+1]}\rangle$. The $m$-degeneracy of $A$ implies that $\langle u, v^{[p+1]}\rangle = 0$ for $1 \le p \le m-2$. But $\langle u, v\rangle \neq 0$, so $\alpha_0 = 0$. Therefore

$$\alpha_1 u{\star}v + \alpha_2 u{\star}v^{[2]} + \cdots + \alpha_{m-2} u{\star}v^{[m-2]} = 0.$$

Take the inner product with $e$, and note that $\langle e, u{\star}v^{[p]}\rangle = \langle u, v^{[p]}\rangle$ by (6.2). Then

$$\alpha_1 \langle u, v\rangle + \alpha_2 \langle u, v^{[2]}\rangle + \cdots + \alpha_{m-2}\langle u, v^{[m-2]}\rangle = 0.$$

By $m$-degeneracy of $A$, this reduces to

$$\alpha_1\langle u, v\rangle = 0,$$

so $\alpha_1 = 0$. Now

$$\alpha_2 u\star v^{[2]} + \alpha_3 u\star v^{[3]} + \cdots + \alpha_{m-2} u\star v^{[m-2]} = 0.$$

The semicancellation law (6.9) implies that

$$\alpha_2 u\star v + \alpha_3 u\star v^{[2]} + \cdots + \alpha_{m-2} u\star v^{[m-3]} = 0$$

and the inner product with $e$ implies that $\alpha_2 = 0$. Inductively, we repeatedly apply the semicancellation law to remove one factor $v$ and take the inner product with $e$ to deduce that $\alpha_3 = \alpha_4 = \cdots = \alpha_{m-2} = 0$. This proves that (6.12) is a linearly independent set, as claimed. ■

Next, we prove the following lemma.

Lemma 6.16. *Assume that $A$ is $(n-1)$-degenerate. Then $v^{[2]}$, $Av^{[2]}$, and $L^{-1}v^{[2]}$ are linearly dependent.*

*Proof.* Observe that $v^{[2]}$, $Av^{[2]}$, and $L^{-1}v^{[2]}$ are orthogonal to $u\star v^{[m]}$ for $0 \le m \le n-3$. When $m = 0$ this follows since

$$\langle u, v^{[2]}\rangle = 0$$

by 2-degeneracy, whence also

$$\langle u, Av^{[2]}\rangle = \langle A^{\mathrm{T}}u, v^{[2]}\rangle = \mu\langle u, v^{[2]}\rangle = 0$$

and finally

$$\langle u, L^{-1}v^{[2]}\rangle = 0$$

since by definition $L : u^\perp \to u^\perp$ and $v^{[2]} \in u^\perp$ by 2-degeneracy.

When $m \ge 1$ we argue as follows:

$$\langle u\star v^{[m]}, v^{[2]}\rangle = \langle u, v^{[2]}\star v^{[m]}\rangle = \langle u, v^{[m+2]}\rangle = 0$$

by $(n-1)$-degeneracy, noting that $m \le n-3$. Similarly

$$\langle u\star v^{[m]}, Av^{[2]}\rangle = \langle u, v^{[2]}\star Av^{[m]}\rangle = 0$$

by $(n-1)$-degeneracy, and

$$\langle u\star v^{[m]}, L^{-1}v^{[2]}\rangle = \langle u, v^{[m]}\star L^{-1}v^{[2]}\rangle = 0$$

by $(n-1)$-degeneracy. Here we use the fact that the reduced map includes terms of the forms $\langle u, v^{[2]}\star Av^{[m]}\rangle$ and $\langle u, v^{[m]}\star L^{-1}v^{[2]}\rangle$. This follows from the classification of polynomial admissible maps in Theorem 4.3.

Since $\dim P = n$, the subspace

$$\mathbb{R}\{u, u\star v, u\star v^{[2]}, \ldots, u\star v^{[n-3]}\}$$

has codimension 2 in $\mathbb{R}$. But its orthogonal complement contains the three vectors $v^{[2]}$, $Av^{[2]}$, $L^{-1}v^{[2]}$. So these must be linearly dependent. ■

Lemma 6.17. *If $v^{[2]}, Av^{[2]}, L^{-1}v^{[2]}$ are linearly dependent, then the subspace $\mathbb{R}\{v^{[2]}, Av^{[2]}\}$ is $A$-invariant.*

*Proof.* Suppose that there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha v^{[2]} + \beta A v^{[2]} + \gamma L^{-1} v^{[2]} = 0.$$

If $\beta = 0$, then $\alpha v^{[2]} + \gamma L^{-1} v^{[2]} = 0$. Since $v^{[2]} \neq 0$ and $L^{-1}v^{[2]} \neq 0$, we have $\alpha, \beta \neq 0$. But $L = A - \mu I$, so we can premultiply by $L$ and rewrite as

$$\alpha(A - \mu I)v^{[2]} + \gamma v^{[2]} = 0.$$

That is,

$$Av^{[2]} = \left(\mu - \frac{\gamma}{\alpha}\right) v^{[2]}.$$

So $v^{[2]}$ is an eigenvector of $A$, and in particular $\mathbb{R}\{v^{[2]}, Av^{[2]}\} = \mathbb{R}\{v^{[2]}\}$ is $A$-invariant.

Otherwise $\beta \neq 0$, so

$$Av^{[2]} = -\frac{\alpha}{\beta} v^{[2]} - \frac{\gamma}{\beta} L^{-1} v^{[2]},$$

which we can premultiply by $A$ and rewrite as

$$A^2 v^{[2]} = \left(\mu - \frac{\alpha}{\beta}\right) A v^{[2]} + \left(\frac{\alpha\mu - \gamma}{\beta}\right) v^{[2]},$$

so again $\mathbb{R}\{v^{[2]}, Av^{[2]}\}$ is $A$-invariant. ■

We introduce a normalization of $A$. With $k$ denoting the valency, as usual, define

$$\tilde{A} = A - kI$$

so that all row-sums of $\tilde{A}$ are zero. All entries of $\tilde{A}$ are nonnegative integers except on the diagonal (where they are nonpositive). The eigenvalues $\nu$ of $A$ shift to $\nu - k$ for $\tilde{A}$, with the same eigenvectors. The same holds for $\tilde{A}^{\mathrm{T}} = A^{\mathrm{T}} - kI$.

The signs of the diagonal entries are of little importance when constructing examples of networks with degenerate bifurcations, because we can always add a positive integer multiple $mI$ of the identity to make the diagonal entries nonnegative. The type of degeneracy is unchanged (since, for example, $\langle u, v_\star \tilde{A} v^{[2]}\rangle = \langle u, v_\star A v^{[2]}\rangle - k\langle u, v^{[3]}\rangle$, and so on), but eigenvalues shift by $m$. So without loss of generality we may assume all row-sums are zero, with no restriction on the signs of diagonal entries. By the Perron–Frobenius theorem, all eigenvalues of $\tilde{A}$ are either 0 (which is simple in the irreducible case) or have negative real parts.

Next, define

$$\mathcal{Y} = \sum_{\nu \neq k} E_\nu(A), \tag{6.13}$$

where $E_\nu(A)$ is the generalized eigenspace of $A$ for eigenvalue $\nu$. The space $\mathcal{Y}$ can be characterized as $w^\perp$, where $w$ is the eigenvector of $A^{\mathrm{T}}$ for eigenvalue $k$, or the image of $A - kI$. When $A$ is normalized to make $k = 0$, these spaces are the kernel and the image of $A$, respectively.

Lemma 6.18. *With the usual notation and assumptions, suppose that the network is path-connected. Then $v^{[2]}$ does not lie in $\mathcal{Y}$.*

*Proof.* Let $w$ be the eigenvector for $A^{\mathrm{T}}$ for eigenvalue $k$ (the valency). Since the network is path-connected, the Perron–Frobenius theorem (2.8) implies that we can choose $w$ so that all $w_j > 0$. If $v^{[2]} \in \mathcal{Y}$, then (2.2) implies that $\langle w, v^{[2]}\rangle = 0$. But

$$\langle w, v^{[2]}\rangle = \sum w_j v_j^2$$

and $w_j > 0$, $v_j^2 \geq 0$. Therefore $v_j = 0$ for all $j$, so $v = 0$, a contradiction. ■

**6.4. Proof of Theorem 6.14.** We are now ready to prove Theorem 6.14.

Suppose for a contradiction that the conclusion is false. Then $A$ is $(n-1)$-degenerate. By Lemma 6.16, the vectors $v^{[2]}, Av^{[2]}, L^{-1}v^{[2]}$ are linearly dependent. By Lemma 6.17, the subspace $\mathbb{R}\{v^{[2]}, Av^{[2]}\}$ is $A$-invariant. Moreover, it lies inside $u^{\perp} = E_k \oplus \mathcal{X}$.

By standard linear algebra, any $A$-invariant subspace of $P$ is of the form $\bigoplus_\nu X_\nu$, where $\nu$ runs through the distinct eigenvalues of $A$ and $X_\nu \subseteq E_\nu$. Since $\dim E_k = 1$, any $A$-invariant subspace of $E_k \oplus \mathcal{X}$ either contains $E_k$ or is contained in $\mathcal{X}$.

If $E_k \subseteq \mathbb{R}\{v^{[2]}, Av^{[2]}\}$, then $e \in \mathbb{R}\{v^{[2]}, Av^{[2]}\}$, so there exist $\rho, \sigma \in \mathbb{R}$ such that

$$e = \rho v^{[2]} + \sigma A v^{[2]}.$$

Take the componentwise product with $v$ to get

$$v = \rho v^{[3]} + \sigma v {\star} A v^{[2]}.$$

Then

$$\langle u, v\rangle = \rho\langle u, v^{[3]}\rangle + \sigma\langle u, v{\star}Av^{[2]}\rangle = 0$$

by 3-degeneracy (this is where we use $n \geq 4$). This is a contradiction.

Therefore $\mathbb{R}\{v^{[2]}, Av^{[2]}\} \subseteq \mathcal{X}$, so $v^{[2]} \in \mathcal{X}$, and hence $v^{[2]} \in \mathcal{Y}$, contrary to Lemma 6.18. ■

## 7. Degeneracy in regular 4-cell networks.

We preview the main result of this section.

*Example* 7.1. There exists a regular 4-cell network of valency 736 with a simple eigenvalue, for which the associated bifurcation is 3-degenerate. That is,

$$\langle u, v^{[2]}\rangle = 0, \tag{7.1}$$

$$\langle u, v^{[3]}\rangle = 0, \tag{7.2}$$

$$\langle u, v{\star}Av^{[2]}\rangle = 0, \tag{7.3}$$

$$\langle u, v{\star}L^{-1}v^{[2]}\rangle = 0. \tag{7.4}$$

The adjacency matrix is

$$A = \begin{bmatrix} 0 & 25 & 171 & 540 \\ 64 & 96 & 576 & 0 \\ 64 & 0 & 32 & 640 \\ 0 & 0 & 0 & 736 \end{bmatrix}. \tag{7.5}$$

The eigenvalues are 736, 176, 16, $-64$. The eigenvectors of $A, A^{\mathrm{T}}$ for eigenvalue $-64$ are

$$v = [3, 6, -2, 0]^{\mathrm{T}}, \qquad u = [-32, 5, 27, 0]^{\mathrm{T}}.$$

Obviously conditions (7.1), (7.2) hold. Further, $v^{[2]}$ is an eigenvector of $A$ (with eigenvalue 176), so (7.3), (7.4) follow from (7.2). Moreover, $\langle u, v^{[4]}\rangle = 4320 \neq 0$, so the bifurcation is 4-determined. ■

We describe the construction of Example 7.1 in more detail, to illustrate the method, which involves "bordering" a matrix to make its row-sums equal. We first construct a $3 \times 3$ nonnegative integer matrix satisfying (7.1), (7.2), (7.3), (7.4) but ignoring the condition that row-sums should be equal. Then we extend this matrix to a $4 \times 4$ nonnegative integer matrix in which all row-sums are equal, preserving (7.1), (7.2), (7.3), (7.4). No network constructed by bordering can be path-connected. We return to the path-connected case, which also permits degenerate bifurcations, in section 6.3.

The construction of Example 7.1 hinges on a simple result.

Proposition 7.2. *Suppose that $A$ is an $n \times n$ matrix of nonnegative integers having a simple real eigenvalue $\mu$. Let $v$ be an eigenvector of $A$ for eigenvalue $\mu$, and let $u$ be an eigenvector of $A^{\mathrm{T}}$ for eigenvalue $\mu$. Write $L = (A - \mu I_n)|_{u^\perp}$. Suppose that $u_1 + \cdots + u_n = 0$ and*

$$\langle u, v^{[2]}\rangle = \langle u, v^{[3]}\rangle = \langle u, v \star A v^{[2]}\rangle = \langle u, v \star L^{-1} v^{[2]}\rangle = 0.$$

*Then there exists an $(n+1) \times (n+1)$ matrix $\hat{A}$ of nonnegative integers, with constant row-sums, having a simple eigenvalue $\mu$, such that if $\hat{v}$ is an eigenvector of $\hat{A}$ for eigenvalue $\mu$, and $\hat{u}$ is an eigenvector of $\hat{A}^{\mathrm{T}}$ for eigenvalue 0, then*

$$\langle \hat{u}, \hat{v}^{[2]}\rangle = \langle \hat{u}, \hat{v}^{[3]}\rangle = \langle \hat{u}, \hat{v} \star \hat{A} \hat{v}^{[2]}\rangle = \langle \hat{u}, \hat{v} \star \hat{L}^{-1} \hat{v}^{[2]}\rangle = 0,$$

*where $\hat{L} = (\hat{A} - \mu I_{n+1})|_{u^\perp}$.*

*Proof.* The result follows by a series of routine calculations in block matrix form and is omitted. ■

We now sketch how Proposition 7.2 leads to Example 7.1. First, we select two vectors

$$v = [3, 6, -2]^{\mathrm{T}}, \qquad u = [-32, 5, 27]^{\mathrm{T}}.$$

These vectors are chosen so that $v, u \in \mathbb{Z}^3$ have small integer entries, and the conditions

$$0 = \langle u, e\rangle = \langle u, v^{[2]}\rangle = \langle u, v^{[3]}\rangle,$$

which are necessary for 3-degeneracy, are valid.

Next, we consider a general $3 \times 3$ matrix $A = (a_{ij})$. We assume that the required eigenvalue is $\mu = 0$, and impose the conditions $Av = 0$, $A^{\mathrm{T}} u = 0$, so that $v$ and $u$ are the appropriate eigenvectors. We also require $\langle u, v \star A v^{[2]}\rangle = 0$. Solving this linear system for the $a_{ij}$ we obtain six equations. (Section 8 gives more details for a similar calculation.) We find a nonnegative rational solution by inspection. Then we multiply $A$ by a positive integer to make the entries nonnegative integers. We also check that the 0 eigenvalue is simple. Experiment leads to the matrix

$$A = \begin{bmatrix} 64 & 25 & 171 \\ 64 & 160 & 576 \\ 64 & 0 & 96 \end{bmatrix}. \tag{7.6}$$

A direct check verifies (7.1), (7.2), (7.3), (7.4). The eigenvalues of $A$ are $240, 80, 0$. If $v$ is the eigenvector for eigenvalue 0, then $v^{[2]}$ is the eigenvector for eigenvalue 240. Finally, bordering (7.6) and subtracting $64I$ (the smallest diagonal entry) to lower the valency yields (7.5).

## 8. Higher degeneracies.

We apply (3.7) to the construction of networks with 4-degenerate bifurcations. The methods used in the quadratic and cubic cases yield the following theorem.

Theorem 8.1. *A necessary and sufficient condition for the Liapunov–Schmidt reduced map to vanish at degrees* 2, 3, *and* 4 *is that* $u$ *is orthogonal to the following expressions:*

$$v^{[2]}, \tag{8.1}$$

$$v^{[3]}, \quad v{\star}Av^{[2]}, \quad v{\star}L^{-1}v^{[2]}, \tag{8.2}$$

$$v^{[4]}, \quad v^{[2]}{\star}Av^{[2]}, \quad v{\star}Av^{[3]}, \quad Av^{[4]}, \qquad (Av^{[2]})^{[2]}, \tag{8.3}$$

$$v^{[2]}{\star}L^{-1}v^{[2]}, \quad (Av^{[2]}){\star}(L^{-1}v^{[2]}), \quad v{\star}(A(v{\star}L^{-1}v^{[2]})), \quad (L^{-1}v^{[2]})^{[2]}, \tag{8.4}$$

$$v{\star}L^{-1}v^{[3]}, \quad v{\star}L^{-1}(v{\star}Av^{[2]}), \quad v{\star}L^{-1}(v{\star}L^{-1}v^{[2]}). \tag{8.5}$$

As usual, $L^{-1}$ makes sense provided the terms to which it is applied are already on the list and we have arranged for them to be orthogonal to $u$. This is how the constructions proceed. To reduce the use of brackets, multiplication by $A$ and $L^{-1}$ takes precedence over $\star$ when interpreting the above expressions.

*Proof.* We prove this theorem in Appendix B. ■

We now exhibit a surprisingly low-valency example of 4-degeneracy for a 5-cell network.

*Example* 8.2. Let $\mathcal{G}$ be the regular 5-cell network of valency 390 with adjacency matrix

$$A = \begin{bmatrix} 54 & 0 & 0 & 54 & 282 \\ 184 & 9 & 5 & 192 & 0 \\ 168 & 9 & 21 & 144 & 48 \\ 52 & 0 & 2 & 48 & 288 \\ 0 & 0 & 0 & 0 & 390 \end{bmatrix}.$$

This has eigenvalues $390, 108, 18, 6, 0$, so all are real and simple. The 0-eigenvectors are

$$v = [-1, -2, 2, 1, 0]^{\mathrm{T}}, \qquad u = [-8, 1, -1, 8, 0]^{\mathrm{T}}.$$

Direct calculation shows that the terms of degrees 2 and 3 in the Liapunov–Schmidt reduction all vanish. (Since $v^{[2]}$ is an eigenvector, with eigenvalue 108, it is enough to check that $v^{[3]}$ is orthogonal to $u$.) The quartic terms that arise from $D^4f(v, v, v, v)$ also vanish. However, there may be other terms in the Liapunov–Schmidt reduction arising from the use of the implicit function theorem. We compute these terms below. Then we return to this example and show that *all* terms of degree 4 in the Liapunov–Schmidt reduction vanish. We also describe a pleasant algebraic feature of this example which explains why this happens.

The corresponding bifurcation is 4-degenerate. Since $\langle u, v^{[5]}\rangle = -48 \neq 0$, it is 5-determined. ■

Having chosen $u, v$ as above, the example is constructed along the usual lines: write down the conditions for $u, v$ to be 0-eigenvectors of $A^{\mathrm{T}}, A$, respectively; then require the five Liapunov–Schmidt reduced quartic terms (8.3) to vanish, along with all quadratic and cubic terms. The remaining terms are taken care of by Corollary 8.3. The resulting equations for the entries of $A$ are then examined, and a solution (not the most general) in positive rationals is derived by making judicious choices of some matrix entries. A suitable integer multiple of $A$ then has integer entries. The valency 390 can be reduced to 381 by subtracting $9I$. The critical eigenvalue then becomes $\mu = -9$.

Now we give the promised proof that the above example is 4-degenerate. We begin with an easy corollary of Theorem 8.1.

Corollary 8.3. (a) *If $v^{[2]}$ is an eigenvector of $A$, then all terms in* (8.3), (8.4) *are orthogonal to $u$ provided the term $v^{[4]}$ in* (8.3) *is orthogonal to $u$. Also, all terms in* (8.5) *similarly reduce to the first term $v_\star L^{-1}v^{[3]}$.*

(b) *If $v^{[2]}$ and $v^{[3]}$ are eigenvectors of $A$, then* (8.3), (8.4), (8.5) *are orthogonal to $u$ if $v^{[4]}$ is.* ■

We can now return to Example 8.2 and show that it is 4-degenerate. Because $v^{[2]} = [1,4,4,1,0]^{\mathrm{T}}$ is an eigenvector of $A$, Corollary 8.3 applies, and the only potentially nonzero quartic terms in the Liapunov–Schmidt reduction are those in the final row of the table. In fact, more is true: all three terms in (8.5) are scalar multiples of the first term $v_\star L^{-1}v^{[3]}$. (This is always true when $v^{[2]}$ is an eigenvector.)

We compute

$$Z = L^{-1}v^{[3]} = \left[-\frac{19}{108}, -\frac{37}{27}, \frac{35}{27}, \frac{17}{108}, 0\right]^{\mathrm{T}}.$$

A quick computation shows that $\langle u, v_\star Z\rangle = 0$.

Note that $v^{[3]}$ is not an eigenvector of $A$ (if it were, then this orthogonality would be obvious). However, it is a linear combination of eigenvectors with nonzero eigenvalues. In fact,

$$Z = -\frac{1}{108}v^{[2]} + \frac{1}{6}v^{[3]}$$

so that

$$v_\star Z = -\frac{1}{108}v^{[3]} + \frac{1}{6}v^{[4]},$$

which is a combination of vectors in $u^\perp$. Thus $A$ has a "triangular eigenstructure" with respect to the powers of $v$.

Triangular eigenstructure provides an effective way to construct feed-forward networks whose generic simple-eigenvalue bifurcations are highly degenerate. We use the following eigenvectors for $A, A^{\mathrm{T}}$:

$$v = [-1,-2,2,1,0]^{\mathrm{T}}, \tag{8.6}$$

$$u = [-8,1,-1,8,0]^{\mathrm{T}}, \tag{8.7}$$

which we choose because $\langle u, v^{[2]}\rangle = \langle u, v^{[3]}\rangle = \langle u, v^{[4]}\rangle = 0$. Now impose the conditions

$$\begin{array}{rcl} Av &=& 0, \\ Av^{[2]} &=& \pi v^{[2]}, \\ Av^{[3]} &=& \theta v^{[3]} + \sigma v^{[2]}, \\ A^{\mathrm{T}}u &=& 0. \end{array} \tag{8.8}$$

Choose $\pi = 1$, $\theta = \frac{1}{6}$, and leave $\sigma$ undetermined. Conditions (8.8) hold provided

$$\begin{aligned}
a_{11} &= -\frac{1}{18} + \frac{1}{8}a_{24} - \frac{1}{8}a_{34} + a_{44} + \frac{1}{3}\sigma,\\
a_{12} &= \frac{7}{48} - \frac{1}{32}a_{24} + \frac{1}{32}a_{34} - \frac{1}{4}a_{44} - \frac{1}{8}\sigma,\\
a_{13} &= \frac{17}{144} - \frac{1}{32}a_{24} + \frac{1}{32}a_{34} - \frac{1}{4}a_{44} + \frac{1}{24}\sigma,\\
a_{14} &= \frac{1}{8}a_{24} - \frac{1}{8}a_{34} + a_{44},\\
a_{21} &= -\frac{4}{9} + a_{24} + \frac{4}{3}\sigma,\\
a_{22} &= \frac{4}{9} - \frac{1}{4}a_{24} + \frac{1}{6}\sigma,\\
a_{23} &= -\frac{4}{9} + a_{34} + \frac{4}{3}\sigma,\\
a_{32} &= \frac{1}{3} - \frac{1}{4}a_{34} - \frac{1}{2}\sigma,\\
a_{33} &= \frac{5}{9} - \frac{1}{4}a_{34} + \frac{1}{6}\sigma,\\
a_{41} &= \frac{1}{18} + a_{44} + \frac{1}{3}\sigma,\\
a_{42} &= \frac{5}{48} - \frac{1}{4}a_{44} - \frac{1}{8}\sigma,\\
a_{43} &= \frac{19}{144} - \frac{1}{4}a_{44} + \frac{1}{24}\sigma.
\end{aligned}$$

These employ parameters $\sigma, a_{24}, a_{34}, a_{44}$. The other $a_{ij}$ will be nonnegative provided these parameters are small enough, except possibly for $a_{11}, a_{14}, a_{21}$. These terms suggest choosing $a_{34} = a_{24}$, $\sigma = \frac{1}{2}$, $a_{44} = \frac{1}{6}$, $a_{24} = \frac{1}{36}$. Scaling to remove denominators, we obtain

$$144A = \begin{bmatrix} 40 & 6 & 14 & 24 \\ 36 & 59 & 75 & 4 \\ 164 & 11 & 91 & 4 \\ 56 & 0 & 16 & 24 \end{bmatrix},$$

which can be bordered to produce an example with valency 270, having degeneracy at degrees 2, 3, 4 but no symmetry. Subtracting $24I$ reduces the valency to 246: explicitly, the matrix is

$$\begin{bmatrix} 16 & 6 & 14 & 24 & 186 \\ 36 & 35 & 75 & 4 & 96 \\ 164 & 11 & 67 & 4 & 0 \\ 56 & 0 & 16 & 0 & 174 \\ 0 & 0 & 0 & 0 & 246 \end{bmatrix}. \tag{8.9}$$

*Example* 8.4. An especially simple example has $\mathbb{Z}_2$ symmetry, which forces all terms of even degree to vanish. The adjacency matrix is

$$\begin{bmatrix} 3 & 1 & 0 & 17 & 63 \\ 32 & 0 & 4 & 48 & 0 \\ 48 & 4 & 0 & 32 & 0 \\ 17 & 0 & 1 & 3 & 63 \\ 0 & 0 & 0 & 0 & 84 \end{bmatrix} \tag{8.10}$$

with valency 84.

The corresponding network has a permutation symmetry (14)(23)(5). This changes the sign of the eigenvectors for eigenvalues $-12, 0$, while acting trivially on those for $-6, 24, 84$. It therefore induces the symmetry $x \mapsto -x$ on the reduced map, so all terms of even degree vanish.

The eigenvalues and corresponding eigenvectors are

$$\begin{aligned} \mu = -12 &: [-1, -2, 2, 1]^{\mathrm{T}}, \\ \theta = -6 &: [-1, -8, 8, 1]^{\mathrm{T}}, \\ \rho = 0 &: [1, -20, -20, 1]^{\mathrm{T}}, \\ \pi = 24 &: [1, 4, 4, 1]^{\mathrm{T}}. \end{aligned}$$

The bifurcation at $\mu = -12$ is degenerate at degree 4, but there is a nonzero degree 5 term $\langle u, v^{[5]} \rangle = -48$, so it is 5-determined.

This is the lowest valency found yet for a strictly 5-determined bifurcation. ■

**9. Quintic degeneracy.** A similar method leads to a 6-cell network that is strictly 6-determined. We record the results and sketch the method.

First we seek a 5-cell adjacency matrix $A$ having a simple 0 eigenvalue with eigenvector $v$, and corresponding $u$ for $A^{\mathrm{T}}$, such that $u$ is orthogonal to all of the vectors

$$e, \qquad v^{[2]}, \qquad v^{[3]}, \qquad v^{[4]}, \qquad v^{[5]}. \tag{9.1}$$

To do this choose $v$ and solve the linear system for $u$. Usually such a system is overdetermined, but there are cases when it is not. The simplest solution we have found is

$$\begin{aligned} v &= [-3, 4, 2, -2, 12]^{\mathrm{T}}, \\ u &= [512, -225, 1512, -1800, 1]^{\mathrm{T}}. \end{aligned}$$

Next, solve the following system (a special case of triangular eigenstructure) for the entries of $A$:

$$\begin{aligned} Av &= 0, \\ A^{\mathrm{T}}u &= 0, \\ Av^{[2]} &= pv^{[2]}, \\ Av^{[3]} &= qv^{[3]} + sv^{[2]}, \\ Av^{[4]} &= tv^{[4]} + mv^{[2]}, \end{aligned}$$

where $p, q, s, t, m \in \mathbb{R}$. Experiment leads to the choices

$$a_{25} = \frac{1}{30}, \qquad a_{35} = 0, \qquad a_{45} = 0, \qquad a_{55} = \frac{1}{30}$$

followed by

$$s = 0, \qquad m = 32, \qquad p = 17, \qquad q = 1, \qquad t = \frac{1}{2}.$$

Multiply $A$ by 3360 to remove denominators (still calling the result $A$):

$$A = \begin{bmatrix} 15760 & 1320 & 53361 & 32655 & 49 \\ 12288 & 4400 & 94080 & 85120 & 112 \\ 6912 & 2720 & 17808 & 12880 & 0 \\ 8960 & 2240 & 18480 & 9520 & 0 \\ 372736 & 233520 & 185472 & 94080 & 112 \end{bmatrix}$$

with eigenvalues $57120, -14560, 3360, 1680, 0$. In particular, 0 is simple and this is the eigenvalue corresponding to $v, u$. Now border $A$ to get

$$\hat{A} = \begin{bmatrix} 15760 & 1320 & 53361 & 32655 & 49 & 782775 \\ 12288 & 4400 & 94080 & 85120 & 112 & 689920 \\ 6912 & 2720 & 17808 & 12880 & 0 & 845600 \\ 8960 & 2240 & 18480 & 9520 & 0 & 846720 \\ 372736 & 233520 & 185472 & 94080 & 112 & 0 \\ 0 & 0 & 0 & 0 & 0 & 885920 \end{bmatrix} \tag{9.2}$$

with the extra eigenvalue (and valency) 885920.

The conditions on $A$ and its triangular eigenstructure imply that the associated bifurcation is 5-degenerate, and $\langle u, v^{[6]} \rangle = 4200$, so the bifurcation is strictly 6-determined.

The valency can be decreased to 885808 by subtracting $112I$. It is not clear whether significantly smaller-valency examples of quintic degeneracy exist: the Diophantine conditions imposed by (9.1) being orthogonal to $u$ seem to lead to fairly large integers, which create large denominators in rational solutions $A$. However, we doubt that the above example is best possible.

On the basis of these examples, we conjecture that for feed-forward networks it is possible to obtain arbitrarily high degeneracies by taking sufficiently many cells.

**10. Degeneracy in path-connected 5-cell networks.** In equivariant dynamics, simple-eigenvalue bifurcations are always 3-determined. We have seen that the analogue is false for feed-forward networks, but nothing yet seen in this paper rules out the possibility that a similar statement might apply to *path-connected* networks. We now prove that it does not.

The example we construct is probably the main result in this paper: there exists a family of path-connected 5-cell networks with 3-degenerate bifurcations. This shows that Theorem 6.14 cannot be improved to 3-determinacy (or to $(n-2)$-determinacy in general). However, we do not know whether the theorem is best possible for more than 5 cells. For example, it is not known whether a 6-cell path-connected regular network can be 4-degenerate.

Begin with a $5 \times 5$ adjacency matrix $A = (a_{ij})$, where $1 \leq i, j \leq 5$, and for convenience normalize $A$ so that rows sum to zero as in section 6.3. Now the $a_{ij}$ with $i \neq j$ are arbitrary and determine the diagonal terms $a_{ii}$. We wish to make all $a_{ij}$ with $i \neq j$ rational and nonnegative. After constructing $A$, we can add a suitable multiple of $I$ to make all entries nonnegative, without changing the degree of degeneracy.

By construction, $A$ has an eigenvalue 0. By the Perron–Frobenius theorem (2.6) all other eigenvalues are nonpositive, and for a path-connected network they are negative.

We choose vectors $u, v$ so that $u$ is orthogonal to $e, v^{[2]}, v^{[3]}$ and arrange for these to be eigenvalues of $A, A^{\mathrm{T}}$ for eigenvalue $\mu$. Experiment leads to the choices

$$v = [1, -2, 0, 2, -1]^{\mathrm{T}}, \qquad u = [-16, 1, 12, 3, 0]^{\mathrm{T}}.$$

To make these into eigenvectors we solve the equations $Av = \mu v$, $A^{\mathrm{T}}u = \mu u$ for $A$. Next, we solve the equation $\langle u, v \star Av^{[2]} \rangle = 0$ and seek to make the off-diagonal entries nonnegative. Then we compute $L^{-1}v^{[2]}$ by solving the equations $\langle u, z \rangle = 0$, $Az - \mu z = v^{[2]}$, so that $z = L^{-1}v^{[2]}$ (recall that $L^{-1}$ acts on $u^{\perp}$). Now choose $z_1$ to make the final cubic coefficient $\langle u, v \star L^{-1}v^{[2]} \rangle$ vanish. The adjacency matrix is now

$$A = \begin{bmatrix} -\frac{175}{192} & \frac{13}{192} & \frac{51}{64} & \frac{1}{32} & \frac{1}{64} \\ \frac{1}{6} & -\frac{11}{12} & \frac{3}{4} & 0 & 0 \\ \frac{1}{12} & \frac{1}{24} & -\frac{1}{8} & 0 & 0 \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{2} & -\frac{5}{6} & \frac{1}{12} \\ \theta & \phi & a_{53} & a_{54} & -(\theta + \phi + a_{53} + a_{54}) \end{bmatrix},$$

where

$$\theta = \frac{1}{624}(389 - 384a_{53} - 888a_{54}),$$
$$\phi = \frac{1}{312}(77 - 72a_{53} + 48a_{54}).$$

This provides a 2-parameter family of solutions provided we make both $\theta$ and $\phi$ nonnegative (diagonal terms do not matter since we can add a multiple of $I$ later). In particular, if $a_{53}, a_{54}$ are small and nonnegative, we get a solution. The choice

$$a_{53} = \frac{1}{32}, \qquad a_{54} = 0$$

leads to

$$A = \begin{bmatrix} -\frac{175}{192} & \frac{13}{192} & \frac{51}{64} & \frac{1}{32} & \frac{1}{64} \\ \frac{1}{6} & -\frac{11}{12} & \frac{3}{4} & 0 & 0 \\ \frac{1}{12} & \frac{1}{24} & -\frac{1}{8} & 0 & 0 \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{2} & -\frac{5}{6} & \frac{1}{12} \\ \frac{29}{48} & \frac{23}{96} & \frac{1}{32} & 0 & -\frac{7}{8} \end{bmatrix}.$$

Multiply by 192 to remove denominators, and add $176I$ to make the diagonal nonnegative. This leads to (we still call it $A$)

$$A = \begin{bmatrix} 1 & 13 & 153 & 6 & 3 \\ 32 & 0 & 144 & 0 & 0 \\ 16 & 8 & 152 & 0 & 0 \\ 16 & 32 & 96 & 16 & 16 \\ 116 & 46 & 6 & 0 & 8 \end{bmatrix}. \tag{10.1}$$

Since all entries of $A^2$ are nonzero, $A$ is path-connected. The eigenvalues of $A$ are 176, $-16$, and the roots of an irreducible cubic, which numerically are $32{\cdot}35$ and $-7{\cdot}67 \pm 5{\cdot}10i$. Direct calculations confirm that

$$\langle u, v^{[2]} \rangle = \langle u, v^{[3]} \rangle = \langle u, v{\star}Av^{[2]} \rangle = \langle u, v{\star}L^{-1}v^{[2]} \rangle = 0,$$

so the bifurcation is 3-degenerate. It must be 4-determined by Theorem 6.14, and in fact $\langle u, v^{[4]} \rangle = 48 \neq 0$.

The method (in particular the role of $z$) makes it clear that the zero entries in the matrix $A$ can be replaced by small positive numbers to create an all-to-all connected example of 3-degeneracy. (A network is *all-to-all connected* if any two distinct cells can be connected, in either direction, by a chain of arrows. Equivalently, all off-diagonal entries of $A$ are nonzero.) Specifically, let

$$a_{24} = a_{25} = a_{34} = a_{35} = a_{53} = a_{54} = \frac{1}{96},$$

and solve for $z$. Multiply by 6594048 and add $6273504I$ to get

$$A = \begin{bmatrix} 178875 & 432162 & 5241753 & 261873 & 158841 \\ 572400 & 0 & 5563728 & 68688 & 68688 \\ 480816 & 274752 & 5380560 & 68688 & 68688 \\ 549504 & 1099008 & 3297024 & 778464 & 549504 \\ 4021904 & 1724512 & 68688 & 68688 & 389712 \end{bmatrix} \tag{10.2}$$

with valency 6273504. The critical eigenvalue concerned becomes $-320544$.

The only 0 entry is on the diagonal, so it does not conflict with the usual definition of "all-to-all connected." In any case, adding $I$ increases all diagonal elements by 1, making all $a_{ij} > 0$. The valency becomes 6273505, and the degree of degeneracy is unchanged.

*Remark* 10.1. By Proposition 6.11, the bifurcating branch has growth rate $|x| \sim \lambda^{1/(r-1)}$ when the normal form is $g_r^{\pm}$, and this growth rate will be observed in any cell $c$ for which $v_c \neq 0$, where $v$ is the critical eigenvector.

In the example, $v$ has one zero component. However, by continuity, we can modify this example to remove the zero entry from $v$. If $v_i \neq 0$ for all $i$, then all 5 cells exhibit the anomalous $\frac{1}{3}$ power growth rate associated with the normal form $g_3$.

A relatively simple example with all entries of $v$ nonzero is

$$A = \begin{bmatrix} 7938 & 4284 & 37611 & 0 & 11907 \\ 13720 & 12740 & 35280 & 0 & 0 \\ 0 & 0 & 46305 & 0 & 15435 \\ 41160 & 20580 & 0 & 0 & 0 \\ 0 & 3054 & 43965 & 2058 & 12663 \end{bmatrix}$$

with valency 61740. Its eigenvalues are 61740, 0, and three roots of an irreducible cubic, so 0 is simple. The corresponding eigenvectors are

$$v = [-3, 6, -1, -6, 3]^{\mathrm{T}}, \qquad u = [-280, 15, 216, 49, 0]^{\mathrm{T}}.$$

This is 3-degenerate. Since $A^3$ has no zero entries, $A$ is path-connected.

Anomalous growth rates have also been found in Hopf bifurcation for feed-forward chains, but here different cells have different growth rates; see Elmhirst and Golubitsky [2]. ■

**Appendix A. Proof of Theorem 5.3.** We compute

$$\langle u, \mathrm{D}^3\Phi(v,v,v)\rangle \quad \text{and} \quad \langle u, \mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v,v))\rangle$$

separately and then substitute in (5.6).

Since $\Phi^3(x)$ is cubic, Taylor's theorem implies that $\mathrm{D}^3\Phi(v,v,v) = 6\Phi^3(v)$. Again $Av = \mu v$. Compute

$$\begin{aligned}
\Phi^3(v) &= Pv^{[3]} + Qv^{[2]}{\star}Av + Rv{\star}Av^{[2]} + Sv{\star}(Av)^{[2]} + TAv^{[3]} + U(Av){\star}(Av^{[2]}) + V(Av)^{[3]} \\
&= Pv^{[3]} + Qv^{[2]}{\star}\mu v + Rv{\star}Av^{[2]} + Sv{\star}(\mu v)^{[2]} + TAv^{[3]} + U(\mu v){\star}(Av^{[2]}) + V(\mu v)^{[3]} \\
&= Pv^{[3]} + \mu Qv^{[3]} + Rv{\star}Av^{[2]} + \mu^2 Sv^{[3]} + TAv^{[3]} + U(\mu v){\star}(Av^{[2]}) + \mu^3 Vv^{[3]} \\
&= (P + \mu Q + \mu^2 S + \mu^3 V)v^{[3]} + TAv^{[3]} + (R + \mu U)v{\star}Av^{[2]}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle u, \Phi^3(v)\rangle &= (P + \mu Q + \mu^2 S + \mu^3 V)\langle u, v^{[3]}\rangle + T\langle u, Av^{[3]}\rangle + (R + \mu U)\langle u, v{\star}Av^{[2]}\rangle \\
&= (P + \mu Q + \mu^2 S + \mu^3 V)\langle u, v^{[3]}\rangle + T\langle A^{\mathrm{T}}u, v^{[3]}\rangle + (R + \mu U)\langle u, v{\star}Av^{[2]}\rangle \\
&= (P + \mu Q + \mu^2 S + \mu^3 V)\langle u, v^{[3]}\rangle + T\langle \mu u, v^{[3]}\rangle + (R + \mu U)\langle u, v{\star}Av^{[2]}\rangle \\
&= (P + \mu Q + \mu T + \mu^2 S + \mu^3 V)\langle u, v^{[3]}\rangle + (R + \mu U)\langle u, v{\star}Av^{[2]}\rangle.
\end{aligned}$$

By (5.5),

$$\mathrm{D}^2\Phi(v,v) = 2\Phi^2(v,v) = 2[(a + \mu b + \mu^2 d)v^{[2]} + cAv^{[2]}].$$

Therefore

$$\begin{aligned}
L^{-1}E\mathrm{D}^2\Phi(v,v) &= 2(a + \mu b + \mu^2 d)L^{-1}Ev^{[2]} + 2cL^{-1}EAv^{[2]} \\
&= 2(a + \mu b + \mu^2 d)L^{-1}v^{[2]} + 2cL^{-1}Av^{[2]}
\end{aligned}$$

because we are assuming that $\langle u, v^{[2]}\rangle = 0$, so both $v^{[2]}$ and $Av^{[2]}$ are in $u^\perp$, on which $E$ is the identity.

Moreover, if $z \in u^\perp$, then $L^{-1}Az = L^{-1}(L + \mu I)z = z + \mu L^{-1}z$, so

$$L^{-1}Av^{[2]} = v^{[2]} + \mu L^{-1}v^{[2]}.$$

Therefore

$$L^{-1}E\mathrm{D}^2\Phi(v,v) = 2(a + \mu b + \mu c + \mu^2 d)L^{-1}v^{[2]} + 2cv^{[2]}.$$

To simplify the computation, temporarily let

$$\alpha = 2(a + \mu b + \mu c + \mu^2 d), \qquad \beta = 2c.$$

Then

$$L^{-1}E\mathrm{D}^2\Phi(v,v) = \alpha L^{-1}v^{[2]} + \beta v^{[2]}.$$

To compute $\mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v,v))$, observe that every quadratic form $q(x)$ over $\mathbb{R}$ determines a unique symmetric bilinear form $b(x,y)$ for which $q(x) = b(x,x)$. The proof follows directly from the polarization identity $2b(x,y) = q(x+y, x+y) - q(x,x) - q(y,y)$ (see, for example, Halmos [10, section 23, Exercise 6, page 38]). Since $\mathrm{D}^2\Phi$ is a symmetric bilinear form, we can write it down without further computation if we can specify a symmetric bilinear form $b(x,y)$ that reduces to $q(x)$ by setting $x = y$. But $\mathrm{D}^2\Phi = 2\Phi^2$ (see (5.2)), so there is an obvious choice:

$$\mathrm{D}^2\Phi(x,y) = 2ax{\star}y + b(x{\star}Ay + y{\star}Ax) + 2cA(x{\star}y) + 2d(Ax){\star}(Ay).$$

Now substitute

$$x = v, \qquad y = \alpha L^{-1}v^{[2]} + \beta v^{[2]}$$

leading to

$$\begin{aligned}\mathrm{D}^2\Phi(x,y) = {} & 2av{\star}(\alpha L^{-1}v^{[2]} + \beta v^{[2]}) + bv{\star}A(\alpha L^{-1}v^{[2]} + \beta v^{[2]}) \\ & + (\alpha L^{-1}v^{[2]} + \beta v^{[2]}){\star}Av + 2cA(v{\star}(\alpha L^{-1}v^{[2]} + \beta v^{[2]})) \\ & + 2d(Av){\star}A(\alpha L^{-1}v^{[2]} + \beta v^{[2]}).\end{aligned}$$

Using $AL^{-1}z = L^{-1}Az = z + \mu L^{-1}z$ and $Av = \mu v$, this expands to yield

$$\mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v,v)) = \gamma v^{[3]} + \delta v{\star}Av^{[2]} + \varepsilon v{\star}L^{-1}v^{[2]} + \zeta Av^{[3]} + \eta A(v{\star}L^{-1}v^{[2]}),$$

where

$$\begin{aligned}\gamma &= 2a\beta + b\alpha + \mu\beta + 2d\mu\beta + 2d\mu\alpha, \\ \delta &= b\alpha + b\beta, \\ \varepsilon &= 2a\alpha + b\alpha\mu + \alpha\mu + 2d\alpha\mu^2, \\ \zeta &= 2c\alpha + 2c\beta, \\ \eta &= 2c\mu\alpha.\end{aligned}$$

Therefore

$$\begin{aligned}\langle u, \mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v,v))\rangle &= \langle u, \gamma v^{[3]} + \delta v{\star}Av^{[2]} + \varepsilon{\star}L^{-1}v^{[2]} + \zeta Av^{[3]} + \eta A(v{\star}L^{-1}v^{[2]})\rangle \\ &= (\gamma + \zeta\mu)\langle u, v^{[3]}\rangle + \delta\langle u, v{\star}Av^{[2]}\rangle + (\varepsilon + \eta\mu)\langle u, v{\star}L^{-1}v^{[2]}\rangle,\end{aligned}$$

where we have used the identity $\langle u, Az\rangle = \langle A^{\mathrm{T}}u, z\rangle = \mu\langle u, z\rangle$.

Combining these results, we obtain

$$\begin{aligned} g_{xxx} &= \langle u, \mathrm{D}^3\Phi(v,v,v)\rangle - 3\langle u, \mathrm{D}^2\Phi(v, L^{-1}E\mathrm{D}^2\Phi(v,v))\rangle \\ &= (P + \mu Q + \mu^2 S + \mu T + \mu^3 V - 3(\gamma + \zeta\mu))\langle u, v^{[3]}\rangle \\ &\quad + (R + \mu U - 3\delta)\langle u, v{\star}Av^{[2]}\rangle - 3(\varepsilon + \eta\mu)\langle u, v{\star}L^{-1}v^{[2]}\rangle \end{aligned}$$

as required. The $\sigma_j$ can be computed explicitly, but the formulas are complicated and not required here. ■

**Appendix B. Proof of Theorem 8.1.** Again we use polarization [22, 23]. If $B$ is a symmetric $p$-linear form, then the corresponding $p$-ic form (symmetric $p$-tensor) is

$$\hat{B}(y) = B(y, \ldots, y).$$

Conversely, given a $p$-ic form $Q$, the corresponding symmetric $p$-linear form is

$$\check{Q}(y_1, \ldots, y_p) = \frac{1}{2^d d!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_p Q\left(\varepsilon_1 y_1 + \cdots + \varepsilon_p y_p\right).$$

Let $\mathcal{L}(Y^p, Z)$ denote the space of symmetric $p$-linear forms $Y \times \cdots \times Y \to Z$, and let $\mathcal{P}(Y, Z)$ denote the space of homogeneous $p$-ic forms $Y \to Z$. Then $\hat{}$ and $\check{}$ define an isomorphism between $\mathcal{L}(Y^p, Z)$ and $\mathcal{P}(Y, Z)$. We use this result to write down the form of higher derivatives of admissible maps. It implies that any symmetric multilinear form of the correct degree, which reduces to the admissible map when all variables are made equal, must be a scalar multiple of the $p$th derivative. It is easy to guess the form of such a map.

Assume that $\Phi$ is admissible. By Corollary 5.5, the terms in the first three rows (8.1), (8.2), (8.3) must be orthogonal to $u$, and from now on we assume this is the case. We have to deal with the terms (8.4), (8.5) involving $L^{-1}$. By admissibility,

$$\begin{aligned} \mathrm{D}^2\Phi(x,y) &= ax{\star}y + b[x{\star}Ay + Ax{\star}y] + cA(x{\star}y) + dAx{\star}Ay, \\ \mathrm{D}^3\Phi(x,y,z) &= P[x{\star}y{\star}z] + Q[x{\star}y{\star}Az + x{\star}Ay{\star}z + Ax{\star}y{\star}z] \\ &\quad + R[x{\star}(A(y{\star}z)) + y{\star}(A(x{\star}z)) + z{\star}(A(x{\star}y))] \\ &\quad + S[x{\star}Ay{\star}Az + Ax{\star}y{\star}Az + Ax{\star}Ay{\star}z] + TA(x{\star}y{\star}z) \\ &\quad + U[Ax{\star}A(y{\star}z) + Ay{\star}A(x{\star}z) + Az{\star}A(x{\star}y)] + V[Ax{\star}Ay{\star}Az], \end{aligned}$$

where $a, b, c, d, P, Q, R, S, T, U, V \in \mathbb{R}$.

Formulas (3.5), (3.6), (3.7) determine the coefficients of degree 2, 3, and 4 terms in the reduced map $g$, and all of these must vanish. A long but routine calculation shows that everything is a linear combination of the terms listed in the table. We describe the main points. We do not compute the exact linear combination that occurs, because we want all terms to be orthogonal to $u$, and this property does not depend on the coefficients. When performing this calculation we assume, inductively, that earlier expressions in the table are orthogonal to $u$. The following facts simplify the calculation:

(1) $Av = \mu v$, so any term involving $Av$ can be replaced by the corresponding one without the $A$.

(2) The identity $AL^{-1} = I + \mu L^{-1}$ replaces any term involving $AL^{-1}$ by a linear combination of two terms; in one we replace $AL^{-1}$ by $L^{-1}$, and in the other we omit it altogether.
(3) The maps $\mathrm{D}^2\Phi$ and $\mathrm{D}^3\Phi$ are symmetric.
(4) When considering terms of the form $L^{-1}EX$, we may assume inductively that $X$ lies in $u^{\perp}$, so the $E$ can be removed.
(5) $\langle u, AX\rangle = \langle A^{\mathrm{T}}u, X\rangle = \mu\langle u, X\rangle$, so the $A$ can be removed at this stage of any calculation.

The full calculation deals with each of the five terms in (3.7) in turn. Here we discuss only the first two.

(a) $\mathrm{D}^4\Phi(v, v, v, v)$.

This is a linear combination of terms in (8.3), so it is orthogonal to $u$ if those terms are, which we are assuming.

(b) $\mathrm{D}^3\Phi(v, v, L^{-1}E\mathrm{D}^2\Phi(v, v))$.

The proof of Theorem 5.1 shows that

$$\mathrm{D}^2\Phi(v, v)) = (a + 2\mu b + \mu^2 d)v^{[2]} + cAv^{[2]}.$$

So $L^{-1}E\mathrm{D}^2\Phi(v, v))$ is a linear combination of the expressions

$$L^{-1}Ev^{[2]}, \qquad L^{-1}EAv^{[2]}.$$

Inductively, $L^{-1}Ev^{[2]} = L^{-1}v^{[2]}$ since (8.1) makes $v^{[2]}$ orthogonal to $u$. Now $u^{\perp}$ is $A$-invariant, so (2) above implies that

$$L^{-1}EAv^{[2]} = L^{-1}Av^{[2]} = v^{[2]} + \mu L^{-1}v^{[2]}.$$

Therefore $\mathrm{D}^3\Phi(v, v, L^{-1}E\mathrm{D}^2\Phi(v, v))$ is a linear combination of the expressions

$$\mathrm{D}^3\Phi(v, v, v^{[2]}), \tag{B.1}$$

$$\mathrm{D}^3\Phi(v, v, L^{-1}v^{[2]}). \tag{B.2}$$

Case (B.1) leads to terms in (8.3).

In case (B.2), $\mathrm{D}^3\Phi(x, y, z)$ is a linear combination of eight terms, and we consider each in turn, substituting $x = v$, $y = v$, $x = v^{[2]}$. Initially we give details:

$$x{\star}y{\star}z = v{\star}v{\star}L^{-1}v^{[2]} = v^{[2]}{\star}L^{-1}v^{[2]},$$

which is new:

$$x{\star}y{\star}Az + x{\star}Ay{\star}z + Ax{\star}y{\star}z = v{\star}v{\star}AL^{-1}v^{[2]} + v{\star}Av{\star}L^{-1}v^{[2]} + Av{\star}v{\star}L^{-1}v^{[2]}.$$

But $AL^{-1}v^{[2]}$ is a combination of $L^{-1}v^{[2]}$ and $v^{[2]}$, and $Av = \mu v$, so no new terms arise:

$$\begin{aligned} &x{\star}(A(y{\star}z)) + y{\star}(A(x{\star}z)) + z{\star}(A(x{\star}y)) \\ &= v{\star}(A(v{\star}L^{-1}v^{[2]})) + v{\star}(A(v{\star}L^{-1}v^{[2]})) + L^{-1}v^{[2]}{\star}(A(v{\star}v)). \end{aligned}$$

There are two new terms $v\star(A(v\star L^{-1}v^{[2]}))$ and $Av^{[2]}\star(L^{-1}v^{[2]})$:

$$\begin{aligned} &x\star Ay\star Az + Ax\star y\star Az + Ax\star Ay\star z \\ &= v\star Av\star AL^{-1}v^{[2]} + Av\star v\star AL^{-1}v^{[2]} + Av\star Av\star L^{-1}v^{[2]} \end{aligned}$$

leading to no new terms:

$$A(x\star y\star z) = A(v\star v\star L^{-1}v^{[2]}) = A(v^{[2]}\star L^{-1}v^{[2]}),$$

which produces nothing new when we take the inner product with $u$ by (5) above:

$$\begin{aligned} &Ax\star A(y\star z) + Ay\star A(x\star z) + Az\star A(x\star y) \\ &= Av\star A(v\star L^{-1}v^{[2]}) + Av\star A(v\star L^{-1}v^{[2]}) + AL^{-1}v^{[2]}\star A(v\star v) \end{aligned}$$

and the only possible new term is $AL^{-1}v^{[2]}\star Av^{[2]}$. But by (2) this is a combination of $v^{[2]}\star Av^{[2]}$ and $L^{-1}v^{[2]}\star Av^{[2]}$, which are not new:

$$Ax\star Ay\star Az = Av\star Av\star AL^{-1}v^{[2]}$$

leads to nothing new.

This case is complete, and we have found three new terms

$$v^{[2]}\star L^{-1}v^{[2]}, \qquad v\star(A(v\star L^{-1}v^{[2]})), \qquad Av^{[2]}\star L^{-1}v^{[2]}$$

which occur in (8.4).

The other cases are similar and are omitted to save space. ■

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