Symmetries on the edge of chaos

Why should train wheels wear down symmetrically at high speeds but not at low speeds? And what has chaos got to do with it?

Mike Field and Martin Golubitsky

THE WORDS "symmetry" and "chaos" have well understood meanings in everyday language. Symmetry suggests balance, equal proportion or repetition—in short, it implies order. Chaos, on the other hand, suggests disorder: anarchy, confusion and turmoil. In their technical usage these words carry similar meanings, so it comes as something of a surprise to find physical systems that can be thought of as simultaneously having characteristics of both.

Nevertheless, mathematicians have recently tried merging symmetry and chaos, with two striking consequences. First, they found that they can generate exceptionally beautiful pictures. Secondly, the combination of symmetry and chaos provides an intriguing new description of how patterns can form in the physical world. These patterns do not appear instantaneously. Instead, they are the product of time averages—we call them "patterns on average". Such patterns can show how complicated motions which scientists describe as chaotic can combine with natural processes such as abrasion to produce an ordered pattern of wear which is an average over time.

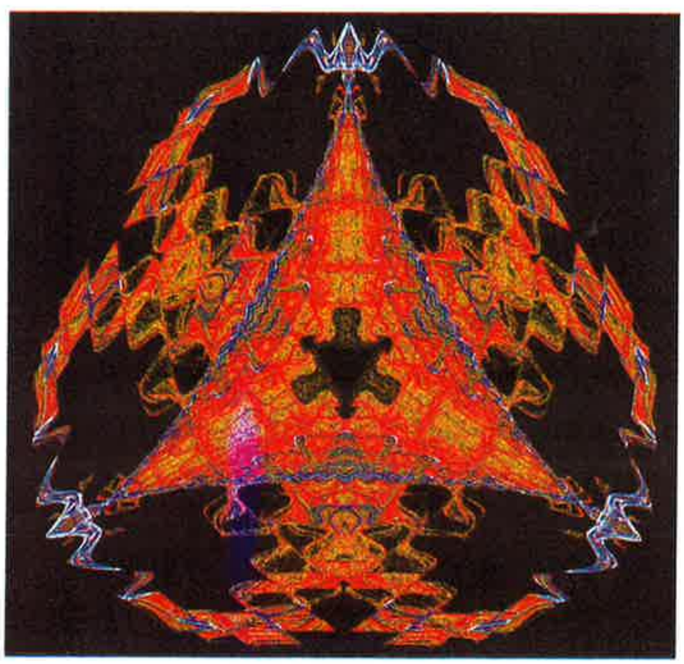
Though the mathematical ideas of symmetry and chaos seem like direct opposites, they both arise in attempts to answer simplified questions about complex scientific issues. Examples of such complex questions are how to describe atmospheric motion, or how to determine the world's population in 10 years' time. But while these questions are extraordinarily difficult, scientists have found that answers to simpler but related questions can give an insight into the more complicated ones. It is in the study of these simpler questions that the ideas of symmetry and chaos have proved useful. Before looking at how these ideas might be combined, it is worth looking briefly at how they have been used separately.

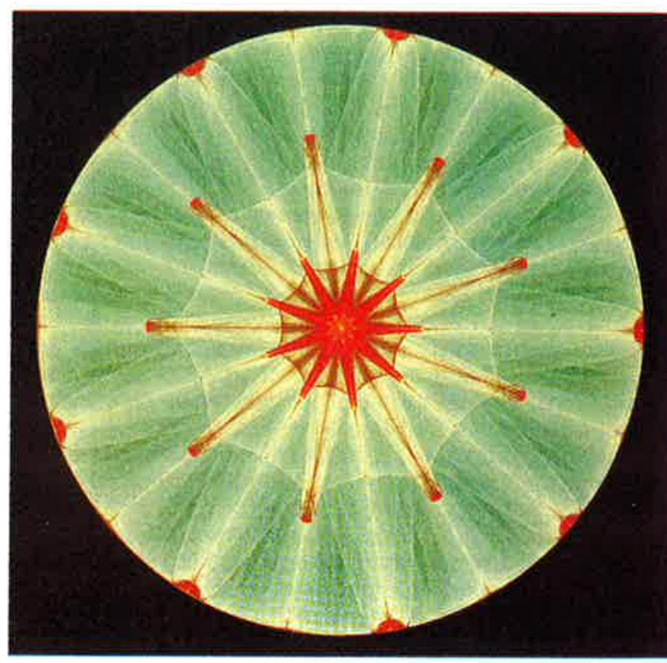
An object, a system, or something as abstract as a mathematical formula can have symmetry. Just as rotating a square through 90 degrees leaves it looking just as it did before, in the same way systematic changes to a mathematical formula can leave it looking the same as it was before. As a simple example, take x^2. Changing x to $-x$ in this formula gives $(-x)^2$, which is the same as x^2.

Impossible questions

The relationship between mathematical symmetries and patterns is suggested by models of the atmosphere. To make accurate predictions of atmospheric motion it would be necessary, as well as understanding how fluids move, to know the exact shape of the Earth, including the location of mountains and the boundaries between water and land, how much energy is transferred to the Earth by the Sun, the effects of the rotation of the Earth, and the gravitational effects of the Moon. Each of these questions is impossibly difficult to answer in practice.

Rather than tackling such difficult problems head-on, scientists often begin with simplified models. For example, it is a fair guess that the atmosphere's high-altitude jet streams arise in some way through the Earth's rotation. To test this hypothesis, scientists can construct a simplified laboratory experiment. Imagine cutting off the polar ice caps and straightening out the Earth to form the surface of a cylinder. Next, imagine sticking the atmosphere (which for convenience can be represented by water) between the Earth cylinder and a second, larger





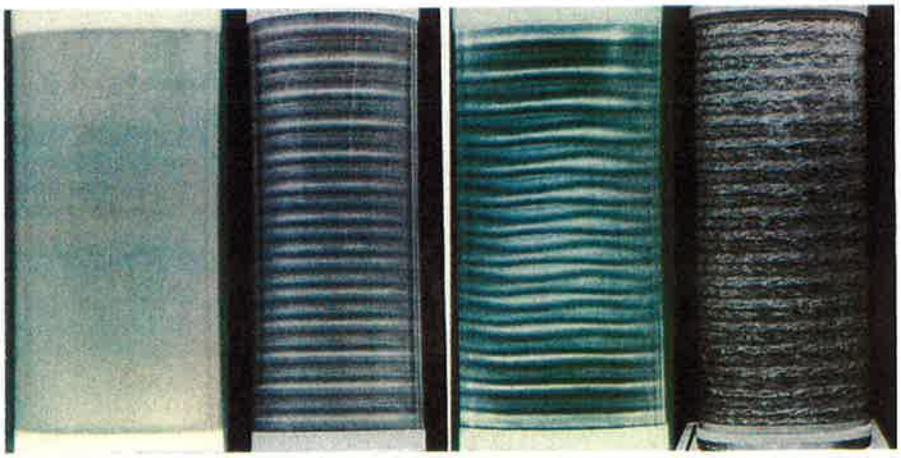

Harry L. Swinney

Figure 1 **Flow patterns formed in a fluid sandwiched between two rotating cylinders, one inside the other, are remarkably regular. As the rotation speed of the inner cylinder is increased (left to right) the flow becomes more turbulent. But even when it reaches a fully turbulent state that is chaotic, the ghost of a regular pattern remains (far right)**

cylinder. Will rotating these cylinders form "jet streams"? There are several examples of experiments that use this geometry—some of them better models of atmospheric motion than others.

One of them is the Couette-Taylor apparatus, which was developed in 1886 in Paris by Maurice Couette to study shear flow. Some of the most interesting experiments with the Couette-Taylor apparatus were done in the mid-1980s by Harry Swinney's team at the University of Texas at Austin. Figure 1 shows four pictures of experiments performed in this laboratory. To visualise the flow patterns, a syrup called kalliroscope, containing many little silver platelets, is added to the water. Because these platelets reflect light, the bright parts of the pictures indicate fluid flow that is mainly vertical while the dark portions indicate flows that are mainly horizontal. How are these remarkably regular patterns formed?

The first picture is of "Couette flow", which appears when the inner cylinder is rotated very slowly. The fluid particles all travel in circles about the cylinder axis, so there are no bright spots. When the speed of the inner cylinder is increased, Couette flow is replaced by "Taylor vortex flow", which resembles a pile of exactly similar doughnuts. In the search for evidence that jet streams are a product of rotating fluids, we seem to have hit the jackpot here. Though this experiment is too simple to enable mathematicians to make a direct link between the patterns seen in Taylor vortices and jet streams in the atmosphere, it does suggest that rotating fluids have a definite tendency to form the right sort of regular patterns.

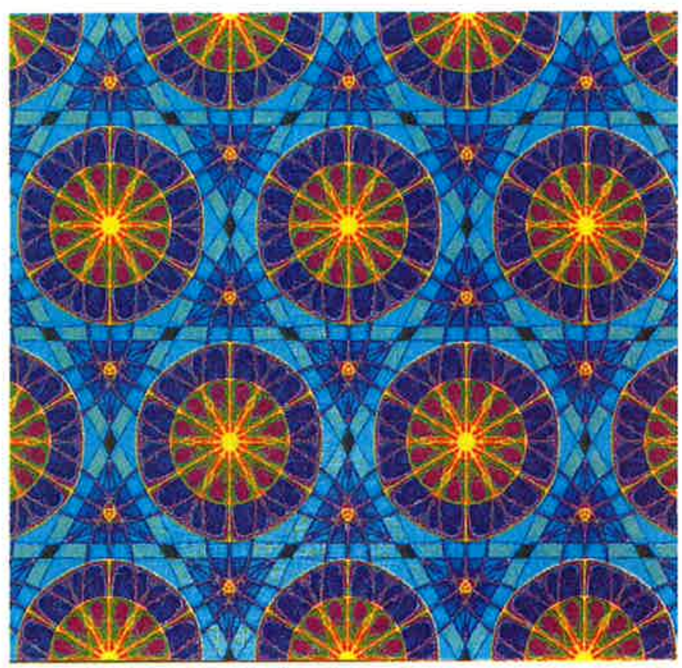
When the inner cylinder is speeded up, Taylor vortex flow is replaced by wavy vortex flow in which the doughnuts are wavy and move around the circumference of the cylinder. When the speed of the inner cylinder is further increased, the doughnuts become more and more irregular until finally they reach the state called "turbulent Taylor vortices". But the key thing to notice is that even though that state is turbulent, the remnants of a perfectly regular pattern appear to persist.

Go with the flow

The pictures in Figure 1 suggest great regularity. Indeed, much mathematical effort has been spent in the past decade showing that the regularity of these flows is intimately tied to the mathematical symmetries in the equations that describe the flows. In this, and in many other physical systems, the formation of regular patterns is found to be virtually synonymous with the presence of symmetry. But these patterns have been produced using rather simple motions. Could patterns be formed using more complicated, or even chaotic, dynamics?

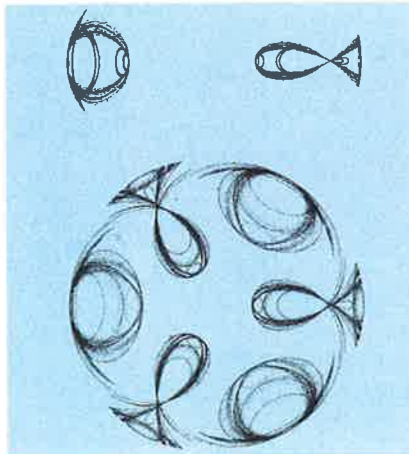
Scientists find chaotic dynamics useful in explaining observed phenomena such as the way populations grow. A major task in population dynamics is to find models that will accurately predict population sizes in the future. The simplest model for studying population dynamics is called the logistic equation. Its

Patterns on average: the outcome of iterating mathematically symmetrical equations can be plotted as points on a computer screen, then given a colour according to how often each point is "hit". The iterations are chaotic—each new point seems to be added at random—but symmetry emerges over a period of time


M. Field & M. Golubitsky

derivation is based on three premises: that next year's population can be computed from the knowledge of this year's; that births each year occur as a percentage of population size; and that there is a maximum population size that the environment can support. In the simplest case these assumptions lead to a quadratic polynomial equation (see Box).

A procedure called iteration is used to create dynamics from this equation. The idea is quite simple. From the population in year 1 the equation is used to compute the population in year 2. Applying the equation again gives the population in year 3, and so on. So the logistic equation can be used to compute the population for every future year, one year at a time. In 1976 Robert May observed that such a scheme can lead to at least three outcomes: populations tend to a fixed number, or they oscillate between a large number and a small, or they keep changing, admitting no pattern whatsoever. The meandering without pattern is called chaotic dynamics and it was a major triumph to observe that such a simple equation could produce such complicated dynamics. And since population evolution is at least as complicated as the logistic equation, we should not be surprised by very complicated changes in population sizes.



M. Field & M. Golubitsky

Figure 2 **Even though the symmetry of the equations does not change, the symmetry of the pictures can. In either case, the more points that are added, the more exact is the symmetry of the picture**

So in one simplified model, patterns form, and the mathematical basis for pattern formation is symmetry; in the other, iterating even simple equations can create very complicated chaotic motion. The picture of turbulent Taylor vortices in Figure 1 (right) shows that this flow has both of the properties already described, at least approximately. Its motion is too complicated to define in rigorous mathematical terms as chaos—though mathematicians are pretty sure that's what it is—but it also has something resembling a well-defined pattern. The properties of symmetry and chaos exist simultaneously in this flow.

But seeing evidence of simultaneous symmetry and chaos in this experiment does not mean the way it arises can be described mathematically. Even though the Couette-Taylor model is relatively simple (at least by comparison with the atmosphere), the accurate numerical simulation and mathematical analysis of turbulent Taylor vortices is beyond the current abilities of mathematicians. Even if such a simulation were possible, it is far from clear whether it would help to explain how symmetry and chaos apparently coexist in the experiment.

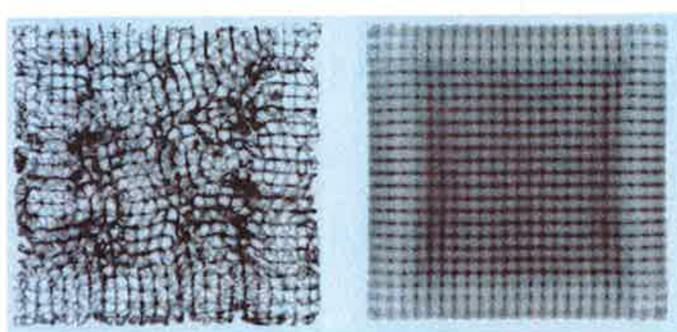
However, it ought to be possible to create a simpler mathematical model in which symmetry and chaos do coexist. Although very complicated, the equations that model Taylor-Couette flow have symmetries that correspond to the symmetries of the experimental apparatus. For example, the apparatus has rotational symmetry about its axis—and so do the mathematical equations. We have constructed simple equations that, like the logistic equation, have built-in symmetries. More specifically, we construct equations that have the symmetry of a regular polygon (see Box for details).

There is an immediate surprise—the symmetry of the pictures can change, even though the symmetry of the equations does not. Iterating these equations on a computer produces pictures that have different types of symmetry, depending on the particular equation used. Figure 2 shows the result of iterating an equation with triangular symmetry. For one value of the parameters in the equation, we obtain the top picture with just a reflectional, "mirror image", symmetry. When we chose a slightly different value of the parameters we obtained the lower picture with full triangular symmetry.

The pictures on the previous page are produced by choosing equations similar to those described in the Box, and iterating them to plot new points on the computer screen. Each pixel on the screen is given a colour depending on the number of times iterates fall onto that pixel. Roughly speaking, the colour of a pixel represents the chance that iterates will fall onto that pixel. Pixels with the same colour have equal chance of being visited during the iteration. This probabilistic interpretation means that the pictures are not exactly regular. However, the more iterations you use to obtain the picture the more exact will be the symmetry.

These computer experiments tell us two things. First, even though it is not apparent from the pictures, the iterations that produced them are chaotic. As the computer builds up the picture, points seem to be added at random. Only as the points accumulate does the overall pattern emerge. Symmetry and structure may arise from this chaos, but this is not seen over a short timescale. Rather, the symmetry of the pictures we see represents the average behaviour over time. Secondly, the resulting averaged picture often has the full symmetry of the equation, although it may have less. This is consistent with what is seen in the Couette-Taylor experiment. The different states pictured in Figure 1 have different symmetries even though the equations always have the same symmetry.

Do these mathematical models just generate pretty pictures or are they indicative of a more basic scientific principle? Theoretical results obtained during the past few years suggest that there is more to these pictures than pleasing shapes. Pascal Chossat of the University of Nice and one of the authors (MG) first discussed how symmetry might be seen in chaotic dynamics and pictures in 1988, and in 1991 Ian Stewart and Greg King at the University of Warwick looked at pictures of symmetric attractors and discussed how symmetry could be be found in chaotic time series. Symmetry and chaos is under study at the Universities of Houston, Warwick, Nice and Hamburg.



Jerry P. Gollub

Figure 3 **Symmetry and chaos can coexist in some physical systems. Averaging the chaotic vibrations of the surface of a fluid (left) over time produces a symmetric pattern (right)**

These ideas are even helping engineers to study the motion of train wheels. For a number of years, engineers have known that trains' wheels may move chaotically. Train wheels are designed to allow sideways motion: this is clearly needed on curves,

Pictures from equations

IF x_n denotes the ratio of the population in year n to the maximum population, then the logistic equation has the form $x_{n+1} = \mu x_n(1-x_n)$. The number μ is called the effective growth rate. (This number μ is restricted to lie between zero and four so that the value of the ratio x never exceeds one.)

To understand how population dynamics can be created from this equation, let's suppose that μ is 2 and that the population in year one, x_1, is 0·4. The equation predicts that the second year's population, x_2, will be 2 x 0·4 x (1–0·4) = 0·48 and that the third year's population, x_3, will be 2 x 0·48 x (1–0·48) = 0·4992. This process, iteration, can be continued indefinitely.

Now, what will happen if we continue iterating? In other words, what is the long-term behaviour of the population? These experiments can be done on a hand calculator. With $\mu = 2$, the population (ratio) x_n will approach 0·5 as the years go by. In short, the population will stabilise.

We can ask the same question for different values of μ. For example, if μ is 2·1, then the population will stabilise at 0·5238 .... However, if we take μ to be 3·1, a new type of behaviour occurs. After a few years, the population alternates between $x = 0·5580\ldots$ and 0·7646 ....

So even in this simple model, a population can oscillate. A high growth rate will result in a small population increasing, but a large population comes too close to the carrying capacity of the land and competition for resources forces the population down. What will happen when the effective growth rate, μ, is further increased?

It was Robert May in 1976, and later Mitchell Feigenbaum in 1978 who made the astonishing observation that when μ is large enough, say about 3·7, the population apparently never settles down. The population takes on various values, large and small, but there is no obvious pattern in the number of years separating transitions between large and small populations; there appears to be almost no correlation between the magnitude of one year's population and the next. Even though the population evolution was determined by this simple formula, it seemed to behave randomly. Worse still, small changes in the population at year one will lead to large changes in the populations predicted for any given year. This kind of motion is called chaotic.

Pictures of symmetric chaos (see previous page) can be obtained by using a formula that is similar to the logistic equation. Using the complex numbers z to denote points, we can define a formula with the symmetry of a regular m-sided polygon by $z_{n+1} = \mu z_n(1-|z|^2)+\gamma(\bar{z}_n)^{m-1}$. In this formula, triangular symmetry corresponds to $m = 3$, square symmetry to $m = 4$, pentagonal symmetry to $m = 5$, and so on. When this equation is used to produce a sequence of points z_n, the result can have a variety of symmetries, depending on the numbers m, μ and γ.

but less obviously it is also needed on straight stretches, as tracks are not always perfectly aligned. Chaotic motion of the wheels does not bode well for the comfort of passengers, nor for the durability of the wheels.

During the past year, three Danish engineers working at the Technical University of Denmark, Carsten Knudsen, Rasmas Feldberg and Hans True, have been studying a detailed model of the side-to-side motion of train wheels. Like all models of train wheels, theirs has a single left/right symmetry—reflectional symmetry about a line perpendicular to the axle that connects the wheels. The equations of motion that describe this system have reflectional symmetry too. But the solutions to these equations may or may not be symmetric—that is, the wheels need not stay "centred" on the rails.

Sideways bias

What the engineers found from their model is that at slow speeds, of about 40 kilometres per hour, the sideways motion of the wheels can be chaotic—but asymmetric. More precisely, the lateral position of the wheels relative to the track is biased either towards the right or towards the left. At such speeds, small imperfections in the track have the tendency to force the train to position its wheels in one preferred direction. This preference can cause one of the wheels, say the left, to wear down more quickly than the right; the difference in the diameters of the wheels then forces the wheels to turn towards the side with the preferential wear, in this case the left, which results in even greater differential wear.

At higher speeds of about 50 or 60 kilometres per hour, their model predicts a transition to motion that looks equally chaotic—if not more so—but that is symmetric on average. This discovery suggests that wheel repairs might be needed more often when trains travel at slower, not at faster, speeds. Of course, once such a phenomenon is identified, the design of the wheels can be adapted to guard against such differential wear. But it is an intriguing example of how symmetry creation in chaotic dynamics can turn up in practical engineering problems.

Another fluid dynamical experiment is based on one suggested by Michael Faraday in the last century as a way of studying how vibration causes surface waves. In this experiment, a layer of fluid in a container is vibrated up and down at constant amplitude and frequency. When the amplitude of the vibration is small, the surface remains flat. From the point of view of pattern formation, this surface is rather uninteresting and analogous to Couette flow. When the amplitude is increased, however, the surface resonates and deforms. The resulting surface waves often form symmetric patterns.

During the past decade, numerous groups have performed the Faraday experiment. The physicist Jerry Gollub and his group at Haverford College near Pennsylvania have studied its symmetry. Working with John David Crawford at the University of Pittsburgh, they have shown that symmetry is important in understanding the relationship between standard mathematical models of fluid dynamics and the Faraday experiment.

Within the past six months, motivated by the pictures of symmetric chaos, Gollub, with Bruce Gluckman, Philippe Marcq and Joshua Bridger have devised a new kind of experiment on the Faraday model to investigate the presence of patterns on average. They increased the amplitude of the vibration to a point where the wave patterns appeared to be varying chaotically. This is shown in Figure 3. When the deformed surface is concave up, transmitted light is dispersed and the area appears dark; when the deformed surface is concave down, transmitted light is focused and the area appears bright. Gollub and his colleagues then averaged the intensity of the transmitted light over time. They found a strikingly regular symmetric pattern appearing—just as was suggested by the simplified mathematical model based on iterations of symmetric equations.

In the past, most studies of pattern formation have relied on very simple dynamics. This merger of symmetry and chaos relies on complicated chaotic motion and so provides a new mechanism for pattern formation. The effect of such patterns will be seen most directly in processes where the time average itself is important such as in the chaotic trains or, indeed, in any process based on wear or growth.

In physics and engineering symmetry has proved important both for understanding observations of the physical world and in relating experiments to theory. Examples range from particle physics and the shape of galaxies through crystallography and the analysis of animal gaits. Until recently most of this work focused on systems that were not chaotic. But it seems that if observations are based on time averages, rather than being time instantaneous, symmetry can play an important role even in the understanding of chaotic systems. Order and chaos can coexist, entwined in the same natural phenomenon. □

Mike Field *and* **Martin Golubitsky** *are professors of mathematics at the University of Houston. Their recent book* Symmetry in Chaos *is published by OUP.*