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A DISCUSSION OF SYMMETRY AND SYMMETRY BREAKING

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There is an intimate relationship between singularity theory and steady state bifurcation theory. For the past several years we have been trying to make this relationship precise (see [**9**, **10**]) and have written several surveys on this material [**11**, **12**, **13**, **21**]. For the most part these reviews have been written for an applied audience as has the review by Ian Stewart [**28**] which includes several of our applications. In this review we want to emphasize those theoretical problems whose resolution would lead to interesting applied mathematics. These are problems about which we have limited knowledge and have made limited calculations. In particular, the problems revolve about the interaction of linear representations of compact Lie groups with the study of singularities of mappings and the notions of "symmetry breaking" that they engender. This review is divided into four parts: one states variable problems (the basic theory), bifurcation problems with symmetry, spontaneous symmetry breaking, and symmetry breaking in the equations. We shall give few proofs. The references include a complete listing of the applications which have followed from this point of view. It seems to us that the study of singularities of mappings which commute with a given representation of a compact Lie group is a rich field in need of further investigation.

1. The one variable theory. A *bifurcation problem* with one state variable is a germ $g(x, \lambda)$ in $\mathcal{E}_{x,\lambda}$, the ring of C^∞ germs based at the origin in $\mathbf{R} \times \mathbf{R}$. The variable x is a *state variable* and the parameter λ is called the *bifurcation parameter*. We shall call the (germ of the) zero set $g(x, \lambda) = 0$ the *bifurcation diagram* of g and a point (x_0, λ_0) where $g(x_0, \lambda_0) = 0$ a *solution* of g. There are two central issues in steady state bifurcation theory. The first issue concerns the computation of the number of solutions to g for each fixed λ, denoted by

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$N(\lambda, g)$. In the classical language a bifurcation occurs when the number $N(\lambda, g)$ changes as λ varies. Note that a necessary condition for bifurcation is $g = g_x = 0$. We call a point satisfying $g = g_x = 0$ a *singularity*.

DEFINITION 1.1. Two bifurcation problems g and h are *equivalent* if

$$g(x, \lambda) = S(x, \lambda)h(X(x, \lambda), \Lambda(\lambda))$$

where $S(0,0) > 0$, $X(0,0) = 0$, $\Lambda(0) = 0$, $X_x(0,0) > 0$, and $\Lambda'(0) > 0$.

Note that equivalence is just a special kind of contact equivalence where the bifurcation parameter λ is treated as distinguished in the change of coordinates. In particular, this leads to the formula

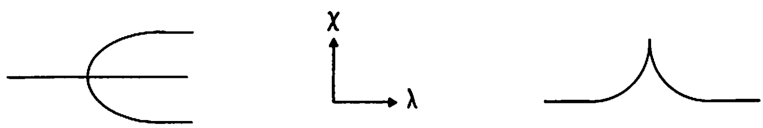
$$N(\lambda, g) = N(\Lambda(\lambda), h). \tag{1.1}$$

The singularity theory approach to the first problem is to show that under certain conditions on the Taylor expansion of g at the origin, g is equivalent to some simple normal form. In the terminology of C. T. C. Wall this is just the *recognition problem* in singularity theory. Simple results are

PROPOSITION 1.2. (a) (*The Pitchfork*). *If* $g = g_x = g_{xx} = g_\lambda = 0$, $g_{xxx} > 0$, *and* $g_{x\lambda} < 0$ *at* $(0,0)$ *then* g *is equivalent to* $x^3 - \lambda x$.

(b) (*The Winged Cusp*). *If* $g = g_x = g_\lambda = g_{xx} = g_{\lambda x} = 0$, $g_{xxx} > 0$, *and* $g_{\lambda\lambda} > 0$ *then* g *is equivalent to* $x^3 + \lambda^2$.

The bifurcation diagrams associated with the normal forms are given in Figure 1. Note that the pitchfork models a problem which transits from 1 to 3 solutions



(a) $x^3 - \lambda x = 0$ (b) $x^3 - \lambda^2 = 0$

FIGURE 1

while the winged cusp has 1 solution x for each λ. By the traditional definition the origin is not a bifurcation point for the winged cusp, yet clearly something singular is happening at $\lambda = 0$.

The second principal issue is the question: Can one classify the perturbed bifurcation diagrams of a given g? In bifurcation theory the problem goes under the name of *imperfect bifurcation* while in singularity theory one uses the name *universal unfolding*. We proceed with a discussion of these issues in the one variable case.

DEFINITION 1.3. The *restricted tangent space* of g, denoted by $RT(g)$, is the ideal generated by g, xg_x, and λg_x in $\mathcal{E}_{x,\lambda}$. We shall use the notation $\langle P_1, \ldots, P_k \rangle$ to indicate the ideal generated by $P_1, \ldots, P_k$ in $\mathcal{E}$. So $RT(g) = \langle g, xg_x, \lambda g_x \rangle$. The restricted tangent space is obtained by considering all germs of the form

$$\frac{d}{dt} S(x, \lambda, t) g(X(x, \lambda, t), \lambda) \Big|_{t=0} \tag{1.2}$$

where $X(0,0,t) = 0$, $S(x, \lambda, 0) = 1$ and $X(x, \lambda, 0) = x$.

REMARK. If one looks at germs f under right equivalence then the ideal corresponding to $RT(g)$ is $\mathfrak{M} \cdot \mathcal{J}(f)$ where $\mathcal{J}(f)$ is the Jacobian ideal of f and $\mathfrak{M}$ is the maximal ideal. The discussion which follows is an attempt to describe the basic determinacy result $\mathfrak{M}^{k+1} \subset \mathfrak{M}^2 \cdot \mathcal{J}(f) \Rightarrow f$ is k-determined with respect to right equivalence in a bifurcation theory format.

The basic analytic result about $RT(g)$ is

THEOREM 1.4. *Let $\mathcal{P} \subset \mathcal{E}_{x,\lambda}$ be a subspace. Suppose that $RT(g + p) = RT(g)$ for all p in $\mathcal{P}$. Then $g + p$ is equivalent to g for all p in $\mathcal{P}$.*

In trying to solve the recognition problem for bifurcation problems one needs to be able to compute a subspace $\mathcal{P}$ which satisfies the hypothesis of Theorem 1.4. There is a natural space to consider. One wants to know which terms in the Taylor expansion of h can be ignored in determining whether h is equivalent to a given g. Moreover, one would like to be able to find these terms by looking only at g. This consideration leads to the following definition.

DEFINITION 1.5. Let g be in $\mathcal{E}_{x,\lambda}$. Define $\mathcal{P}(g) = \{p \in \mathcal{E}_{x,\lambda} \mid RT(h + p) = RT(h)$ for all h equivalent to $g\}$.

It turns out that a notion of intrinsic ideals due to B. L. Keyfitz leads to a method of computing $\mathcal{P}(g)$ rather explicitly. In particular, one can prove that $\mathcal{P}(g)$ is a subspace of $\mathcal{E}_{x,\lambda}$ and apply Theorem 1.4 to $\mathcal{P}(g)$. We now explain this method.

DEFINITION 1.6. An ideal $\mathcal{I} \subset \mathcal{E}_{x,\lambda}$ is *intrinsic* if whenever g is in $\mathcal{I}$ and h is equivalent to g then h is in $\mathcal{I}$.

Note that the only intrinsic ideals relative to ordinary contact equivalence of finite codimension are $\mathfrak{M}^k$, the powers of the maximal ideal $\mathfrak{M}$. What makes this concept interesting is the existence of the distinguished parameter λ. For example $\langle \lambda \rangle$ is an intrinsic ideal. Since sums and products of intrinsic ideals are intrinsic one has that any ideal of the form

$$\mathfrak{M}^k + \mathfrak{M}^{k_1}\langle \lambda^{l_1} \rangle + \cdots + \mathfrak{M}^{k_s}\langle \lambda^{l_s} \rangle \tag{1.3}$$

where $0 < l_1 < \cdots < l_s < k$ and $k > k_1 + l_1 > \cdots > k_s + l_s$ are intrinsic and of finite codimension. Note that we make the convention $\mathfrak{M}^0 = \mathcal{E}_{x,\lambda}$ and observe that the restrictions on the l's and k's are made to avoid trivial redundancies. It is not difficult to prove

LEMMA 1.7. *$\mathcal{I}$ is an intrinsic ideal of finite codimension in $\mathcal{E}_{x,\lambda}$ iff $\mathcal{I}$ has the form (1.3).*

Given any ideal $\mathcal{J}$ in $\mathcal{E}_{x,\lambda}$ there is a largest intrinsic ideal contained in $\mathcal{J}$ (since sums of intrinsic ideals are intrinsic). We call this subideal the *intrinsic part* of $\mathcal{J}$ and denote it by $\mathrm{Itr}(\mathcal{J})$.

We can now state the main general result about $\mathcal{P}(g)$.

THEOREM 1.8. *Let g be in $\mathcal{E}_{x,\lambda}$. Assume that $RT(g)$ has finite codimension. Then $\mathcal{P}(g)$ is an intrinsic ideal and*

$$\mathrm{Itr}(\mathfrak{M} \cdot RT(g)) \subset \mathcal{P}(g) \subset \mathrm{Itr}(RT(g)). \tag{1.4}$$

We now compute several examples using Theorem 1.8.

$$\begin{aligned}\mathfrak{M}^4 + \mathfrak{M}^2 \cdot \langle \lambda \rangle &\subset \mathcal{P}(x^3 - \lambda x) \subset \mathfrak{M}^3 \cdot \langle \lambda \rangle, \\ \mathfrak{M}^4 + \mathfrak{M} \cdot \langle \lambda^2 \rangle &\subset \mathcal{P}(x^3 + \lambda^2) \subset \mathfrak{M}^3 + \langle \lambda^2 \rangle.\end{aligned} \tag{1.5}$$

Using the following lemma one can compute $\mathcal{P}(g)$ precisely for the examples in (1.5).

LEMMA 1.9. *If* $\lambda^2 g_{xx}$ *and* $\lambda x^2 g_{xx}$ *are in* $\mathfrak{M} \cdot RT(g)$ *then* $\operatorname{Itr}(\langle xg, \lambda g, x^2 g_x, \lambda g_x \rangle) \subset \mathcal{P}(g)$.

It follows from Lemma 1.9 that

$$\begin{aligned}\mathcal{P}(x^3 - \lambda x) &= \mathfrak{M}^4 + \mathfrak{M}^2 \cdot \langle \lambda \rangle + \langle \lambda^2 \rangle, \\ \mathcal{P}(x^3 + \lambda^2) &= \mathfrak{M}^4 + \mathfrak{M}^2 \cdot \langle \lambda \rangle.\end{aligned} \tag{1.6}$$

Now using (1.6) it is straightforward to prove Proposition 1.2. We indicate the details for the pitchfork. We want to find conditions on h to determine whether or not h is equivalent to the pitchfork $x^3 - \lambda x$. Using Taylor's Theorem one shows that

$$h = a + bx + c\lambda + dx^2 + ex\lambda + fx^3 + p(x, \lambda)$$

where $p \in \mathcal{P}(x^3 - \lambda x)$. It is clear that the constant and linear terms in h must vanish if h is equivalent to g. Also the first nonvanishing power of x is an invariant of equivalence. So we must assume $a = b = c = d = 0$ and $e \neq 0 \neq f$. In order to preserve orientation one must assume $f > 0$ and $e < 0$. After rescaling x and λ one has that h is equivalent to $x^3 - \lambda x + p$. The result follows from Theorem 1.4.

In order to resolve the issue of imperfect bifurcation, one must be able to find universal unfoldings relative to equivalence. The relevant definition is

DEFINITION 1.10. The *formal tangent space* of g, denoted by $T(g)$, is the subspace

$$\langle g, g_x \rangle + \mathcal{E}_\lambda\{g_\lambda\}$$

where $\mathcal{E}_\lambda$ is the space of smooth germs in the one variable λ.

The elements of $T(g)$ are obtained by computing all possible germs of the form

$$\frac{d}{dt} S(x, \lambda, t) g(X(x, \lambda, t), \Lambda(x, \lambda)) \Big|_{t=0}$$

where $S(x, \lambda, 0) = 1$, $X(x, \lambda, 0) = x$, and $\Lambda(\lambda, 0) = \lambda$.

Note that $T(g)$ differs from $RT(g)$ in two regards. First we have allowed proper changes of coordinates in the λ-variable (respecting the notion of equivalence, of course) and we have permitted the change of coordinates to move the origin when $t \neq 0$. Observe that $T(g)$ is not, in general, an ideal.

The basic result about unfoldings is

THEOREM 1.11. *Let $f(x, \lambda, \alpha) \in \mathcal{E}_{x,\lambda,\alpha}$ be a k-parameter unfolding of $g(x, \lambda)$ (i.e. $g(x, \lambda) = f(x, \lambda, 0)$ as germs and $\alpha \in \mathbf{R}^k$). Then f is a universal unfolding of g if and only if*

$$\mathcal{E}_{x,\lambda} = T(g) + \mathbf{R}\left\{\frac{\partial f}{\partial \alpha_1}(x, \lambda, 0), \ldots, \frac{\partial f}{\partial \alpha_k}(x, \lambda, 0)\right\}$$

where the notation $\mathbf{R}\{p_1, \ldots, p_k\}$ means the vector subspace of $\mathcal{E}_{x,\lambda}$ generated by $p_1, \ldots, p_k$.

The following proposition due to J. Damon resolves the issue of when a germ g has finite codimension.

PROPOSITION 1.12. codim $T(g)$ *is finite if and only if* codim $RT(g)$ *is finite.*

Now define the *codimension of* g to be the codimension of $T(g)$ in $\mathcal{E}_{x,\lambda}$. Theorem 1.11 implies that a germ has a universal unfolding precisely when it has finite codimension.

It is easy to compute

$$\begin{aligned} T(x^3 - \lambda x) &= \mathfrak{M}^3 + \mathfrak{M} \cdot \langle \lambda \rangle + \mathbf{R}\{3x^2 - \lambda, x\}, \\ T(x^3 + \lambda^2) &= \mathfrak{M}^3 + \langle \lambda^2 \rangle + \mathbf{R}\{x^2, \lambda\}. \end{aligned} \tag{1.7}$$

It follows from (1.7) that

PROPOSITION 1.13. (a) $x^3 - \lambda x + \beta x^2 + \alpha$ *is a universal unfolding of* $x^3 - \lambda x$.
(b) $x^3 + \lambda^2 + \alpha + \beta x + \gamma x \lambda$ *is a universal unfolding of* $x^3 + \lambda^2$.

We have one theoretical issue left to resolve. Suppose that one has a universal unfolding $f(x, \lambda, \alpha)$ of a germ $g(x, \lambda)$. (WLOG, one can assume that f and g are polynomials.) How does one actually solve $f(x, \lambda, \alpha) = 0$ for various choices of α? Part of the answer is given by the following theorem. First we make some definitions.

$\mathfrak{B} = \{\alpha \mid \exists x_0, \lambda_0 \text{ with } f = f_x = f_\lambda = 0 \text{ at } (x_0, \lambda_0, \alpha)\}$,
$\mathfrak{H} = \{\alpha \mid \exists x_0, \lambda_0 \text{ with } f = f_x = f_{xx} = 0 \text{ at } (x_0, \lambda_0, \alpha)\}$,
$\mathfrak{DL} = \{\alpha \mid \exists x_1 \neq x_2, \lambda_0 \text{ with } f = f_x = 0 \text{ at } (x_i, \lambda_0, \alpha)\}$,
$\Sigma = \mathfrak{B} \cup \mathfrak{H} \cup \mathfrak{DL}$.

THEOREM 1.14. *Σ is (the germ of) a codimension one semialgebraic variety in $\mathbf{R}^k$. If α_1 and α_2 are in the same connected component of $\mathbf{R}^k \sim \Sigma$, then $f(\cdot, \cdot, \alpha_1)$ and $f(\cdot, \cdot, \alpha_2)$ are equivalent.*

To make this theorem precise, one would have to specify definite neighborhoods of the origin on which the equivalences hold (see **[9]**).

In Figures 2 and 3 we present the computation of Σ for the two examples of universal unfoldings given in Proposition 1.13.

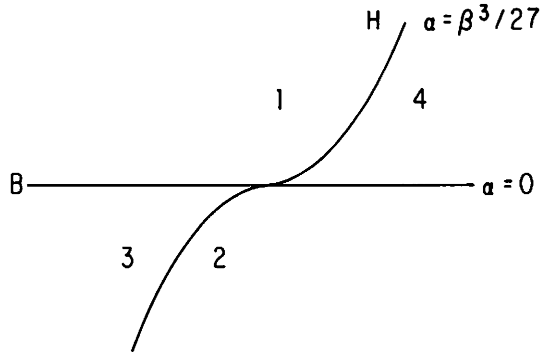


FIGURE 2. Σ for the pitchfork

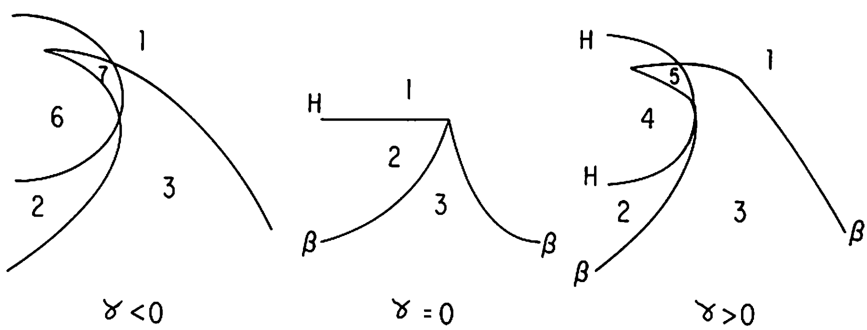


FIGURE 3. Σ for the winged cusp

The drawing for the winged cusp in Figure 3 is given by sections of γ = constant. In each figure the connected components of the complement of Σ are enumerated. The perturbed bifurcation diagrams which correspond to the various components are given in Figures 4 and 5.

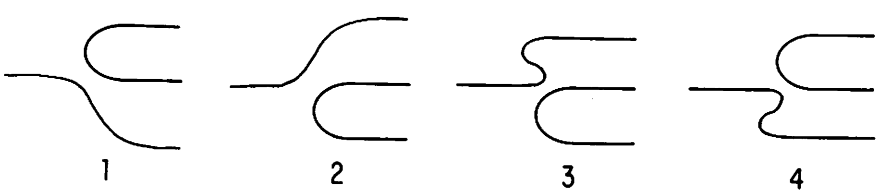


FIGURE 4. Perturbations of the pitchfork

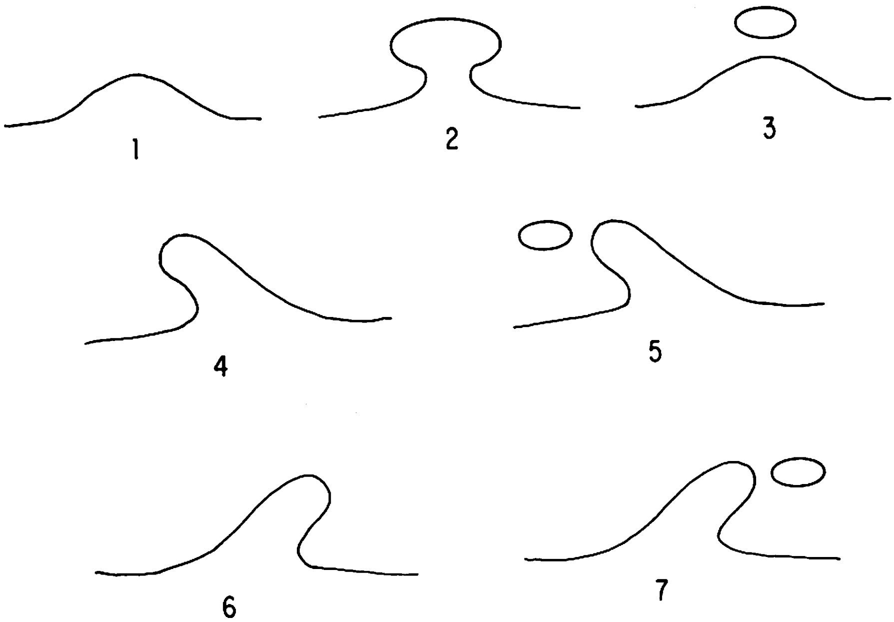


FIGURE 5. Perturbations of the winged cusp

For an application of the winged cusp in a chemical system see Golubitsky and Keyfitz [**6**] or the surveys [**12**, **13**, **28**].

2. Bifurcation with symmetry. The imposition of symmetry is a common thread in the idealization of many problems in classical applied mathematics. For example, in the planar theory of buckling of a column one implicitly imposes a reflection ($\mathbf{Z}_2$) symmetry in the assumption that buckling to the left or right has the same potential energy. If one allows the column to buckle in three dimensional space then one obtains a symmetry group $O(2)$. As another example thermal convection in the molten inner layer of the Earth is first approximated by convective fluid flow between two spherical shells. If the rotation of the Earth is neglected (after scaling it is a "small" parameter) one obtains a problem with $O(3)$ symmetry. If the rotation is included then the resulting mathematical problem has $\mathrm{SO}(2) + \mathbf{Z}_2$ symmetry. A third example is the buckling of a rectangular plate. One obtains a problem with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry while in the buckling of a triangular beam one has the permutation group S_3 (acting as symmetries of an equilateral triangle) as a group of symmetries. The list goes on (see Sattinger [**19**, **20**]).

The local steady state bifurcation problems which are derived from the above examples can, after the imposition of some classical applied mathematics—the Liapunov-Schmidt method—be reduced to the following situation:

Let g be a germ of a mapping of $\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ based at the origin. Let Γ be a compact Lie group acting linearly on $\mathbf{R}^n$—in fact, we may assume that $\Gamma \subset O(n)$

is a subgroup. Then g is a *bifurcation problem with symmetry group* Γ if

$$g(\gamma x, \lambda) = \gamma g(x, \lambda) \quad \forall \gamma \in \Gamma;$$

that is, g commutes with Γ. Let $\vec{\mathcal{E}}(\Gamma)$ denote the space of all germs g which commute with the action of Γ.

The first observation about $\vec{\mathcal{E}}(\Gamma)$ is that it is a module over the ring of invariant functions $\mathcal{E}(\Gamma)$. More precisely, $f(x, \lambda)$ is in $\mathcal{E}(\Gamma)$ if $f(\gamma x, \lambda) = f(x, \lambda)$ for all $\gamma \in \Gamma$. Also, if f is invariant and g is in $\vec{\mathcal{E}}(\Gamma)$ then $f \cdot g$ is in $\vec{\mathcal{E}}(\Gamma)$. Using theorems of Schwarz **[25]** and Poenaru **[18]** it is possible to calculate this module structure explicitly for a number of examples.

(1) $\Gamma = \mathbf{Z}_2$ acting on $\mathbf{R}$. $\mathcal{E}(\mathbf{Z}_2)$ consists of even functions and $\vec{\mathcal{E}}(\mathbf{Z}_2)$ consists of odd functions. Hence g has the form $g(x, \lambda) = a(u, \lambda)x$ where $u = x^2$.

(2) $\Gamma = O(2)$ acting on $\mathbf{R}^2$. $\mathcal{E}(O(2))$ consists of all smooth functions of the form $f(x^2 + y^2)$ and mappings $g: \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ in $\vec{\mathcal{E}}(O(2))$ having the form $g(x, y, \lambda) = a(u, \lambda)(x, y)$ where $u = x^2 + y^2$.

(3) $\Gamma = S_3$ acting on $\mathbf{R}^2 \cong \mathbf{C}$ as symmetries of the triangle. The action of S_3 is generated by $z \to \bar{z}$ and $z \to e^{i\theta}z$ where $\theta = 2\pi/3$. Invariant functions have the form $f(u, v)$ where $u = |z|^2$ and $v = \mathrm{Re}(z^3)$. Mappings g in $\vec{\mathcal{E}}(S_3)$ have the form $g(z, \lambda) = a(u, v, \lambda)z + b(u, v, \lambda)\bar{z}^2$ where a and b are (real-valued) smooth germs.

In a way analogous to the theory outlined in §1, one can define equivalence and prove unfolding theories for bifurcation problems with symmetries (see **[10]**.) The basic definition is

DEFINITION 2.1. Let g and h be bifurcation problems with symmetric group Γ, i.e. $g, h \in \vec{\mathcal{E}}(\Gamma)$. Then g and h are *Γ-equivalent*, i.e.

$$g(x, \lambda) = S(x, \lambda)h(X(x, \lambda), \Lambda(\lambda)),$$

and the equivalence respects Γ. Here $S(x, \lambda)$ is an $n \times n$ matrix satisfying $S(\gamma x, \lambda)\gamma = \gamma S(x, \lambda)$ where $\gamma \in \Gamma \subset O(n)$ is viewed as a matrix and $X(\gamma x, \lambda) = \gamma X(x, \lambda)$. Note that $S(0, 0)$ commutes with Γ. We put a further restriction on S as follows. Let $L(\Gamma)$ be the group of invertible $n \times n$ matrices which commute with Γ and let $L^0(\Gamma)$ be the connected component of the identity in $L(\Gamma)$. Assume that $S(0, 0)$ and $(d_x X)(0, 0)$ are in $L^0(\Gamma)$ where $d_x X$ is the $n \times n$ Jacobian matrix of X obtained by differentiation in the x-directions only.

As examples, we describe the bifurcation problems with Γ-codimension 0 in the above cases.

(1) $\Gamma = \mathbf{Z}_2$, $g = (\pm u \pm \lambda)x = \pm x^3 \pm \lambda x$, the pitchfork.

(2) $\Gamma = O(2)$, $g = (\pm u \pm \lambda(x, y) = (\pm(x^2 + y^2) \pm \lambda)(x, y)$, the pitchfork of revolution.

(3) $\Gamma = S_3$; $g(z, \lambda) = \pm\lambda z \pm \bar{z}^2$.

Note that if g commutes with Γ then g vanishes on orbits of Γ. It is therefore desirable to draw—schematically—the bifurcation diagrams $g = 0$ in the space of orbits (see Figure 6). Below each diagram in orbit space we give the diagram in the appropriate $\mathbf{R}^n \times \mathbf{R}$ space. The diagrams for $\mathbf{Z}_2$ and $O(2)$ symmetry are fairly obvious. We concentrate on the S_3 case.

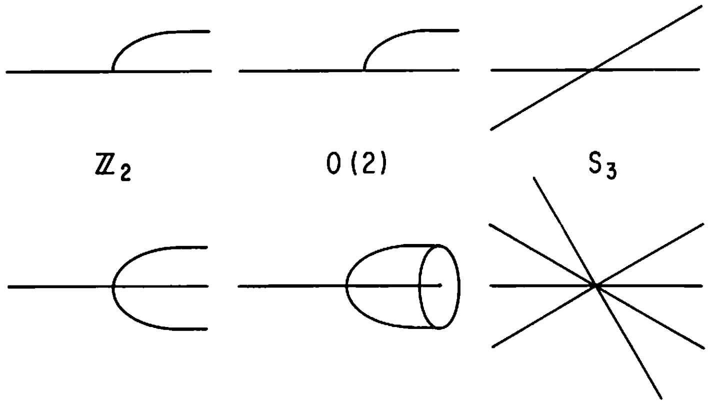


FIGURE 6. Various codim $\Gamma = 0$ bifurcation problems

We assume that $g\colon \mathbf{C} \times \mathbf{R} \to \mathbf{C}$ has the form

$$g(z, \lambda) = a(u, v, \lambda)z + b(u, v, \lambda)\bar{z}^2,$$

where $u = |z|^2$ and $v = \operatorname{Re}(z^3)$. There are three types of solutions:

(I) The trivial solution $z = 0$.

(II) Solutions where z and $\bar{z}^2$ are nonzero and parallel in the complex plane; that is z^3 is real. Note that z^3 is real on the three lines containing the cube roots of unity. These lines are mapped onto each other by the group S_3. Since g vanishes on orbits one need only find those solutions when $\operatorname{Im}(z) = 0$. Writing $z = x + iy$ and setting $y = 0$ one finds such solutions by solving $a + xb = 0$.

(III) Solutions where z and $\bar{z}^2$ are not parallel are given by $a = b = 0$. Recall that the isotropy group of an x in $\mathbf{R}^n$ is the subgroup $\Sigma_x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\}$. The isotropy subgroups of solutions of type (I), (II), (III) are S_3, $\mathbf{Z}_2$, $\{1\}$ respectively. Note that $\mathbf{Z}_2$ is generated by the element $z \to \bar{z}$ in S_3.

For the normal form $g = \bar{z}^2 - \lambda z$ there are no solutions of type (III) as $b \equiv 1 \neq 0$ and there is one nontrivial solution of type (II) given by $x = \lambda$, $y = 0$. Thus the orbit space bifurcation diagram is the one in Figure 6. Since each solution of type (II) has isotropy group $\mathbf{Z}_2$, it follows that there are three solutions in each orbit. Hence the actual bifurcation diagram $g = 0$ has 4 lines in $\mathbf{R}^2 \times \mathbf{R}$, the trivial line $z = 0$ and the 3 symmetry related nontrivial lines. The S_3 symmetric bifurcation problem given here along with a more degenerate g is described in **[14]**. These bifurcation problems also appear in work of Ball and Schaeffer **[1]** and Buzano, Geymonat, and Poston **[2]**.

This example leads to some interesting observations about the nature of spontaneous symmetry breaking which we describe in the next section.

There is one unresolved question which comes directly from the definition of Γ-equivalence.

In applications one would like to make as many germs as possible equivalent while not changing the inherent structure of the problem. In the case of Γ-equivalence, the following question arises: "Are there more equivalences of bifurcation problems which leave $\vec{\mathcal{E}}(\Gamma)$ invariant than just Γ-equivalences?" The answer is surely yes. Let $S(x, \lambda)$ be an $n \times n$ matrix in Γ for each (x, λ). Then

$$S(x, \lambda)^{-1} g(S(x, \lambda)x, \lambda) = g(x, \lambda) \tag{2.1}$$

for every g commuting with Γ. Of course such Γ-equivalences do not make more g's equivalent than were already equivalent by Γ-equivalences.

In his thesis [16] Hummel observed that there are equivalences of the form

$$h(x, \lambda) = T(x, \lambda)g(x, \lambda) \tag{2.2}$$

such that h is in $\vec{\mathcal{E}}(\Gamma)$ whenever g is in $\vec{\mathcal{E}}(\Gamma)$ yet T does not satisfy the equivariance condition $T(\gamma x, \lambda)\gamma = \gamma T(x, \lambda)$. That is, T is not a Γ-equivalence. However, he also observed that if h and g are related as in (2.2) and Γ is compact then (by integrating over Γ) one can replace T by an equivariant S. So no new germs are equivalent.

The question remains: Suppose there is an equivalence pair $T(x, \lambda)$, $Y(x, \lambda)$ such that

$$h(x, \lambda) = T(x, \lambda)g(Y(x, \lambda), \lambda) \tag{2.3}$$

is in $\vec{\mathcal{E}}(\Gamma)$ for every g in $\vec{\mathcal{E}}(\Gamma)$. Is there a Γ-equivalence S, X such that

$$h(x, \lambda) = S(x, \lambda)g(X(x, \lambda), \lambda)?$$

We conjecture that the answer is yes. More precisely, let $\mathcal{K}(\Gamma)$ be the group of Γ-equivalences and let $\hat{\mathcal{K}}(\Gamma)$ be the group of equivalences as in (2.3) which leave $\vec{\mathcal{E}}(\Gamma)$ invariant. Let $\hat{\Gamma}$ be the group of smooth mappings of $(x, \lambda) \to \Gamma$ which act on g trivially as in (2.1). Let $\hat{S}(\Gamma)$ be the group of invertible matrices found by Hummel which leave $\vec{\mathcal{E}}(\Gamma)$ invariant.

Conjecture 1. $\hat{\mathcal{K}}(\Gamma)$ is the (semidirect) product $\tilde{\mathcal{K}}(\Gamma)$ where $\tilde{\mathcal{K}}(\Gamma) \equiv \mathcal{K}(\Gamma) \times \hat{\Gamma} \times \hat{S}(\Gamma)$.

We can more or less prove this conjecture in the case of $\Gamma = O(2)$ acting on $\mathbf{R}^2$ and will use this fact in §4.

We end this section with a discussion of linearized stability. Consider the system of ordinary differential equations

$$\frac{dx}{dt} + g(x, \lambda) = 0.$$

An equilibrium solution $g(x_0, \lambda_0)$ is *stable* if all of the eigenvalues of the Jacobian matrix $(d_x g)(x_0, \lambda_0)$ lie in the right half plane, and *unstable* otherwise. We ask the question: "To what extent are the signs of the real parts of the eigenvalues of $(d_x g)$ at solutions invariants of equivalence?" The answer is: To a greater extent than one might think. Consider the general equivalence as a three stage process:

(a) $g(x, \lambda) \to g(x, \Lambda(\lambda))$,
(b) $g(x, \lambda) \to g(X(x, \lambda)\lambda)$,
(c) $g(x, \lambda) \to S(x, \lambda)g(x, \lambda)$.

In stage (a) one just reparametrizes λ and does not change the linearized stability of g. We claim that stage (b) can be converted into stage (c). Consider the vector field change of coordinates

$$X_*(g) = (dX)^{-1}g(X(x,\lambda),\lambda).$$

At solutions the Jacobian $d_x(X_*g)$ is similar (as matrices) to d_xg and the eigenvalues are the same. Now write $g(X(x,\lambda),\lambda) = (dX)\cdot X_*(g)$. Since the stability assignment to solution of X_*g is the same as that of g we have shown that stability assignment are invariants of equivalence precisely when there are invariants of stage (c).

Next observe that if $h = Sg$, then $h = 0$ iff $g = 0$. Moreover, one can compute

$$(d_xh)|_{h=0} = S\cdot(d_xg)|_{g=0}.$$

So stability assignments are invariants if whenever one multiplies by an allowable matrix S, one does not change the signs of the real parts of the eigenvalues of (d_xg).

LEMMA 2.2. *When $n = 1$, linearized stability assignments are invariants of solutions.*

PROOF. See Definition 1.1 to recall that when $n = 1$ $S > 0$.

In general, when $n \geqslant 2$, the linearized stability assignment is *not* an invariant of equivalence. It is, however, often an invariant of Γ-equivalence.

Observe that the fact that $g(\gamma x,\lambda) = \gamma g(x,\lambda)$ implies that

$$(d_xg)(\gamma x,\lambda)\gamma = \gamma(d_xg)(x,\lambda). \tag{2.4}$$

Hence, if γ is in the isotropy subgroup Σ_x then $(d_xg)(x,\lambda)$ commutes with γ. The same is true for $S(x,\lambda)$ satisfying

$$S(\gamma x,\lambda)\gamma = \gamma S(x,\lambda).$$

Suppose now that Γ acts irreducibly. Then $(d_xg)(0,\lambda)$ commutes with Γ. Suppose that Γ acts *absolutely irreducibly*; that is, the only matrices commuting with Γ are multiplies of the identity. Then $(d_xg)(0,\lambda)$ is a constant multiple of the identity. The restrictions on S in the definition of Γ-equivalence (Definition 2.1) suffice to show that S is a positive multiple of the identity. Hence the stability assignment of the trivial solution is an invariant of Γ-equivalence.

We consider now the example of S_3 acting on $\mathbf{R}^2 \cong \mathbf{C}$. For solutions of type (II) the isotropy group is $\mathbf{Z}_2$ where $\mathbf{Z}_2$ is generated by $z \to \bar{z}$. The matrix representation of $z \to \bar{z}$ is $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ which has distinct eigenvalues. So if (d_xg) and S commute with $z \to \bar{z}$ (as follows from (2.2)) they must be diagonal matrices. Since $S(0,0)$ is constrained to be a positive multiple of the identity, the stability assignments of solutions of type (II) are invariants of S_3-equivalence. We do not know if stability assignments for solutions of type (III) are always invariants. However, for certain singularities they are. (see [**14**]).

What one sees here is that the singularity theory of equivariant mappings has invariants which are present because of the existence of a symmetry group. To our knowledge, this information has not been exploited in a coherent way.

Problem 2. For which groups Γ and which representations of Γ do there exist codimension 0 bifurcation problems?

A necessary condition is that the representation of Γ be absolutely irreducible. However, this condition is not sufficient since there are no bifurcation problems with codimension 0 commuting with the standard action of the dihedral group D_4 on $\mathbf{R}^2$. One should note that there is a bifurcation problem in $\vec{\mathcal{E}}(D_4)$ with codimension 1, modality 1, and topological codimension 0. Perhaps the above problem should state that if Γ acts absolutely irreducibly then there exist bifurcation problems with topoplogical codimension 0.

Conjecture 3. The stability assignment of branches emanating from (topological) codimension 0 bifurcation problems are invariants of Γ-equivalence.

See [3] for another example of a group for which these conjectures hold.

3. Spontaneous symmetry breaking. Suppose g is in $\vec{\mathcal{E}}(\Gamma)$ where Γ is a subgroup of $O(n)$ acting strongly irreducibly on $\mathbf{R}^n$. The symmetry of a solution x is usually defined to be the isotropy subgroup Σ_x. Note that $\Sigma_0 = \Gamma$; that is, the trivial solution $x = 0$ enjoys the full symmetry of the group, and since Γ acts irreducibly, $g(0, \lambda) \equiv 0$. One usually speaks of *spontaneous symmetry breaking* in the following context. Suppose there is a singularity of g at the origin and suppose the trivial solution is stable (in the dynamic sense) for $\lambda < 0$ and is unstable for $\lambda > 0$. Then, as λ is varied through 0, the system jumps to a new state $x \neq 0$ (which we assume to be an equilibrium point of the system). This new state will have an isotropy subgroup which is not Γ; that is, the new solution will have less symmetry than the old. The symmetry has broken spontaneously.

One can see this process occurring in the simplest situations. Consider the buckling of an Euler column. In the planar model the vertical column enjoys $\mathbf{Z}_2$ symmetry before it buckles and no symmetry after. In the 3-dimensional model, the column enjoys $O(2)$ symmetry before it buckles and no symmetry after.

It is an interesting problem to determine in advance what the likely symmetries are after bifurcation. The singularity theory response is somewhat surprising. The suggestion is that one will bifurcate to a solution which has an isotropy group corresponding to a solution appearing on a branch from a codimension zero bifurcation problem. The following example should help in understanding this last statement.

Consider the implication of this statement for the example of S_3 acting on $\mathbf{C}$. The set of points $z \in \mathbf{C}$ which have isotropy group $\{1\}$ is open and dense in $\mathbf{C}$ (being the complement of the three lines $\mathrm{Im}(z^3) = 0$). Yet the prediction here is that the initial bifurcation will be to a solution which enjoys $\mathbf{Z}_2$ symmetry. The reader is referred to the four papers [**14**, **1**, **2**, **4**] where exactly this fact is observed in different physical contexts.

The problem to which this observation leads is: Classify the isotropy groups which correspond to solutions of (topological) codimension 0 bifurcation problems. A natural conjecture given the examples we have computed is the following. Consider the lattice of isotropy subgroups of Γ ordered by inclusion. Then solutions to (topological) codimension 0 bifurcation problems have isotropy subgroups which are maximal subgroups in this lattice.

4. Symmetry breaking in the equation. At the beginning of §2 we listed several examples where the process of idealization introduces an exact symmetry into the mathematical formulation and where that exact symmetry is not present in the original problem. For example, a column is never completely symmetric; the Earth is not really spherical; and rotation of the Earth is a small but not entirely negligible parameter. In such a situation it is not sufficient to just analyse the ideal symmetric problem; one also has to analyse the perturbed problem where the perturbation breaks symmetry. Exactly how to proceed with such a theory for symmetry breaking is not clear. The main point here is that when a bifurcation problem commutes with a continuous group Γ then it has infinite codimension with respect to perturbations which destroy this symmetry. In this section we outline a suggestion for how to handle such problems. In addition, we present the results of one calculation which indicates that the suggestion is reasonable.

When Γ is a finite group then bifurcation problems in $\vec{\mathcal{E}}(\Gamma)$ which have finite Γ-codimension have finite codimension when the group structure is ignored. For example, the pitchfork $x^3 - \lambda x$ has $\mathbf{Z}_2$-codimension 0 and codimension 2. In §1 we described the universal unfolding of the pitchfork (see Figures 2 and 4). It should be clear from Figure 2 that although regions (1)–(4) occur as small perturbations of the pitchfork, there should be some sense in which regions (1) and (2) are more likely. That is, the most frequently observed perturbations of the pitchfork should be those which come from regions (1) and (2) (see Figure 7). We have inserted the appropriate stability assignments on the diagrams in Figure 7, having proved in §2 that they are invariants of equivalence. One way to formalize this observation is to note that if $g(x, \lambda, \mu)$ is a 1-parameter family of bifurcation problems, depending on μ, such that at $\mu = 0$ $g(x, \lambda, 0) = x^3 - \lambda x$, then g corresponds (via the unfolding theorem) to a curve through the origin in the $\alpha\beta$ plane of Figure 2. If this curve satisfies the genericity condition that it is transverse to $\alpha = 0$ then the bifurcation diagrams observed when $\mu \neq 0$ will be those of Figure 7. Also note that any such curve will be stable to small perturbation—as long as it is forced to go through the origin.

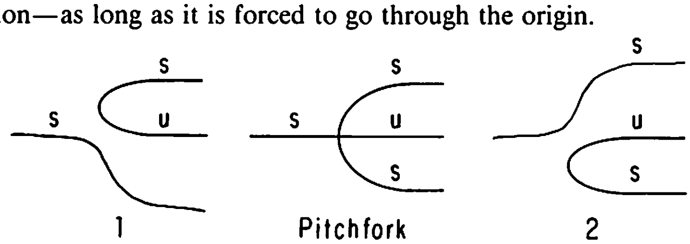


FIGURE 7

Now for a continuous group Γ one does not have the universal picture of Figure 2 since the symmetric problem has infinite codimension. So one purpose of the following definition is to finesse the problem of infinite codimension. We consider families of bifurcation problems $g(x, \lambda, \mu)$, $g\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ where for $\mu = 0$ g commutes with a group Γ and for $\mu \neq 0$ g commutes with a subgroup Δ of Γ. Two such families g and h are *equivalent* if

$$(4.1) \qquad g(x, \lambda, \mu) = S(x, \lambda, \mu) h(X(x, \lambda, \mu), \Lambda(\lambda, \mu), \mu M(\lambda, \mu))$$

where for $\mu = 0$ (4.1) defines a Γ-equivalence and for $\mu \neq 0$ (4.1) defines a Δ-equivalence. We make two observations about (4.1). First, the form of the change of coordinates in μ, namely, $\mu \to \mu M(\lambda, \mu)$, is given so that $\mu = 0$ is preserved. Second, we really want to assume that $g(x, \lambda, 0) = h(x, \lambda, 0)$ though we need not demand that the equivalence at $\mu = 0$ is the identity equivalence.

For the case of the pitchfork, we have been able to show that the formal tangent space to $g(x, \lambda, \mu) = x^3 - \lambda x + \mu g(x, \lambda, \mu)$ where $g(0,0,0) \neq 0$ relative to the equivalence in (4.1) has codimension 0. This suggests the phrase that the "stable ways to break symmetry in the pitchfork" are given in Figure 7. Unfortunately if one makes the same computations for $\Gamma = O(2)$ and the pitchfork of revolution, one finds that the formal tangent space always has infinite codimension.

One can remedy this situation by returning to the enlarged group of equivalences for $\vec{\mathcal{E}}(\Gamma)$ described in §2; namely $\tilde{\mathcal{K}}(\Gamma)$. For example, consider the rotation of the plane through angle θ, R_θ. Let

$$(4.2) \qquad g(x, y, \lambda, \mu) = (u - \lambda)(x, y) + \mu q(x, y, \lambda, \mu)$$

where $u = x^2 + y^2$; that is; g is a family which breaks the $O(2)$ symmetry of the pitchfork of revolution. Observe that $R_{-\theta} g(R_\theta(x, y), \lambda, \mu) = (u - \lambda)(x, y) + \mu R_{-\theta} q(R_\theta(x, y), \lambda, \mu)$. The action of R_θ on q may be nontrivial even though the action of R_θ on the pitchfork of revolution is trivial as in (2.1). This computation leads to the following definition.

DEFINITION 4.1. Two families $g(x, \lambda, \mu)$ and $h(x, \lambda, \mu)$ are *equivalent* if

$$(4.3) \qquad g(x, \lambda, \mu) = S(x, \lambda, \mu) h(X(x, \lambda, \mu), \Lambda(\lambda, \mu), \mu M(\lambda, \mu))$$

where for $\mu = 0$ the equivalence (4.3) is in $\tilde{\mathcal{K}}(\Gamma)$ and for $\mu \neq 0$ the equivalence (4.3) is in $\tilde{\mathcal{K}}(\Delta)$.

With this generalized definition of equivalence one can show that the formal tangent space corresponding to (4.2) has codimension 0 if $q(0,0,0,0)$ is a nonzero vector in $\mathbf{R}^2$.

Calculations by J. Damon [30] prove that the unfolding theory for such families is valid. In particular, this fact implies that there is a stable way to break symmetry for the pitchfork of revolution—even though this bifurcation problem has infinite codimension.

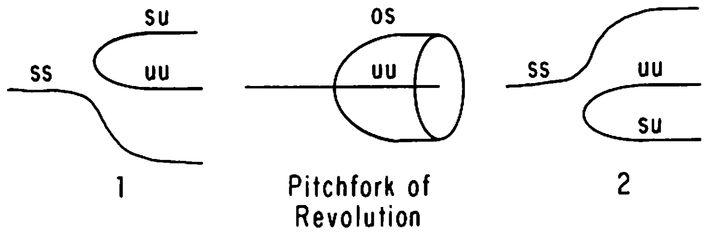


FIGURE 8

One can compute the bifurcation diagrams associated to this stable g for $\mu \neq 0$ and they are given in Figure 8. Note that the perturbed diagrams are the same as those for the pitchfork *but* the stability assignments are different. In particular, there must be a zero eigenvalue corresponding to a solution on the pitchfork of revolution. (In general, for $g \in \vec{\mathcal{E}}(\Gamma)$ there are $\dim(\Gamma/\Sigma_x)$ zero eigenvalues for the Jacobian $(d_x g)(x, \lambda)$ if $g(x, \lambda) = 0$.) Also, on the trivial branch two eigenvalues cross zero. We have indicated the signs of the eigenvalues in the diagrams in Figure 8 by s for stable (real part > 0), u for unstable (real part < 0), and 0 for real part equal to 0.

Note that one now has an interesting difference between the planar theory for column buckling ($\mathbf{Z}_2$ symmetry) and the three-dimensional theory ($O(2)$ symmetry). In the first case after symmetry is broken there are two stable equilibrium solutions corresponding to buckling left or buckling right. If the column buckles left one can "kick" it through to the other stable state. In the $O(2)$ case there is only one stable equilibrium after symmetry is broken. Suppose the column buckles in some direction, let us call it left. Now if you "kick" the column through to the buckled right state it should slowly rotate back to its original equilibrium (following the ghost of the pitchfork of revolution).

There are some surprising consequences of this calculation which were brought to our attention through discussions with J Guckenheimer. Consider (4.2) with

$$q(x, y, \lambda, \mu) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + M\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.4}$$

where M is a constant. Since $q(0,0,0,0) \neq 0$, the bifurcation diagrams of this problem are those depicted in Figure 8, with three equilibria for $\lambda > 0$. However, for λ sufficiently large there is only one rest point of the flow, a source, which is encircled by a periodic orbit along the ghost of the original pitchfork of revolution. The two flows are sketched in Figure 9, and the global bifurcation diagram for $\mu > 0$ in Figure 10. (N.B. There are no self-intersections in Figure 10.) The transition between flows at $\lambda = \lambda_2$ (notation of Figure 10) occurs when the saddle and the sink in Figure 9a approach one another, merge, and finally disappear, thereby transforming the saddle-sink connections of Figure 9(a) into the periodic orbit of Figure 9(b). The surprising fact is that $\lambda_2 \to 1/M^2$ as $\mu \to 0$; λ_2 *does not tend to* 0 as $\mu \to 0$, so the periodic orbit lies outside the scope of a local theory. Of course if M is large, λ_2 is close to zero; in our opinion, the natural way to treat such a problem with local methods is to break symmetry in two steps, wth separate parameters for each step. The first perturbation, here the $M\binom{0\ 1}{-1\ 0}\binom{x}{y}$ term, would preserve SO(2) symmetry. The second, the $\binom{1}{0}$ term, would destroy all symmetry but would also be much smaller than the first. We have not pursued these ideas yet.

One calculation does not make a theory. Nevertheless, these calculations suggest that Definition 4.1 is a good place to start in developing a theory for symmetry breaking of the equation.

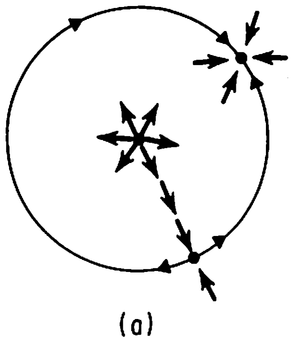


FIGURE 9

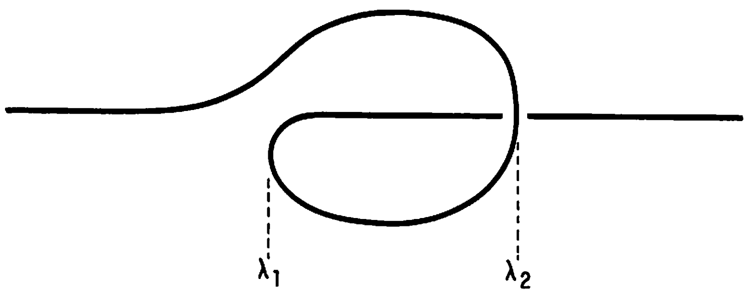


FIGURE 10

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