Nonlinearity **9** (1996) 575–599. Printed in the UK

Coupled cells with internal symmetry: II. Direct products

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Received 27 May 1994, in final form 17 November 1995
Recommended by R S MacKay

Abstract. We continue the study of arrays of coupled identical cells that possess both global and internal symmetries, begun in part I. Here we concentrate on the 'direct product' case, for which the symmetry group of the system decomposes as the direct product $\mathcal{L} \times \mathcal{G}$ of the internal group $\mathcal{L}$ and the global group $\mathcal{G}$. Again, the main aim is to find general existence conditions for symmetry-breaking steady-state and Hopf bifurcations by reducing the problem to known results for systems with symmetry $\mathcal{L}$ or $\mathcal{G}$ separately.

Unlike the wreath product case, the theory makes extensive use of the representation theory of compact Lie groups. Again the central algebraic task is to classify axial and **C**-axial subgroups of the direct product and to relate them to axial and **C**-axial subgroups of the two groups $\mathcal{L}$ and $\mathcal{G}$. We demonstrate how the results lead to efficient classification by studying both steady state and Hopf bifurcation in rings of coupled cells, where $\mathcal{L} = \mathbf{O}(2)$ and $\mathcal{G} = \mathbf{D}_n$. In particular we show that for Hopf bifurcation the case $n = 4$ modulo 4 is exceptional, by exhibiting two extra types of solution that occur only for those values of n.

AMS classification scheme numbers: 20xx, 57T05

1. Introduction

This paper continues the study of symmetric networks of coupled identical oscillators, each having its own internal symmetries, begun in [6, 11]. There we identified two natural types of symmetric coupling, leading to symmetry groups that are either the *wreath product* $\mathcal{L} \wr \mathcal{G}$ or the *direct product* $\mathcal{L} \times \mathcal{G}$ of the internal symmetry group $\mathcal{L}$ and the global symmetry group $\mathcal{G}$ of the network. We developed a general theory of steady-state and Hopf bifurcation in the wreath product case.

We now develop an analogous theory for the direct product. The analysis is more delicate, and relies more heavily on the general machinery of group representation theory. The results apply to *any* system with direct product symmetry, but we have found it convenient to motivate the ideas in terms of a network of coupled symmetric oscillators.

Alexander and Fiedler [3], building on results of Alexander and Auchmuty [2], consider coupled systems having direct product of internal and global symmetries. Some physical systems whose models possess direct product symmetry are described in [11]. They include hierarchical neural networks, discretizations of PDEs with range symmetries, and the Couette–Taylor system. Other authors have studied specific examples of direct product symmetry. Dangelmayr *et al* [7, 8] study a hierarchical network with $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry, finding that in Hopf bifurcation there are 11 types of periodic solution whose isotropy

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subgroups have two-dimensional fixed-point spaces. Wegelin [19] studies Hopf bifurcation in the cases $\mathbf{O}(2) \times \mathbf{O}(2), \mathbf{D}_m \times \mathbf{O}(2)$ (m not divisible by 4), $\mathbf{D}_m \times \mathbf{D}_n$ (n not divisible by 4). He finds, respectively, 6, 7 and 11 branches of solutions whose isotropy subgroups have two-dimensional fixed-point spaces. He also studies the stability of these branches, and more complex dynamics including heteroclinic cycles, quasiperiodic oscillations and (possibly symmetric) chaos. Oppenländer [16] studies $\mathbf{D}_m \times \mathbf{D}_n$ symmetry (mainly when $m = n = 3$). He also mentions that some models of arrays of Josephson junctions possess $\mathbf{S}_m \times \mathbf{S}_n$ symmetry, where $\mathbf{S}_n$ is the symmetric group of degree n. (But see [11] for an argument that wreath product symmetries may also arise in models of such arrays.)

1.1. Internal and global symmetries

In [6] we observed that a natural form for systems of N identical cells with identical coupling is

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = f(X_j) + \sum_{i=1}^{N} C(i,j)h(X_i, X_j) \tag{1.1}$$

for $1 \leqslant j \leqslant N$ where

(a) $X_j \in \mathbf{R}^k$ are the state variables for the jth cell,
(b) $f : \mathbf{R}^k \to \mathbf{R}^k$ represents the internal dynamics of each cell,
(c) $h(X_i, X_j)$ represents the coupling from cell i to cell j, and
(d) the $N \times N$ *connection* matrix is

$$C(i,j) = \begin{cases} 1 & \text{if cell } i \text{ is coupled to cell } j \\ 0 & \text{otherwise}. \end{cases}$$

The assumption of identical cells implies that f is independent of j and the assumption of identical coupling implies that h is independent of both i and j. The vector $X = (X_1, \ldots, X_N) \in (\mathbf{R}^k)^N$ denotes points in the state space for this system. Abstractly, we shall refer to the system of differential equations as

$$\dot{X} = F(X).$$

We now discuss the symmetries of F. There are two types of symmetries that we consider: *internal* and *global*. The global symmetries are symmetries forced on (1.1) by the pattern of coupling. Let $\sigma \in \mathbf{S}_N$ be a permutation. The action of σ on state space is:

$$\sigma \cdot X = (X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(N)}).$$

Observe that σ is a symmetry of (1.1) if

$$\sigma C \sigma^{-1} = C, \tag{1.2}$$

where σ is viewed as an $N \times N$ permutation matrix in (1.2). The global symmetry group $\mathcal{G}$ consists precisely of all of these permutation symmetries. It follows that

$$F(\sigma \cdot X) = \sigma \cdot F(X)$$

for all $\sigma \in \mathcal{G}$. This equivariance condition encodes the information that these symmetries permute the cells so that the differential equations do not change.

Next we discuss the local internal symmetry group $\mathcal{L} \subset \mathbf{O}(k)$. To be an internal symmetry we require that $\ell \in \mathcal{L}$ satisfy

$$f(\ell X_j) = \ell f(X_j).$$

Whether internal symmetries are symmetries of (1.1) depends on properties of the coupling term h. As a minimum we require that when ℓ acts simultaneously on each cell, then it is a symmetry of the coupled cell system. That is, we require that

$$h(\ell X_i, \ell X_j) = \ell h(X_i, X_j)\,.$$

If we define

$$\ell \cdot X = (\ell X_1, \ldots, \ell X_N)$$

then

$$F(\ell \cdot X) = \ell \cdot F(X)$$

and ℓ is a symmetry of (1.1).

It follows that the group $\mathcal{L} \times \mathcal{G}$ consists of symmetries of (1.1) where $\mathcal{L}$ is viewed as the diagonal subgroup of $\mathcal{L}^N$. Note that if the coupling term h is diagonal linear, that is

$$h(X_i, X_j) = X_i - X_j, \tag{1.3}$$

then the direct product is a symmetry group of (1.1).

In [6] we also consider coupled systems where the action of ℓ on each cell individually is a symmetry of (1.1). That is, we suppose

$$h(X_i, \ell X_j) = \ell h(X_i, X_j) \qquad h(\ell X_i, X_j) = h(X_i, X_j)\,.$$

In this case, the group $\mathcal{L}^N$ is a symmetry group of (1.1). The *wreath product* $\mathcal{L} \wr \mathcal{G}$ is the symmetry group generated by the groups $\mathcal{L}^N$ and $\mathcal{G}$; under these assumptions it is a symmetry group of (1.1). In this paper we focus only on the direct product couplings such as (1.3) which lead to the symmetry group $\Gamma = \mathcal{L} \times \mathcal{G}$. As in [6] our results apply to any system with this symmetry group, and not just the special form that occurs in (1.1).

In order to simplify the analysis we shall assume that the global symmetries act transitively on the cells, that is, we assume

(H_T) $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$.

If the action of $\mathcal{G}$ is intransitive, consideration of group orbits of cells under $\mathcal{G}$ reduces the analysis to a finite list of cases in each of which (H_T) holds.

In this paper we continue to develop a theory of how patterns formed through steady state and Hopf bifurcations in such systems depend upon both the internal and global symmetries. As noted in [6], it is well known in steady-state bifurcations that when isotropy subgroups have one-dimensional fixed-point subspaces then generically the equivariant branching lemma [12] guarantees the existence of solutions with that symmetry. We call a subgroup $\Sigma \subset \Gamma$ *axial* if it is an isotropy subgroup having a one-dimensional fixed-point subspace.

Similarly, when studying Hopf bifurcations, the equivariant Hopf theorem [12] states that branches of periodic solutions having symmetry Σ occur generically whenever Σ has a two-dimensional fixed-point subspace. We call a subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ **C**-*axial* if it is an isotropy subgroup having a two-dimensional fixed-point subspace.

Finding axial and **C**-axial subgroups when the coupling yields direct product symmetry groups requires detailed information concerning the generalities of real irreducible representations. In section 2 we discuss the linear theory of bifurcations based on this representation theory. In section 3 we develop criteria for subgroups of direct products to be axial. We then study the example of a ring of N cells ($\mathcal{G} = \mathbf{D}_N$) when the internal symmetry is $\mathcal{L} = \mathbf{O}(2)$ in section 4. We discuss the group theory for Hopf bifurcation for tensor product representations in section 5 and **C**-axial subgroups for tensor product representations in section 6. Finally we apply the theory to $\mathbf{O}(2) \times \mathbf{D}_N$ Hopf bifurcation in section 7.

1.2. Hopf bifurcation for four cell rings with $\mathbf{O}(2)$ *symmetry*

The remainder of this introduction is devoted to previewing our general results in the case of Hopf bifurcation in a ring of four cells when the internal symmetry group in each cell is $\mathcal{L} = \mathbf{O}(2)$. The symmetry group for this cell system is then $\Gamma = \mathbf{O}(2) \times \mathbf{D}_4$. We assume that $X = 0$ is an equilibrium in (1.1) and we imagine varying a parameter in (1.1) so that the linearization $(\mathrm{d}F)_0$ has eigenvalues on the imaginary axis at $\pm\omega\mathrm{i}$. Symmetry may force these eigenvalues to be multiple.

For example, in systems with $\mathbf{O}(2)$ symmetry, the critical eigenvalues may be forced to be double. When this happens there are two $\mathbf{C}$-axial subgroups $\hat{A}_1$ and $\hat{A}_2$ corresponding to rotating and standing waves [9, 12]. A *standing wave* is a periodic solution that is fixed by a reflection $\kappa \in \mathbf{O}(2)$ for all time. A *rotating wave* is a periodic solution in which time evolution is the same as spatial rotation.

Similarly, in systems with $\mathbf{D}_4$ symmetry critical eigenvalues may be forced to be double and when this happens there are three $\mathbf{C}$-axial subgroups $\hat{B}_1$, $\hat{B}_2$ and $\hat{B}_3$ [10, 12]. The first two are discrete standing waves and the third is a discrete rotating wave or a *pony on a merry-go-round*. The pattern associated with each of these solutions may be described in terms of the four-cell coupled cell system where each cell has no internal symmetry. Solutions of type $\hat{B}_1$ have two pairs of adjacent cells oscillating in-phase with cells in different pairs oscillating a half-period out-of-phase. Solutions of type $\hat{B}_2$ have one diagonal pair of cells oscillating in-phase and the other pair of diagonal cells oscillating a half-period out-of-phase. The in-phase cells oscillate at twice the frequency of the out-of-phase cells. Finally, the discrete rotating wave solution has each cell oscillating according to the same wave form with a quarter-period phase shift between adjacent cells.

As discussed in section 5, the linear theory of Hopf bifurcation for product groups is driven by tensor products of representations of the individual groups. It is therefore possible that the centre subspace for these coupled systems with $\mathbf{O}(2) \times \mathbf{D}_4$ symmetry will have Hopf bifurcations where the critical eigenvalues $\pm\omega\mathrm{i}$ each have multiplicity four yielding an eight-dimensional centre subspace. When this happens, our results show that there are nine $\mathbf{C}$-axial groups and nine families of periodic solutions. See proposition 7.1. In proposition 6.4 we show that pairing each $\mathbf{C}$-axial subgroup $\hat{A}$ for $\mathcal{L}$ with a $\mathbf{C}$-axial subgroup $\hat{B}$ for $\mathcal{G}$ yields a $\mathbf{C}$-axial 'twisted product' subgroup for $\mathcal{L} \times \mathcal{G}$ which is denoted by $\hat{A} \dot{\times} \hat{B}$. In this example we find six twisted product $\mathbf{C}$-axial subgroups $\hat{A}_i \dot{\times} \hat{B}_j$. Our calculations show that there are three additional $\mathbf{C}$-axial groups $\tilde{\mathbf{D}}_4$, $\hat{\mathbf{D}}_4[\kappa]$ and $\hat{\mathbf{D}}_4[\frac{\pi}{2}\kappa]$.

We now discuss the patterns of oscillation of each of these nine solutions. We view these solutions in the following way. In each cell we project the motion $X_j(t)$ into a plane in which $\mathbf{O}(2)$ acts by its standard action. We can then view the oscillations of each cell as a trajectory $z_j(t) \in \mathbf{C}$. Finally, we can draw each of these trajectories in the same plane (using different colours to distinguish the four individual projections). With this presentation of the periodic trajectories we can describe the patterns of oscillation forced by symmetry.

We first describe the motions associated with the rotating wave $\hat{A}_1$. In $\hat{A}_1 \dot{\times} \hat{B}_3$ the four cells traverse the same circle with adjacent cells a quarter-period out-of-phase. In $\hat{A}_1 \dot{\times} \hat{B}_2$ one pair of diagonal cells traverse the same circle a half-period out of phase while the other diagonal pair of cells are forced by symmetry to be at the origin for all time. (The double frequency motion is forced to zero by the additional symmetry.) In $\hat{A}_1 \dot{\times} \hat{B}_1$ the cells divide into two pairs of adjacent cells. The motion in each pair is identical and in a circle and the motions of cells in different pairs are a half-period out-of-phase.

The motions corresponding to the $\hat{A}_2 \dot{\times} \hat{B}_j$ are similar. Here, however, the motions of the cells are all in the same line rather than on circles.

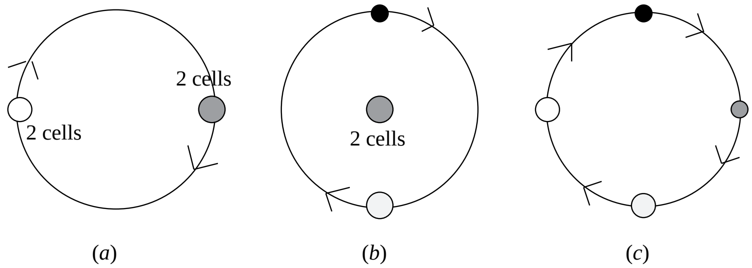

Figure 1. Rotating wave axials $\hat{A}_1 \dot{\times} \hat{B}$: ($a$) $\hat{B} = \hat{B}_1$, (b) $\hat{B} = \hat{B}_2$, (c) $\hat{B} = \hat{B}_3$.

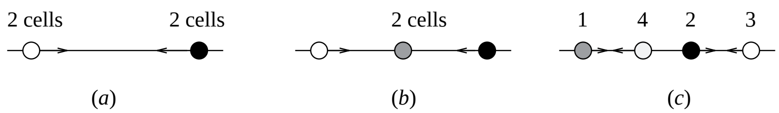

Figure 2. Standing wave axials $\hat{A}_2 \dot{\times} \hat{B}$: ($a$) $\hat{B} = \hat{B}_1$, (b) $\hat{B} = \hat{B}_2$, (c) $\hat{B} = \hat{B}_3$.

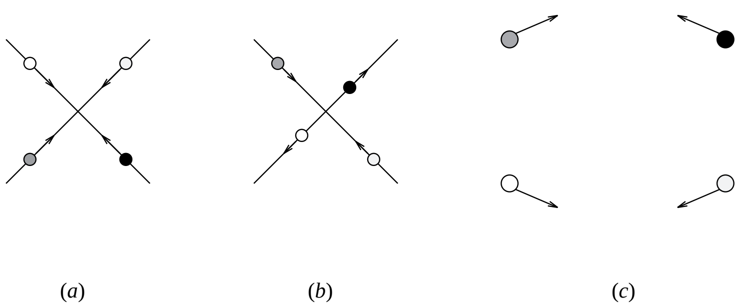

Figure 3. Exceptional axials: (a) $\tilde{\mathbf{D}}_4$, (b) $\hat{\mathbf{D}}_4[\frac{\pi}{2}\kappa]$, ($c$) $\hat{\mathbf{D}}_4[\kappa]$.

The three exceptional groups provide the most interesting patterns of oscillation. The group $\tilde{\mathbf{D}}_4$ generates a motion where diagonally opposite cells move on the same line at points z and $-z$, the two pairs of cells travel on lines at right angles, and of the two adjacent cells one is in-phase and the other is a half-period out-of-phase. Moreover, after a half-period the point z moves to the point $-z$. The seven patterns of oscillation just described have analogous patterns in rings of N cells with $\mathbf{O}(2)$ symmetry. The last two solution types only occur when $N = 0 \pmod 4$.

The group $\hat{\mathbf{D}}_4[\frac{\pi}{2}\kappa]$ also generates a motion where diagonally opposite cells move on the same line at points z and $-z$, and the two pairs of cells travel on lines at right angles. For this group, however, adjacent cells are a quarter-period out-of-phase so that there are no four-way collisions at the origin.

The last group $\hat{\mathbf{D}}_4[\kappa]$ generates a motion where the four cells are always at vertices of a rectangle defined by points z, $-\bar{z}$, $-z$ and $\bar{z}$. Moreover, $z(t)$ itself is a discrete rotating wave; that is, $-\mathrm{i}z(t) = z(t + \frac{T}{4})$ where T is the period of the motion.

2. The linear theory for direct products

The theory of Γ-equivariant bifurcations proceeds by first identifying the irreducible representations of Γ on state space. In this section we consider this issue when $\Gamma = \mathcal{L} \times \mathcal{G}$ and the state space is V^N where $V = \mathbf{R}^k$. Indeed, this state space is just $V \otimes W$ where $W = \mathbf{R}^N$. Here $\otimes$ denotes the tensor product over the reals. We later also refer to tensor products over the complex numbers, which we denote by $\otimes_{\mathrm{c}}$. In this notation V is an $\mathcal{L}$-invariant space and W is a $\mathcal{G}$-invariant space, so that $V \otimes W$ is a $\Gamma = \mathcal{L} \times \mathcal{G}$-invariant space. The structure of this phase space involves subtleties in the theory of irreducible representations over the reals, and we will be forced to consider these.

We begin by decomposing V into a direct sum of $\mathcal{L}$-irreducible subspaces

$$V = V_1 \oplus \cdots \oplus V_p$$

and $\mathbf{R}^N$ into a direct sum of $\mathcal{G}$-irreducible subspaces

$$W = W_1 \oplus \cdots \oplus W_q\ .$$

Then the state space $V \otimes W$ decomposes into the following direct sums of Γ-invariant subspaces

$$V \otimes W = \bigoplus_{ij} V_i \otimes W_j\ .$$

If we were dealing with representation theory over $\mathbf{C}$, then $V_i \otimes W_j$ would be irreducible, and we would have written state space as a sum of Γ-irreducible representations. However, over $\mathbf{R}$, the tensor product of irreducibles is not necessarily irreducible, and we digress to describe what actually happens.

We begin by defining an isomorphism-invariant of a representation X. It is called the *algebra of commuting linear maps*, and is defined to be

$$\mathcal{D}_\Gamma(X) = \{\alpha : X \to X | \alpha \text{ is linear and } \alpha(\gamma x) = \gamma(\alpha(x))\ \forall \gamma \in \Gamma\}.$$

The real vector space $\mathcal{D}_\Gamma(X)$ is closed under composition of maps, and is thus an associative $\mathbf{R}$-algebra. Denote by $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ the $\mathbf{R}$-algebras of real numbers, complex numbers, and quaternions. These are division algebras over the reals, of dimensions 1, 2, 4 respectively. Indeed:

Lemma 2.1. *(Real version of Schur's lemma). If X is irreducible, then $\mathcal{D}_\Gamma(X)$ is a division algebra over* $\mathbf{R}$*. Such algebra are (isomorphic to) either* $\mathbf{R}$*,* $\mathbf{C}$ *or* $\mathbf{H}$*.*

Proof. See Kirillov [13], section 8.2, theorem 2 p 119. □

Accordingly, we say that an irreducible representation X is of *real*, *complex*, or *quaternionic type* or, equivalently, of type $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, respectively. Being of type $\mathbf{R}$ is the same as being *absolutely irreducible*. A *nonabsolutely irreducible* representation is an irreducible representation of complex or quaternionic type.

The algebra of commuting linear maps behaves nicely with respect to tensor products, as follows:

Lemma 2.2. *Let U and V be representations of $\mathcal{L}$ and $\mathcal{G}$, respectively. Then*

$$\mathcal{D}_{\mathcal{L}\times\mathcal{G}}(U \otimes V) \cong \mathcal{D}_{\mathcal{L}}(U) \otimes \mathcal{D}_{\mathcal{G}}(V) \tag{2.1}$$

where $\cong$ denotes $\mathbf{R}$*-algebra isomorphism.*

Proof. Note that $\mathcal{L} \times \mathcal{G}$ is generated by $\mathcal{L} \times \mathbf{1}$ and $\mathbf{1} \times \mathcal{G}$. It follows that

$$\begin{aligned}\mathcal{D}_{\mathcal{L}\times\mathcal{G}}(U \otimes V) &= \mathcal{D}_{\mathcal{L}\times\mathbf{1}}(U \otimes V) \cap \mathcal{D}_{\mathbf{1}\times\mathcal{G}}(U \otimes V) \\ &\cong (\mathcal{D}_{\mathcal{L}}(U) \otimes V) \cap (U \otimes \mathcal{D}_{\mathcal{G}}(V)) \\ &= \mathcal{D}_{\mathcal{L}}(U) \otimes \mathcal{D}_{\mathcal{G}}(V)\end{aligned}$$

as claimed. □

We now prove the following:

Theorem 2.3. *Let U and V be irreducible representations of compact Lie groups $\mathcal{L}$ and $\mathcal{G}$, respectively. Consider $U \otimes V$ as a representation of $\Gamma = \mathcal{L} \times \mathcal{G}$. Then the type of this representation is given in table 1.*

Table 1. Decomposition of tensor product representations.

Type of U	Type of V	$U \otimes V$	Remarks
R	**R**	W	W is type **R**
R	**C**	W	W is type **C**
R	**H**	W	W is type **H**
C	**C**	$W_1 \oplus W_2$	W_j is type **C**, $W_1 \ncong W_2$
C	**H**	$W \oplus W$	W is type **C**
H	**H**	$W \oplus W \oplus W \oplus W$	W is type **R**

Proof. The proof is a consequence of lemma 2.2 and the algebra isomorphisms proved in Porteous [17]. They are:

$$\mathbf{R} \otimes \mathbf{R} \cong \mathbf{R}, \quad \mathbf{R} \otimes \mathbf{C} \cong \mathbf{C}, \quad \mathbf{R} \otimes \mathbf{H} \cong \mathbf{H},$$

$$\mathbf{C} \otimes \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}, \quad \mathbf{C} \otimes \mathbf{H} \cong sl_2(\mathbf{C}), \quad \mathbf{H} \otimes \mathbf{H} \cong sl_4(\mathbf{R})\,.$$

□

We now make the following observation: see Golubitsky *et al* [12] XIII, proposition 3.2. Generically steady-state bifurcations correspond to kernels $\mathcal{K}$ of linearized equations on which the action of Γ on $\mathcal{K}$ is absolutely irreducible. Theorem 2.3 implies that absolutely irreducible representations can appear in state space in one of two ways.

Proposition 2.4. *With the above notation, and in the generic case, the kernel $\mathcal{K}$ is an absolutely irreducible representation if and only if one of the following cases holds:*

(a) V_i and W_j are absolutely irreducible representations of $\mathcal{L}$ and $\mathcal{G}$, respectively, and $\mathcal{K} \cong V_i \otimes W_j$ is a representation of $\Gamma = \mathcal{L} \times \mathcal{G}$.
(b) V_i and W_j are both irreducible representations of type **H** *and*

$$V_i \otimes W_j \cong U \oplus U \oplus U \oplus U, \tag{2.2}$$

where $\mathcal{K} \cong U$.

Proof. Generically, $\mathcal{K}$ is an irreducible component of type **R** of some $V_i \otimes W_j$. By theorem 2.3 either V_i and W_j are both irreducible of type **R** and (a) holds, or they are both of type **H** and (b) holds. □

We note that case (b) can occur:

Example 2.5. Let $\mathbf{SU}(2)$ denote the group of unit quaternions. This acts on $\mathbf{H}$ on the left,

$$\gamma \cdot h = \gamma h \quad (\gamma \in \mathbf{SU}(2), h \in \mathbf{H}) .$$

The action is irreducible of type $\mathbf{H}$: the commuting linear maps are just right multiplication by elements of $\mathbf{H}$, see Montaldi *et al* [15]. Let $\mathcal{L} = \mathcal{G} = \mathbf{SU}(2), U = V = \mathbf{H}$, so that $\mathbf{SU}(2)\times\mathbf{SU}(2)$ acts diagonally on $\mathbf{H}\otimes\mathbf{H}$. By theorem 2.3 , we have $\mathbf{H}\otimes\mathbf{H} \cong W\oplus W\oplus W\oplus W$ where W is irreducible of type $\mathbf{R}$.

The same argument applies to the diagonal subgroup of $\mathcal{L}\times\mathcal{L}$ acting on $U\otimes U$ whenever U is an $\mathbf{H}$-type irreducible for $\mathcal{L}$. For instance $\mathcal{L}$ can be taken to be the quaternion group Q_8 of order 8, which is a finite group of global symmetries suitable for a network of oscillators. See Ashwin and Stork [4]. Indeed, we can also take $\mathcal{L} = \mathbf{SU}(2)$ and $\mathcal{G} = Q_8$.

Similarly, generically, Hopf bifurcations occur when the generalized eigenspace corresponding to the complex conjugate purely imaginary eigenvalues is Γ-simple. This is possible whenever an absolutely irreducible representation is repeated or when a nonabsolutely irreducible representation occurs. From theorem 2.3 we see that all combinations are possible.

3. Axial subgroups

We now make a more detailed study of conditions under which a subgroup can be proved to be axial. By proposition 2.4 there are two cases (a) and (b). For applications, case (a) is by far the commonest. We divide this section into three subsections: the first is applicable only to case (a); the second is applicable to both cases; and the third is applicable only to case (b).

3.1. Tensor product of real irreducibles

In this subsection we assume that $\mathcal{L} \times \mathcal{G}$ acts on $U \otimes V$ where U and V are absolutely irreducible representations of $\mathcal{L}$ and $\mathcal{G}$, respectively. We will prove theorems about axial subgroups and in so doing we will compute fixed-point subspaces using the following result:

Lemma 3.1. *Let $A \subset \mathcal{L}$ and $B \subset \mathcal{G}$ be subgroups. Then*

$$\mathrm{Fix}_{U\otimes V}(A \times B) = \mathrm{Fix}_U(A) \otimes \mathrm{Fix}_V(B) .$$

Proof. Observe that $A \times B$ is generated by $A \times \mathbf{1}$ and $\mathbf{1} \times B$. Hence

$$\begin{aligned}\mathrm{Fix}_{U\otimes V}(A \times B) &= \mathrm{Fix}_{U\otimes V}(A \times \mathbf{1}) \cap \mathrm{Fix}_{U\otimes V}(\mathbf{1} \times B)\\ &= \mathrm{Fix}_U(A) \otimes V \cap U \otimes \mathrm{Fix}_V(B)\\ &= \mathrm{Fix}_U(A) \otimes \mathrm{Fix}_V(B)\end{aligned}$$

as claimed. □

Let $A \subset \mathcal{L}$ and $B \subset \mathcal{G}$ be axial. Since $\dim \mathrm{Fix}_U(A) = 1$, either $N_{\mathcal{L}}(A) = A$ or $N_{\mathcal{L}}(A)/A = \mathbf{Z}_2$ where $N_{\mathcal{L}}(A)$ is the normalizer of A in $\mathcal{L}$. In the latter case, elements of $N_{\mathcal{L}}(A) - A$ act as $-I$ on $\mathrm{Fix}_U(A)$. A similar conclusion holds for B. Define

$$A\dot{\times}B = A \times B \cup (N_{\mathcal{L}}(A) - A) \times (N_{\mathcal{G}}(B) - B) .$$

If either $N_{\mathcal{L}}(A) = A$ or $N_{\mathcal{G}}(B) = B$ then the second term in the union is empty and can be neglected. Thus either $A \times B$ is equal to $A\dot{\times}B$ or it is an index two subgroup of $A\dot{\times}B$. In either case

$$\mathrm{Fix}_{U\otimes V}(A\dot{\times}B) = \mathrm{Fix}_{U\otimes V}(A \times B). \tag{3.1}$$

This is trivial if $A\dot{\times}B = A \times B$. If not, elements in $A\dot{\times}B - A \times B$ have the form (s, t) where $s \in N_{\mathcal{L}}(A) - A$ and $t \in N_{\mathcal{G}}(B) - B$. Then $(s, t)(u \otimes v) = (-u) \otimes (-v) = u \otimes v$, as required.

Lemma 3.2. *Suppose that $\ell \in \mathcal{L}$, $g \in \mathcal{G}$, $u \in U$ is nonzero, and $v \in V$ is nonzero. Then*

$$(\ell, g)(u \otimes v) = u \otimes v$$

if and only if either

$$\ell u = u \qquad \text{and} \qquad gv = v \tag{3.2}$$

or

$$\ell u = -u \qquad \text{and} \qquad gv = -v\,. \tag{3.3}$$

Proof. The sufficiency of this condition is clear. To prove necessity, let $\{u, u_2, \ldots, u_s\}$ be a basis for U and write

$$\ell u = au + \sum_{j=2}^{s} b_j u_j\,.$$

Then

$$0 = (\ell u) \otimes (gv) - u \otimes v = u \otimes (agv - v) + \sum_{j=2}^{s} u_j \otimes (b_j gv)\,.$$

Since $u, u_2, \ldots, u_s$ are linearly independent, it follows that $agv = v$ and $b_j gv = 0$. Since $v \neq 0$ it follows that both $a \neq 0$ and $gv \neq 0$. Thus $b_j = 0$, $\ell u = au$ and $gv = \frac{1}{a}v$. Finally, since the linear mapping ℓ is orthogonal, $a = \pm 1$, which proves necessity. $\square$

Proposition 3.3. *Let $A \subset \mathcal{L}$ and $B \subset \mathcal{G}$ be axial. Then $A\dot{\times}B$ is axial.*

Proof. We begin by showing that $\mathrm{Fix}_{U\otimes V}(A\dot{\times}B)$ is one-dimensional. Lemma 3.1 and (3.1) imply

$$\mathrm{Fix}_{U\otimes V}(A\dot{\times}B) = \mathrm{Fix}_{U\otimes V}(A \times B) = \mathrm{Fix}_U(A) \otimes \mathrm{Fix}_V(B)\,.$$

Axiality of A and B implies that the space on the right is one-dimensional.

We must also show that $A\dot{\times}B$ is an isotropy subgroup. Let $\Delta \supset A\dot{\times}B$ be the isotropy subgroup of a point $u \otimes v \in \mathrm{Fix}_{U\otimes V}(A\dot{\times}B)$. Since Δ leaves $\mathrm{Fix}_{U\otimes V}(A \times B)$ invariant it is a subgroup of the normalizer of $A \times B$ in $\mathcal{L} \times \mathcal{G}$. This normalizer is just $N_{\mathcal{L}}(A) \times N_{\mathcal{G}}(B)$. It is easy to check, using lemma 3.2, that the elements in $N_{\mathcal{L}}(A) \times N_{\mathcal{G}}(B)$ that are not in $A\dot{\times}B$ act as $-I$ on $\mathrm{Fix}_{U\otimes V}(A \times B)$. Thus $\Delta = A\dot{\times}B$. $\square$

Proposition 3.4. *Let $P \subset \mathcal{L} \times \mathcal{G}$ be axial, let $A = P \cap (\mathcal{L} \times \mathbf{1})$ and let $B = P \cap (\mathbf{1} \times \mathcal{G})$. Suppose*

(i) $\dim \mathrm{Fix}_U(A) = 1$,
(ii) there is an element $\omega \in \mathcal{G}$ such that ω acts as $-I$ on V.

Then $P = A\dot{\times}B$ where A and B are axial.
Proof. Since $P \supset A \times B$, it follows that

$$\mathrm{Fix}_U(A) \otimes \mathrm{Fix}_V(B) \supset \mathrm{Fix}_{U\otimes V}(P) \neq \{0\}\,.$$

Therefore $\mathrm{Fix}_V(B) \neq \{0\}$. In addition, since $\dim \mathrm{Fix}_U(A) = 1$, we can choose $u \neq 0$ in $\mathrm{Fix}_U(A)$ and write any vector $w \in \mathrm{Fix}_U(A) \otimes \mathrm{Fix}_V(B)$ as $w = u \otimes v$ where $v \in \mathrm{Fix}_V(B)$. Next we choose a nonzero $w \in \mathrm{Fix}_{U\otimes V}(P)$ and write $w = u \otimes v$, where $v \neq 0$ in $\mathrm{Fix}_V(B)$.

The isotropy subgroup Σ_u of u must be A. Certainly, $\Sigma_u \supset A$. But if $\ell \in \Sigma_u$, then $(\ell, 1)(u \otimes v) = (\ell u) \otimes v = u \otimes v$. Hence $(\ell, 1) \in P$ and $\ell \in A$. Similarly, B is the isotropy subgroup of $v \in \mathcal{G}$. We have now proved that A is axial.

Let (ℓ, g) be in P. Either (3.2) holds, in which case $(\ell, g) \in A \times B$, or (3.3) hold. In the latter case, $\ell \in N_{\mathcal{L}}(A) - A$, and ωg fixes v since ω acts as $-I$ on V. Thus $\omega g \in B$ since B is the isotropy subgroup of v. Therefore

$$P \subset ((N_{\mathcal{L}}(A) - A) \times \omega B) \cup (A \times B)\,.$$

Now suppose $\hat{v} \in \mathrm{Fix}_V(B)$; then P fixes $\hat{w} = u \otimes \hat{v}$. This is true since $\hat{w}$ is certainly fixed by any element of $A \times B$ and a calculation shows that $\hat{w}$ is fixed by $(N_{\mathcal{L}}(A) - A) \times \omega B$ and hence by P. Since P is axial, $\hat{w}$ is a scalar multiple of w and $\hat{v}$ is a scalar multiple of v. It follows that $\dim \mathrm{Fix}_V(B) = 1$, $N_{\mathcal{G}}(B) = B \cup \omega B$, and B is axial. □

Remark 3.5. Suppose that $-I \notin \mathcal{G}$ and that P is an axial subgroup of $\mathcal{L} \times \mathcal{G}$ that fixes the vector $u \otimes v$. We can still use proposition 3.4 to determine the form of P by using the following trick. Extend $\mathcal{G}$ to $\tilde{\mathcal{G}} = \mathcal{G} \oplus \mathbf{Z}_2(-I)$ and let $\tilde{P}$ be the isotropy subgroup of $u \otimes v$ in $\mathcal{L} \times \tilde{\mathcal{G}}$. Let $\tilde{B} = P \cap (\mathbf{1} \times \mathcal{G})$. Proposition 3.4 states that $\tilde{B}$ is an axial subgroup of $\tilde{\mathcal{G}}$ and that $\tilde{P} = A \dot{\times} \tilde{B}$. We can now compute P by the relationship $P = \tilde{P} \cap (\mathcal{L} \times \mathcal{G})$.

3.2. Representation-theoretic criteria for axiality

We now develop criteria for a subgroup to be axial that make more explicit use of representation theory. We begin by stating another representation-theoretic result. Let U and V be representations of a compact Lie group Γ. Define $\mathcal{D}_\Gamma(U, V)$ to be the vector space of linear mappings of U to V that commute with the actions of Γ. Note that $\mathcal{D}_\Gamma(U, U) = \mathcal{D}_\Gamma(U)$, which was defined previously.

Lemma 3.6. *Let U and V be irreducible representations of a compact Lie group Γ. Then*

(a) $\dim \mathrm{Fix}_{U\otimes V}(\Gamma) = \dim \mathcal{D}_\Gamma(U, V)$.
(b) $\mathcal{D}_\Gamma(U, V) = 0$ *if and only if U and V are nonisomorphic representations.*

Proof. Part (b) is just one version of the standard Schur's lemma. Begin the proof of (a) by noting that since Γ is compact, the representations of Γ on U and U^* are isomorphic. This point is proved by choosing a Γ-invariant inner product $\langle \cdot, \cdot \rangle$ and constructing an isomorphism from U to U^* by $u^*(w) = \langle u, w \rangle$. Observe that

$$\gamma \cdot u^*(w) = \langle u, \gamma^{-1} w \rangle = \langle u, \gamma^t w \rangle$$

since γ is orthogonal. Therefore

$$\gamma \cdot u^*(w) = \langle \gamma u, w \rangle = (\gamma u)^*(w)\,.$$

Next let $L(U, V)$ be the space of (real) linear mappings of U to V. The group Γ acts on $L(U, V)$ by $\gamma \cdot A(u) = \gamma^t A(\gamma u)$. Note that matrices that are fixed by this action are precisely the matrices in $\mathcal{D}_\Gamma(U, V)$. To prove (a), recall that $U^* \otimes V \cong L(U, V)$. Moreover, the isomorphism is given by $u^* \otimes v \mapsto A(u) \equiv u^*(w)v$. A calculation shows that this isomorphism is a Γ-equivariant isomorphism. Hence $L(U, V) \cong U^* \otimes V \cong U \otimes V$ as Γ representations. □

We assume now that $\mathcal{L} \times \mathcal{G}$ acts on $U \otimes V$ where U is a representation of $\mathcal{L}$ and V is a representation of $\mathcal{G}$. Let

$$\Pi_{\mathcal{L}} : \mathcal{L} \times \mathcal{G} \to \mathcal{L} \qquad \text{and} \qquad \Pi_{\mathcal{G}} : \mathcal{L} \times \mathcal{G} \to \mathcal{G}$$

be projections. Given a subgroup $P \subset \mathcal{L} \times \mathcal{G}$, we define representations of P on U and V as follows:

$$\rho_U = \eta_U \circ \Pi_{\mathcal{L}} \qquad \text{and} \qquad \rho_V = \eta_V \circ \Pi_{\mathcal{G}} .$$

Proposition 3.7. *Let P be an axial subgroup of $\mathcal{L} \times \mathcal{G}$. Then there is precisely one irreducible representation of the action ρ_U of P on U that is isomorphic to precisely one irreducible representation of the action ρ_V of P on V. Moreover, these representations of P are absolutely irreducible.*

Proof. Let $P \subset \mathcal{L} \times \mathcal{G}$ be a subgroup. Suppose that

$$U = U_1 \oplus \cdots \oplus U_s \qquad \text{and} \qquad V = V_1 \oplus \cdots \oplus V_t$$

where the U_j and V_k are irreducible representations of P on U and V, respectively. Observe that

$$\mathrm{Fix}_{U \otimes V}(P) = \bigoplus_{ij} \mathrm{Fix}_{U_i \otimes V_j}(P). \tag{3.4}$$

Since P is axial, $\dim \mathrm{Fix}_{U \otimes V}(P) = 1$. For the right-hand side to sum to 1, it is necessary that $\mathrm{Fix}_{U_i \otimes V_j}(P) = 0$ for all pairs i, j except one. Lemma 3.6(a) implies that for all of these pairs $\mathcal{D}_\Gamma(U_i, V_j) = 0$ and lemma 3.6(b) implies that U_i and V_j are nonisomorphic representations. Lemma 3.6 also implies that for this one exceptional pair $U_i \cong V_j$. Since $\dim(\mathcal{D}_\Gamma(U_i)) = 1$, it follows from theorem 2.3 that U_i is absolutely irreducible. □

Corollary 3.8. *Let $P \subset \mathcal{L} \times \mathcal{G}$ be axial and let $A = \Pi_{\mathcal{L}}(P)$. Write $U = U_1 \oplus \cdots \oplus U_s$ as a direct sum of A-irreducible representations. Then at least one of the U_j is A-absolutely irreducible and distinct from the other U_i.*

Proof. The main point in the proof of this corollary is that the matrices in the representation η_U of A are identical with the matrices in the representation ρ_U of P. Therefore if U is irreducible (or absolutely irreducible) for one of these representations, then it is irreducible (or absolutely irreducible) for the other.

Suppose that all of the U_j are nonabsolutely irreducible for A. Since $\mathrm{Fix}_{U \otimes V}(P)$ is nonzero by assumption, proposition 3.7 implies that U_j is isomorphic to some P-irreducible representation in V. Then lemma 3.6 and theorem 2.3 imply that $\dim \mathrm{Fix}_{U \otimes V}(P) > 1$. Thus some U_j must be A-absolutely irreducible. If all absolutely irreducible representations have multiplicity greater than one, then proposition 3.7 also implies that $\dim \mathrm{Fix}_{U \otimes V}(P) > 1$, which again contradicts the assumption that P is axial. □

Corollary 3.9. *Let $P \subset \mathcal{L} \times \mathcal{G}$ be axial and let $A \subset \mathcal{L}$ and $B \subset \mathcal{G}$ be the projections of P. Suppose that B acts faithfully on each irreducible representation V_j. Then B is isomorphic to a quotient group of A.*

Proof. Let U_i be the irreducible representation of A that matches up with the irreducible representation V_j of B to produce a one-dimensional fixed-point subspace for P as in proposition 3.7. The corresponding representations of P are isomorphic, so that

$$P/\ker(\rho_{U_i}) \cong P/\ker(\rho_{V_j}) . \tag{3.5}$$

We make the following group-theoretic observation. Suppose that $\eta : A \to X$ is a group homomorphism. Recall that $\Pi_{\mathcal{L}} : P \to A$ is a surjective homomorphism and note that $\rho = \eta \circ \Pi_{\mathcal{L}} : P \to X$ is a homomorphism. Since $\Pi_{\mathcal{L}}$ is surjective,

$$A/\ker(\eta) \cong \mathrm{image}(\eta) = \mathrm{image}(\rho) \cong P/\ker(\rho) .$$

In this case we let η_{U_i} denote the representation of A on U_i and take X to be $\mathbf{gl}_{U_i}(\mathbf{R})$. Observe that $\rho_{U_i} = \eta_{U_i} \circ \Pi_{\mathcal{L}}$. Since the same construction works for B and $\Pi_{\mathcal{G}}$, we conclude from (3.5) that

$$A/\ker(\eta_{U_i}) \cong B/\ker(\eta_{V_j})\,. \tag{3.6}$$

Since the actions of B are faithful, $\ker(\eta_{V_j})$ is trivial, so B is isomorphic to a quotient of A. □

Remark 3.10. We will use corollary 3.9 in the following way. We fix a finite group $B \subset \mathcal{G}$ and ask whether for each subgroup $A \subset \mathcal{L}$ there is an axial group $P \subset A \times \mathcal{G}$ such that $\Pi_{\mathcal{G}}(P) = B$ and $\Pi_{\mathcal{L}}(P) = A$. Corollary 3.9 implies that the only A that we need check are those with a quotient equal to B. Moreover, if the actions of these A on the various U_i are also faithful, then the isomorphism (3.6) implies that $A \cong B$, since A must also be a quotient group of B.

3.3. Tensor product of quaternionic irreducibles

Throughout this section U and V are irreducible representations of type $\mathbf{H}$ of $\mathcal{L}$ and $\mathcal{G}$ respectively, and the zero eigenspace W satisfies (2.2), which we repeat for convenience in the form:

$$U \otimes V \cong W \oplus W \oplus W \oplus W$$

where the representation W is irreducible of type $\mathbf{R}$. Suppose that $P \subset \mathcal{L} \times \mathcal{G}$ is axial. Then $\dim \mathrm{Fix}_W(P) = 1$ and $\dim \mathrm{Fix}_{U\otimes V}(P) = 4$. The representation-theoretic result that we need in this section is:

Lemma 3.11. *Suppose that* Γ *acts irreducibly on* X*, with type* $\Lambda = \mathbf{R}, \mathbf{C}$*, or* $\mathbf{H}$*. Then* $\dim \mathrm{Fix}(\Sigma)$ *is a multiple of* $\dim \Lambda$.

Proof. Since X is a representation of Γ over Λ, it restricts to a representation of Σ over Λ. Therefore, $\mathrm{Fix}(\Sigma)$ is a Λ-vector subspace of X and hence of real dimension a multiple of $\dim \Lambda$. □

Lemma 3.12. *Let* $A \subset \mathcal{L}$ *and* $B \subset \mathcal{G}$ *be subgroups.*

(a) If $P \subset A \times B$ *then either* $\dim \mathrm{Fix}_U(A) = 0$ *or* $\dim \mathrm{Fix}_V(B) = 0$.
(b) If $P \supset A \times B$ *then* $\dim \mathrm{Fix}_U(A) \geqslant 4$ *and* $\dim \mathrm{Fix}_V(B) \geqslant 4$.

Proof. (a) Since $A \times B \supset P$, $\mathrm{Fix}_U(A) \otimes \mathrm{Fix}_V(B) \subset \mathrm{Fix}_{U\otimes V}(P)$, whence

$$\dim \mathrm{Fix}_U(A) \cdot \dim \mathrm{Fix}_V(B) \leqslant \dim \mathrm{Fix}_{U\otimes V}(P) = 4\,.$$

But both factors on the left-hand side are multiples of 4, by lemma 3.11. Therefore at least one is zero.

(b) Since $A \times B \subset P$,

$$\dim \mathrm{Fix}_U(A) \cdot \dim \mathrm{Fix}_V(B) \geqslant \dim \mathrm{Fix}(P) = 4\,.$$

Therefore both factors on the left-hand side are nonzero, and since they are both multiples of 4, each is at least 4. □

Corollary 3.13. *The axial subgroup* P *cannot be of the form* $A \times B$ *where* $A \subset \mathcal{L}$ *and* $B \subset \mathcal{G}$.

Recall example 2.5 where we noted that representations of quaternionic type do occur in oscillator systems. Let Σ be a group that acts irreducibly on U as a representation of quaternionic type. Let $\mathcal{L} = \mathcal{G} = \Sigma$ and let $U = V$, so that $\Sigma \times \Sigma$ acts diagonally on $U \otimes U$. Recall from theorem 2.3 that $U \otimes U = W^4$ where W is a representation of $\Sigma \times \Sigma$ of type $\mathbf{R}$. Let $P = \{(g, g) | g \in \Sigma\}$ be the diagonal subgroup. Then $\dim \mathrm{Fix}_W(P) = 1$ and P is an axial subgroup of the representation of $\Sigma \times \Sigma$ on W. The verification of this dimension calculation can be done with character theory.

4. Steady-state bifurcation in rings of coupled cells

We now consider an example where $\mathcal{L} = \mathbf{O}(2)$ and $\mathcal{G} = \mathbf{D}_N$ both act irreducibly on two-dimensional spaces which we identify with $\mathbf{C}$. (This implies that $N \geqslant 3$.) We assume that these actions are the standard ones for these groups. We use the following notation for elements of the 'standard' $\mathbf{D}_N$: we write ζ for the rotation $R_{2\pi/N}$ and κ_1 for complex conjugation. The elements ζ and κ_1 generate $\mathbf{D}_N$. When N is odd all reflections are conjugate to κ_1. When N is even there are two conjugacy classes of reflections; the second one is generated by $\kappa_2 = \zeta\kappa_1$. In $\mathbf{O}(2)$ all reflections are conjugate to the standard reflection κ, which acts in the same way as κ_1.

Proposition 4.1. *Assume* $N \geqslant 3$. *Then there are precisely three conjugacy classes of axial groups* $P \subset \mathbf{O}(2) \times \mathbf{D}_N$ *acting on* $\mathbf{C} \otimes \mathbf{C}$. *Representatives are:*

(1) $\mathbf{D}_1[\kappa]\dot{\times}\mathbf{D}_1[\kappa_1]$,
(2 *even*) $\mathbf{D}_1[\kappa]\dot{\times}\mathbf{D}_1[\kappa_2]$ *if* N *is even*
(2 *odd*) $\{(I, I), (\kappa, I), (-I, \kappa_2), (-\kappa, \kappa_2)\}$ *if* N *is odd,*
(3) $\tilde{\mathbf{D}}_N = \{(\gamma, \gamma) : \gamma \in \mathbf{D}_N\} \cong \mathbf{D}_N$.

Proof. Since $-I \in \mathbf{O}(2)$ we can use proposition 3.3 to classify the axial subgroups P where $\dim \mathrm{Fix}_V(B) = 1$. Here we use the notation of the previous section, in which $A = P \cap (\mathbf{O}(2) \times \mathbf{1})$ and $B = P \cap (\mathbf{1} \times \mathbf{D}_N)$. Note that this proposition implies that we can assume that B is axial. Up to conjugacy the possible subgroups B are $\mathbf{D}_1[\kappa_1]$ (for all N) and $\mathbf{D}_1[\kappa_2]$ (when N is even). Similarly, proposition 3.3 guarantees that A is axial and hence, in this case, up to conjugacy $A = \mathbf{D}_1[\kappa]$. Thus $\mathbf{D}_1[\kappa]\dot{\times}\mathbf{D}_1[\kappa_1]$ is axial (for all N) and $\mathbf{D}_1[\kappa]\dot{\times}\mathbf{D}_1[\kappa_2]$ is axial (when N is even). Thus we may assume that $\dim \mathrm{Fix}_V(B) = 2$.

When N is even, $-I \in \mathbf{D}_N$ and we may reverse the roles of A and B. When we do this, we find no new axial subgroups P and we may assume that $\dim \mathrm{Fix}_U(A) = 2$. When N is odd, we use remark 3.5 to complete the analysis of this case. Note that $\tilde{\mathcal{G}} = \mathbf{D}_N \oplus \mathbf{Z}_2(-I) = \mathbf{D}_{2N}$. Now there is a new axial group $\tilde{P} = \mathbf{D}_1[\kappa]\dot{\times}\mathbf{D}_1[\kappa_2]$. It is easy to check that $P = \tilde{P} \cap (\mathcal{G} \times \mathcal{L})$ is the group listed in (2 odd). Thus we may also assume that $\dim \mathrm{Fix}_U(A) = 2$ when N is odd.

There are no axial subgroups when $\Pi_{\mathcal{G}} = \mathbf{Z}_k$ since the representations of $\mathbf{Z}_k$ on $\mathbf{C}$ are either nonabsolutely irreducible ($k \geqslant 3$) or the direct sum of two isomorphic irreducible representations ($k = 1$ or $k = 2$). Here we use corollary 3.8.

We can therefore write P as a twisted group of the form

$$P^k(\phi) = \{(\phi(\gamma), \gamma) : \gamma \in \mathbf{D}_k\}$$

where $\phi : \mathbf{D}_k \to \mathbf{O}(2)$ is an isomorphism onto the standard $\mathbf{D}_k \subset \mathbf{O}(2)$. Observe that $\phi(\kappa)$ is of order two and is conjugate to κ in $\mathbf{O}(2)$. Suppose this conjugacy is given by γ so that $\gamma\phi(\kappa)\gamma^{-1} = \kappa$. Then conjugating $P^k(\phi)$ by $(\gamma, 1)$ puts ϕ in the form

$$\phi(\kappa) = \kappa \qquad \text{and} \qquad \phi(\zeta) = \zeta^{\ell}$$

where k and ℓ are coprime. Moreover, we can conjugate $P^k(\phi)$ by κ to obtain $1 \leqslant \ell \leqslant \frac{k}{2}$. Let $\phi^1(\gamma) = \gamma$. If $\phi \neq \phi^1$, then the irreducible representations ρ_U of $P^k(\phi)$ on U and ρ_V of $P^k(\phi)$ on V are not isomorphic. To see this observe that $P^k(\phi) \cong \mathbf{D}_k$ and $U \cong V \cong \mathbf{C}$. The action ρ_V of $\mathbf{D}_k$ on $\mathbf{C}$ is the standard one while the action ρ_U of $\mathbf{D}_k$ on $\mathbf{C}$ is generated by

$$\rho_U(\kappa) = \kappa \qquad \text{and} \qquad \rho_U(\zeta) = \zeta^{\ell}\,.$$

A direct calculation shows that ρ_U are ρ_V are not isomorphic unless $\ell = 1$. Then proposition 3.7 implies that these twisted $P^k(\phi)$ are not axial. We can assume

$$P = P^k(I) \equiv \{(\gamma, \gamma) : \gamma \in \mathbf{D}_k\},$$

whence $P \subset \tilde{\mathbf{D}}_n$. Now maximality of P implies that $P = \tilde{\mathbf{D}}_n$. □

5. Hopf bifurcation

We now review the general theory for equivariant Hopf bifurcation. We emphasize the natural complex structure that occurs in these bifurcations.

Let Γ be a compact Lie group, acting on a finite-dimensional real vector space X, and let $f : X \times \mathbf{R} \to X$ be a Γ-equivariant C^∞ mapping. Consider the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \tag{5.1}$$

where $x \in X$, and $\lambda \in \mathbf{R}$ is a bifurcation parameter. Suppose that $f(0,0) = 0$ and that at $\lambda = 0$ the linearization $(\mathrm{d}f)_0$ has a nonresonant complex conjugate pair of purely imaginary eigenvalues $\pm\omega\mathrm{i}$ where $\omega \neq 0$. Then (see Golubitsky and Stewart [9], Golubitsky *et al* [12]) periodic solutions to (5.1) of period near $2/\pi\omega$ are in one-to-one correspondence with zeros of a reduced bifurcation equation

$$g(y, \lambda) = 0$$

where Y is the $\pm\mathrm{i}\omega$ real eigenspace of $(\mathrm{d}f)_0$, and where $g : Y \times \mathbf{R} \to Y$ is C^∞ and $\Gamma \times \mathbf{S}^1$-equivariant. Here the circle group action is induced by the linear flow of $(\mathrm{d}f)_0$ on Y.

Periodic solutions in Hopf bifurcations are identified using the group structure of $\Gamma^* = \Gamma \times \mathbf{S}^1$ acting on Y as follows. As shown in [9, 12] there are branches of periodic solutions corresponding to $\mathbf{C}$-axial subgroups of Γ^*. A subgroup $\Sigma \subset \Gamma^*$ is $\mathbf{C}$-*axial* if Σ is an isotropy subgroup satisfying $\dim \mathrm{Fix}_Y(\Sigma) = 2$. Note that if $(\gamma, \theta) \in \Sigma$ then the corresponding periodic solution $x(t)$ to (5.1) satisfies $x(t + \theta) = \gamma x(t)$, thus yielding a mixed spatio-temporal symmetry.

5.1. *Complex structure*

We denote elements of $\mathbf{S}^1$ by $\theta \in [0, 2\pi)$. By nonresonance this action is *fixed-point free*, that is, if $\theta \cdot y = y$ for $\theta \in \mathbf{S}^1$ and $y \in Y$ then $\theta = 0$ or $y = 0$.

By Golubitsky *et al* [12], XVI, proposition 1.4 the centre subspace Y is generically Γ-*simple*, that is, either

$Y \cong W \oplus W$ where W is absolutely irreducible under Γ or
Y is nonabsolutely irreducible under Γ.

We can use the $\mathbf{S}^1$-action to give Y the structure of a complex vector space. To do this, let $z \in \mathbf{C}$, where $z = r\mathrm{e}^{\mathrm{i}\theta}$, $r \geqslant 0$, and $\theta \in \mathbf{S}^1$. Define

$$zy = \theta \cdot (ry) = r(\theta \cdot y)\,. \tag{5.2}$$

Because the $\mathbf{S}^1$-action is fixed-point free, it follows that if $z, y \neq 0$ then $zy \neq 0$. The remaining properties of a complex vector space are easily verified. We call this complex vector space $Y_{\mathbf{C}} \cong Y \otimes \mathbf{C}$.

Lemma 5.1. *The following statements are equivalent:*

(a) Y is Γ-simple as a real representation.
(b) $Y \otimes \mathbf{C}$ is a real irreducible representation of $\Gamma \times \mathbf{S}^1$ of complex type.
(c) $Y_{\mathbf{C}}$ is a complex irreducible representation of Γ.

Proof. See Golubitsky *et al* [12], XVI, proposition 3.5. □

We can now redefine $\mathbf{C}$-axial subgroups using the complex structure on $Y_{\mathbf{C}}$. A subgroup Σ is $\mathbf{C}$-*axial* if it is an isotropy subgroup with a complex one-dimensional fixed-point subspace. Note that $\mathbf{C}$-axial subgroups are maximal isotropy subgroups.

It is shown in Golubitsky *et al* [12] that, in the case of Γ-simple centre subspaces, isotropy subgroups of Γ^* always have the form of a twisted subgroup. A *twisted subgroup* is a subgroup $\Sigma = A^{\phi} \subset \Gamma \times \mathbf{S}^1$ where $A \subset \Gamma$ is the projection of Σ into Γ, the map $\phi : A \to \mathbf{S}^1$ is a homomorphism, and

$$A^{\phi} = \{(a, \phi(a)) : a \in A\}\,.$$

In short, in a twisted subgroup there are no elements of the form $(1, \theta)$ where $\theta \neq 0$ and this point follows from the assumption in Hopf bifurcation that the critical centre subspace is Γ-simple.

5.2. Complex tensor product representations

We now specialize to the case of interest in this paper, Hopf bifurcation in a $\mathcal{G}$-symmetric network of $\mathcal{L}$-symmetric cells. Then $\Gamma = \mathcal{L} \times \mathcal{G}$ and $X = U \otimes V$ as before. By lemma 5.1, generically the action of $\mathcal{L} \times \mathcal{G}$ on the imaginary eigenspace $Y_{\mathbf{C}}$ is a complex irreducible representation of $\mathcal{L} \times \mathcal{G}$, and we henceforth assume this. A crucial simplification occurs in this case, as follows:

Lemma 5.2. *As a complex representation of $\Gamma = \mathcal{L} \times \mathcal{G}$,*

$$Y_{\mathbf{C}} \cong U' \otimes_{\mathrm{c}} V' \tag{5.3}$$

where U' is a $\mathbf{C}$-irreducible representation of $\mathcal{L}$ and V' is a $\mathbf{C}$-irreducible representation of $\mathcal{G}$.

Proof. Every complex irreducible for $\mathcal{L}\times\mathcal{G}$ is a tensor product of a complex irreducible for $\mathcal{L}$ and a complex irreducible for $\mathcal{G}$, see Bröcker and tom Dieck [5], chapter 2, proposition 4.14. □

Suppose that the Γ-simple real representation has the form $Y = (U\otimes V)\oplus(U\otimes V)$ where U and V are absolutely irreducible representations of $\mathcal{L}$ and $\mathcal{G}$, respectively. Then U' and V' in lemma 5.2 can easily be identified. They are $U' = U\otimes\mathbf{C}$ and $V' = V\otimes\mathbf{C}$; that is, U' and V' are the complexifications of U and V. When the Γ-simple representation is obtained in a different way (the full list of ways can be found by consulting theorem 2.3), then the identification of U' and V' is more difficult to describe. From the point of view of the discussion here, the main point is that the complex representations U' and V' always exist and we can proceed with a general discussion of Hopf bifurcation under this assumption.

For simplicity of notation we henceforth omit the primes on U' and V', so that $Y_{\mathbf{C}} = U \otimes V$ where U is **C**-irreducible for $\mathcal{L}$ and V is **C**-irreducible for $\mathcal{G}$.

5.3. Circle actions and tensor products

We aim to relate Hopf bifurcation for the symmetry group $\mathcal{L} \times \mathcal{G}$ to Hopf bifurcation for $\mathcal{L}$ and $\mathcal{G}$ separately—in part because much is known about Hopf bifurcation for particular groups such as $\mathbf{O}(2)$, $\mathbf{SO}(2)$, $\mathbf{SO}(3)$, $\mathbf{D}_N$, $\mathbf{Z}_N$, and $\mathbf{S}_N$. To do this it turns out to be convenient to work in a group slightly larger than $\mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$, defined as follows. Let

$$\begin{aligned} \mathcal{L}^* &= \mathcal{L} \times \mathbf{S}^1 \\ \mathcal{G}^* &= \mathcal{G} \times \mathbf{S}^1 \\ \Omega &= \mathcal{L}^* \times \mathcal{G}^* = \mathcal{L} \times \mathbf{S}^1 \times \mathcal{G} \times \mathbf{S}^1 . \end{aligned}$$

There is a homomorphism

$$\begin{aligned} &\Theta : \Omega \to \mathcal{L} \times \mathcal{G} \times \mathbf{S}^1 \\ &\Theta(\ell, \phi, g, \psi) = (\ell, g, \phi + \psi) . \end{aligned} \tag{5.4}$$

The kernel of Θ is the antidiagonal subgroup

$$\Delta = \{(1, \phi, 1, -\phi)\} \cong \mathbf{S}^1 .$$

We define an action of Ω on $Y_{\mathbf{C}}$ by

$$(\ell, \phi, g, \psi) \cdot y = \Theta(\ell, \phi, g, \psi) \cdot y = (\ell, g, \phi + \psi) \cdot y = \mathrm{e}^{\mathrm{i}(\phi+\psi)}(\ell, g) \cdot y . \tag{5.5}$$

This is the action induced from the $\mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$-action via the homomorphism Θ. Therefore, $Y_{\mathbf{C}}$ is Ω-irreducible.

Formula (5.5) implies that the complex tensor product is compatible with the $\mathbf{S}^1$-actions of $\mathcal{L}^* = \mathcal{L} \times \mathbf{S}^1$ and $\mathcal{G}^* = \mathcal{G} \times \mathbf{S}^1$ in the following sense:

$$\begin{aligned} (\ell, \phi, g, \psi) \cdot (u \otimes_{\mathrm{c}} v) &= \mathrm{e}^{\mathrm{i}(\phi+\psi)}(\ell, g) \cdot (u \otimes_{\mathrm{c}} v) \\ &= \mathrm{e}^{\mathrm{i}(\phi+\psi)}((\ell \cdot u) \otimes_{\mathrm{c}} (g \cdot v)) \\ &= (\mathrm{e}^{\mathrm{i}\phi}\ell \cdot u) \otimes_{\mathrm{c}} (\mathrm{e}^{\mathrm{i}\psi} g \cdot v) \\ &= ((\ell, \phi) \cdot u) \otimes_{\mathrm{c}} ((g, \psi) \cdot v) . \end{aligned}$$

In particular, the antidiagonal group Δ acts trivially on $U \otimes_{\mathrm{c}} V$, as by definition it must do.

6. C-axial subgroups

In this section we adapt the results of section 2 to the case when $\mathcal{L}$ and $\mathcal{G}$ are compact Lie groups acting on complex vector spaces U and V, respectively. The real tensor product $U \otimes V$ will now be replaced by the complex tensor product $U \otimes_{\mathrm{c}} V$. Until further notice 'one-dimensional' will mean 'one-dimensional as a vector space over **C**.' For clarity we use the notation $\dim_{\mathbf{C}}$ for the complex dimension.

By (5.5), if $P \subset \mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$ and $Q = \Theta^{-1}(P)$ where Θ is defined in (5.4), then

$$\mathrm{Fix}_Y(P) = \mathrm{Fix}_Y(Q) . \tag{6.1}$$

Thus we have:

Lemma 6.1. *There is a one-to-one correspondence between* $\mathbf{C}$*-axial subgroups* $P \subset \mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$ *and* $\mathbf{C}$*-axial subgroups* $Q \subset \Omega$*. This correspondence is given by* $Q = \Theta^{-1}(P)$*.*

Proof. Since P and Q fix the same set of points in Y, we see that if one is an isotropy subgroup, then so is the other. Similarly, if one has a one-dimensional fixed-point subspace, then so does the other. □

Note that Q is not a twisted subgroup.

6.1. Twisted subgroups in $\mathcal{L} \times \mathcal{G}$

There is a simple way to combine twisted subgroups $A^\phi \subset \mathcal{L}^*$ and $B^\Psi \subset \mathcal{G}^*$ to get the *twisted product* subgroup $A^\phi \dot{\times} B^\Psi \subset \mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$ as follows.

Twist the direct product $A \times B \subset \mathcal{L} \times \mathcal{G}$ using the homomorphism $\phi + \psi$ defined by $(\phi + \psi)(a, b) = \phi(a) + \psi(b)$. Equivalently,

$$A^\phi \dot{\times} B^\Psi = \Theta(A^\phi \times B^\Psi) . \tag{6.2}$$

Note that

$$\Theta^{-1}(A^\phi \dot{\times} B^\Psi) = (A^\phi \times B^\Psi) \times \Delta . \tag{6.3}$$

We now have the complex analogue of lemma 3.1:

Lemma 6.2.

$$\dim_{\mathbf{C}} \mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(A^\phi \dot{\times} B^\Psi) = \dim_{\mathbf{C}} \mathrm{Fix}_U(A^\phi) \cdot \dim_{\mathbf{C}} \mathrm{Fix}_V(B^\Psi) .$$

Proof. By (6.1)

$$\mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(A^\phi \dot{\times} B^\Psi) = \mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(\Theta^{-1}(A^\phi \dot{\times} B^\Psi))$$

and by (6.3) this is equal to

$$\mathrm{Fix}_{U \otimes_{\mathbf{C}} V}((A^\phi \times B^\Psi) \times \Delta)$$

which in turn is the same as

$$\mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(A^\phi \times B^\Psi)$$

since Δ acts trivially. The proof now follows from the complex analogue of lemma 3.1. □

This leads to a simple way to obtain $\mathbf{C}$-axial subgroups of Ω from $\mathbf{C}$-axial subgroups of $\mathcal{L}^*$ and $\mathcal{G}^*$. But first we prove a lemma. We assume that $\mathcal{L}^*$ and $\mathcal{G}^*$ have unitary actions on the complex vector spaces U and V, respectively. It follows that there is a notion of length for vectors in these spaces using the group-invariant (Hermitian) inner product.

Lemma 6.3. *Let* U *and* V *be complex vector spaces with nonzero equal length vectors* $u_1, u_2 \in U$ *and* $v_1, v_2 \in V$*. Then*

$$u_1 \otimes_{\mathrm{c}} v_1 = \mathrm{e}^{\mathrm{i}\phi} u_2 \otimes_{\mathrm{c}} v_2$$

if and only if

$$u_1 = \mathrm{e}^{\mathrm{i}\Psi_u} u_2 \qquad \textit{and} \qquad v_1 = \mathrm{e}^{\mathrm{i}\Psi_v} v_2,$$

where $\Psi_u + \Psi_v = \phi$*.*

Proof. Suppose v_1 and v_2 are linearly independent in V. Then $u_1 \otimes_{\mathrm{c}} v_1 - (\mathrm{e}^{\mathrm{i}\phi} u_2) \otimes_{\mathrm{c}} v_2 = 0$ implies that $u_1 = 0$ contradicting the assumption that these vectors are nonzero. Thus v_1 and v_2 are dependent which implies that $v_1 = c v_2$ for some complex scalar c. Since $|v_1| = |v_2|$, $|c| = 1$ and $v_1 = \mathrm{e}^{\mathrm{i}\Psi_v} v_2$. It follows that $(\mathrm{e}^{\mathrm{i}\Psi_v} u_1 - \mathrm{e}^{\mathrm{i}\phi} u_2) \otimes_{\mathrm{c}} v_2 = 0$. Hence $u_1 = \mathrm{e}^{\mathrm{i}(\phi - \Psi_v)} u_2$. So we may choose $\Psi_u = \phi - \Psi_v$. □

Proposition 6.4. *Suppose that* $A^\phi \subset \mathcal{L}^*$ *and* $B^\Psi \subset \mathcal{G}^*$ *are* **C**-*axial. Then* $P = A^\phi \dot{\times} B^\Psi \subset \mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$ *is* **C**-*axial.*

Proof. Lemma 6.2 states that P has a complex one-dimensional fixed-point subspace. Therefore we need show only that P is an isotropy subgroup. By the proof of lemma 6.1 it is equivalent to show that $Q = \Theta^{-1}(P)$ is an isotropy subgroup. Let $u_0 \in U$ be a nonzero fixed point of A^ϕ and let $v_0 \in V$ be a nonzero fixed point of B^Ψ. Then $w_0 = u_0 \otimes_{\mathrm{C}} v_0$ is a nonzero fixed point of P, so P is contained is the isotropy subgroup of w_0. We now suppose that $(\ell, \phi, g, \psi) \in \mathcal{L}^* \times \mathcal{G}^*$ fixes w_0; that is,

$$
\begin{aligned}
u_0 \otimes_{\mathrm{C}} v_0 &= \mathrm{e}^{\mathrm{i}\phi}(\ell \cdot u_0) \otimes_{\mathrm{C}} \mathrm{e}^{\mathrm{i}\psi}(g \cdot v_0) \\
&= \mathrm{e}^{\mathrm{i}(\phi+\psi)}(\ell \cdot u_0) \otimes_{\mathrm{C}} (g \cdot v_0)\,.
\end{aligned}
$$

By lemma 6.3

$$\ell \cdot u_0 = \mathrm{e}^{-\mathrm{i}\theta_\ell} u_0 \tag{6.4}$$

$$g \cdot v_0 = \mathrm{e}^{-\mathrm{i}\theta_g} v_0 \tag{6.5}$$

for some θ_ℓ and θ_g such that $\theta_\ell + \theta_g = \phi + \psi$. Therefore $(\ell, \theta_\ell) \in A^\phi$ since A^ϕ is the isotropy subgroup of u_0. Similarly, $(g, \theta_g) \in B^\Psi$. Since A^ϕ and B^Ψ are twisted subgroups, we see that $\theta_\ell = \phi(\ell)$, $\theta_g = \psi(g)$. Hence $(\ell, \theta_\ell, g, \theta_g) \in A^\phi \times B^\Psi$. Since $\theta_\ell + \theta_g = \phi + \psi$ it follows that $(\ell, g, \phi + \psi) \in A^\phi \dot{\times} B^\Psi$. $\square$

There is a partial converse to proposition 6.4 given as follows.

Proposition 6.5. *Let* $P \subset \mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$ *be* **C**-*axial. Let* $Q = \Theta^{-1}(P)$. *Define* $\hat{A} = Q \cap \mathcal{L}^*$ *and* $\hat{B} = Q \cap \mathcal{G}^*$. *Suppose that* $\dim_{\mathbf{C}} \mathrm{Fix}_U(\hat{A}) = 1$. *Then* $\hat{A}$ *and* $\hat{B}$ *are* **C**-*axial and* $P = \hat{A} \dot{\times} \hat{B}$.

Proof. By lemma 6.2, $\mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(Q)$ is one-dimensional. Since $Q \supset \hat{A} \times \hat{B}$ we have

$$\mathrm{Fix}_U(\hat{A}) \otimes_{\mathrm{C}} \mathrm{Fix}_V(\hat{B}) \supset \mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(Q) \neq \{0\},$$

so that $\mathrm{Fix}_V(\hat{B}) \neq \{0\}$. Since $\dim_{\mathbf{C}} \mathrm{Fix}_U(\hat{A}) = 1$ we may choose a vector $u \in U$ that spans $\mathrm{Fix}_U(\hat{A})$ over **C**. Let w span $\mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(Q)$ over **C**; then w is of the form $w = u \otimes_{\mathrm{C}} v$ for some $v \in \mathrm{Fix}_V(\hat{B})$.

First, we claim that the isotropy subgroup Σ_u of u in $\mathcal{L}^*$ is equal to $\hat{A}$. Certainly $\Sigma_u \supset \hat{A}$. But if $\sigma \in \Sigma_u$, so that $(\sigma, 1) \in \mathcal{L}^* \times \mathcal{G}^*$, then $(\sigma, 1)(u \otimes_{\mathrm{C}} v) = (\sigma u) \otimes_{\mathrm{C}} v = u \otimes_{\mathrm{C}} v$. Hence $(\sigma, 1) \in P$ and $\sigma \in \hat{A}$. Similarly, $\hat{B}$ is the isotropy subgroup of v in $\mathcal{G}^*$. Note that we have not yet proved that $\hat{B}$ is axial, since we have not yet computed $\dim_{\mathbf{C}} \mathrm{Fix}_V(\hat{B})$. We have however shown that $Q \supset \hat{A} \times \hat{B}$ from which it follows that $P \supset \hat{A} \dot{\times} \hat{B}$.

Now suppose that $(\ell, g) \in Q$ where $\ell \in \mathcal{L}^*$, $g \in \mathcal{G}^*$. Then

$$(\ell, g) \cdot (u \otimes_{\mathrm{C}} v) = u \otimes_{\mathrm{C}} v,$$

or equivalently

$$(\ell \cdot u) \otimes_{\mathrm{C}} (g \cdot v) = u \otimes_{\mathrm{C}} v\,.$$

By lemma 6.3, there exists $\theta \in \mathbf{S}^1$ such that

$$\ell \cdot u = \mathrm{e}^{\mathrm{i}\theta} u \qquad g \cdot v = \mathrm{e}^{-\mathrm{i}\theta} v\,.$$

That is,

$$(\ell, -\theta) \in \Sigma_u = \hat{A} \qquad (g, \theta) \in \Sigma_v = \hat{B}\,.$$

Therefore,

$$(\ell, -\theta, g, \theta) \in (\hat{A} \times \hat{B}) \times \Delta\,.$$

Hence $Q \subset (\hat{A} \times \hat{B}) \times \Delta$ and $P \subset \hat{A} \dot{\times} \hat{B}$. Since

$$\dim_{\mathbf{C}} \mathrm{Fix}_{U \otimes_{\mathbf{C}} V}(P) = \dim_{\mathbf{C}} \mathrm{Fix}_U(\hat{A}) \cdot \dim_{\mathbf{C}} \mathrm{Fix}_V(\hat{B}),$$

it follows that $\dim_{\mathbf{C}} \mathrm{Fix}_{V'}(\hat{B}) = 1$ and $\hat{B}$ is axial. □

6.2. Representation-theoretic criteria for **C***-axiality*

We now present the analogue of proposition 3.7. Let

$$\Pi_{\mathcal{L}^*} : \Omega \to \mathcal{L}^* \qquad \text{and} \qquad \Pi_{\mathcal{G}^*} : \Omega \to \mathcal{G}^*$$

be the canonical projections. Given a subgroup $Q \subset \Omega$, we define representations of Q on U and V as follows:

$$\rho_U = \eta_U \circ \Pi_{\mathcal{L}^*} \qquad \text{and} \qquad \rho_V = \eta_V \circ \Pi_{\mathcal{G}^*},$$

where η_U and η_V are the given representations of $\mathcal{L}^*$ and $\mathcal{G}^*$ on U and V, respectively.

Proposition 6.6. *Let $Q \subset \Omega$ be a maximal isotropy subgroup. The Q has a one-dimensional fixed-point subspace if and only if there is precisely one irreducible representation of the action ρ_U of Q on U that is isomorphic to precisely one irreducible representation of the action ρ_V of Q on V.*

Proof. The proof is just like the proof of proposition 3.7, but with one simplification. Over **C** Schur's lemma implies that $\dim \mathcal{D}_{\Gamma(U,V)} = 1$ if U and V are isomorphic irreducible representations. See (3.4). □

6.3. Complexification of irreducibles of real type

In many applications the representations of the groups $\mathcal{L}$ and $\mathcal{G}$ on U and V are of real type, that is,

(H1) $\quad U \cong U_0 \oplus U_0 \cong U_0 \otimes \mathbf{C} \quad$ and $\quad V \cong V_0 \oplus V_0 \cong V_0 \otimes \mathbf{C}$.

If this hypothesis holds, then we can add extra detail. Observe the following isomorphism of complex vector spaces:

$$(U_0 \otimes V_0) \otimes \mathbf{C} \cong (U_0 \otimes \mathbf{C}) \otimes_{\mathrm{c}} (V_0 \otimes \mathbf{C})\,. \tag{6.6}$$

Henceforth we omit the subscripts on U_0 and V_0, so that what we have previously called U and V now become $U \oplus U \cong U \otimes \mathbf{C}$ and $V \oplus V \cong V \otimes \mathbf{C}$. We introduce the notation

$$Y_{\mathbf{C}} \equiv (U \otimes \mathbf{C}) \otimes_{\mathrm{c}} (V \otimes \mathbf{C})\,.$$

Next, we wish to state and prove the corollary analogous to corollary 3.8. In order to do this we need to discuss the relationship between real irreducible representations and their complexifications. We summarize this discussion in the following lemma.

Lemma 6.7. *Let the group A act irreducibly on the real vector space U and let $\Sigma = A \times \mathbf{S}^1$. Then*

(a) Σ acts irreducibly on $U \otimes \mathbf{C}$ as a complex representation if and only if A acts absolutely irreducibly on U.
(b) If U is of complex type, then $U \otimes \mathbf{C} = W \oplus \overline{W}$, where W is a representation of Σ of complex type and W and $\overline{W}$ are distinct.
(c) If U is of quaternionic type, then $U \otimes \mathbf{C} = W \oplus W$, where W is a representation of Σ of complex type.

Proof. With one exception the proof follows directly from theorem 2.3 using the observation that $\mathbf{S}^1$ acts on $\mathbf{C}$ as an irreducible representation of complex type.

(a) An irreducible representation U tensored with an irreducible representation of complex type can be irreducible only if U is of real type.
(b) The tensor product of two irreducible representations of complex type has the desired form. Since U is of complex type there exits a linear mapping $J : U \to U$ that commutes with A and for which $J^2 = -I$. Then $W = \{u + \mathrm{i}Ju : u \in U\}$ and $\overline{W} = \{u - \mathrm{i}Ju : u \in U\}$ where we identify $U \otimes \mathbf{C} = U \oplus \mathrm{i}U$. A calculation shows that there are no linear isomorphisms from W to $\overline{W}$ that commute with Σ.
(c) The tensor product of two irreducible representations, one of quaternionic type and one of complex type, has the desired form. □

Corollary 6.8. *Assume hypothesis (H1). Suppose that $Q = \Theta^{-1}(P)$ is an isotropy subgroup with a (complex) one-dimensional fixed-point subspace. Let $A = \Pi_{\mathcal{L}}(Q)$ and*

$$U = U_1 \oplus \cdots \oplus U_s,$$

where the U_j are A-irreducible subspaces. Then at least one of the U_j is A-absolutely irreducible and distinct from the other U_j.

Proof. The matrices of Q acting on $U \otimes \mathbf{C}$ by $\rho_{U_j}|Q$ are the same as those of $\Sigma = A \times \mathbf{S}^1$ acting on $U \otimes \mathbf{C}$ by $\eta_{U_j}|\Sigma$. Therefore, a subspace of $U \otimes \mathbf{C}$ is Q-irreducible if and only if it is Σ-irreducible.

Similarly, we define $B = \Pi_{\mathcal{G}}(Q)$, write $V = V_1 \oplus \cdots \oplus V_t$ as a direct sum of B-irreducible representations, and note that the matrices of Q acting on $V \otimes \mathbf{C}$ by $\rho_{V_j}|Q$ are the same as those of $T = B \times \mathbf{S}^1$ acting on $V \otimes \mathbf{C}$ by $\eta_{V_j}|T$.

The dimension of the fixed-point subspace of Q is given by the number of common representations of Q using ρ_U and ρ_V. See proposition 6.6.

We can write $U \otimes \mathbf{C}$ as

$$U \otimes \mathbf{C} = (U_1 \otimes \mathbf{C}) \oplus \cdots \oplus (U_s \otimes \mathbf{C}) \tag{6.7}$$

a direct sum of Σ-invariant subspaces. Similarly for V.

We first prove that at least one of the U_j is A-absolutely irreducible. Suppose that all of the U_j are nonabsolutely irreducible. Then $U_j \otimes \mathbf{C}$ is the sum of two Σ-irreducible representations of complex type. If one of these representations is isomorphic to a $V_i \otimes \mathbf{C}$ representation, then there must be two pairs of isomorphic representations (see lemma 6.7) and the complex dimension of $\mathrm{Fix}_{Y_{\mathbf{C}}}(Q) > 1$.

If all of the absolutely irreducible representations U_j have multiplicity greater than one, then the Σ-irreducible representations $U_j \otimes \mathbf{C}$ have multiplicity greater than one, and again $\dim_{\mathbf{C}} \mathrm{Fix}_{Y_{\mathbf{C}}}(Q) > 1$. □

Corollary 6.9. *Suppose that $Q = \Theta^{-1}(P)$ is an isotropy subgroup with a one-dimensional fixed-point subspace. Let $A = \Pi_{\mathcal{G}^*}(Q)$ and*

$$U = U_1 \oplus \cdots \oplus U_s$$

where the U_j are $\mathbf{C}$-irreducible subspaces for A. Then at least one of the U_j is distinct from the other U_j.

Proof. As in the real case, corollary 3.8. □

7. Hopf bifurcation in rings of coupled cells

In this section we derive the $\mathbf{C}$-axial subgroups of $\mathbf{O}(2) \times \mathbf{D}_N$ where $\mathbf{O}(2)$ and $\mathbf{D}_N$ each have critical eigenvalues that are double. More precisely, we assume that $\mathbf{O}(2)$ acts on $U = \mathbf{C}$ by the standard representation; that is, the action of $\mathbf{O}(2)$ is generated by

$$\kappa z = \bar{z} \qquad \phi \cdot z = \mathrm{e}^{\mathrm{i}\phi} z \, .$$

Similarly, we assume that $\mathbf{D}_N$ acts on $V = \mathbf{C}$ by its standard representation generated by κ and

$$\zeta \cdot z = \mathrm{e}^{2\pi \mathrm{i}/N} z \, .$$

We consider Hopf bifurcation of $\mathbf{O}(2) \times \mathbf{D}_N$ corresponding to the action on $U \otimes V = \mathbf{C} \otimes \mathbf{C}$. The critical eigenspace for this bifurcation will be eight-dimensional (over $\mathbf{R}$) and have the form $(U \otimes V) \otimes \mathbf{C}$.

We begin by recalling from Golubitsky and Stewart [9] and Golubitsky *et al* [12] that $\mathbf{O}(2)$ Hopf bifurcation, when the critical eigenvalues are double, leads to two branches of periodic solutions: rotating waves $\hat{A}_1$ and standing waves $\hat{A}_2$ where

$$\hat{A}_1 = \{(\theta, 1, \theta) \in \mathbf{O}(2) \times \mathbf{D}_N \times \mathbf{S}^1\}$$
$$\hat{A}_2 = \mathbf{D}_1(\kappa, 1, 0) \oplus \mathbf{Z}_2(\pi, 1, \pi) \, .$$

With later discussion in mind, we have written these groups as subgroups of $\mathbf{O}(2) \times \mathbf{D}_N \times \mathbf{S}^1$. Similarly, Golubitsky and Stewart [10] and Golubitsky *et al* [12] study Hopf bifurcation in the presence of $\mathbf{D}_N$ symmetry. When the critical eigenvalues are double, there are three $\mathbf{C}$-axial subgroups for each N—the precise form of these axial subgroups depends on the parity of N. See table 2. These $\mathbf{C}$-axial subgroups represent discrete rotating waves $\hat{B}_3$ and two types of discrete standing waves $\hat{B}_1$ and $\hat{B}_2$. Define

$$\tilde{\mathbf{Z}}_N = \left\{ \left(1, \zeta^k, \frac{-2\pi}{N} k\right) \in \mathbf{O}(2) \times \mathbf{D}_N \times \mathbf{S}^1 : 0 \leqslant k \leqslant N-1 \right\}.$$

For each N Hopf bifurcation of $\mathbf{O}(2) \times \mathbf{D}_N$ produces one $\mathbf{C}$-axial group in addition to the expected twisted product groups. Let

$$\tilde{\mathbf{D}}_N = \{(g, g, 0) \in \mathbf{O}(2) \times \mathbf{D}_N \times \mathbf{S}^1 : g \in \mathbf{D}_N\}.$$

When $N = 0 \bmod 4$, Hopf bifurcation produces two additional $\mathbf{C}$-axial subgroups. Let $\tau \in \mathbf{D}_4 \subset \mathbf{D}_N$ be a reflection and let $\hat{\mathbf{D}}_4[\tau] \subset \mathbf{O}(2) \times \mathbf{D}_N \times \mathbf{S}^1$ be the subgroup generated by $(\frac{\pi}{2}, \frac{\pi}{2}\tau, \frac{\pi}{2})$ and $(\tau, \tau, 0)$.

Proposition 7.1. *When the critical eigenvalues have multiplicity four in an* $\mathbf{O}(2) \times \mathbf{D}_N$ *Hopf bifurcation problem, then up to conjugacy the* $\mathbf{C}$*-axial groups are enumerated as follows:*

(a) $\hat{A}_i \dot{\times} \hat{B}_j$ *where* $i = 1, 2$ *and* $j = 1, 2, 3$.
(b) $\tilde{\mathbf{D}}_N \oplus \mathbf{Z}_2(\pi, 1, \pi)$.
(c) $\hat{\mathbf{D}}_4[\kappa] \oplus \mathbf{Z}_2(\pi, \pi, 0)$ *and* $\hat{\mathbf{D}}_4[\frac{\pi}{2}\kappa] \oplus \mathbf{Z}_2(\pi, \pi, 0)$ *when* $N = 0 \bmod 4$.

Table 2. Isotropy subgroups in $\mathbf{D}_N$ Hopf bifurcation.

	N odd	N even
$\hat{B}_1$	$\mathbf{D}_1(1, \kappa, 0)$	$\mathbf{D}_1(1, \kappa, 0) \oplus \mathbf{Z}_2(1, \pi, \pi)$
$\hat{B}_2$	$\mathbf{D}_1(1, \kappa, \pi)$	$\mathbf{D}_1(1, \kappa\zeta, 0) \oplus \mathbf{Z}_2(1, \pi, \pi)$
$\hat{B}_3$	$\tilde{\mathbf{Z}}_N$	$\tilde{\mathbf{Z}}_N$

The groups in (b) and (c) are not conjugate when $N = 4$.

Proof. Proposition 6.4 proves that the twisted product groups in (a) are all **C**-axial. Let $P \subset \mathbf{O}(2) \times \mathbf{D}_N \times \mathbf{S}^1$ be a **C**-axial subgroup. Let $\hat{B} = P \cap (\mathbf{1} \times \mathbf{D}_N \times \mathbf{S}^1)$ and let $\hat{A} = P \cap (\mathbf{O}(2) \times \mathbf{1} \times \mathbf{S}^1)$. If either $\dim_{\mathbf{C}} \mathrm{Fix}_{V\otimes\mathbf{C}}(\hat{B}) = 1$ or $\dim_{\mathbf{C}} \mathrm{Fix}_{U\otimes\mathbf{C}}(\hat{A}) = 1$, then proposition 6.5 implies that $P = \hat{A} \dot{\times} \hat{B}$.

The other **C**-axial groups P must satisfy $\dim_{\mathbf{C}} \mathrm{Fix}_{V\otimes\mathbf{C}}(\hat{B}) = 2$ and $\dim_{\mathbf{C}} \mathrm{Fix}_{U\otimes\mathbf{C}}(\hat{A}) = 2$. This implies that $\hat{A} = \mathbf{Z}_2(\pi, 1, \pi)$ and

$$\hat{B} = \begin{cases} \mathbf{1} & \text{if } N \text{ is odd} \\ \mathbf{Z}_2(1, \pi, \pi) & \text{if } N \text{ is even}\,. \end{cases}$$

Proposition 6.6 implies that the subgroup in (b) has a one-dimensional fixed-point space. The corresponding isotropy subgroup would have to contain the group in (b). However, any larger group would introduce extra elements into either $\hat{A}$ or $\hat{B}$. Hence the group in (b) is an isotropy subgroup and is **C**-axial. Similarly proposition 6.6 implies that the subgroups in (c) have a one-dimensional fixed-point subspaces.

The remainder of the proof is devoted to showing that up to conjugacy there are no additional **C**-axial subgroups. The group $B = \Pi_{\mathbf{D}_N}(P)$ is isomorphic to $P/\mathbf{Z}_2(\pi, 1, \pi)$. Since $B \subset \mathbf{D}_N$, either $B = \mathbf{Z}_k$ or $B \cong \mathbf{D}_k$ where k divides N. Corollary 6.8 implies that $B = \mathbf{Z}_k$ cannot occur. Similarly, $A = \Pi_{\mathbf{O}(2)}(P) \subset \mathbf{O}(2)$ is a finite subgroup and hence is isomorphic to either $\mathbf{Z}_{k'}$ or $\mathbf{D}_{k'}$. Again, corollary 6.8 implies that $\mathbf{Z}_{k'}$ cannot occur. Finally, by counting the order of P we see that

$$k' = \begin{cases} 2k & \text{if } N \text{ is odd} \\ k & \text{if } N \text{ is even}\,. \end{cases}$$

We claim that, after a conjugacy of P, there is a reflection κ in $\mathbf{D}_k \subset \mathbf{D}_N$ so that $(\kappa, \kappa, 0) \in P$. There is an element $(\tau, \kappa, \theta) \in P$. If τ is a reflection, then, since all reflections in $\mathbf{O}(2)$ are conjugate, we may conjugate P by an element in $\mathbf{O}(2)$ so that τ is conjugated to κ, as claimed. Suppose τ is not a reflection. Since P projects onto $\mathbf{D}_{k'} \subset \mathbf{O}(2)$, there is an element in P whose first coordinate κ' is a reflection. If the second coordinate is also a reflection, we can use the previous argument to show that, after a conjugacy, an element of the form (κ, κ, θ) is in P. Hence, if $(\kappa, \sigma, \psi) \in P$ where σ is a rotation, then $(\tau\kappa, \kappa\sigma, \theta + \psi) \in P$ and again the first two coordinates are reflections. Finally, since $(\kappa, \kappa, \theta)^2 \in P$ it follows that either $\theta = 0$ (which verifies the claim) or $\theta = \pi$. In the latter case, recall that $(\pi, 1, \pi) \in P$ and hence that $(\pi\kappa, \kappa, 0) \in P$. Now conjugate P by $\pi/2 \in \mathbf{O}(2)$ to see that $(\kappa, \kappa, 0) \in P$, as required.

It follows that P is contained in one of the groups listed in (a) when $k = 1$. Hence we may assume that $k \geqslant 2$.

Suppose that N is odd. Then

$$P \cong A \cong \mathbf{D}_{2k} \cong \mathbf{D}_k \oplus \mathbf{Z}_2(\pi, 1, \pi)\,.$$

Indeed, let $\tau \in \mathbf{D}_{2k}$ be a rotation that generates $\mathbf{Z}_{2k} \subset \mathbf{D}_{2k}$ and choose $p_\tau = (\tau, \sigma, \phi) \in P$. Then $P = \mathbf{D}_k \oplus \mathbf{Z}_2(\pi, 1, \pi)$ where $\mathbf{D}_k$ is generated by p_τ^2 and $(\kappa, \kappa, 0)$. We verify this point as follows. If σ is a reflection, then $p_\tau^2 = (\tau^2, 1, 2\phi) \in P$ which implies that $p_\tau^2 \in \mathbf{Z}_2(\pi, 1, \pi)$. Hence either $k = 2$ or $k = 4$ which is not possible since k divides N and N is odd. It follows that σ is a rotation and hence that $\sigma = \tau^m$ for some integer m. We can conjugate $\mathbf{D}_k \subset \mathbf{D}_N$ so that $1 \leqslant m \leqslant k/2$. Finally, we apply the proof of proposition 6.6 to see that $\mathrm{Fix}(P) = \{0\}$ unless $m = 1$. It follows that $P = \{(g, g, 0) : g \in \mathbf{D}_k\} \oplus \mathbf{Z}_2(\pi, 1, \pi)$. Note that $k \geqslant 3$ (which follows since $k \geqslant 1$ is odd) and P is contained inside the group listed in (b). This completes the enumeration of the **C**-axial subgroups when N is odd.

When N is even, we proceed as above, noting that $k' = k$ in this case. Now when $k = 2$ it also follows that P is contained in one of the groups listed in (a). Hence we may assume that $k \geqslant 3$. Again we can choose $\tau \in \mathbf{D}_k \subset \mathbf{O}(2)$ to be a rotation that generates $\mathbf{Z}_k$ and we can choose $p_\tau = (\tau, \sigma, \phi) \in P$. We show that when $N = 2 \bmod 4$ that $P = \mathbf{D}_k \oplus \mathbf{Z}_2(\pi, 1, \pi)$ where $\mathbf{D}_k$ is generated by p_τ and $(\kappa, \kappa, 0)$. As above, when σ is a reflection, we find that either $k = 2$ or $k = 4$. Since $k \geqslant 3$, k divides N, and $N = 2 \bmod 4$; neither of these is possible. We can now proceed as above. Note that the case $N = 2 \bmod 4$ may be solved using proposition 7.3 below.

When $N = 0 \bmod 4$, then the argument is more complicated. When $k \neq 4$, we can proceed as above. However, now we must consider the case $k = 4$ more carefully. We choose $\tau \in \mathbf{Z}_4 \subset \mathbf{O}(2)$ to be a generator. That is, we choose $\tau = \frac{\pi}{2}$. Choose $(\frac{\pi}{2}, \sigma, \phi) \in P$. If σ is a rotation, then proceed as above. Suppose now that σ is a reflection. This possibility cannot be eliminated when $N = 0 \bmod 4$. Observe that $(\frac{\pi}{2}, \sigma, \phi)^2 = (\pi, 1, 2\phi) \in P$. It follows that $\phi = \pm\frac{\pi}{2}$. Since $(1, \pi, \pi) \in P$ we can assume that $\phi = \frac{\pi}{2}$ and that $(\frac{\pi}{2}, \sigma, \frac{\pi}{2}) \in P$. We know that an element of the form $(\kappa, \kappa, 0) \in P$ where κ is a reflection in $\mathbf{D}_4$. Up to conjugacy there are two types of reflection in $\mathbf{D}_4$ and this fact leads to the two different $\mathbf{C}$-axial subgroups in (c). It follows that $(\frac{\pi}{2}\kappa, \sigma\kappa, \frac{\pi}{2}) \in P$ where $\frac{\pi}{2}\kappa$ is a reflection. Squaring yields $(1, (\sigma\kappa)^2, \pi) \in P$. Thus $(\sigma\kappa)^2 = \pi$ and $\sigma\kappa = \pm\frac{\pi}{2}$ which we may rewrite as $\sigma = \pm\frac{\pi}{2}\kappa$. The two reflections $\frac{\pi}{2}\kappa$ and $-\frac{\pi}{2}\kappa$ are conjugate by κ in D_N since N is even. Hence we may conjugate P by $(0, \kappa, 0)$ to show that P is generated by $(\kappa, \kappa, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2}\kappa, \frac{\pi}{2})$ and $(\pi, \pi, 0)$, as claimed. The $\mathbf{C}$-axial subgroups obtained in (b) and (c) are not conjugate—since the twisted reflections in these subgroups are of nonconjugate types. □

Remark 7.2. We can now explain how the figures in the introduction illustrating the nine patterns of oscillation in $\mathbf{O}(2) \times \mathbf{D}_4$ systems were derived. We consider the two subgroups listed in proposition 7.1(c) and leave the remaining seven as an exercise. We number the four cells as in figure 4.

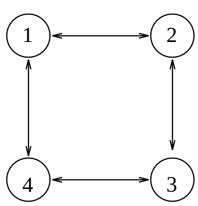

Figure 4. Four cells in a ring.

The Hopf bifurcation considered in proposition 7.1(c) occurs only if the state space of each individual cell contains a standard two-dimensional irreducible representation of $\mathbf{O}(2)$; moreover, we can identify this two-dimensional space with $\mathbf{C}$. We project the 2π-periodic trajectory $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$ into $\mathbf{C}^4$ and identify these coordinates as $(z_1(t), z_2(t), z_3(t), z_4(t))$. The symmetry $\mathbf{Z}_2(\pi, \pi, 0)$ implies that $z_3(t) = -z_1(t)$ and $z_4(t) = -z_2(t)$.

The symmetry $(\kappa, \kappa, 0)$ in the first subgroup implies that $z_4(t) = \bar{z}_1(t)$ and $z_3(t) = \bar{z}_2(t)$. Thus solutions of the first subgroup will have the form

$$(z(t), -\bar{z}(t), -z(t), \bar{z}(t))\,.$$

Finally, the symmetry $(\frac{\pi}{2}, \frac{\pi}{2}\kappa, \frac{\pi}{2})$ implies the spatio-temporal symmetry $z(t + \frac{\pi}{2}) = -\mathrm{i}z(t)$. See figure 3($c$).

The symmetry $(\frac{\pi}{2}\kappa, \frac{\pi}{2}\kappa, 0)$ in the second group implies $z_1(t) = \mathrm{i}\bar{z_1}(t)$ (so that $z_1(t) = x(t)(1+\mathrm{i})$ where $x(t) \in \mathbf{R}$) and $z_4(t) = \mathrm{i}\bar{z}_2(t)$. Since $z_4(t) = -z_2(t)$ we see that $z_2(t) = y(t)(1-\mathrm{i})$ where $y(t) \in \mathbf{R}$. Finally, the spatio-temporal symmetry $(\frac{\pi}{2}, \pi\kappa, \frac{\pi}{2})$ implies that $y(t) = x(t+\frac{\pi}{2})$. Thus, in these coordinates, trajectories for the second group will have the form

$$(x(t)(1+\mathrm{i}), x\left(t+\frac{\pi}{2}\right)(1-\mathrm{i}), -x(t)(1+\mathrm{i}), -x\left(t+\frac{\pi}{2}\right)(1-\mathrm{i}))\,.$$

See figure 3(*b*).

Proposition 7.3. *Let $\mathcal{L}$ act U and let $\mathbf{Z}_2(\tau)$ act nontrivially on $\mathbf{R}$. Then the $\mathbf{C}$-axial subgroups $P \subset \mathcal{L} \times \mathbf{Z}_2(\tau) \times \mathbf{S}^1$ all have the form $P_0 \oplus \mathbf{Z}_2(1, \tau, \pi)$ where $P_0 \subset \mathcal{L} \times \mathbf{1} \times \mathbf{S}^1$ is a $\mathbf{C}$-axial subgroup.*

Proof. Let $\hat{B} = P \cap (\mathbf{1} \times \mathbf{Z}_2(\tau) \times \mathbf{S}^1)$. Since P is $\mathbf{C}$-axial, $\dim \mathrm{Fix}_{\mathbf{R}}(\hat{B}) = 1$ and $\hat{B} = \mathbf{Z}_2(1, \tau, \pi)$ since the group $\mathbf{Z}_2(1, \tau, \pi)$ is the kernel of the representation of $\mathbf{Z}_2(\tau) \times \mathbf{S}^1$ on $\mathbf{R} \otimes \mathbf{C}$. Proposition 6.5 implies that $P = P_0 \dot{\times} \mathbf{Z}_2(1, \tau, \pi)$. Since $\mathbf{Z}_2(1, \tau, \pi)$ commutes with P_0, we can write $P = P_0 \dot{\times} \mathbf{Z}_2(1, \tau, \pi) = P_0 \oplus \mathbf{Z}_2(1, \tau, \pi)$. □

We end this section by describing how our general results make it possible to recover the results of Dangelmayr *et al* [7] on $\mathbf{D}_3 \times \mathbf{D}_3$ Hopf bifurcation. Given our discussion on $\mathbf{O}(2) \times \mathbf{D}_3$ Hopf bifurcation it is a simple matter to recover the existence of the eleven families of periodic solutions found in [7]. We hasten to add, however, that Dangelmayr *et al* do much more than find the existence of these periodic solutions—they also determine their stability for a reduced centre manifold vector field in normal form. They also discuss solutions in this normal form vector field that are more complicated than periodic.

Proposition 7.4. *(Dangelmayr et al [7]). Consider a Hopf bifurcation in the presence of $\mathbf{D}_3 \times \mathbf{D}_3$-symmetry, where each of the critical eigenvalues has multiplicity four. Let B_1, B_2 and B_3 be the three isotropy subgroups of $\mathbf{D}_3 \times \mathbf{S}^1$ acting on $\mathbf{C} \otimes \mathbf{C}$ obtained in Hopf bifurcation with $\mathbf{D}_3$-symmetry. Then up to conjugacy there are eleven families of periodic solutions:*

(a) $\hat{B}_i \dot{\times} \hat{B}_j$ *where* $i = 1, 2, 3$ *and* $j = 1, 2, 3$.
(b) $\tilde{\mathbf{D}}_3 = \{(\ell, \ell, 0) : \ell \in \mathbf{D}_3\}$.
(c) $\hat{\mathbf{D}}_3 = \{(\ell, \ell, \psi(\ell)) : \ell \in \mathbf{D}_3\}$ *where*

$$\psi(\ell) = \begin{cases} 0 & \text{if } \ell \in \mathbf{Z}_3 \\ \pi & \text{if } \ell \in \mathbf{D}_3 - \mathbf{Z}_3\,. \end{cases}$$

Proof. The structure of the proof is identical to that of proposition 7.1. The groups in statement (a) are obtained as products of $\mathbf{C}$-axial groups for $\mathbf{D}_3$ while the remaining two groups are found by a similar argument to that in proposition 7.1. Indeed it is simpler because in this case a $\mathbf{C}$-axial subgroup P projects isomorphically onto both $\mathbf{D}_3 \times \mathbf{S}^1$ factors. □

A similar approach also allows us to recover the existence results of Wegelin [19] for Hopf bifurcation in the cases $\mathbf{O}(2) \times \mathbf{O}(2)$, $\mathbf{D}_m \times \mathbf{O}(2)$, and $\mathbf{D}_m \times \mathbf{D}_n$.

Acknowledgments

We wish to thank the referee whose comments have helped simplify the representation theory described here. We also wish to thank Mike Field for helpful discussions, in particular, for suggesting that we exploit more explicitly the complex structure associated with Hopf bifurcation. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada (BD), NSF Grant DMS-9101836 (MG), the Texas Advanced Research Program (003652037) (MG), a grant from the Science and Engineering Research Council of the UK (INS), and a European Community Laboratory Twinning grant (INS). The research of INS was carried out under the auspices of the European Bifurcation Theory Group. All three authors were supported by the Fields Institute for Research in the Mathematical Sciences, and are grateful for the hospitality extended to them.

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