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Planforms in two and three dimensions

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In honor of Klaus Kirchgässner on the occasion of his sixtieth birthday

1. Introduction

Many systems of partial differential equations are posed on all of $\mathbf{R}^n$ and have Euclidean symmetry. These include the Navier–Stokes equations, the Boussinesq equations, the Kuramoto–Sivashinsky equation and reaction-diffusion systems (with constant diffusion coefficients). In many applications, where these and related Euclidean equivariant equations are used, time independent, spatially periodic solutions are sought; and, typically, they are obtained by bifurcation from an invariant equilibrium. In this paper we attempt to classify, by symmetry, spatially periodic solutions that can be obtained through bifurcation. Our main result is a partial classification of such solutions in two and three spatial dimensions obtained using symmetry methods and equation independent genericity considerations. The remainder of this Introduction is devoted to making precise the kind of classification theorem we intend to prove. We show that a certain class of planforms may be found by solving an algebraic problem whose data is based on the irreducible representations of the symmetry groups of n-dimensional lattices.

The planar planforms are classified in Section 2 (see Theorem 2.1, whose proof is given in Section 3). The main theoretical results (valid for all n) are also presented in Section 2. The classification of planforms in three dimensions is more complicated than in two. In Section 4 we describe our results for the standard cubic lattices; that is, for the standard spatially triply periodic planforms. Details for the other three-dimensional lattices may be found in Dionne [6].

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(a) Reduction to an algebraic problem

The standard method used to find spatially periodic, steady solutions, which we call *planforms*, may be abstracted as follows. Write the system of PDE for steady solutions in operator form between function spaces $\mathscr{X}$ and $\mathscr{Y}$;

$$F(u, \lambda) = 0 \tag{1.1}$$

where $F: \mathscr{X} \times \boldsymbol{R} \to \mathscr{Y}$, λ is a bifurcation parameter, and $u: \boldsymbol{R}^n \to \boldsymbol{R}$. (We have simplified the general situation by assuming that u is real-valued, rather than vector-valued, but when considering bifurcations the general situation can be reduced to this case.)

We assume that there is a *trivial solution* $u = 0$; that is,

$$F(0, \lambda) = 0. \tag{1.2}$$

To find spatially periodic solutions by bifurcation from the trivial solution, one fixes a lattice $\mathscr{L}$ in $\boldsymbol{R}^n$ and demands that

$$u(x + \ell) = u(x) \tag{1.3}$$

for all $\ell \in \mathscr{L}$. Mappings satisfying (1.3) are called *$\mathscr{L}$-periodic*. We denote by $\mathscr{X}_{\mathscr{L}}$ the space of all $\mathscr{L}$-periodic functions in $\mathscr{X}$. The Euclidean invariance of F implies that

$$F: \mathscr{X}_{\mathscr{L}} \times \boldsymbol{R} \to \mathscr{Y}_{\mathscr{L}}. \tag{1.4}$$

Finally one performs a bifurcation analysis on (1.4). (Note that (1.4) may be stated in another language. Restrict the system of PDE to a fundamental cell of the lattice and assume periodic boundary conditions on the boundary of this cell.)

We observe that there is a natural compact group of symmetries acting on $\mathscr{L}$-periodic mappings that is derived from the action of the Euclidean group. Recall that the Euclidean group $\boldsymbol{E}_n$ is a semidirect sum of $\boldsymbol{O(n)}$ with the group of translations $\boldsymbol{R}^n$, and $g \in \boldsymbol{E}_n$ acts on $u: \boldsymbol{R}^n \to \boldsymbol{R}$ by

$$(g \cdot u)(x) = u(g^{-1}x). \tag{1.5}$$

The action of $\boldsymbol{E}_n$ on the space of $\mathscr{L}$-periodic mappings is best understood by considering the translations and the rotations separately. It is easy to see that translations leave the space of $\mathscr{L}$-periodic mappings invariant. Of course, by definition (1.3), translations in $\mathscr{L}$ fix all $\mathscr{L}$-periodic functions. Thus, the effective action of the group translations on the space of $\mathscr{L}$-periodic functions is as the n-torus $\boldsymbol{T}^n = \boldsymbol{R}^n/\mathscr{L}$ which is compact. Similarly, for the action of rotations recall that the *holohedry* H of the lattice $\mathscr{L}$ is the largest subgroup of $\boldsymbol{O(n)}$ that leaves $\mathscr{L}$ invariant. It follows from (1.5) that H leaves the space of $\mathscr{L}$-periodic functions invariant. Thus, the largest group that can be constructed from $\boldsymbol{E}_n$ that acts on $\mathscr{L}$-periodic functions is the compact semidirect sum:

$$\Gamma = H \dot{+} \boldsymbol{T}^n. \tag{1.6}$$

Finally, we note that F in (1.4) is Γ-equivariant.

Suppose that there is a steady-state bifurcation at $\lambda = 0$ in (1.4); that is, assume that

$$V \equiv \ker(dF)_{0,0} \neq \{0\}.$$

Note that the kernel V is always Γ-invariant. Branches of planforms may be found using the Equivariant Branching Lemma, as follows. Fix a subgroup $\Sigma \subset \Gamma$ and compute $\dim \mathrm{Fix}_V(\Sigma)$ where the *fixed-point subspace* is defined by

$$\mathrm{Fix}_V(\Sigma) \equiv \{v \in V : \sigma v = v \ \forall \sigma \in \Sigma\}. \tag{1.7}$$

When the choice of V is clear we write $\mathrm{Fix}_V(\Sigma)$ as $\mathrm{Fix}(\Sigma)$. In its simplest form the Equivariant Branching Lemma states that if

$$\dim \mathrm{Fix}(\Sigma) = 1, \tag{1.8}$$

then generically there is a unique branch of steady-state solutions to (1.4) with symmetries Σ. The genericity condition states that the eigenvalues that go through zero do so with nonzero speed. We note that the only subgroups that we need consider are *isotropy* subgroups of the action of Γ on V.

This theorem allows us to find a well defined class of planforms, namely, those solutions whose isotropy subgroups satisfy (1.8). In this paper, we classify when $n = 2$ all planforms satisfying (1.8); the corresponding classification for $n = 3$ is given in Dionne [6]. To indicate the complexity of the classification when $n = 3$ we present part of the results in Section 4. In general, generically, there may exist solutions whose isotropy satisfies $\dim \mathrm{Fix}(\Sigma) > 1$. It is for this reason that our classification of planforms is only partial. We note, however, that almost all results exhibiting specific solutions for equations actually assume (1.8). For exceptions see Busse [2], Chossat [4], and Chossat et al. [5].

(b) Procedure for classifying planforms

We now continue with a more precise statement of the three steps needed to classify planforms.

1. Choose a lattice $\mathcal{L}$.
2. Determine V, the kernel of $(dF)_{0,0}$.
3. Find those isotropy subgroups Σ of $\Gamma = H \dot{+} \boldsymbol{T}^n$ that satisfy (1.8).

As we shall see these steps are interrelated. We begin our discussion by fixing a lattice $\mathcal{L}$ and addressing step (2). In Sections 2 and 4 we will describe explicitly the five planar lattices and the fourteen three-dimensional lattices, the *Bravais* lattices.

Since the symmetry group Γ is compact, we expect V to be finite-dimensional. (This point will be discussed further.) Since V is Γ-invariant, and Γ is compact, we may write V as a direct sum of Γ-irreducible subspaces

$$V = V_1 \oplus \cdots \oplus V_p.$$

A simple calculation shows that

$$\mathrm{Fix}_V(\Sigma) = \mathrm{Fix}_{V_1}(\Sigma) \oplus \cdots \oplus \mathrm{Fix}_{V_p}(\Sigma).$$

Hence, if $\dim \mathrm{Fix}_V(\Sigma) = 1$, then $\mathrm{Fix}_V(\Sigma) = \mathrm{Fix}_{V_j}(\Sigma)$ for some V_j. Thus, the first step in classifying the planforms associated with a fixed lattice $\mathscr{L}$ is to enumerate each irreducible subspace of Γ that can occur in the action of Γ on $\mathscr{X}_{\mathscr{L}}$. This we do in Section 2 for the planar lattices and in Section 4 for the primitive cubic lattice.

Next we classify for each of the irreducible representations the isotropy subgroups Σ in Γ for which (1.8) is valid. There are two simplifications:

(a) We need only classify conjugacy classes of isotropy subgroups satisfying (1.8).
(b) We may assume

$$\Sigma \cap \boldsymbol{T}^n = 1. \tag{1.9}$$

We call those subgroups Σ of Γ that satisfy (1.9) *translation free*.

The first simplification is standard; the second requires some comment. If an $\mathscr{L}$-periodic solution to (1.4) has a translation symmetry that is not in $\mathscr{L}$, then there is a finer lattice (if $\Sigma \cap \boldsymbol{T}^n$ is finite) or a lower dimensional lattice (if $\Sigma \cap \boldsymbol{T}^n$ is continuous) that supports this solution. In either case the solution will appear on a lattice $\mathscr{L}'$ as a solution associated with an isotropy subgroup Σ' satisfying (1.9). In Section 2(a) we verify this statement. It is also true that if Σ satisfies (1.8) then so does Σ'; see Dionne [6].

(c) *Multiplicity of solutions*

In the preceding discussion we assumed that a lattice $\mathscr{L}$ was fixed. There are two analytic simplifications that are obtained by assuming $\mathscr{L}$-periodicity and they are worth noting here. (This discussion also shows how such an $\mathscr{L}$ might be fixed for a specific partial differential equation and addresses step (1).) The two difficulties are: infinite dimensional eigenspaces and continuous spectra. We comment on each in turn.

Let $L_\lambda : \mathscr{X} \to \mathscr{Y}$ denote the linearization of the PDE about the trivial solution. We define a *plane wave* as a complex-valued function of the form

$$w_{\boldsymbol{k}}(x) = e^{i\boldsymbol{k} \cdot x} \tag{1.10}$$

where $\boldsymbol{k}$ is a *wave vector* in $\boldsymbol{R}^n$ and $\kappa = |\boldsymbol{k}|$ is the *wave number*. There is a large class of linear PDE admitting plane waves (with a specified wave number) as solutions; examples of such PDE can be found in [9] and [13]. In particular, reaction-diffusion equations in n dimensions are examples.

Suppose that we relax the condition that we are looking only for $\mathscr{L}$-periodic solutions. We assume that there exist a smallest value of λ_c and a nonzero *critical wave number* κ_c such that the plane waves $w_{\boldsymbol{k}}$ with wave numbers $|\boldsymbol{k}| = \kappa_c$ and $\boldsymbol{k} \in \mathscr{L}^*$ are null vectors for L_{λ_c}. This gives an eigenvalue λ_c. For each of the PDE listed in the Introduction (for instance), one can find such λ_c and κ_c. Now Euclidean equivariance of L_{λ_c} guarantees that if $w_{\boldsymbol{k}}$ is an eigenfunction, then so is $w_{\boldsymbol{k}'}$, for every $\boldsymbol{k}'$ having the same wave number as $\boldsymbol{k}$. These $w_{\boldsymbol{k}}$ generate an infinite dimensional function space. Hence, if there is a plane wave in the kernel of L_{λ_c}, Euclidean equivariance implies that this kernel is infinite dimensional (when periodic boundary conditions are omitted).

Recall that we have fixed a lattice $\mathscr{L}$. For this lattice the spectrum of L_{λ_c} contains at most a finite number of $\mathscr{L}$-periodic plane waves with wave numbers $|\boldsymbol{k}| = \kappa_c$. Hence the kernel of the linearization of (1.4) is finite dimensional, whereas the kernel of the linearization of (1.1) is infinite dimensional. This observation is related to the fact that the symmetry group Γ of (1.4) is compact while the symmetry group of (1.1), the Euclidean group, is noncompact.

If we think of the bifurcation problem as one where we are looking for instability of the trivial solution (to time-independent perturbations), as λ is increased, then the value λ_c is the value where linear instability first occurs and that instability is to plane waves of wave number κ_c. We choose the lattice $\mathscr{L}$ so that there are some critical plane waves that are $\mathscr{L}$-periodic. In fact, up to equivalence of lattices, we can choose $\mathscr{L}$ to be of any lattice type. We now discuss this point in more detail.

We begin by introducing the *dual lattice* $\mathscr{L}^*$. Let

$$\mathscr{L}^* = \{\boldsymbol{k} \in \boldsymbol{R}^n : e^{2\pi i \boldsymbol{k} \cdot x} \text{ is } \mathscr{L}\text{-periodic}\}. \tag{1.11}$$

It follows easily from (1.11) that the wave vectors $\boldsymbol{k} \in \mathscr{L}^*$ form a lattice. We assume that $\mathscr{X}$ and $\mathscr{Y}$ can be chosen so that the $\mathscr{L}$-periodic functions of $\mathscr{X}_{\mathscr{L}}$ and $\mathscr{Y}_{\mathscr{L}}$ have Fourier expansions in terms of plane waves whose wave vectors are in $\mathscr{L}^*$.

As we discussed previously, the eigenfunctions of the linearization L_{λ_c} are generated by plane waves whose wave vectors have wave number κ_c. Thus, for a critical plane wave to be $\mathscr{L}$-periodic, there must be some vector $\boldsymbol{k} \in \mathscr{L}^*$ such that $|\boldsymbol{k}| = \kappa_c$. When that happens, all wave vectors $\boldsymbol{k} \in \mathscr{L}^*$ having that critical wave number are critical. Since $\mathscr{L}^*$ is a lattice the possible lengths of wave vectors in $\mathscr{L}^*$ form a countable discrete set which we denote by

$$\{0, M_1, M_2, M_3, \ldots\}$$

where $M_j < M_{j+1}$ for all j. We call $M_1 = |\mathscr{L}^*|$ the length of $\mathscr{L}^*$.

We claim that for each j we can always choose $\mathscr{L}$ up to equivalence so that $M_j = \kappa_c$. Thus, up to equivalence, we can choose $\mathscr{L}$ such that any given length M_j of wave vectors in $\mathscr{L}^*$ is the critical wave number. It follows, therefore, that *all* of the isotropy subgroups that we enumerate according to the previous discussion correspond to planforms that simultaneously bifurcate for our given PDE as branches of solutions from the trivial solution at $\lambda = \lambda_c$. Moreover, in order for this statement to be valid, all we need is for the PDE to have Euclidean invariance and for the eigenvalues of $\mathscr{L}_\lambda$ that go through zero to do so with nonzero speed.

Next we prove the stated claim. Note that in order for $e^{2\pi i k \cdot x}$ to be $\mathscr{L}$-periodic we need $\boldsymbol{k} \cdot \ell$ to be an integer for all $\ell \in \mathscr{L}$. Let $\{\ell_1, \ldots, \ell_n\}$ be a basis for $\mathscr{L}$. Then the dual basis $\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\}$ is a basis for $\mathscr{L}^*$ where for all ℓ_i.

$$\boldsymbol{k}_j \cdot \ell_i = \delta_{ij}.$$

We can always find a lattice $\mathscr{L}'$ equivalent to $\mathscr{L}$ by multiplying all of the vectors in $\mathscr{L}$ by a fixing positive scalar c. Then $(\mathscr{L}')^*$ is obtained from $\mathscr{L}^*$ by dividing all of the wave vectors in $\mathscr{L}^*$ by c. Choosing c appropriately verifies the claim.

Remark. (a) In general, if we want the critical wave number κ_c to correspond to the j-th wave vector length M_j for some large j, then we have to take the length of $\mathscr{L}$ to be very large.

(b) In this discussion we have not mentioned the stability of the bifurcating branches. We just note that in most applications it can be proved that the solutions corresponding to all lengths M_j except one are definitely unstable. Which M_j corresponds to solutions that are possibly stable depends on the particular PDE.

2. Planforms in dimension two

As indicated in the Introduction, our classification of planforms requires three major steps.

1. The enumeration of lattices $\mathscr{L}$.
2. The enumeration of the irreducible representations V of the action of $\Gamma = H \dot{+} \boldsymbol{T}^n$ on $\mathscr{X}_{\mathscr{L}}$.
3. The enumeration of conjugacy classes of translation free (see (1.9)) isotropy subgroups Σ satisfying (1.8).

We list the results of each of these steps in Subsections (b)–(d). The enumeration of lattices is taken from Armstrong [1]. The enumeration of irreducible representations is simply explained and the proof is sketched in Subsection (c). The list of planforms is given in Subsection (d). This list leads to the statement of Theorem 2.3, which is the main result of this paper concerning planforms for planar systems of PDE. We begin in Subsection (a) with a discussion of translation free isotropy subgroups.

(a) Translation free subgroups

Our approach to finding planforms is to presume in advance both the periodicity ($\mathscr{L}$) and the symmetries (Σ) of the planform P and try to prove the existence of $\mathscr{L}$-periodic solutions with symmetry Σ. As noted in the Introduction we can, without loss of generality, assume that Σ contains no translations. Let $M = \Sigma \cap \boldsymbol{T}^n$. We discuss this point in more detail here. We consider two cases: M finite and M infinite.

When M is finite we can form a new lattice $\mathscr{L}'$ from $\mathscr{L}$ by adjoining to $\mathscr{L}$ those vectors in $\boldsymbol{R}^n$ that are obtained from 0 by translation using elements in M. Since $M \subset \boldsymbol{T}^n \equiv \boldsymbol{R}^n/\mathscr{L}$, it follows that $\mathscr{L}'$ is a lattice in $\boldsymbol{R}^n$. Moreover, any $\mathscr{L}$-periodic planform which is also M-invariant will be $\mathscr{L}'$-periodic. Hence, we can find that solution supported on the lattice $\mathscr{L}'$.

Similarly suppose $M \subset \boldsymbol{T}^n$ is a continuous subgroup. Since isotropy subgroups are closed, we see that M^0—the connected component of the identity in M—is a torus $\mathscr{T}^m$ of dimension $m > 0$. We can write $\mathscr{T}^m$ as the projection of a subspace $N \subset \boldsymbol{R}^n$ into $\boldsymbol{T}^n = \boldsymbol{R}^n/\mathscr{L}$. Next, we observe that $\mathscr{L}' = \mathscr{L} \cap N^\perp$ is an $(n-m)$-dimensional lattice in $N^\perp$. By assumption, the planform P may be thought of as being $\mathscr{L}'$-periodic on $N^\perp$ and constant in the directions in N. This lowers the dimension of the problem we were considering; what remains of the symmetry of the planform is Σ/M^0 whose intersection with $\boldsymbol{T}^n$ is finite. If necessary, we may have to refine the lattice $\mathscr{L}'$ in $N^\perp$ (as in the previous paragraph) to obtain a lattice on which the planform is translation free.

Thus we have shown that any planform on a lattice $\mathscr{L}$ may be thought of as a solution on a refined lattice $\mathscr{L}'$, perhaps of lower dimension, on which the isotropy subgroup of that planform is translation free.

We end this section with a remark. If there is a nontrivial translation $t \in \boldsymbol{T}^n$ that acts trivially on V, then every isotropy subgroup of the action of Γ on V contains t. So all planforms obtained from this V will be supported on another lattice, and we can ignore this V in our classification. We say that the representation of Γ on V is *translation free* if the only translation in $\boldsymbol{T}^n$ that acts trivially on V is the identity in $\boldsymbol{T}^n$.

(b) The lattices

On the line, ($n = 1$) there is, up to scaling, one lattice $\mathscr{L}$ with a basis vector $\ell = 1$. The holohedry of this lattice is $\boldsymbol{Z}_2$ (generated by $x \mapsto -x$) and the group of symmetries is $\Gamma = \boldsymbol{Z}_2 \dot{+} \boldsymbol{S}^1 \equiv \boldsymbol{O}(2)$.

Armstrong [1] lists the Bravais lattices in dimension $n = 2$ (cf. [1], p. 149); that list is reproduced in Table 1. We denote by $\boldsymbol{D}_m$ the *dihedral* group of order $2m$.

(c) Irreducible representations

In this section we enumerate all of the irreducible subspaces V of $\mathscr{X}_{\mathscr{L}}$ under the action of $\Gamma = H \dot{+} \boldsymbol{T}^n$ where H is the holohedry of the lattice $\mathscr{L}$. As we indicated in the Introduction we assume that the functions in $\mathscr{X}_{\mathscr{L}}$ are regular enough to have Fourier expansions in terms of the plane waves $w_{\boldsymbol{k}}(x) = e^{2\pi i \boldsymbol{k} \cdot x}$ where $\boldsymbol{k} \in \mathscr{L}^*$.

We begin by noting that V must be a direct sum of $\boldsymbol{T}^n$-irreducible subspaces. Since $t \in \boldsymbol{T}^n$ acts through the translation $x \to x + t$, we have

$$t \cdot w_{\boldsymbol{k}}(x) = w_{\boldsymbol{k}}(x - t) = w_{\boldsymbol{k}}(-t) w_{\boldsymbol{k}}(x). \tag{2.1}$$

It follows that the two-dimensional subspace

$$V_{\boldsymbol{k}} \equiv \{\mathrm{Re}(z w_{\boldsymbol{k}}(x)) : z \in \boldsymbol{C}\} \cong \boldsymbol{C}$$

Table 1
Lattices in two dimensions

Name	Holohedry	Basis of $\mathscr{L}$	Basis of $\mathscr{L}^*$
Hexagonal	$\boldsymbol{D}_6$	$\ell_1 = \left(\frac{1}{\sqrt{3}}, 1\right)$ $\ell_2 = \left(\frac{2}{\sqrt{3}}, 0\right)$	$\boldsymbol{k}_1 = (0, 1)$ $\boldsymbol{k}_2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
Square	$\boldsymbol{D}_4$	$\ell_1 = (1, 0)$ $\ell_2 = (0, 1)$	$\boldsymbol{k}_1 = (1, 0)$ $\boldsymbol{k}_2 = (0, 1)$
Rhombic	$\boldsymbol{D}_2$	$\ell_1 = (1, -\cot\theta)$ $\ell_2 = (0, \csc\theta)$ $0 < \theta < \frac{\pi}{2}, \theta \neq \frac{\pi}{3}$	$\boldsymbol{k}_1 = (1, 0)$ $\boldsymbol{k}_2 = (\cos\theta, \sin\theta)$
Rectangular	$\boldsymbol{D}_2$	$\ell_1 = (1, 0)$ $\ell_2 = (0, c)$ $0 < c < 1$	$\boldsymbol{k}_1 = (1, 0)$ $\boldsymbol{k}_2 = \left(0, \frac{1}{c}\right)$
Oblique	$\boldsymbol{Z}_2$	$\lvert\ell_1\rvert \neq \lvert\ell_2\rvert$ $\ell_1 \cdot \ell_2 \neq 0$	

is $\boldsymbol{T}^n$ invariant. Note that $V_{-\boldsymbol{k}} = V_{\boldsymbol{k}}$. Indeed, a calculation shows that $\boldsymbol{T}^n$ acts irreducibly on $V_{\boldsymbol{k}}$ and that the representation of $\boldsymbol{T}^n$ on $V_{\boldsymbol{k}}$ and $V_{\boldsymbol{k}'}$ are distinct unless $\boldsymbol{k} = \pm \boldsymbol{k}'$. It follows that the irreducible representation V of Γ must have the form

$$V = V_{\boldsymbol{K}_1} \oplus \cdots \oplus V_{\boldsymbol{K}_s} \cong \boldsymbol{C}^s \tag{2.2}$$

for some set of dual wave vectors $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_s$.

Proposition 2.1. The space V in (2.2) is Γ-irreducible if and only if the set of $2s$ dual vectors $\{\pm \boldsymbol{K}_1, \ldots, \pm \boldsymbol{K}_s\}$ is an orbit in $\mathscr{L}^*$ of the action of the holohedry H.

Proof. Note that an element h in the holohedry is an orthogonal matrix and acts on the plane wave $w_{\boldsymbol{k}}$ by

$$\begin{aligned} h \cdot w_{\boldsymbol{k}}(x) &= w_{\boldsymbol{k}}(h^{-1}x) \\ &= e^{2\pi i \boldsymbol{k} \cdot h^{-1}x} \\ &= e^{2\pi i (h\boldsymbol{k}) \cdot x} \\ &= w_{h\boldsymbol{k}}(x). \end{aligned} \tag{2.3}$$

In particular, the holohedry H always contains the reflection $r(x) = -x$ and $r \cdot w_{\boldsymbol{k}} = w_{-\boldsymbol{k}}$.

The proposition is now easily verified. □

Remark. It follows from Proposition 2.1 that the number s of summands $V_{\boldsymbol{K}}$ in (2.2) divides $|H|$, where $|H|$ denotes the *order* of H. In fact, s divides $|H|/2$ because $r : V_{\boldsymbol{k}} \to V_{\boldsymbol{k}}$. (Indeed, from the definition of $V_{\boldsymbol{k}} \cong \boldsymbol{C}$ in (2.1) we see that r acts as complex conjugation on $V_{\boldsymbol{k}}$.)

It is now instructive to discuss the case $n = 1$. Since the holohedry when $n = 1$ is $\boldsymbol{Z}_2$, it follows from the remark that $s = 1$ in (2.2). So the irreducible representations of $\boldsymbol{O}(2)$ are just $V_k = \{\mathrm{Re}(ze^{2\pi ikx})\}$ where $k = 0, 1, 2, \ldots$. Note that the action of Γ on V_k is translation free only when $k = 1$ since $x \to x + (1/k)$ acts trivially on V_k. As we noted previously we need consider only translation free actions, and the only irreducible representation of $\boldsymbol{O}(2)$ that we need consider when finding planforms is (the standard action on) V_1.

Next we discuss the planar lattices listed in Table 1. A short calculation shows that none of the irreducible representations V of the rectangular and oblique lattices are translation free. For example, it follows from the remark above that all irreducible representation of the symmetries of the oblique lattice have $V = V_k$ for some wave vector $\boldsymbol{k}$. Since $\boldsymbol{T}^2$ acts orthogonally on V_k and dim $V_k = 2$, there is a circle in $\boldsymbol{T}^2$ that acts trivially on V_k.

The list of distinct translation free irreducible representations for the three remaining planar lattices is given in Table 2. It is worth commenting here on the results enumerated in Table 2. It is straightforward to show that the only four- and six-dimensional translation free representations are the ones listed here. For instance, the six-dimensional representations are of the form (2.2) with

$$\boldsymbol{K}_1 = \alpha \boldsymbol{k}_1 + \alpha \boldsymbol{k}_2$$
$$\boldsymbol{K}_2 = -\alpha \boldsymbol{k}_2$$
$$\boldsymbol{K}_3 = -\alpha \boldsymbol{k}_1$$

where α is a positive integer. If $\alpha > 1$, the vector $(1/\alpha)\ell_1$ is a nontrivial translation acting trivially on the representation.

Table 2
Translation free irreducible representations

$\mathscr{L}$	Basis for $\mathscr{L}^*$	dim	$V_{K_1} \oplus \cdots \oplus V_{K_s}$
Rhombic $\boldsymbol{D}_2$	$\boldsymbol{k}_1 = (1, 0)$ $\boldsymbol{k}_2 = (\cos\theta, \sin\theta)$ $0 < \theta < \frac{\pi}{2}, \theta \neq \frac{\pi}{3}$	4	$\boldsymbol{K}_1 = \boldsymbol{k}_1$ $\boldsymbol{K}_2 = \boldsymbol{k}_2$
Square $\boldsymbol{D}_4$	$\boldsymbol{k}_1 = (1, 0)$ $\boldsymbol{k}_2 = (0, 1)$	4	$\boldsymbol{K}_1 = \boldsymbol{k}_1$ $\boldsymbol{K}_2 = \boldsymbol{k}_2$
		8	$\boldsymbol{K}_1 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$ $\boldsymbol{K}_2 = -\beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2$ $\boldsymbol{K}_3 = \beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2$ $\boldsymbol{K}_4 = -\alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$ α and β are integers, $\alpha > \beta > 0$, $\alpha + \beta$ is odd and $(\alpha, \beta) = 1$.
Hexagonal $\boldsymbol{D}_6$	$\boldsymbol{k}_1 = (0, 1)$ $\boldsymbol{k}_2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	6	$\boldsymbol{K}_1 = \boldsymbol{k}_1 + \boldsymbol{k}_2$ $\boldsymbol{K}_2 = -\boldsymbol{k}_2$ $\boldsymbol{K}_3 = -\boldsymbol{k}_1$
		12	$\boldsymbol{K}_1 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$ $\boldsymbol{K}_2 = (-\alpha + \beta)\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2$ $\boldsymbol{K}_3 = -\beta\boldsymbol{k}_1 + (\alpha - \beta)\boldsymbol{k}_2$ $\boldsymbol{K}_4 = \alpha\boldsymbol{k}_1 + (\alpha - \beta)\boldsymbol{k}_2$ $\boldsymbol{K}_5 = -\beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2$ $\boldsymbol{K}_6 = (-\alpha + \beta)\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$ α and β are integers, $\alpha > \beta > \alpha/\beta > 0$, $(\alpha, \beta) = 1$ and $(3, \alpha + \beta) = 1$.

It is also straightforward to enumerate the eight-dimensional representations of the square lattice and the twelve-dimensional representations of the hexagonal lattice by the two integers α and β. That these integers must be relatively prime in order to generate translation free irreducible representations is also easy to show. What is less clear is which of the remaining irreducible representations are actually translation free.

For the eight-dimensional irreducible representations where both integers α and β are odd, one can explicitly find a translation (by $t = (\frac{1}{2}, \frac{1}{2})$) showing that the representation is not translation free. Thus we may assume that one of α and β is even and the other is odd. That these remaining eight-dimensional representations are in fact translation free is verified in Lemma 2.2.

Similarly, the translation free, twelve-dimensional irreducible representations are completely characterized by the conditions $(3, \alpha + \beta) = 1$ and $(\alpha, \beta) = 1$. When 3 divides $\alpha + \beta$, the translation t defined by $t \cdot \boldsymbol{k}_1 = t \cdot \boldsymbol{k}_2 = \frac{1}{3}$ (namely, $t = \frac{1}{3}\ell_1 + \frac{1}{3}\ell_2$) acts trivially on the twelve-dimensional representations. So we may assume that $(3, \alpha + \beta) = 1$. Similarly, if $\gamma > 1$ is a common factor of α and β, then translation by $(1/\gamma)(\ell_1 + \ell_2)$ acts trivially. Hence we can assume $(\alpha, \beta) = 1$. The fact that the remaining twelve-dimensional irreducible representations are translation free is verified in Lemma 2.2.

Lemma 2.2. (a) The eight-dimensional irreducible representations on the square lattice are translation free when $(\alpha, \beta) = 1$ and $\alpha + \beta$ is odd.

(b) The twelve-dimensional irreducible representations on the hexagonal lattice are translation free when $(\alpha, \beta) = 1$ and $(3, \alpha + \beta) = 1$.

Proof. (a) To show that the stated conditions are sufficient to characterize the translation free eight-dimensional irreducible representations on the square lattice, we show that if t is a translation acting trivially on an eight-dimensional representation, then $t = 0$ in $\boldsymbol{T}^2$. Let $u = \boldsymbol{k}_1 \cdot t$ and $v = \boldsymbol{k}_2 \cdot t$. We may assume that $0 \leq u, v < 1$; if not, we can always add to t a linear combination of ℓ_1 and ℓ_2 with integer coefficients to make that so. From $\boldsymbol{K}_1 \cdot t - \boldsymbol{K}_4 \cdot t \in \boldsymbol{Z}$ and $\boldsymbol{K}_2 \cdot t + \boldsymbol{K}_3 \cdot t \in \boldsymbol{Z}$, we get that $2\alpha u \in \boldsymbol{Z}$ and $2\alpha v \in \boldsymbol{Z}$. Therefore, $u = p/(2\alpha)$ and $v = q/(2\alpha)$ for some positive integers p and q. Substituting these expressions for u and v into $\boldsymbol{K}_j \cdot t \in \boldsymbol{Z}$ for $j = 1, 2, 3$ and 4, we obtain:

$$p\alpha + q\beta \equiv 0 \pmod{2\alpha} \tag{2.4}$$

$$q\alpha - p\beta \equiv 0 \pmod{2\alpha} \tag{2.5}$$

$$q\alpha + p\beta \equiv 0 \pmod{2\alpha} \tag{2.6}$$

$$-p\alpha + q\beta \equiv 0 \pmod{2\alpha} \tag{2.7}$$

From (2.4) and (2.7), we find that $2q\beta \equiv 0 \pmod{2\alpha}$. Since $(\alpha, \beta) = 1$, it follows that

$$q \equiv 0 \pmod{\alpha}.$$

Similarly, from (2.5) and (2.6), we get that

$$p \equiv 0 \pmod{\alpha}.$$

Hence, $u, v = 0$ or $\frac{1}{2}$. Since $\alpha + \beta$ is odd, only $t = (u, v) = (0, 0)$ satisfies $\boldsymbol{K}_j \cdot t \in \boldsymbol{Z}$ for $j = 1$, 2, 3 and 4.

(b) To show that the stated conditions are sufficient to characterize the translation-free twelve-dimensional irreducible representations, we now show that if t is a translation acting trivially on a twelve-dimensional representation, then $t = 0$ in $\boldsymbol{T}^2$. Let $u = \boldsymbol{k}_1 \cdot t$ and $v = \boldsymbol{k}_2 \cdot t$. As in (a), we may assume that $0 \leq u, v < 1$. From $\boldsymbol{K}_4 \cdot t - \boldsymbol{K}_3 \cdot t \in \boldsymbol{Z}$ and $\boldsymbol{K}_6 \cdot t - \boldsymbol{K}_2 \cdot t \in \boldsymbol{Z}$, we get that $(\alpha + \beta)u \in \boldsymbol{Z}$ and $(\alpha + \beta)v \in \boldsymbol{Z}$. Therefore, $u = p/(\alpha + \beta)$ and $v = q/(\alpha + \beta)$ for some positive integers p and q. Substituting these expressions for u and v into $\boldsymbol{K}_j \cdot t \in \boldsymbol{Z}$ for $j = 1, 2, 4, 5$, we obtain:

$$p\alpha + q\beta \equiv 0 \pmod{\alpha + \beta} \tag{2.8}$$

$$-(p + q)\alpha + p\beta \equiv 0 \pmod{\alpha + \beta} \tag{2.9}$$

$$(p + q)\alpha - q\beta \equiv 0 \pmod{\alpha + \beta} \tag{2.10}$$

$$p\beta + q\alpha \equiv 0 \pmod{\alpha + \beta} \tag{2.11}$$

From (2.9) and (2.10), we find that $p\beta \equiv q\beta \pmod{\alpha + \beta}$. Since $(\alpha, \beta) = 1$ and $(\alpha + \beta, \beta) = 1$ it follows that

$$p \equiv q \pmod{\alpha + \beta}.$$

Similarly, from (2.8), (2.9) and (2.11), we get that

$$p \equiv -(p + q) \pmod{\alpha + \beta}.$$

Hence, $3p \equiv 0 \pmod{\alpha + \beta}$. Since $(3, \alpha + \beta) = 1$, we get that

$$p \equiv q \equiv 0 \pmod{\alpha + \beta}.$$

Finally, since u and v are positive integers less than 1, $u = v = 0$ and $t = 0$ in $\boldsymbol{T}^2$. □

(d) The planforms in two dimensions

We now enumerate the translation free isotropy subgroups of the translation free irreducible representations listed in Table 2. We also describe the form that the resulting planforms must have.

On the line, the only irreducible representation that occurs is the standard action of $\boldsymbol{O}(2)$ on $\boldsymbol{C}$. The only nontrivial isotropy subgroup of this action is $\boldsymbol{Z}_2$, generated by a reflection on the line. This isotropy subgroup has a one-dimensional fixed-point subspace and generates a solution known as *rolls* in the convection literature.

For the two-dimensional lattices the classification is more complicated. Our results are given in Table 3. We comment on these results here. In his thesis, Swift [12] enumerated the isotropy subgroups of the four-dimensional representations on the rhombic and square lattices. He showed that in each case there are two maximal isotropy subgroups having one-dimensional fixed-point subspaces, one corresponding to rolls and the other corresponding to rectangular or square symmetry, depending on the lattice. Swift also showed that generically steady-state bifurcations with these symmetries produce no other solutions. Similarly, Buzano and Golubitsky [3] show that the only isotropy subgroups corresponding to the six-dimensional irreducible representation on the hexagonal lattice having one-dimensional fixed-point subspaces correspond to rolls and hexagons, and that generically bifurcations on this lattice produce no other solutions. (It should be noted here that degenerate bifurcation problems on the hexagonal lattice can lead to solutions with submaximal symmetry [3], but that issue is not pursued here.)

Previous results for higher dimensional irreducible representations on the square and hexagonal lattices are more limited. Kirchgässner [10] studies the twelve-dimensional irreducible representations finding a number of solutions (corresponding to one-dimensional fixed-point subspaces). The only one that is translation free has hexagonal symmetry. To our knowledge no one has previously considered the eight-dimensional representations of the square lattice. Here we find two non-conjugate isotropy subgroups that are isomorphic, both having square symmetry and one-dimensional fixed-point subspaces. The first group ($\boldsymbol{D}_4^+$) corresponds to solutions that are invariant under both rotation by 90° and reflections across the axes. The second group ($\boldsymbol{D}_4^-$) corresponds to solutions that are invariant under the rotation, but are taken to a shift of themselves by $(\frac{1}{2}, \frac{1}{2})$ when reflected across an axis.

Table 3
Isotropy subgroups having one-dimensional fixed-point subspaces

$\mathscr{L}$	dim	Isotropy Subgroups Σ	Planform
Rhombic	4	$\boldsymbol{D}_2$	Rectangles
Square	4	$\boldsymbol{D}_4$	Simple squares
	8	$\boldsymbol{D}_4^+$	Squares
		$\boldsymbol{D}_4^-$	Anti-squares
Hexagonal	6	$\boldsymbol{D}_6$	Simple Hexagons
	12	$\boldsymbol{D}_6$	Hexagons

In the next section we discuss in more detail our method of proof of these statements. Now we summarize our bifurcation results.

Theorem 2.3. Given a system of PDE in the plane depending on a bifurcation parameter λ and satisfying:

(a) Euclidean equivariance.
(b) A trivial translation invariant equilibrium for each λ.
(c) This equilibrium loses stability at $\lambda = \lambda_c$; that is, the linearized PDE has solutions at $\lambda = \lambda_c$ with nontrivial spatial dependence (and no nonzero constant solution).
(d) The spaces $\mathscr{X}$ and $\mathscr{Y}$ are chosen so that a Liapunov-Schmidt reduction to the kernel of the linearized equations defined on $\mathscr{X}_{\mathscr{L}}$ and $\mathscr{Y}_{\mathscr{L}}$ is possible.

Then there are branches of (Euclidean group orbits of) planforms bifurcating from the trivial solution at $\lambda = \lambda_c$ that correspond to each of the following:

1. Rolls
2. Rectangles (a continuum, one for each θ)
3. Simple Squares
4. Simple Hexagons
5. Squares (a countable number, one for each of the specified α, β)
6. Anti-squares (a countable number, one for each of the specified α, β)
7. Hexagons (a countable number, one for each of the specified α, β)

We remark that this theorem is incomplete in two respects. We do not know whether generically solutions with other isotropies are possible in the eight-dimensional irreducibles for the square lattice and the twelve-dimensional irreducibles for the hexagonal lattice. In addition, there will be kernels of the linearized equations restricted to certain lattices for which the corresponding representations are reducible. In these cases planforms, in addition to those that we have enumerated, can be expected.

In the next section we will discuss the proof of Theorem 2.3. In the remainder of this section we discuss the symmetries and form of the seven types of planforms whose existence is asserted by Theorem 2.3. We also include pictures of the planforms. These figures are obtained as follows: each planform in Table 3 corresponds to an $\mathscr{L}$-periodic mapping that is a linear combination of exponentials with wave vectors listed in Table 2. In our figures we give contour plots and 3-D graphs of these functions.

It is worth noting the difference between Simple squares and Squares and between Simple hexagons and Hexagons. In the 3-D graphs one can see that the number of minima along the right-hand side of the graph increases as you go from the planform associated with the lower dimensional repre-

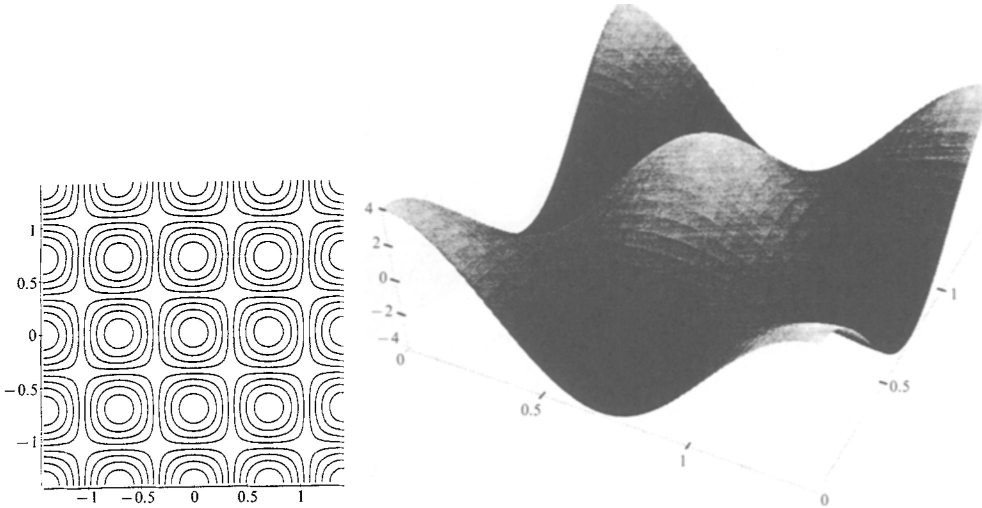

Figure 1
Simple Squares. (a) contour plot with four cells (b) 3D plot of one cell.

sentation to the one associated with the higher dimensional representation. This happens even though the symmetries of the pairs of planforms are identical. One should also note the symmetry of Anti-Squares. These planforms are invariant under rotation by 90° but not under any reflection of the square. Another symmetry of Anti-Squares is obtained by reflecting

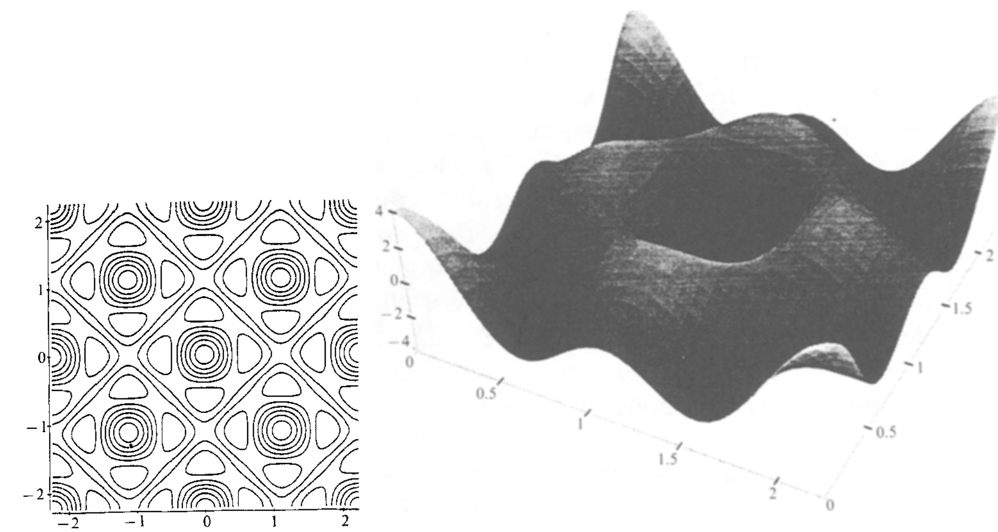

Figure 2
Squares with $\alpha = 2$, $\beta = 1$. (a) contour plot with four cells (b) 3D plot of one cell.

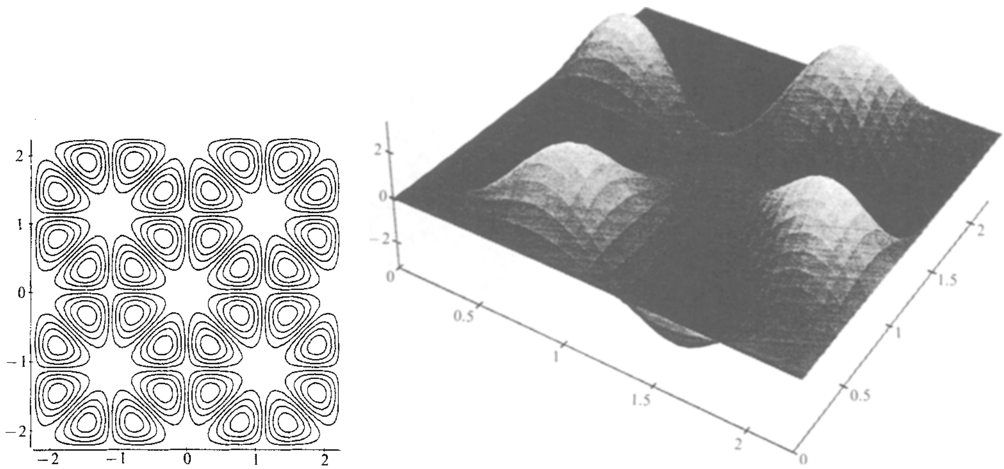

Figure 3
Anti-Squares with $\alpha = 2$, $\beta = 1$. (a) contour plot with four cells (b) 3D plot of one cell.

about the diagonal of the square and then translating along that diagonal by half a cell. This symmetry is most easily seen on the contour plot in Fig. 3(a).

Finally, as suggested by Figs. 1 to 5, Theorem 2.3 may be used to study problems posed on square or hexagonal domain with boundary conditions other than periodic (for instance, with Neumann boundary conditions). We will not elaborate on this topic here.

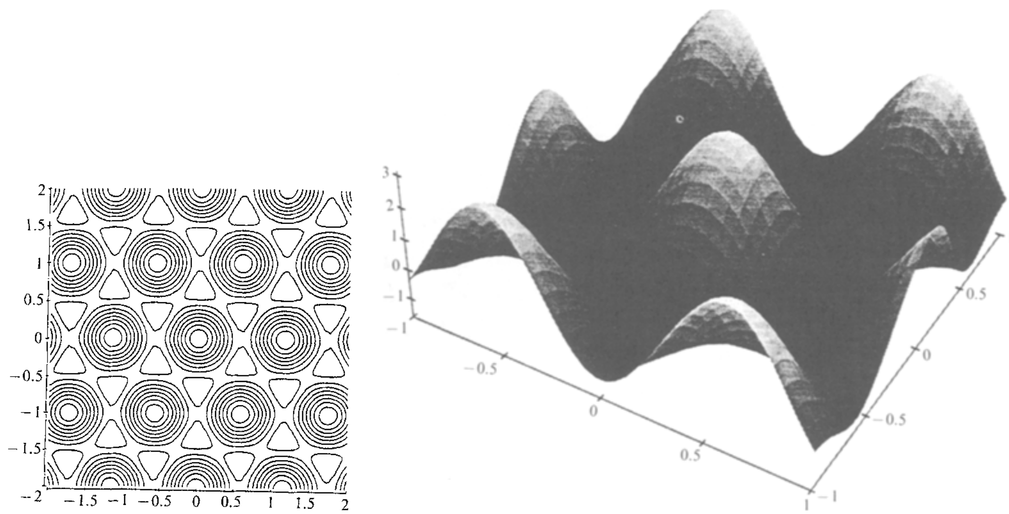

Figure 4
Simple Hexagons. (a) contour plot including seven cells (b) 3D plot including one cell.

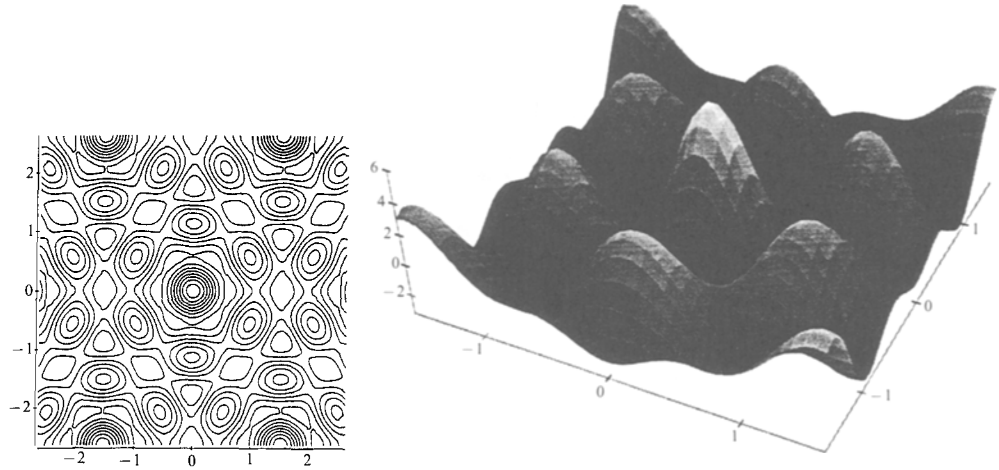

Figure 5
Hexagons with $\alpha = 3$, $\beta = 1$. (a) contour plot including one cell (b) 3D plot including one cell.

3. Proof of the classification theorem

In this section we complete the proof of Theorem 2.3 by computing up to conjugacy all translation free isotropy subgroups having a one-dimensional fixed-point subspace. In the first subsection, we explain our general procedure. The details of the proof are given in the last two sections.

As mentioned in Section 2(d), the existence of branches of planforms corresponding to Rolls, Rectangles, Simple Squares and Simple Hexagons has been proved before. Here we prove the existence of branches of planforms corresponding to Squares, Anti-Squares and Hexagons.

(a) Procedure

As in the previous section, $\mathscr{L}$ is an n-dimensional lattice in $\boldsymbol{R}^n$ with holohedry H and $\Gamma = H \dot{+} \boldsymbol{T}^n$ where $\boldsymbol{T}^n = \boldsymbol{R}^n/\mathscr{L}$, and V is a Γ-absolutely irreducible subspace of the form (2.2). Before discussing our procedure we prove two preliminary results.

The dimension of the fixed-point subspace of a finite subgroup Σ of Γ may be computed using the *trace formula* (cf. [8], p. 76), which is:

$$\dim \mathrm{Fix}(\Sigma) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \mathrm{tr}(\sigma) \tag{3.1}$$

where $|\Sigma|$ is the *order* of Σ. Using (2.1) and (2.3) we compute the trace of

$(h, t) \in \Gamma$ as:

$$\operatorname{tr}(h, t) = \sum_j 2\cos(2\pi\theta_j) \tag{3.2}$$

where the sum is over all j such that K_j is fixed by h and $\theta_j = -K_j \cdot h^{-1}t$.

Let Π_H be the projection (a group epimorphism) from Γ to H defined by

$$\Pi_H(h, t) = h.$$

Since the kernel of Π_H is T^n, the translation free (isotropy) subgroups of Γ are isomorphic by Π_H to subgroups of H.

Lemma 3.1. Assume that $\dim V = |H|$. Then all translation free isotropy subgroups Σ of Γ having a one-dimensional fixed-point subspace are isomorphic by Π_H to H.

Proof. Irreducibility (see Proposition 2.1) implies that H acts transitively on the wave vector $\pm K_j$ generating V and there are $|H|$ of them. Hence there does not exist a K_j which is fixed by some $h \in H$ with $h \neq 1$. It follows from (3.2) that $\operatorname{tr}(\sigma) = 0$ for all $\sigma \in \Sigma$, $\sigma \neq 1$, and from the trace formula that

$$\dim \operatorname{Fix}(\Sigma) = \frac{\operatorname{tr}(1)}{|\Sigma|} = \frac{\dim V}{|\Sigma|}.$$

Therefore $\dim \operatorname{Fix}(\Sigma) = 1$ only if Σ and H are of the same order; namely, Σ is isomorphic by Π_H to H. □

Before stating the second result we introduce some notation. Let Σ_z denote the isotropy subgroup for a vector $z = (z_1, \ldots, z_s) \in V$. Let A_z be the set of wave vectors $\{\pm K_j : z_j \neq 0\}$ and let A'_z be the corresponding set of wave vector pairs $\{(K_j, -K_j) : z_j \neq 0\}$. Note that H acts naturally on wave vector pairs.

Proposition 3.2. Let Σ_z be an isotropy subgroup having a one-dimensional fixed-point subspace. Then $\Pi_H(\Sigma_z)$ acts transitively on A_z.

Proof. We begin by showing that $\Pi_H(\Sigma_z)$ acts transitively on the set A' of wave vector pairs. We first assume that there exists a nonempty subset $B' \subset A'_z$ which is invariant under the action of $\Pi_H(\Sigma_z)$.

Define $x \in C^s$ by

$$x_j = \begin{cases} z_j & \text{if } K_j \in B' \\ 0 & \text{otherwise.} \end{cases}$$

The vectors x and z must be linearly dependent since they are fixed by Σ_z and $\dim \operatorname{Fix}(\Sigma_z) = 1$. Hence $x = z$ and $B' = A'$.

Next we assume that $\Pi_H(\Sigma_z)$ acts transitively on the set A'_z of wave vector pairs. Let $B \subset A_z$ be nonempty and $\Pi_H(\Sigma_z)$-invariant; to complete the proof we must show that $B = A_z$. Without loss of generality we may assume $\boldsymbol{K}_1 \in B$.

Note that since $\Pi_H(\Sigma_z)$ acts transitively on A'_z either $\Pi_H(\Sigma_z)$ acting on $\boldsymbol{K}_1$ is all of A_z (in which case $B = A_z$ and the proof is complete) or half of A_z. In the latter case, the action of $\Pi_H(\Sigma_z)$ on $\boldsymbol{K}_1$ reaches $\kappa_j \boldsymbol{K}_j$ for each $\boldsymbol{K}_j \in A_z$ where $\kappa_1 = 1$ and $\kappa_j = 1$ or -1 for all $j \neq 1$. Now define $\boldsymbol{x} \in \boldsymbol{C}^s$ by:

$$x_j = \begin{cases} \kappa_j z_j & \text{if } \boldsymbol{K}_j \in A_z \\ 0 & \text{otherwise.} \end{cases}$$

Since the element $\boldsymbol{x}$ is in Fix Σ_z, it must be a multiple of $\boldsymbol{z}$. In fact, we must have that $\boldsymbol{x} = \boldsymbol{z}$. □

To compute (representatives of the conjugacy classes of) translation free isotropy subgroups of Γ having one-dimensional fixed-point subspaces, we proceed as follows.

First, we eliminate certain conjugacy classes of subgroups of H altogether. To do this, we use the trace formula, equations (3.1) and (3.2), to find a lower bound for dim Fix(Σ) over all subgroups Σ isomorphic by Π_H to G. The main point is that (3.2) implies that $\mathrm{tr}(h \cdot t) \geq -2N$ where N is the number of wave vectors $\boldsymbol{K}_j$ fixed by h.

A consequence of Proposition 3.2 is that we can also eliminate those G that do not act transitively on any set of wave vectors of the form $\{\pm \boldsymbol{K}_j : j \in J\}$ where J is a subset of $\{1, 2, \ldots, s\}$. Moreover, since we are looking for translation free subgroups, we can also eliminate those G such that when they do act transitively on a set of wave vectors of the form $\{\pm \boldsymbol{K}_j : j \in J\}$ where J is a subset of $\{1, 2, \ldots, s\}$, there is always a nontrivial translation acting trivially on the coordinates associated to these wave vectors.

Second, for those representatives G of conjugacy classes of subgroups of H that remain, we compute the possible generators of the subgroups of Γ isomorphic by Π_H to G. Let $g_1, g_2, \ldots, g_r$ be the generators of G, the generators of a subgroup Σ of Γ isomorphic by Π_H to G are of the form $(g_1, \boldsymbol{t}_1), (g_2, \boldsymbol{t}_2), \ldots, (g_r, \boldsymbol{t}_r)$ where the t_j's are determined by the order of the elements of Σ. (The computations done here are similar to the computations needed to classify the crystallographic space groups.)

Third, for the subgroups Σ obtained in the second step, we compute Fix(Σ) to eliminate those that do not have one-dimensional fixed-point subspace. We also delete those that are not isotropy subgroups.

(b) 12-Dimensional representations of the hexagonal lattices

It is relatively easy to compute the translation free isotropy subgroups of $\Gamma = D_6 \dotplus \boldsymbol{T}^2$ having a one-dimensional fixed-point subspace where Γ is acting on a twelve dimensional, translation free, absolutely irreducible subspace V. This subspace V is of the form (2.2) where $s = 6$ and the wave vectors $\boldsymbol{K}_j$'s are given in Table 2.

Since $\dim V = |D_6|$, the translation free isotropy subgroups of Γ having one dimensional fixed-point subspaces are isomorphic by Π_H to D_6. The subgroups of Γ isomorphic by Π_H to D_6 are generated by $(\varrho_{\pi/3}, \boldsymbol{t}_1)$ and $(\tau_x, \boldsymbol{t}_2)$ where $\varrho_{\pi/3}$ is the rotation by $\pi/3$ about the origin and τ_x is the flip across the x-axis. The elements $\varrho_{\pi/3}$ and τ_x are generators of D_6.

Note that after an initial conjugacy using an element of $\boldsymbol{T}^2$ we may assume that $t_1 = 0$. Hence we only have to compute the subgroups Σ of Γ generated by $\varrho_{\pi/3}$ and $(\tau_x, \boldsymbol{t})$, and isomorphic by Π_H to D_6.

Since Σ is isomorphic by Π_H to D_6, the element $(\tau_x, \boldsymbol{t})$ is of order two because τ_x is. Hence

$$1 = (\tau_x, \boldsymbol{t})^2 = (1, \tau_x \boldsymbol{t} + \boldsymbol{t})$$

and $\tau_x \boldsymbol{t} + \boldsymbol{t} \in \mathscr{L}$. If we substitute $\boldsymbol{t} = t_1 \boldsymbol{k}_1 + t_2 \boldsymbol{k}_2$, we get that $(t_1 + 2t_2)\boldsymbol{k}_2 \in \mathscr{L}$ and $t_1 + 2t_2 \equiv 0 \pmod 1$.

The product of $(\tau_x, \boldsymbol{t})$ and $(\varrho_{\pi/3}, 0)$, namely $(\tau_x \varrho_{\pi/3}, \boldsymbol{t})$, is an element of order two in Σ since $\tau_x \varrho_{\pi/3}$ is of order two. Hence

$$1 = (\tau_x \varrho_{\pi/3}, \boldsymbol{t})^2 = (1, \tau_x \varrho_{\pi/3} \boldsymbol{t} + \boldsymbol{t})$$

and $\tau_x \varrho_{\pi/3} \boldsymbol{t} + \boldsymbol{t} \in \mathscr{L}$. Again, if we substitute $\boldsymbol{t} = t_1 \boldsymbol{k}_1 + t_2 \boldsymbol{k}_2$, we get that $-t_2 \boldsymbol{k}_1 + 2t_2 \boldsymbol{k}_2 \in \mathscr{L}$ and $t_2 \equiv 0 \pmod 1$.

Therefore, the only translation free subgroup of Γ isomorphic by Π_H to D_6 is D_6 itself.

It is now easy to check that D_6 is an isotropy subgroup with

$$\mathrm{Fix}(D_6) = \boldsymbol{R}\{(1, 1, 1, 1, 1, 1)\}.$$

To see this, we note that the actions of $\varrho_{\pi/3}$ and τ_x on V induce the following actions of $\varrho_{\pi/3}$ and τ_x on $\boldsymbol{C}^6$.

$$\varrho_{\pi/3}(z_1, z_2, \ldots, z_6) = (\bar{z}_2, \bar{z}_3, \bar{z}_1, \bar{z}_5, \bar{z}_6, \bar{z}_4)$$

$$\tau_x(z_1, z_2, \ldots, z_6) = (z_6, z_5, z_4, z_3, z_2, z_1).$$

Moreover, since the representation of Γ on V is translation free, D_6 is a translation free isotropy subgroup having a one-dimensional fixed-point subspace.

(c) Eight-dimensional representations for the square lattices

The computations of the translation free isotropy subgroups of $\Gamma = D_4 \dotplus \boldsymbol{T}^2$ having one-dimensional fixed-point subspaces are similar to the computations of the previous subsection. Here Γ is acting on an eight-dimensional, translation free absolutely irreducible subspace V of the form (2.2) where $s = 4$ and the wave vectors $\boldsymbol{K}_j$'s are given in Table 2.

Since $|D_4| = \dim V$, we have that the translation free subgroups of Γ having one-dimensional fixed-point subspace are isomorphic by Π_H to D_4. Moreover, as in the previous subsection, we only have to compute the subgroups Σ of Γ generated by $\varrho_{\pi/2}$ and $(\tau_x, \boldsymbol{t})$, and isomorphic by Π_H to D_4.

Since τ_x is an element of order two, the element $(\tau_x, \boldsymbol{t})$ is of order two. Hence, $\tau_x \boldsymbol{t} + \boldsymbol{t} \in \mathscr{L}$. If we substitute $\boldsymbol{t} = t_1 \boldsymbol{k}_1 + t_2 \boldsymbol{k}_2$ into this expression, we find that $2t_1 \boldsymbol{k}_1 \in \mathscr{L}$ and, therefore, $2t_1 \equiv 0 \pmod 1$.

The product of $(\tau_x, \boldsymbol{t})$ and $(\varrho_{\pi/2}, 0)$, namely $(\tau_x \varrho_{\pi/2}, \boldsymbol{t})$, is an element of order two in Σ. Hence, $\tau_x \varrho_{\pi/2} \boldsymbol{t} + \boldsymbol{t} \in \mathscr{L}$ and, after having substituted $\boldsymbol{t} = t_1 \boldsymbol{k}_1 + t_2 \boldsymbol{k}_2$, we get that $(t_1 - t_2)\boldsymbol{k}_1 + (-t_1 + t_2)\boldsymbol{k}_2 \in \mathscr{L}$. Therefore, $t_1 \equiv t_2 \pmod 1$.

We conclude that there are two subgroups of Γ which are isomorphic by Π_H to D_4: the group D_4^+, which is the group D_4 itself, and the group D_4^- generated by $\varrho_{\pi/2}$ and $(\tau_x, (1/2, 1/2))$.

We now show that both D_4^+ and D_4^- are translation free isotropy subgroups of Γ having one-dimensional fixed-point subspaces. Recall that α and β in the definition of the $\boldsymbol{K}_j$'s in Table 2 are not both odd nor both even. Hence,

$$(1/2, 1/2) \cdot z = -z.$$

Moreover, the actions of $\varrho_{\pi/2}$ and τ_x on V induce the following actions of $\varrho_{\pi/2}$ and τ_x on C^4.

$$\varrho_{\pi/2}(z_1, z_2, z_3, z_4) = (\bar{z}_2, z_1, \bar{z}_4, z_3)$$
$$\tau_x(z_1, z_2, z_3, z_4) = (\bar{z}_4, \bar{z}_3, \bar{z}_2, \bar{z}_1)$$

It is now easy to check that

$$\mathrm{Fix}(D_4^+) = R\{(1, 1, 1, 1)\}$$
$$\mathrm{Fix}(D_4^-) = R\{(1, 1, -1, -1)\}.$$

Finally, D_4^+ and D_4^- are translation free isotropy subgroups of Γ because the representation of Γ on V is translation free. These groups are not conjugate since no element of Γ maps $(1, 1, 1, 1)$ to a scalar multiple of $(1, 1, -1, -1)$.

4. Planforms in dimension three

From the Introduction, we recall that the major steps required to classify the planforms are.

1. The enumeration of the three-dimensional *Bravais* lattices $\mathscr{L}$.
2. The enumeration of the translation free, absolutely irreducible representations V of the action of $\Gamma = H \dot{+} \boldsymbol{T}^n$ on $\mathscr{X}_{\mathscr{L}}$.
3. The enumeration of conjugacy classes of translation free (see 1.9)) isotropy subgroups Σ satisfying (1.8).

The enumeration of the Bravais lattices can be found in Miller ([11], p. 51). We reproduce this list in Table 4. We adopt the following convention. A rotation by θ radians about a line l is denoted by $\varrho_{\theta,l}$. The particular case of a rotation by π about a line l (or, equivalently, a flip across this line) is denoted by τ_l. The orientation of a rotation is given by the right hand rule where the axis of rotation is pointing in the half space $x > 0$ or in the first quadrant of the y, z plane if the axis is in this plane.

Remark. Since we know the classification of planforms associated to the one- and two-dimensional lattices, we may ignore all absolutely irreducible representations which are only supported by one wave vector $\boldsymbol{K}_j$ (the two-dimensional representations) or by two coplanar wave vectors $\boldsymbol{K}_j$'s. The solutions that we get from the representations supported by coplanar wave vectors correspond to planforms classified in Section 2 except for the fact that they are constant along the lines perpendicular to the plane containing the wave vectors. We can make a similar observation for the solutions obtained from two-dimensional representations.

Moreover, a Bravais lattice may not support translation free irreducible representations of all acceptable dimensions (since the dimension of the representation must be divisible by $|H|/2$).

In the next two subsections, we elaborate on steps (2) and (3) above in the special case when the lattice $\mathscr{L}$ is the primitive cubic lattice. We prove:

Theorem 4.1. Given a system of PDE in $\boldsymbol{R}^3$ depending on a bifurcation parameter λ and satisfying the hypotheses of Theorem 2.3. Then there are branches of (Euclidean group orbits of) planforms bifurcating from the trivial solution at $\lambda = \lambda_c$ that are periodic with respect to the primitive cubic lattice and have the isotropy subgroups of Table 7 as symmetry groups.

The complete classification of the planforms in three dimensions can be found in Dionne [6] and [7].

(a) Irreducible representations

Basic vectors for the dual lattice of the primitive cubic lattice are:

$\boldsymbol{k}_1 = (1, 0, 0)$

$\boldsymbol{k}_2 = (0, 1, 0)$

$\boldsymbol{k}_3 = (0, 0, 1)$.

In Table 5, we list the translations free irreducible representations supported by this lattice. For the cubic lattices, there are two types of

Table 4
Bravais lattices

Name	Holohedry	Generators	Basis of $\mathscr{L}$
Primitive Cubic	$\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,x}$ $\varrho_{\pi/2,y}$ $-\mathrm{Id}$	$\ell_1 = (1, 0, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (0, 0, 1)$
Body Centered Cubic			$\ell_1 = (1, 0, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (1/2, 1/2, 1/2)$
Face Centered Cubic			$\ell_1 = (1/2, 1/2, 0)$ $\ell_2 = (-1/2, 1/2, 0)$ $\ell_3 = (0, 1/2, 1/2)$
Hexagonal	$\boldsymbol{D}_6 \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/3,z}$ τ_y $-\mathrm{Id}$	$\ell_1 = (\sqrt{3}/2, 1/2, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (0, 0, a)$ $a > 0$
Primitive Tetragonal	$\boldsymbol{D}_4 \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,z}$ τ_y $-\mathrm{Id}$	$\ell_1 = (1, 0, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (0, 0, a)$ $a > 0$ and $a \neq 1$
Body Centered Tetragonal			$\ell_1 = (1, 0, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (1/2, 1/2, a/2)$ $a > 0$ and $a \neq 1$
Rhombohedral	$\boldsymbol{D}_3 \oplus \boldsymbol{Z}_2^c$	$\varrho_{2\pi/3,z}$ τ_y $-\mathrm{Id}$	$\ell_1 = (\sqrt{3}/2, -1/2, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (\sqrt{3}/6, -1/2, a/3)$ $a \neq 0$
Primitive Orthorhombic	$\boldsymbol{D}_4 \oplus \boldsymbol{Z}_2^c$	τ_z τ_y $-\mathrm{Id}$	$\ell_1 = (a, 0, 0)$ $\ell_2 = (0, b, 0)$ $\ell_3 = (0, 0, c)$ $a, b, c > 0$ $a \neq b \neq c \neq a$
Body Centered Orthorhombic			$\ell_1 = (a, 0, 0)$ $\ell_2 = (0, b, 0)$ $\ell_3 = (a/2, b/2, c/2)$ $a, b, c > 0$ and $a \neq b$
Based Centered Orthorhombic		τ_z τ_u $-\mathrm{Id}$	$\ell_1 = (a, b, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (0, 0, c)$ $a, b, c \geq 0$; $a^2 + b^2 = 1$ $a \neq \sqrt{3}/2$
Face Centered Orthorhombic			$\ell_1 = (a, b, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (a/2, (b+1)/2, c/2)$ $a, b, c > 0$ and $a^2 + b^2 = 1$
Primitive Monoclinic	$\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2^c$	τ_z $-\mathrm{Id}$	$\ell_1 = (a, b, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (0, 0, c)$ $a, b, c > 0$ and $a^2 + b^2 \neq 1$

Table 4 (continued)

Name	Holohedry	Generators	Basis of $\mathscr{L}$
Based Centered Orthorhombic			$\ell_1 = (a, b, 0)$ $\ell_2 = (0, 1, 0)$ $\ell_3 = (0, 1/2, c/2)$ $a, b, c > 0$ and $a^2 + b^2 \neq 1$
Triclinic Orthorhombic	$\boldsymbol{Z}_2^c$	$-\mathrm{Id}$	ℓ_1, ℓ_2 and ℓ_3 do not satisfy any of the previous cases.

We denote by the letter u the line containing the bisector of the angle between ℓ_1 and ℓ_2

Table 5
Translation free irreducible representations for the primitive cubic lattice

dim	$V_{K_1} \oplus \cdots \oplus V_{K_s}$	dim	$V_{K_1} \oplus \cdots \oplus V_{K_s}$
24	$\boldsymbol{K}_1 = \beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$	48	$\boldsymbol{K}_1 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 - \gamma\boldsymbol{k}_3$
1st type	$\boldsymbol{K}_2 = \beta\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_2 = \alpha\boldsymbol{k}_1 + \gamma\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_3 = \beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$		$\boldsymbol{K}_3 = \alpha\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 + \gamma\boldsymbol{k}_3$
	$\boldsymbol{K}_4 = \beta\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 - \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_4 = \alpha\boldsymbol{k}_1 - \gamma\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_5 = \beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$		$\boldsymbol{K}_5 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 + \gamma\boldsymbol{k}_3$
	$\boldsymbol{K}_6 = \beta\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_6 = \alpha\boldsymbol{k}_1 - \gamma\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_7 = \beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$		$\boldsymbol{K}_7 = \alpha\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 - \gamma\boldsymbol{k}_3$
	$\boldsymbol{K}_8 = \beta\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 - \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_8 = \alpha\boldsymbol{k}_1 + \gamma\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_9 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$		$\boldsymbol{K}_9 = \beta\boldsymbol{k}_1 + \gamma\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$
	$\boldsymbol{K}_{10} = \alpha\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$		$\boldsymbol{K}_{10} = \beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2 + \gamma\boldsymbol{k}_3$
	$\boldsymbol{K}_{11} = \alpha\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$		$\boldsymbol{K}_{11} = \beta\boldsymbol{k}_1 - \gamma\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$
	$\boldsymbol{K}_{12} = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$		$\boldsymbol{K}_{12} = \beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2 + \gamma\boldsymbol{k}_3$
	α and β are integers.		$\boldsymbol{K}_{13} = \beta\boldsymbol{k}_1 - \gamma\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$
	$(\alpha, \beta) = 1$		$\boldsymbol{K}_{14} = \beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2 - \gamma\boldsymbol{k}_3$
	α odd and β even.		$\boldsymbol{K}_{15} = \beta\boldsymbol{k}_1 + \gamma\boldsymbol{k}_2 - \alpha\boldsymbol{k}_3$
			$\boldsymbol{K}_{16} = \beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2 + \gamma\boldsymbol{k}_3$
24	$\boldsymbol{K}_1 = \alpha\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$		$\boldsymbol{K}_{17} = \gamma\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$
2nd type	$\boldsymbol{K}_2 = -\beta\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_{18} = \gamma\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_3 = \alpha\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$		$\boldsymbol{K}_{19} = \gamma\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 - \alpha\boldsymbol{k}_3$
	$\boldsymbol{K}_4 = \beta\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_{20} = \gamma\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_5 = \beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2$		$\boldsymbol{K}_{21} = \gamma\boldsymbol{k}_1 - \beta\boldsymbol{k}_2 + \alpha\boldsymbol{k}_3$
	$\boldsymbol{K}_6 = \beta\boldsymbol{k}_1 + \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_{22} = \gamma\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2 - \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_7 = \beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_2$		$\boldsymbol{K}_{23} = \gamma\boldsymbol{k}_1 + \beta\boldsymbol{k}_2 - \alpha\boldsymbol{k}_3$
	$\boldsymbol{K}_8 = \beta\boldsymbol{k}_1 - \alpha\boldsymbol{k}_3$		$\boldsymbol{K}_{24} = \gamma\boldsymbol{k}_1 + \alpha\boldsymbol{k}_2 + \beta\boldsymbol{k}_3$
	$\boldsymbol{K}_9 = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$		α, β, γ are integers.
	$\boldsymbol{K}_{10} = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_3$		$\alpha > \beta > \gamma > 0$
	$\boldsymbol{K}_{11} = \alpha\boldsymbol{k}_1 - \beta\boldsymbol{k}_2$		α, β, γ do not have
	$\boldsymbol{K}_{12} = \alpha\boldsymbol{k}_1 - \beta\boldsymbol{k}_3$		a common factor
	α and β are integers.		other than 1.
	$(\alpha, \beta) = 1$		Only one of α, β
	α odd and β even.		and γ is odd.
6	$\boldsymbol{K}_1 = \boldsymbol{k}_1$		
	$\boldsymbol{K}_1 = \boldsymbol{k}_2$		
	$\boldsymbol{K}_3 = \boldsymbol{k}_3$		

Table 6
Subgroups that may have one-dimensional fixed-point subspaces.

Σ	Generator
$\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,\pi x}$, $\varrho_{\pi/2,y}$, $-\mathrm{Id}$
$\boldsymbol{T} \oplus \boldsymbol{Z}_2^c$	$\varrho_{2\pi/3,v}$, $\varrho_{2\pi/3,w}$, $-\mathrm{Id}$
$\boldsymbol{O}$	$\varrho_{\pi/2,x}$, $\varrho_{\pi/2,y}$
$\boldsymbol{O}^-$	$\varrho_{2\pi/3,v}$, $-\mathrm{Id}\varrho_{\pi/2,z}$
$\boldsymbol{D}_4 \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,z}$, τ_u, $-\mathrm{Id}$
$\boldsymbol{D}_3 \oplus \boldsymbol{Z}_2^c$	$\varrho_{2\pi/3,v}$, τ_u, $-\mathrm{Id}$

The line $x = y, z = 0$ is denoted by the letter u.
The line $-x = y = z$ is denoted by the letter v.
The line $x = y = z$ is denoted by the letter w.

24-dimensional, translation free absolutely irreducible representations. They differ by the action of the holohedry on the wave vectors supporting the representation. The primitive cubic lattice does not support translation free irreducible representations of dimensions other than 6, 24 and 48. In Table 5, we give necessary and sufficient conditions to have translation free representations.

(b) The planforms

In Table 7, we list for each translation free, absolutely irreducible representation the translation free, isotropy subgroups having one-dimensional, fixed-point subspaces.

Table 7
Isotropy subgroups having one-dimensional fixed-point subspaces for the primitive cubic lattice.

Dim	Isotropy Subgroup Σ	Generator
6	$\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,x}$, $\varrho_{\pi/2,y}$, $-\mathrm{Id}$
24	$\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,x}$, $\varrho_{\pi/2,y}$, $-\mathrm{Id}$
1st type		
	$\boldsymbol{O}^{\mathrm{b}} \oplus \boldsymbol{Z}_2^c$	$(\varrho_{\pi/2,x}, (\frac{1}{2}, \frac{1}{2}, 0))$, $(\varrho_{\pi/2,y}, (0, \frac{1}{2}, \frac{1}{2}))$, $-\mathrm{Id}$
	$\boldsymbol{O}^{\#}$	$(\varrho_{\pi/2,x}, (\frac{1}{4}, \frac{-1}{2}, 0))$, $(\varrho_{\pi/2,y}, (0, \frac{1}{4}, 0)$
	$\boldsymbol{T}^{\mathrm{b}} \oplus \boldsymbol{Z}_2^c$	$(\varrho_{2\pi/3,v}, (\frac{1}{2}, 0, \frac{1}{2}))$, $\varrho_{2\pi/3,w}$, $-\mathrm{Id}$
	$\boldsymbol{D}_3^{\mathrm{b}} \oplus \boldsymbol{Z}_2^c$ when $\alpha = 1$.	$\varrho_{2\pi/3,v}$, $(\tau_u, (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}))$, $-\mathrm{Id}$
24	$\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,x}$, $\varrho_{\pi/2,y}$, $-\mathrm{Id}$
2nd type		
	$\boldsymbol{O}^{\#} \oplus \boldsymbol{C}_2^c$	$(\varrho_{\pi/2,x}, (\frac{1}{2}, 0, 0))$, $(\varrho_{\pi/2,y}, (0, \frac{1}{2}, 0))$, $-\mathrm{Id}$
	$\boldsymbol{O}^{\#}$	$(\varrho_{\pi/2,x}, (\frac{1}{4}, \frac{-1}{2}, 0))$, $(\varrho_{\pi/2,y}, (0, \frac{1}{4}, 0))$
	$\boldsymbol{T}^{\#} \oplus \boldsymbol{Z}_2^c$	$(\varrho_{2\pi/3,v}, (0, \frac{1}{2}, \frac{1}{2}))$, $\varrho_{2\pi/3,w}$, $-\mathrm{Id}$
48	$\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$	$\varrho_{\pi/2,x}$, $\varrho_{\pi/2,y}$, $-\mathrm{Id}$
	$\boldsymbol{O}^{\mathrm{b}} \oplus \boldsymbol{Z}_2^c$	$(\varrho_{\pi/2,x}, (\frac{1}{2}, \frac{1}{2}, 0))$, $(\varrho_{\pi/2,y}, (0, \frac{1}{2}, \frac{1}{2}))$, $-\mathrm{Id}$
	$\boldsymbol{O}^{\#} \oplus \boldsymbol{Z}_2^c$	$(\varrho_{\pi/2,x}, (\frac{1}{2}, 0, 0))$, $(\varrho_{\pi/2,y}, (0, \frac{1}{2}, 0))$, $-\mathrm{Id}$
	$\boldsymbol{O}^{\dagger} \oplus \boldsymbol{Z}_2^c$	$(\varrho_{\pi/2,x}, (0, \frac{1}{2}, 0))$, $(\varrho_{\pi/2,y}, (0, 0, \frac{1}{2}))$, $-\mathrm{Id}$

The line $x = y, z = 0$ is denoted by the letter u.
The line $-x = y = z$ is denoted by the letter v.
The line $x = y = z$ is denoted by the letter w.

To produce this table, we follow the procedure outlined in Section 3(a). We only sketch the computations for the first type of 24-dimensional representation; the details can be found in Dionne [6]. Let H be the holohedry $\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$ of the primitive cubic lattice and $\Gamma = H \dotplus T^3$. From the trace formula 3.1, we get that a subgroup $\Sigma \subset \Gamma$ may have a one-dimensional fixed-point subspace if Σ is isomorphic by Π_H to one of the subgroups of H given in Table 6.

Two of the subgroups in Table 6 can be ignored. From Proposition 3.2, no subgroup $\Sigma \subset \Gamma$ isomorphic by Π_H to $\boldsymbol{O}^-$ can have a one-dimensional fixed-point subspace. Since $\boldsymbol{O}^-$ acts transitively on the set of wave vector pairs $\{(\boldsymbol{K}_j, -\boldsymbol{K}_j) : j = 1, 2, \ldots, 12\}$, all the coordinates of $z \in \mathrm{Fix}(\Sigma)$ are either zero or nonzero. Since $\boldsymbol{O}^-$ does not act transitively on the set of wave vectors $\{\pm\boldsymbol{K}_j : j = 1, 2, \ldots, 12\}$, Proposition 3.2 can not be satisfied.

Moreover, no isotropy subgroup $\Sigma \subset \Gamma$ isomorphic by Π_H to $\boldsymbol{D}_4 \oplus \boldsymbol{Z}_2^c$ can have a one-dimensional fixed-point subspace and be translation free. Suppose that Σ_z is an isotropy subgroup having a one-dimensional fixed-point subspace and isomorphic by Π_H to $\boldsymbol{D}_4 \oplus \boldsymbol{Z}_2^c$. To satisfy Proposition 3.2, the nonzero coordinates of z must be associated to a set of wave vectors of the form $A = \{\pm\boldsymbol{K}_j : j \in J\}$ where J is a subset of $\{1, 2, \ldots, 12\}$ and $\boldsymbol{D}_4 \oplus \boldsymbol{Z}_2^c$ acts transitively on A. The only two subsets of wave vectors of this form are:

$$\{\pm\boldsymbol{K}_j : j = 1, 3, 5, 7\}$$

and

$$\{\pm\boldsymbol{K}_j : j \neq 1, 3, 5, 7\}.$$

But, $(1/\alpha, 1/\alpha, 0)$ (respectively, $(0, 0, 1/\alpha)$) is a nontrivial translation that acts trivially on the coordinates z_j's associated with the wave vectors of the first (respectively, second) set above. In either case, Σ_z is not translation free. For similar reasons, when $\alpha \neq 1$, no isotropy subgroups $\Sigma \subset \Gamma$ isomorphic by Π_H to $\boldsymbol{D}_3 \oplus \boldsymbol{Z}_2^c$ can have a one-dimensional fixed-point subspace and be translation free.

For the three remaining subgroups G of H, we proceed as in Sections 3(b) and (c) to compute the subgroups Σ of Γ that are isomorphic by Π_H to G. For instance, there are four subgroups of Γ that are isomorphic by Π_H to $\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$; they are generated by

$$\left(\varrho_{\pi/2,x}, \left(\frac{n}{2}, \frac{m}{2}, 0\right)\right), \quad \left(\varrho_{\pi/2,y}, \left(0, \frac{n}{2}, \frac{m}{2}\right)\right), \quad \text{and } -\mathrm{Id}.$$

for $n, m = 0$ or 1.

Finally, a direct calculation determines which of the subgroups Σ are translation free isotropy subgroups having one-dimensional fixed-point subspaces. For instance, of the four groups isomorphic by Π_H to $\boldsymbol{O} \oplus \boldsymbol{Z}_2^c$, only two have one-dimensional fixed-point subspaces. They are given by

$n = m = 0$ and $n = m = 1$. These subgroups are translation free isotropy subgroups when the representation is translation free.

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Summary

When solving systems of PDE with two space dimensions it is often assumed that the solution is spatially doubly periodic. This assumption is usually made in systems such as the Boussinesq equation or reaction-diffusion equations where the equations have Euclidean invariance. In this article we use group theoretic techniques to determine a large class of spatially doubly periodic solutions that are forced to existence near a steady-state bifurcation from a translation-invariant equilibrium.

This type of bifurcation problem has been considered by many authors when studying a number of different systems of PDE. Typically, these studies focus at the beginning on equilibria that are spatially periodic with respect to a fixed planar lattice type—such as square or hexagonal. Our focus is different in that we attempt to find all spatially periodic equilibria that bifurcate on all lattices. This point of view leads to some technical simplifications such as being able to restrict to translation free irreducible representations.

Of course, many of the types of solutions that we find are well-known—such as hexagon and roll solutions on a hexagonal lattice. This coordinated group theoretic approach does lead, however, to solutions which seem not to have been discussed previously (antisquare solutions on a square lattice) as well as to a more complete classification of the symmetry types of possible solutions. Moreover, our methods extend to triply periodic solutions of PDE with three spatial variables. Some of these results, namely those concerned with primitive cubic lattices, are presented here. The complete results on triply periodic solutions may be found in [6, 7].

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