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Detecting the symmetry of attractors

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This paper addresses the issue of how to determine numerically the symmetry of an attractor for dynamical systems. (The symmetries of attractors in phase space are related to patterns in the time-average of the solution.) Our approach to this question proceeds in two parts. First, we prove a general theorem, based on group-theoretic and differential topological ideas, which states that generically the symmetry of a (thickened) attractor can be computed from the symmetries of a point in an auxiliary space. This theorem proceeds by integrating an equivariant mapping over the thickened attractor.

Once this is done, the numerical computation of symmetries reduces to showing that a certain nonnegative number is zero. Numerically, demonstrating that this number is zero can be difficult. Thus the second part of the algorithm is to consider how this number varies with parameters and noting that sudden jumps towards zero can be associated with increases in symmetry. The paper is divided into two parts. In the first we prove the general theorem and in the second we illustrate how the numerical techniques work on several examples including discrete dynamical systems with tetrahedral symmetry in $\mathbb{R}^3$ and systems of three coupled cells. In high dimensions the integral mentioned previously is difficult to compute. For such examples, we assume that an ergodic theorem is valid and that symmetries can be computed using a time-average. We compare both of these methods on the low-dimensional examples as well as detect points of symmetry creation for a reaction–diffusion equation on an interval. This technique can also be used in principle to compute the symmetries of an attractor in an experiment from a time-series.

1. Introduction

Let Γ be a finite group and let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuous and Γ-equivariant. We view f as a discrete dynamical system and assume that the compact set L is an attractor for f with an open basin of attraction. The question we address is: "How can we determine the symmetry group of L?" We denote that symmetry group by

$$\Sigma(L) = \{\gamma \in \Gamma: \gamma L = L\}.$$

The reasons for asking this question are discussed in [4]. Roughly speaking, we view f as representing the dynamics of an equation in *phase space* while our interest in the symmetry of an attractor lies in *physical space*. For equilibria and for periodic states there is a well understood connection between symmetries in phase space and symmetries in physical space [7]. In particular, symmetries of equilibria have been identified with patterns in solutions in a number of physically interesting situations including Rayleigh–Bénard convection, the Taylor–Couette experiment and Turing patterns in reaction–diffusion systems. Rather little attention has been paid, however, to the physical space interpretation of the symmetry of a chaotic attractor. It is shown by example in [4] that those symmetries are related to patterns that appear in the time-aver-

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age of a solution (see section 10) – even though that pattern is never present at any particular moment in time.

The existence of symmetries of attractors is demonstrated clearly through pictures for planar maps [2,6,9] but rather less is known about the symmetries of attractors in higher dimensions, in part because of the difficulty of visualizing these symmetries. Even in three dimensions it is sometimes difficult to determine the exact symmetries of a cloud of points – which is what a chaotic attractor of a discrete dynamical system resembles in this dimension.

Our approach to determining numerically the symmetries of an attractor L is a three step process. First, we thicken L to an open set A having the same symmetries as L. Second, we transfer the symmetries of an open set $A \subset \mathbb{R}^n$ to the symmetries of a point in an associated space W by integrating an equivariant map $\phi: \mathbb{R}^a \to W$ over A. This point is denoted by $K_\phi(A)$. We call ϕ an *observable* and $K_\phi(A)$ an *observation*. In lemma 3.2 we show that the symmetries of A fix the point $K_\phi(A)$. Hence, by definition, $\Sigma(A)$ is contained in the isotropy subgroup of the point $K_\phi(A)$ which we denote by $\Sigma_\phi(A)$. Finally, we show that for certain ϕ generically the symmetry group $\Sigma(A)$ actually equals $\Sigma_\phi(A)$ whose numerical computation is, in principle, a straightforward task.

The notion of genericity that we use here is a natural one for dynamical systems. Observe that if $\psi: \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant diffeomorphism, then the set $\psi(L)$ is an attractor for the mapping $\psi \circ f \circ \psi^{-1}$, which is just the map f viewed in a new coordinate system, and the symmetries of A and $\psi(A)$ are equal. Fix the equivariant map ϕ. What we prove is that if ϕ satisfies certain easily verified conditions, then for any A there is an open dense set of near identity diffeomorphisms ψ so that $\Sigma(A) = \Sigma_\phi(\psi(A))$. These ϕ, which we call *detectives*, each generate a method for detecting symmetries which works, in principle, for almost any open set A.

In theorem 5.2 we prove that if W is a representation of Γ that contains all of the nontrivial irreducible representations of Γ and if the observable $\phi: \mathbb{R}^n \to W$ is a polynomial mapping whose components in each of these irreducible representations is nonzero, then ϕ is a detective – from which we conclude that detectives always exist. It also follows that for most finite groups that one is likely to consider, it is possible to construct detectives. The general theorems concerning the existence and construction of detectives are presented in sections 2–5 of this paper.

The remaining sections are devoted to illustrating the use and explicit construction of detectives. In section 6 we discuss the symmetries of attractors for a certain parametrized family of mappings on $\mathbb{R}^3$ having tetrahedral symmetry. We show how, using detectives, the computer can determine, almost automatically, the symmetries of attractors as parameters are varied. Indeed, this variation of parameters is more or less necessary for the method to work. The difficulty concerns the numerical computation of $K_\phi(A)$, which – as in any numerical computation – can be computed only approximately. Indeed, we may reformulate the question: "Is the group element γ a symmetry of the point $K_\phi(A)$ in W?" by computing the distance of the computed $K_\phi(A)$ to the fixed-point subspace $\mathrm{Fix}(\gamma)$. In theory γ is in the isotropy subgroup $\Sigma_\phi(A)$ precisely when this computed distance is zero. So the numerical difficulty in determining whether γ is a symmetry of A reduces to determining whether a certain nonnegative number is actually zero.

Our strategy for determining when this distance is zero is to compute the distance as a function of parameters and call the distance zero when there is a jump in the distance to a number close to zero. This test is based on the experience obtained by simulation in [2] which suggests that when the symmetry of an attractor changes as parameters are varied, the size of the attractor also changes dramatically. Thus this numerical method seems to be well suited to determining approximate parameter values where symmetry increasing bifurcations occur.

The result of our particular computations is a phase diagram showing regions in parameter space where attractors with various symmetries have been found. See fig. 2.

We note that in low dimensions or in the presence of reflections the symmetry groups of attractors cannot be just any subgroup of Γ. In certain cases some subgroups are excluded [5,1].

The numerical method that we describe theoretically is based on computing numerically an integral over the thickened attractor. When the state space $\mathbb{R}^n$ is of large dimension (even infinite-dimensional in the case of PDEs), the computation of this integral becomes impractical. Then our only recourse is to presume that an ergodic theorem for the attractor is valid and to compute an ergodic sum rather than the integral. In section 7 we describe this ergodic sum more fully and compare it to the integral test for certain low-dimensional examples.

The remaining sections are devoted to describing how our method works for systems of ODEs and PDEs and how it might be used in experiments. Systems of identical coupled oscillators are described in sections 8 and 9. In these sections we show that for rings of oscillators which have D_p symmetry, the matrix outer product $\phi(x) = x \cdot x^t$ mapping the state space into the space of symmetric matrices is always a detective, and we illustrate this fact by computing the symmetries of attractors for a system of three coupled oscillators.

In the last section, section 10, we apply the method to the Brusselator on the line, which has only a reflectional symmetry. We also illustrate how these methods would apply to PDEs or even to experiments defined on a domain with square geometry.

2. Thickened attractors

Let Γ be a finite group. We assume that we have a mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ that is continuous and Γ-equivariant. We view f as a discrete dynamical system and suppose that the compact set L is an attractor for f with an open basin of attraction. The question we ask is: "How can we determine the symmetry group of L?" We let $\Sigma(L)$ denote the group of symmetries of L. One consequence of the open basin assumption is that for each $\gamma \in \Gamma$ either $\gamma L = L$ or $\gamma L \cap L = \emptyset$. See proposition 1.1 of [2].

In general, it is impossible to know precisely the set L. What one computes graphically on a computer is the set A defined as follows. Choose a small positive number τ and let A be the set of all points whose distance to L is less than τ. Since L is compact, A is in the basin of attraction for L for small enough τ and A has the same symmetry group as L. We call A a *thickened attractor* and note that thickened attractors have the property that either $\gamma A = A$ or $\gamma A \cap A = \emptyset$ for all $\gamma \in \Gamma$.

Later on, we shall need A to have a boundary that is sufficiently regular to apply Stokes theorem. Indeed we may assume that A is an open set with the same symmetries as L, $\bar{A}$ is compact, and A has a piecewise smooth boundary. To construct such an A cover L by a finite number of τ-balls and let A be the union of these τ-balls along with all images of these τ-balls under $\Sigma(L)$.

The mathematical problem that we address in this paper is the following. Let $\mathscr{A}$ be the class of all open subsets of $\mathbb{R}^n$ with piecewise smooth boundary that satisfy the dichotomy $\gamma A = A$ or $\gamma A \cap A = \emptyset$ for all $\gamma \in \Gamma$. Find a procedure for generically determining the symmetries of sets in $\mathscr{A}$.

Our basic approach is to transfer the problem of finding the symmetries of a set in $\mathscr{A}$ to finding the symmetries of a point in an associated space W. We do this by averaging an observable over the set, as we now explain. We refer to the subgroup of symmetries of the set A as $\Sigma(A)$.

3. Estimates for $\Sigma(A)$

Definition 3.1. An observable is a C^∞ Γ-equivariant mapping $\phi: \mathbb{R}^n \to W$ where W is some (finite-dimensional) representation of Γ. An observation is:

$$K_\phi(A) = \int_A \phi \, \mathrm{d}\mu ,$$

where μ is Lebesgue measure.

Note that the observation $K_\phi(A)$ is a vector in the space W since the observation is just the integral of a W-valued function. Moreover, this integral can be nonzero since A is an open set and has positive Lebesgue measure. This should be contrasted with integrating over the set L which itself might have zero Lebesgue measure. Define $\Sigma_\phi(A)$ to be the isotropy subgroup of $K_\phi(A)$ in W, that is,

$$\Sigma_\phi(A) = \{\gamma \in \Gamma : \gamma K_\phi(A) = K_\phi(A)\} .$$

Lemma 3.2. For each observable ϕ

$$\Sigma(A) \subset \Sigma_\phi(A) .$$

Proof. Suppose that $\sigma \in \Sigma(A)$. We use the Γ-equivariance of ϕ to see that

$$\sigma K_\phi(A) = \sigma \int_A \phi(x)\, \mathrm{d}\mu(x) = \int_A \phi(\sigma x)\, \mathrm{d}\mu(x) .$$

Since Σ acts orthogonally on $\mathbb{R}^n$ it follows that

$$\int_A \phi(\sigma x)\, \mathrm{d}\mu(x) = \int_A \phi(\sigma x)\, \mathrm{d}\mu(\sigma x) .$$

Then the change of variables formula for integration implies that

$$\sigma K_\phi(A) = \int_{\sigma A} \phi(x)\, \mathrm{d}\mu(x) .$$

Finally, the fact that $\sigma A = A$ implies that

$$\sigma K_\phi = K_\phi(A) ,$$

and $\sigma \in \Sigma_\phi(A)$. □

Proposition 3.3. For every open set $A \in \mathscr{A}$ there exists a representation W of Γ and an observable $\phi: \mathbb{R}^n \to W$ such that

$$\Sigma(A) = \Sigma_\phi(A) .$$

Proof. The representation that we use here is the *left regular representation* V_Γ consisting of all real-valued functions on Γ. V_Γ is a vector space of dimension $|\Gamma|$. The action of $\gamma \in \Gamma$ on $h \in V_\Gamma$ is defined by

$$(\gamma \cdot h)(\delta) = h(\gamma^{-1}\delta) .$$

We can choose an open set $U \in \mathscr{A}$ such that $\bar{U} \subset A$ and the symmetries of U are the same as those for A. Moreover, given $\varepsilon > 0$ we can choose U so that $\mu(A - U) < \varepsilon$.

Next we define $\hat{\phi}: \mathbb{R}^n \to V_\Gamma$. Let $h: \Gamma \to \mathbb{R}$ be defined by

$$h(\gamma) = \begin{cases} 1 & \text{for } \gamma \in \Sigma(A) , \\ 0 & \text{otherwise} . \end{cases}$$

Let $\rho: \mathbb{R}^n \to [0, 1]$ be a smooth bump function that is 1 on U and 0 off A. Then define $\hat{\phi}(x) = \rho(x)\, h$ so that

$$\hat{\phi}(x) = \begin{cases} 0 & \text{if } x \notin A , \\ h & \text{if } x \in U . \end{cases}$$

Next define ϕ by averaging $\hat{\phi}$ over Γ as follows:

$$\phi(x) = \sum_{\gamma \in \Gamma} \gamma^{-1} \hat{\phi}(\gamma x) .$$

It is easy to check that $\phi: \mathbb{R}^n \to V_\Gamma$ is Γ-equivariant.

We now complete the proof by showing that if $\delta \notin \Sigma(A)$, then $\delta \notin \Sigma_\phi(A)$. Begin by noting that if $x \in U$, then

$$\phi(x) = |\Sigma(A)| h .$$

This equality may be verified as follows. Observe that if $x \in U$ then either $\gamma x \in U$ if $\gamma \in \Sigma(A)$ or $\gamma x \notin A$ if $\gamma \notin \Sigma(A)$. Hence $\gamma^{-1} h = h$ for all

$\gamma \in \Sigma(A)$ and is zero otherwise. Now compute

$$\begin{aligned}\phi(x) &= \sum_{\gamma \in \Gamma} \gamma^{-1}\hat{\phi}(\gamma x) \\ &= \sum_{\gamma \in \Sigma(A)} \gamma^{-1} h \\ &= |\Sigma(A)| h .\end{aligned}$$

It follows that

$$\begin{aligned}K_\phi(A) &= \int_A \phi \, \mathrm{d}\mu \\ &= \int_U \phi \, \mathrm{d}\mu + O(\varepsilon) \\ &= |\Sigma(A)| \mu(U)\, h + O(\varepsilon) .\end{aligned}$$

Hence $\delta K_\phi(A) = |\Sigma(A)| \mu(U)\, \delta h + O(\varepsilon)$. Since $\delta h \neq h$ when $\delta \not\in \Sigma(A)$ we see that $K_\phi(A) \neq \delta K_\phi(A)$. So $\delta \not\in \Sigma_\phi(A)$, as desired. □

4. Distinguishing subgroups

Definition 4.1. Let $\Sigma \subset \Gamma$ be a subgroup and let W be a representation of Γ. We say that W distinguishes Σ if

$$\dim \mathrm{Fix}_W(\Delta) < \dim \mathrm{Fix}_W(\Sigma) < \dim \mathrm{Fix}_W(Y)$$

whenever Δ and Y are subgroups not equal to Σ and $\Delta \supset \Sigma \supset Y$.

Definition 4.2. Two representations V and W of Γ are lattice equivalent if there exists a linear isomorphism $L: V \to W$ such that

$$L(\mathrm{Fix}_V(\Sigma)) = \mathrm{Fix}_W(\Sigma)$$

for every subgroup $\Sigma \subset \Gamma$.

Clearly, isomorphic representations are lattice equivalent, but inequivalent representations can be lattice equivalent. For example, consider the two distinct two-dimensional irreducible representations of D_5.

We now state the basic result of this section. Let $W_1, \ldots, W_s$ be, up to lattice equivalence, all of the nontrivial irreducible representations of Γ. Define

$$W(\Gamma) = W_1 \oplus \cdots \oplus W_s .$$

Theorem 4.3. Let Γ be a finite group and let $V \supset W(\Gamma)$. Then V distinguishes all subgroups of Γ.

We begin the proof of theorem 4.3 with two lemmas.

Lemma 4.4. Suppose that V distinguishes Σ and that W is a representation of Γ. Then $V \oplus W$ distinguishes Σ.

Proof. Since

$$\mathrm{Fix}_{V \oplus W}(\Sigma) = \mathrm{Fix}_V(\Sigma) \oplus \mathrm{Fix}_W(\Sigma) \tag{4.1}$$

all you need to verify the strict inequality in dimension is the inequality in dimension on fixed-point subspaces of V. □

Lemma 4.5. Suppose that V distinguishes Σ.

(a) Suppose that $V = V_1 \oplus V_2 \oplus W$ where V_1, V_2, and W are representations of Γ and V_1 and V_2 are lattice equivalent. Then $V_1 \oplus W$ distinguishes Σ.

(b) Suppose that $V = V_1 \oplus W$ where W is the trivial representation of Γ. Then V_1 distinguishes Σ.

This lemma is easily proved using (4.1).

Proof of theorem 4.3. Recall that V_Γ, the left regular representation of Γ, consists of all functions from Γ into $\mathbb{R}$. Let $\Sigma \subset \Gamma$ be a subgroup. Then

$$\mathrm{Fix}_{V_\Gamma}(\Sigma) = \{h \in V_\Gamma : h(\sigma^{-1}\delta) = h(\delta) \quad \forall \sigma \in \Sigma\} .$$

Thus if h is in $\mathrm{Fix}_{V_\Gamma}(\Sigma)$ then h is constant on Σ. It follows that if $\Sigma \subset \Delta$ but $\Sigma \neq \Delta$, then $\dim \mathrm{Fix}_{V_\Gamma}(\Delta) < \dim \mathrm{Fix}_{V_\Gamma}(\Sigma)$. Certainly, $\mathrm{Fix}_{V_\Gamma}(\Delta) \subset \mathrm{Fix}_{V_\Gamma}(\Sigma)$. However, it is easy to construct a function that is in $\mathrm{Fix}_{V_\Gamma}(\Sigma)$ that is not in $\mathrm{Fix}_{V_\Gamma}(\Delta)$. Define

$$h(\gamma) = \begin{cases} 1 & \gamma \in \Delta - \Sigma \\ 0 & \text{otherwise}. \end{cases}$$

It follows that V_Γ distinguishes every subgroup of Γ. A standard theorem states (see [10], p. 77) that up to isomorphism every irreducible representation appears in V_Γ. Hence lemma 4.5 implies that $W(\Gamma)$ distinguishes all subgroups. Now apply Lemma 4.4 to prove that V distinguishes all subgroups of Γ. □

We note that proposition 3.3 can be strengthened to show the following. Fix $A \in \mathscr{A}$. Then $\Sigma(A) = \Sigma_\phi(A)$ for almost all observables $\phi: \mathbb{R}^n \to W \supset W(\Gamma)$. The idea is to show that the linear mapping $\Phi(\phi) = K_\phi(A)$ is onto $\mathrm{Fix}_W(\Sigma(A))$, which can be done using bump functions. We will not pursue this result since there is a stronger and more useful version of this theorem which we present in the next section.

5. Detectives

Let $\phi: \mathbb{R}^n \to W$ be an observable. Roughly speaking we call ϕ a detective if for almost every open set $A \subset \mathbb{R}^n$ the subgroup $\Sigma_\phi(A)$ (the isotropy subgroup of the observation $K_\phi(A)$) and $\Sigma(A)$ (the group of symmetries of A) are equal. To make this definition precise we must define more accurately what we mean by 'almost every', and to do that we must indicate how we can perturb an open set in such a way as to preserve its group of symmetries.

Perturbations of sets are defined using diffeomorphisms in the following way. Let ψ be in $\mathrm{Diff}_\Gamma(\mathbb{R}^n)$, where $\mathrm{Diff}_\Gamma(\mathbb{R}^n)$ is the group of Γ-equivariant C^∞ diffeomorphisms on $\mathbb{R}^n$. It is easy to check that the group of symmetries of $\psi(A)$ equals $\Sigma(A)$. Moreover, if ψ is near identity, then $\psi(A)$ is a small perturbation of A.

We note that if $f: \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant dynamical system with an attractor L, then $\psi(L)$ is an attractor for the dynamical system $\psi \circ f \circ \psi^{-1}$. Thus the type of perturbations of A that we consider here are natural from the point of view of dynamical systems, as they correspond to making a smooth change of coordinates in the original dynamical system.

Definition 5.1. The observable ϕ is a detective if for each subset $A \in \mathscr{A}$, almost all near identity diffeomorphisms $\psi \in \mathrm{Diff}_\Gamma(\mathbb{R}^n)$ satisfy $\Sigma_\phi(\psi(A)) = \Sigma(A)$.

Recall from lemma 3.2 that $\Sigma_\phi(\psi(A)) \supset \Sigma(\psi(A))$. Since ψ is a Γ-equivariant diffeomorphism it follows that $\Sigma(\psi(A)) = \Sigma(A)$. Thus to prove that ϕ is a detective we must show that for almost all near identity ψ, $\Sigma_\phi(\psi(A)) \subset \Sigma(\psi(A))$.

For ϕ to be a detective we need to know that there are observations $K_\phi(\psi(A))$ that lie in $\mathrm{Fix}_W(\Sigma(A)) - \mathrm{Fix}_W(\Delta)$ for all subgroups $\Delta \supset \Sigma(A)$ with $\Delta \neq \Sigma(A)$. This is not possible if $\dim \mathrm{Fix}_W(\Sigma(A)) = \dim \mathrm{Fix}_W(\Delta)$. Thus ϕ may be a detective only if W distinguishes all subgroups of Γ. Thus a necessary condition for ϕ to be a detective is that W distinguishes all subgroups.

We now state our main theorem. Recall that $W(\Gamma)$ is the sum of all nontrivial lattice inequivalent irreducible representations of Γ. We can write $W(\Gamma) = W_1 \oplus \cdots \oplus W_s$. Should a representation W of Γ contain $W(\Gamma)$, then we can decompose $W = W(\Gamma) \oplus W^\perp$ for some representation $W^\perp$.

Theorem 5.2. Let $W \supset W(\Gamma)$ and let $\phi: \mathbb{R}^n \to W$ be a polynomial observable where $\phi = (\phi_1, \ldots, \phi_s, \phi^\perp)$ in coordinates. Suppose that $\phi_j \neq 0$ for $1 \leq j \leq s$. Then ϕ is a detective.

Corollary 5.3. Every finite subgroup $\Gamma \subset \mathrm{O}(n)$ has a detective.

Proof. To prove this corollary we apply theorem 5.2 and to apply this theorem we need only show that there exists a nonzero Γ-equivariant polynomial map from $\mathbb{R}^n$ to W where W is any representation of Γ. This we do by averaging.

Since Γ is a finite group the principal orbit type is the trivial group. Hence we can choose a nonzero vector $x \in \mathbb{R}^n$ that has trivial isotropy. Next choose a nonzero vector $w \in W$. Let $\hat{f}: \mathbb{R}^n \to W$ be a polynomial mapping such that $\hat{f}(x) = w$ and $\hat{f}(\gamma x) = 0$ for all nonidentity $\gamma \in \Gamma$. Now define

$$f(z) = \sum_{\gamma \in \Gamma} \gamma^{-1} \hat{f}(\gamma z),$$

and observe that f is a Γ-equivariant polynomial. Finally compute $f(x) = w$ from which one can conclude that f is nonzero. □

We begin our proof of theorem 5.2 with the observation that the bigger the range space W the more likely it is that ϕ is a detective.

Lemma 5.4. Let $\rho: W_1 \to W_2$ be a Γ-equivariant projection. Let $\phi_1: \mathbb{R}^n \to W_1$ be an observable and let $\phi_2 = \rho\phi_1$. If the observable ϕ_2 is a detective, then ϕ_1 is also a detective.

Proof. The assumption that ρ is Γ-equivariant and onto implies that $W_1 = W_2 \oplus V$ for some Γ-invariant subspace V of W_1. Hence, the isotropy subgroup of a point (w_2, v) is the intersection of the isotropy subgroups of $w_2 \in W_2$ and $v \in V$. In particular, the isotropy subgroup of a point in W_1 is contained in the isotropy of the projection of that point in W_2. Since $\rho(K_{\phi_1}(A)) = K_{\phi_2}(A)$, it follows that $\Sigma_{\phi_1}(A) \subset \Sigma_{\phi_2}(A)$.

On the other hand, if ϕ_2 is a detective, then generically $\Sigma_{\phi_2}(A) = \Sigma(A)$. But lemma 3.2 states that $\Sigma(A) \subset \Sigma_{\phi_1}(A)$ always. Therefore, generically $\Sigma_{\phi_2}(A) \subset \Sigma_{\phi_1}(A)$. Thus generically $\Sigma_{\phi_1}(A) = \Sigma_{\phi_2}(A) = \Sigma(A)$ and ϕ_1 is a detective, as asserted. □

It follows that it is sufficient to prove theorem 5.2 when $W = W(\Gamma)$.

We divide the proof of theorem 5.2 into three main steps. First, we show that ϕ is a detective if a certain nonlinear map Ψ_ϕ^A is locally onto (lemma 5.5). Next we show how to prove that Ψ_ϕ^A is onto by linearizing Ψ_ϕ^A (proposition 5.6). Finally, we show how to verify that the linearization of Ψ_ϕ^A is onto by using the fact that ϕ is a polynomial mapping (proposition 5.8). We begin by defining the mapping Ψ_ϕ^A.

Let Ψ_ϕ^A: $\mathrm{Diff}_{\Gamma}(\mathbb{R}^n) \to \mathrm{Fix}_W(\Sigma(A))$ be defined by

$$\Psi_\phi^A(\psi) = K_\phi(\psi^{-1}(A)).$$

Lemma 5.5. Let $\phi: \mathbb{R}^n \to W$ be an observable and assume that W distinguishes all subgroups. If for each set $A \in \mathscr{A}$ there exists an open neighborhood U of the identity in $\mathrm{Diff}_{\Gamma}(\mathbb{R}^n)$ such that the observations $\Psi_\phi^A(U)$ cover an open neighborhood $\mathcal{O}$ of $K_\phi(A)$ in $\mathrm{Fix}_W(\Sigma(A))$, then ϕ is a detective.

Proof. Let V be the algebraic variety $\cup \mathrm{Fix}_W(\Delta)$ where the union is taken over all subgroups Δ containing but not equal to $\Sigma(A)$. Since W distinguishes all subgroups, V is a variety of codimension at least one in $\mathrm{Fix}_W(\Sigma(A))$. It follows that the set $\mathcal{O}' = \mathcal{O} - V$ is an open dense subset of $\mathcal{O}$ in W whose closure includes $K_\phi(A)$.

Since Ψ_ϕ^A is smooth, $(\Psi_\phi^A)^{-1}(\mathcal{O}') \cap U$ is an open dense subset of U in $\mathrm{Diff}_{\Gamma}(\mathbb{R}^n)$ whose closure contains the identity. It follows that for most near identity diffeomorphisms ψ the observations $\Psi_\phi^A(\psi) = K_\phi(\psi^{-1}(A))$ are not in V and are arbitrarily close to $K_\phi(A)$. This proves the lemma since observations not in the variety V have the correct isotropy subgroup. □

Next we want to show that for each open set A, Ψ_ϕ^A is onto a neighborhood of $K_\phi(A)$. We do this by using the implicit function theorem. We assume that ψ_t is a one-parameter family in $\mathrm{Diff}_{\Gamma}(\mathbb{R}^n)$ with $\psi_0(x) = x$, and we let X be the infinitesimal generator for ψ_t, that is, $X = (\mathrm{d}/\mathrm{d}t)\psi_t|_{t=0}$. Then we differentiate $\Psi^A_\phi(\psi_t)$ with respect to t and evaluate at $t=0$, to get a linear mapping $\mathscr{L}^A_\phi(X)$. If we can show that $\mathscr{L}^A_\phi$ is onto $\mathrm{Fix}_W(\Sigma(A))$, then we can apply the implicit function theorem and lemma 5.5 to conclude that ϕ is a detective. Let $C_\Gamma(\mathbb{R}^n, \mathbb{R}^n)$ be the space of C^∞ Γ-equivariant mappings on $\mathbb{R}^n$.

Proposition 5.6. Let $\phi: \mathbb{R}^n \to W$ be an observable and assume that W distinguishes all subgroups. Suppose that for every set $A \in \mathscr{A}$ the linear mapping

$$\mathscr{L}^A_\phi: C_\Gamma(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{Fix}_W(\Sigma(A))$$

is onto, then ϕ is a detective. Moreover,

$$\mathscr{L}^A_\phi(X) = \int_{\partial A} \phi X \cdot N \,\mathrm{d}\nu ,$$

where N is the unit outward normal on ∂A and ν is the natural measure induced by Lebesgue measure on ∂A.

Proof. The discussion preceding the statement of proposition 5.6 shows that ϕ is a detective. Thus, we need only verify the computation of the linear mapping $\mathscr{L}^A_\phi$. Using change of variables in integration, observe that

$$\Psi^A_\phi(\psi_t) = \int_{\psi_t(A)} \phi(x)\,\mathrm{d}\mu(x) = \int_A \phi(\psi_t(x)) \det(\mathrm{d}\psi_t)_x \,\mathrm{d}\mu(x) .$$

Differentiating with respect to t and evaluating at $t=0$ leads to

$$\mathscr{L}^A_\phi(X) = \int_A \left[(\mathrm{d}\phi)_x(X(x)) + \phi \frac{\mathrm{d}}{\mathrm{d}t} \det(\mathrm{d}\psi_t)|_{t=0} \right] \mathrm{d}\mu(x) .$$

Next observe that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(\mathrm{d}\psi_t)|_{t=0} = \mathrm{tr}(\mathrm{d}X)_x = \nabla \cdot X .$$

Finally, use Stokes theorem to conclude that

$$\mathscr{L}^A_\phi(X) = \int_{\partial A} \phi X \cdot N \,\mathrm{d}\nu . \tag{5.1}$$

□

It follows from (5.1) that if X has support in the interior of A, then $\mathscr{L}^A_\phi(X) = 0$. Indeed, this fact could have been anticipated in the following way. If ψ were a diffeomorphism on $\mathbb{R}^n$ with support in A, then $\psi(A) = A$. Thus the observations K_ϕ over A and $\psi(A)$ are equal. So no infinitesimal change occurs for such deformations.

We now use the explicit computation of $\mathscr{L}^A_\phi$ in (5.1) to compute explicitly the image of $\mathscr{L}^A_\phi$. Let F^A_ϕ be the subspace of W spanned by $\phi(x)$ where x is in the smooth part of ∂A. Let $P: W \to \mathrm{Fix}_W(\Sigma(A))$ be orthogonal projection. It follows from the *trace formula* [7] that

$$P(w) = \frac{1}{|\Sigma(A)|} \sum_{\sigma \in \Sigma(A)} \sigma(w) .$$

Since the integral in (5.1) may be taken over the smooth part of ∂A, we see that $\mathrm{Im}(\mathscr{L}^A_\phi) \subset P(F^A_\phi)$. In fact, we prove:

Lemma 5.7. $\mathrm{Im}(\mathscr{L}^A_\phi) = P(F^A_\phi)$.

Proof. Let $x \in \partial A$ be a point on the smooth part of the boundary of A and let δ_x be a delta measure supported at $x \in \partial A$ with value $[1/|\Sigma(A)|]N(x) \in \mathbb{R}^n$. Let $\rho = \Sigma_{\gamma\in\Gamma} \delta_{\gamma x}$. Formally, ρ is a Γ-invariant mapping, and formally

$$\mathscr{L}^A_\phi(\rho) = \frac{1}{|\Sigma(A)|} \sum_{\gamma \in \Sigma(A)} \phi(\gamma x) ,$$

since only when $\gamma \in \Sigma(A)$ is $\delta_{\gamma x}$ supported in ∂A. Now we use the trace formula to conclude that $P(\phi(x))$ is in the image of $\mathscr{L}_\phi^A$. By linearity, the subspace $P(F_\phi^A)$ is in the image of $\mathscr{L}_\phi^A$, which yields the desired equality.

We now discuss how to make this formal computation rigorous. Let U be a small ball centered at x. Choose a vector field X supported in U, pointing in the direction N at x with magnitude $1/|\Sigma(A)|$ and equivariant with respect to the isotropy subgroup of x. Use the group action of Γ to extend X to a Γ equivariant vector field on $\mathbb{R}^n$ supported on the balls γU (which we can assume are either disjoint or equal, if U is small enough).

Observe that $\mathscr{L}_\phi^A(X)$ points approximately in the direction of $P(\phi(x))$. Indeed, an appropriate limit of vector fields U will converge to δ_x; and the integrals will converge to $P(\phi(x))$, as desired. □

Next we show how to use the fact that ϕ is a polynomial and the explicit form of $\mathscr{L}_\phi^A$ to show that this linear map is onto. To do this we need to introduce a new subspace of W. Let $W_\phi \subset W$ be the subspace generated by the vectors $\phi(x)$ for $x \in \mathbb{R}^n$.

Proposition 5.8. Let $\phi: \mathbb{R}^n \to W$ be a polynomial observable. Assume that $W_\phi = W$ and that W distinguishes all subgroups. Then ϕ is a detective.

Proof. It follows that we can apply proposition 5.6 precisely when $P(F_\phi^A)$ equals $\mathrm{Fix}_W(\Sigma(A))$. We begin by showing that we can deform A to A' by a near identity diffeomorphism so that the corresponding $P(P_\phi^{A'})$ equals $\mathrm{Fix}_W(\Sigma(A))$. Then we use the implicit function theorem to deform A' to $\hat{A}$ which has the correct symmetry when observed by ϕ. The composition of two near identity diffeomorphisms is still near identity – so ϕ is a detective as claimed.

We extend the space F_ϕ^A to F' where $F' = \langle \phi(x): x \text{ is near } \partial A\rangle$. We use the fact that ϕ is a polynomial mapping to show that $P(F') = \mathrm{Fix}_W(\Sigma(A))$. Since ϕ is a polynomial mapping, $P\phi$ is also a polynomial mapping of $\mathbb{R}^n$ into $\mathrm{Fix}_W(\Sigma(A))$. If an open set of all images of $P\phi$ end up in a subspace, then all vectors in $P\phi(\mathbb{R}^n)$ are in that subspace. But this contradicts the assumption that $W_\phi = W$. For if this assumption is valid, then $P(F') = P(W_\phi) = \mathrm{Fix}_W(\Sigma(A))$.

Next we observe that since $\mathrm{Fix}_W(\Sigma(A))$ is finite-dimensional, there exist points $x_1, \ldots, x_s$ such that $\phi(x_1), \ldots, \phi(x_s)$ is a basis for $\mathrm{Fix}_W(\Sigma(A))$. Since ϕ is continuous, there are neighborhoods U_j of x_j for $1 \le j \le s$ such that $\phi(y_1), \ldots, \phi(y_s)$ is a basis for $\mathrm{Fix}_W(\Sigma(A))$ whenever $y_j \in U_j$ for $1 \le j \le s$. It follows that we can choose the x_j's to have trivial isotropy and so that no two are on the same group orbit. Next choose $a_1, \ldots, a_s$ on ∂A so that a_j is near x_j, a_j has trivial isotropy, and no two a_j are on the same group orbit. We can now construct a near identity Γ-equivariant diffeomorphism ψ which moves a_j to x_j for each j. Let $A' = \psi(A)$. In this way we can deform A by a diffeomorphism ψ just a little near the boundary so that the points x_j are in $\partial A'$. Now we can apply proposition 5.6 since the new $P(D_\phi^{A'})$ is equal to $\mathrm{Fix}_W(\Sigma(A))$. □

Proof of theorem 5.2. Theorem 4.3 implies that W distinguishes all subgroups. Next let W_{ϕ_j} be the subspace of W_j spanned by all vectors in $\phi_j(\mathbb{R}^n)$. The equivariance of ϕ_j guarantees that the space W_{ϕ_j} is Γ-invariant. The irreducibility of W_j implies that $W_{\phi_j} = W_j$ since ϕ_j is nonzero. Similarly, the space W_ϕ is a Γ-invariant subspace of W and the projection of W into W_j whose kernel is spanned by the other W_k's takes W_ϕ onto W_{ϕ_j}. Therefore, there is a subspace of W_ϕ that is Γ-isomorphic to W_j. Since all of the representations of Γ on the W_j's are distinct, this implies that $W_\phi = W$. Now we can apply proposition 5.8 to conclude that ϕ is a detective. □

6. Example: tetrahedral symmetry

To illustrate the foregoing considerations, and to investigate what might comprise typical behavior for symmetric maps, we study an example with the tetrahedral group T acting on $\mathbb{R}^3$ in the usual way. Truncating arbitrarily at third order, the general tetrahedral equivariant map can be written

$$f(x, y, z) = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \alpha(x^2 + y^2 + z^2) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \beta \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} + \gamma \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix} + \delta \begin{pmatrix} xy^2 \\ yz^2 \\ zx^2 \end{pmatrix} .$$

See [11] for a complete discussion of the polynomial tensors of the point groups.

Recall that T has two nontrivial irreducible representations, one of dimension three (the standard representation) and one of dimension two (where the D_2 subgroup acts trivially). Thus we can construct a detective $\phi: \mathbb{R}^3 \to \mathbb{R}^5$, which we take to be

$$\phi(x, y, z) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \\ 2x^2 - y^2 - z^2 \\ 2y^2 - x^2 - z^2 \end{pmatrix} ,$$

where the first three and last two components span irreducible subspaces. For convenience, we write $W = \mathbb{R}^5 = W_3 \oplus W_2$. It follows from the disjoint union decomposition ($\mathrm{T} = \dot{\cup}^4 \mathbb{Z}_3 \dot{\cup}^3 \mathbb{Z}_2$; see [7]) that there are seven distances to calculate: four to $\mathbb{Z}_3$ fixed-point subspaces and three to $\mathbb{Z}_2$ fixed-point subspaces. The $\mathbb{Z}_3$ fixed-point spaces are one-dimensional and are given by

$$\mathrm{Fix}_W(\mathbb{Z}_3^{(1)}) = \mathbb{R}\{(1, 1, 1, 0, 0)\} ,$$
$$\mathrm{Fix}_W(\mathbb{Z}_3^{(2)}) = \mathbb{R}\{(1, -1, 1, 0, 0)\} ,$$
$$\mathrm{Fix}_W(\mathbb{Z}_3^{(3)}) = \mathbb{R}\{(1, 1, -1, 0, 0)\} ,$$
$$\mathrm{Fix}_W(\mathbb{Z}_3^{(4)}) = \mathbb{R}\{(-1, 1, 1, 0, 0)\} ,$$

while the $\mathbb{Z}_2$ fixed-point subspaces are three-dimensional and are given by

$$\mathrm{Fix}_W(\mathbb{Z}_2^{(1)}) = \mathbb{R}\{(1, 0, 0, 0, 0)\} \oplus W_2 ,$$
$$\mathrm{Fix}_W(\mathbb{Z}_2^{(2)}) = \mathbb{R}\{(0, 1, 0, 0, 0)\} \oplus W_2 ,$$
$$\mathrm{Fix}_W(\mathbb{Z}_2^{(3)}) = \mathbb{R}\{(0, 0, 1, 0, 0)\} \oplus W_2 .$$

Writing $|\phi|^2_{W_3} = \phi_1^2 + \phi_2^2 + \phi_3^2$ and $|\phi|^2_{W_2} = \phi_4^2 + \phi_5^2$, the corresponding distance formulas are

$$d(\phi, \mathrm{Fix}(\mathbb{Z}_3^{(1)}))^2 = |\phi|^2_{W_2} + \tfrac{2}{3}(|\phi|^2_{W_3} - \phi_1\phi_2 - \phi_1\phi_3 - \phi_2\phi_3) ,$$
$$d(\phi, \mathrm{Fix}(\mathbb{Z}_3^{(2)}))^2 = |\phi|^2_{W_2} + \tfrac{2}{3}(|\phi|^2_{W_3} - \phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3) ,$$
$$d(\phi, \mathrm{Fix}(\mathbb{Z}_3^{(3)}))^2 = |\phi|^2_{W_2} + \tfrac{2}{3}(|\phi|^2_{W_3} + \phi_1\phi_2 + \phi_1\phi_3 - \phi_2\phi_3) ,$$
$$d(\phi, \mathrm{Fix}(\mathbb{Z}_3^{(4)}))^2 = |\phi|^2_{W_2} + \tfrac{2}{3}(|\phi|^2_{W_3} + \phi_1\phi_2 - \phi_1\phi_3 + \phi_2\phi_3) ,$$
$$d(\phi, \mathrm{Fix}(\mathbb{Z}_2^{(1)}))^2 = \phi_2^2 + \phi_3^2 ,$$
$$d(\phi, \mathrm{Fix}(\mathbb{Z}_2^{(2)}))^2 = \phi_1^2 + \phi_3^2 ,$$
$$d(\phi, \mathrm{Fix}(\mathbb{Z}_2^{(3)}))^2 = \phi_1^2 + \phi_3^2 .$$

The symmetry of the attractor is given by the distances as follows. If all distances vanish, the attractor has full tetrahedral symmetry. If any one of the $\mathbb{Z}_3$ distances vanish, with all other distances nonzero, the attractor has $\mathbb{Z}_3$ symmetry (all copies of $\mathbb{Z}_3$ are conjugate in T). Similarly, if any one of the $\mathbb{Z}_2$ distances vanish the attractor has $\mathbb{Z}_2$ symmetry (again, all copies of $\mathbb{Z}_2$ are conjugate), while if all three $\mathbb{Z}_2$ distances vanish but the $\mathbb{Z}_3$ distances are nonzero then the attractor has D_2 symmetry. If all distances are nonzero the attractor has trivial symmetry. Generically, the only other possibility is for the map to "blowup" with the orbit diverging to infinity.

We have made a rather rough investigation of the symmetry of attractors of this map; nevertheless, we have found chaotic attractors of all symmetry types. An advantage of these methods is that it is possible to automate the search, and so determine the symmetry over whole regions of parameter space. The remainder of this section will describe the methodology and results of such a search over a particularly interesting two-parameter region.

In the notation above, the values of three of the five parameters were fixed at $\alpha = 1.0$, $\beta = 1.0$ and $\delta = -1.0$, while the remaining two covered the region $-2.1 < \lambda < -1.6$ and $0.4 < \gamma < 1.4$. The search was conducted "quasistatically" both in parameter space and with respect to initial conditions. The initial parameter values were taken to be $(\lambda, \gamma) = (-2.1, 0.6)$, then λ was held fixed while γ was incremented by 0.01 at a time until it reached its maximum value of 1.4. Then λ was incremented by 0.005 with γ held at 1.4, and γ was then decremented by steps of 0.01 until its minimum value was reached. Then λ was incremented again and the γ process repeated so that the parameter region was covered by a snaking path.

For each pair of parameter values the map was iterated 1000 times to eliminate transients and set the scale of the attractor (or to check for blowup). Then the grid was defined to dice the region covering the attractor into boxes and the map was iterated 1000 iterates at a time until convergence was achieved. Convergence was defined to have occurred if the net increase in the number of occupied boxes in one cycle of 1000 iterates was 0.001 or less of the total number of occupied boxes at the end of the previous cycle. We stress that this feature of automatic detection of convergence is extremely useful, especially in light of the very poor asymptotic convergence properties of the ergodic sum (see section 7). It is also worth noting that the number of iterates required for convergence contains interesting dynamical information. The simplest dynamics (fixed points and finite n-cycles) result in convergence on the first pass (1000 iterates), more complicated but still nonchaotic attractors such as invariant curves and also small chaotic attractors are indicated by convergence times of a few thousand while large chaotic attractors can take hundreds of thousands of iterates to converge.

The initial conditions in $\mathbb{R}^3$ were arbitrarily taken to be (0.75, 0.5, 0.66), but were subsequently chosen quasistatically in the following sense. When convergence to an attractor occurred, the final value was taken to be the initial condition for the transient cycle of the next set of parameters. In order to avoid being artificially caught in invariant subspaces, the value was perturbed by adding a small random number ($O(10^{-2})$) to each component. If a blowup occurred, the previous initial condition was reused.

The final issue to be addressed is the way in which the distance values were interpreted to yield symmetry types. The fundamental question is essentially: "What is zero?", and it must be admitted that our treatment is somewhat ad hoc. Operationally, zero is any quantity smaller than the difference caused in an observation by changing the number of occupied boxes by one. Unfortunately, this quantity will depend on the attractor and the observable, so it is difficult to apply this rule in practice though it is clear that the finer the grid chosen the smaller this minimal quantity will be. We chose our grid so that each side was broken into 51 intervals; this number represents a compromise between precision (size of zero) and quick convergence. Checks were performed with 101 interval grids to verify that the results were not sensitive to the grid choice. Finally, by trial and error, we chose a value of 0.005 to be the zero scale of distance squared. A much larger value caused some parameter regions to appear to have more symmetry (e.g. $\mathbb{Z}_3$ attractors might be labelled T symmetric), and a much smaller one caused some regions to be erroneously labelled as having only trivial symmetry.

Our results are summarized in fig. 1 and fig. 2 which should be read in the same manner as thermodynamic phase diagrams. Figure 1 is an exact representation of our results where there is

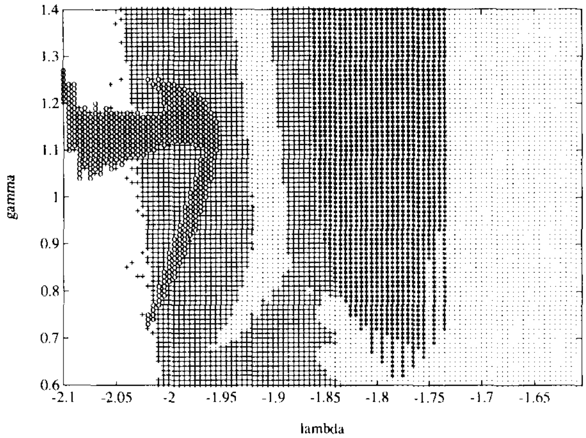


Fig. 1. Symmetry types. See text for discussion.

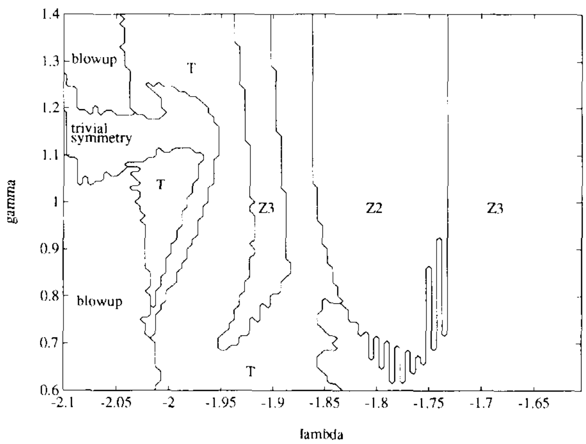


Fig. 2. Symmetry phase diagram. See text for discussion.

a symbol printed for each pair of parameter values. Plus signs indicate T symmetry, dots $\mathbb{Z}_3$, asterisks $\mathbb{Z}_2$, circles indicate trivial symmetry and blank spaces indicate regions of blowup. It so happens that there are no D_2 symmetric attractors in this particular region. Figure 2 is essentially the same diagram but with the boundaries of the symmetry regions plotted to make it easier to read. In practice, calculations near the boundaries can be somewhat problematic and can produce apparently spurious symmetry types.

Roughly speaking, the dynamics goes from simple to complicated as one goes from right to left in the diagrams. The $\mathbb{Z}_3$ symmetric dynamics at the far right consist of two-cycles each point of which becomes first an invariant curve and eventually chaotic as λ decreases. At the vertical boundary between the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ regions the $\mathbb{Z}_3$ limit cycles lose stability to $\mathbb{Z}_2$ two-cycles, these two-cycles also become limit cycles and eventually chaotic much as the $\mathbb{Z}_3$ case. The serrated appearance of the lower $\mathbb{Z}_2$–$\mathbb{Z}_3$ boundary is clear evidence of hysteresis in the symmetry transitions. The tetrahedrally symmetric dynamics are all chaotic and result from collisions of the lower symmetry chaotic attractors. The thin peninsula of $\mathbb{Z}_3$ symmetry represents chaotic dynamics of a different sort than in the other region. The dynamics with only trivial symmetry are particularly interesting, and seem to result from fracture of the symmetric attractor into shards. Asymmetric attractors tend to have small support in $\mathbb{R}^3$. They tend to appear as three disconnected regions; indeed, for certain parameters these attractors are just asymmetric three-cycles.

7. A method for observation: the ergodic sum

Given an observable ϕ and a set $A \in \mathscr{A}$, we define the mapping $g: \Gamma \to \mathbb{R}$ by taking g to be the distance of the observation $K_\phi(A)$ to $\mathrm{Fix}_W(\gamma)$. The isotropy subgroup of $K_\phi(A)$ is just the set of $\gamma \in \Gamma$ for which $g(\gamma) = 0$. From this information we get the symmetry of the set A.

As an alternative to computing the integral $K_\phi(A)$, one can presume that the ergodic theorem is valid and compute (approximately)

$$K_\phi^{\mathrm{E}}(x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N} \phi(f^j(x_0)) .$$

To use $K_\phi^{\mathrm{E}}(x_0)$ we must show, in analogy to lemma 3.2, that $\Sigma(L)$, the symmetry group of the attractor L, is contained in the isotropy group of this ergodic observation. It follows from the ergodic theorem (see [3]) that there is an (map) invariant measure ν that is also $\Sigma(L)$ invariant. (Just average an invariant ergodic measure over the group of symmetries.) The Bowen–Ruelle–Sinai box-counting measure (when it exists) is an example of such a symmetric ergodic measure. For these measures ν it can be shown, using the ergodic theorem, that symmetries σ of L fix K_{ϕ}^{E}. In symbols

$$\sigma K_{\phi}^{\mathrm{E}}(x_0) = \int_{L} \phi(\sigma x)\,\mathrm{d}\nu(x) = \int_{\sigma L} \phi(x)\,\mathrm{d}\nu(x) = K_{\phi}^{\mathrm{E}}(x_0)\,.$$

The same theorem suggests that when we use a detective function ϕ this calculation will generically produce the actual symmetry of the attractor for the discrete dynamical system f. The main difficulty is the numerical issue of deciding when the distance g is actually zero. This difficulty is accentuated by the fact that the ergodic sums converge slowly. In fact, in the numerical examples below we will see that the direct computation of the integral $K_{\phi}(A)$ is – at least for low-dimensional attractors – of advantage in comparison to the computation of the sum $K_{\phi}^{\mathrm{E}}(x_0)$. In higher dimensions, however, the computation of the integral $K_{\phi}(A)$ becomes impractical and then we have no recourse but to compute the ergodic sum $K_{\phi}^{\mathrm{E}}(x_0)$.

As a final point in this section, we illustrate the sense in which our numerical experiments seem to show that the convergence properties of the integral method are superior to those of the ergodic sum. (It is known that in general the convergence rate of the ergodic sum can be proved to be no better than $1/N$.) Moreover, in practice, the ergodic sum tends to exhibit large oscillations while converging, making a test for convergence difficult. We will illustrate this difficulty with examples of attractors of D_3-equivariant planar mappings, and compare the results with the integral test.

On the other hand, numerical approximations to the integral method as described in section 6 must converge to its final value in a *finite* number of iterations, because no additional boxes will be filled after the attractor is covered. Moreover, this saturation will occur faster when a coarse grid is chosen, making it possible for the calculation to proceed quite briskly in comparison with the ergodic method. Of course, we pay a price for this acceleration of convergence in the form of some loss of precision. As mentioned previously, a coarse grid causes more uncertainty in the interpretation of a distance value as zero or nonzero.

We illustrate these points with a simple example with D_3 symmetry acting on $\mathbb{C}$, choosing a situation in which it is known that a fully D_3 symmetric chaotic attractor exists (see [2]). With $z \in \mathbb{C}$, the general equivariant map truncated at third order is

$$f(z) = (\alpha z\bar{z} + \lambda)z + \gamma\bar{z}^2\,,$$

and a D_3 symmetric attractor appears to exist for the parameter values $\alpha = -1.0$, $\gamma = -0.5$ and $\lambda = 2.3$. The results are presented in fig. 3 where we plot the value of the distance to the fixed-point space of the rotation in D_3. The detective is three-dimensional – a two-dimensional irreducible component z and a one-dimensional nontrivial irreducible component $\mathrm{Im}(z^3)$.

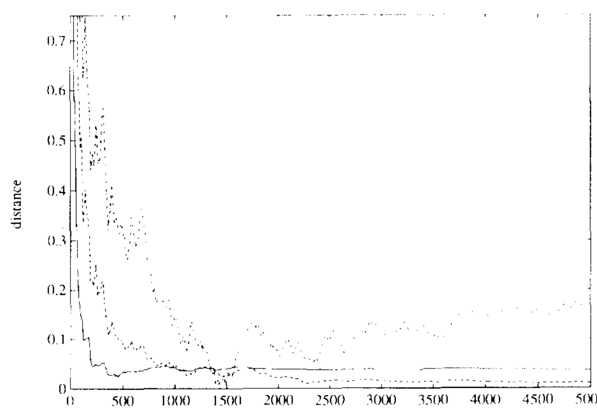


Fig. 3. Convergence properties: 51×51 grid – solid line; 201×201 grid – dashed line; 1001×1001 grid – dotted line; ergodic sum – dash-dotted line.

Two points are immediately clear from the diagram. First, the ergodic sum converges more slowly than any of the integral results; second, the coarser the grid the faster is the convergence for the integrals. Put another way, the finer the grid the more closely the ergodic behavior is mimicked by the integral. Comparison of the results for the 51×51 grid and the 201×201 grid illustrates the price of the quick convergence: the asymptotic value of the distance for the coarser grid is 0.036 and for the finer grid 0.010.

8. A detective for coupled cells

The results that we obtained in the previous sections apply to attractors in ordinary differential equations as well. Then the attractor is given by $L = \overline{\{x(t): t \geq 0\}}$ and the time-average becomes

$$K_{\phi}^{\mathrm{E}}(x(0)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(x(t))\, \mathrm{d}t .$$

In this section we will apply theorem 5.2 to rings of coupled cells, which are D_p symmetric systems of the form

$$\dot{z}_j = f(z_{j-1}, z_j, z_{j+1}, \lambda) \quad (j = 1, \ldots, p) , \tag{8.1}$$

where $z_j \in \mathbb{R}^m$ and $f(x, y, z, \lambda) = f(z, y, x, \lambda)$. (We use the convention here that $z_0 = z_p$ and $z_{p+1} = z_1$.) We set $n = mp$.

We now present a detective for such systems and, in the next section, we explore numerically an example of three coupled cells ($p = 3$) consisting of two equations each ($m = 2$) so that $n = 6$. The representation space W that we use for this detective is the space of $n \times n$ real symmetric matrices where $\gamma \in \mathrm{D}_p$ acts by similarity transformations on W:

$$\gamma \cdot w = \gamma w \gamma^{\mathrm{t}} \quad \text{for all } w \in W .$$

We will prove the following:

Theorem 8.1. Assume that the number of cells is $p \geq 3$ and the number of equations governing each cell is $m \geq 2$. Then the mapping

$$\phi(x) = x \cdot x^{\mathrm{t}}$$

is a detective.

It is easy to check that with respect to this action of D_p on W, ϕ is D_p-equivariant; hence, ϕ is a polynomial observable. We will use theorem 5.2 to prove theorem 8.1. There are two points that must be checked. We must show that W contains every (lattice equivalence class of) nontrivial irreducible representations of D_p and that this particular ϕ is nonzero on each of these representations. In fact, we will show using the theory of characters that W contains every nontrivial irreducible representation.

It is easy to show that ϕ will then be nonzero on each of these representations by showing that $W_\phi = W$ where $W_\phi = \langle \phi(x): x \in \mathbb{R}^n \rangle$. (The vector space W_ϕ was introduced in the proof of theorem 5.2. To prove that ϕ is a detective in this case we find it easier to verify the hypotheses of proposition 5.8.) To verify the claim let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$ and define the vectors $x_{i,j}$ by

$$x_{i,j} = \begin{cases} e_i & i = j \\ e_i + e_j & i \neq j \end{cases} \qquad i, j = 1, \ldots, n .$$

Then one can check that the set of matrices $\phi(x_{i,i+k})$ ($k = 0, 1, \ldots, n-1$, $i = 1, 2, \ldots, n-k$) defines a basis of W, which implies $W_\phi = W$.

To check the first assumption of theorem 5.2 we compute the multiplicities of the irreducible representations of D_p in W using characters. Recall that the *character* of a representation W is the mapping $\chi_W: \Gamma \to \mathbb{R}$ defined by $\chi_W(\gamma) = \mathrm{trace}(\gamma)$ where this trace refers to the trace of the action of γ on W.

The multiplicity of an irreducible representation V in the representation W is the number of independent isomorphic copies of V that appears in W. Since we fix W we denote this multiplicity by c_V.

The theory of characters states that (see Miller [10]) c_V may be computed by

$$c_V = \frac{1}{2p} \sum_{\gamma \in \mathrm{D}_p} \chi_V(\gamma)\chi(\gamma)\,, \tag{8.2}$$

where χ is the character of W and χ_V is the character of V.

To compute the character of D_p acting on W we need to compute trace(γ) for each $\gamma \in \mathrm{D}_p$. This we can do by a direct combinatorial argument. We denote the basic rotation in D_p by R. The reflections fall into three types – all reflections interchange some cells in pairs and fix other cells. Each reflection in D_p (p odd) fixes exactly one cell while the reflections in D_p (p even) fix either 0 or 2 cells. According to this we denote them by S_1, S_0 and S_2 respectively.

Lemma 8.2.

$$\mathrm{trace}(I) = \tfrac{1}{2}mp(mp+1)\,,$$

$$\mathrm{trace}(R^j) = \begin{cases} \tfrac{1}{2}mp & (j = \tfrac{1}{2}p) \\ 0 & (\text{otherwise})\,, \end{cases}$$

$$\mathrm{trace}(S_0) = \tfrac{1}{2}mp\,,$$

$$\mathrm{trace}(S_1) = \tfrac{1}{2}m(p+m)\,,$$

$$\mathrm{trace}(S_2) = \tfrac{1}{2}m(p+4m)\,.$$

Proof. Let A be the $p \times p$ matrix (a_{ij}) and let $\gamma \in \mathrm{D}_p$ act on A by $\gamma \cdot A = \gamma A \gamma^{\mathrm{t}}$. Each $\gamma \in \mathrm{D}_p$ is a permutation matrix. A short calculation shows that permutation matrices act by just permuting indices, that is,

$$\gamma \cdot (a_{ij}) = (a_{\gamma(i)\gamma(j)})\,, \tag{8.3}$$

where, by abuse of notation, we also denote the permutation on indices by γ. Similarly, when A is an $(mp) \times (mp)$ matrix and each a_{ij} is an $m \times m$ block matrix, we see that (8.3) is still valid, though here block matrices rather than individual elements are permuted.

Now we suppose that $A = (a_{ij})$ is a symmetric matrix, so that $a_{ij}^{\mathrm{t}} = a_{ji}$ and, in particular, a_{ii} is itself a symmetric matrix. We can see from (8.3) that the only contributions to trace(γ) come when $(\gamma(i), \gamma(j))$ equals either (i, j) or (j, i).

There are three possibilities. If $(\gamma(i), \gamma(i)) = (i, i)$ then there is a contribution of $\frac{1}{2}m(m+1)$ to trace(γ), since a_{ii} is an $m \times m$ symmetric matrix. Should $(\gamma(i), \gamma(j)) = (i, j)$ where $i \neq j$, then the contribution to trace(γ) is m^2 since a_{ij} is an arbitrary $m \times m$ matrix. Finally if $(\gamma(i), \gamma(j)) = (j, i)$ where $i \neq j$, then the contribution to trace(γ) is only m since only the diagonal elements of a_{ij} contribute to the trace.

This remark can now be used to compute trace (γ) for $\gamma \in \mathrm{D}_p$. If $\gamma = R$ then $(R(i), R(j)) = (i+1, j+1) \bmod p$ and $(R(i), R(j))$ is never equal to (i, j) or (j, i). So trace$(R) = 0$.

Now consider R^l $(1 \leq l \leq p-1)$. Then $(R^l(i), R^l(j)) = (i+l, j+l) \bmod p$. The only possibility for a contribution to the trace is when $(i+l, j+l) = (j, l) \bmod p$. Then p must be even, $l = \frac{1}{2}p$ and $j = i+1$. Thus trace$(R^l) = 0$ unless $l = \frac{1}{2}p$ in which case trace$(R^{p/2}) = \frac{1}{2}mp$ since the $\frac{1}{2}p$ blocks

$$a_{1,l+1}, a_{2,l+2}, \ldots, a_{l,p}$$

each contribute m to the trace.

The elements of D_p remaining to be discussed are all reflections. Exchanging one pair of cells contributes m to the trace while fixing a cell contributes $\frac{1}{2}m(m+1)$ to the trace. Finally, when two cells are fixed say cells 1 and $\frac{1}{2}p+1$ then an additional block $a_{1,\frac{1}{2}p+1}$ is also fixed and an extra m^2 is contributed to the trace. Therefore

$$\mathrm{trace}(S_0) = \tfrac{1}{2}pm = \tfrac{1}{2}mp \quad (p \text{ even})\,,$$

$$\mathrm{trace}(S_1) = \tfrac{1}{2}(p-1)m + \tfrac{1}{2}m(m+1)$$
$$= \tfrac{1}{2}m(p+m) \quad (p \text{ odd})\,,$$

$$\mathrm{trace}(S_2) = \tfrac{1}{2}(p-2)m + \tfrac{2}{2}m(m+1) + m^2$$
$$= \tfrac{1}{2}m(p+4m) \quad (p \text{ even})\,.$$

Since trace (I) equals the dimension of the space,

$$\mathrm{trace}(I) = \tfrac{1}{2}mp(mp+1)\,. \qquad \square$$

Proof of theorem 8.1. It remains to show that W distinguishes all the subgroups if $m \geq 2$ and $p \geq 3$. For this we use theorem 4.3 by showing that all the nontrivial irreducible representations of D_p are present in W.

In the case that p is even there exist three distinct nontrivial one-dimensional representations which we denote by $W_{\mathbb{Z}_p}$, $W_{\mathrm{D}_{\frac{1}{2}}^1}$ and $W_{\mathrm{D}_{\frac{1}{2}}^2}$. The subscripts indicate the kernels of those representations. Using (8.2) and lemma 8.2 we compute their multiplicities in W. For the irreducible representation $W_{\mathbb{Z}_p}$ we obtain

$$\begin{aligned}2p \cdot c_{\mathbb{Z}_p} &= \tfrac{1}{2}mp(mp+1) + \tfrac{1}{2}mp - \tfrac{1}{2}p \cdot \tfrac{1}{2}mp \\ &\quad - \tfrac{1}{2}p \cdot \tfrac{1}{2}m(p+4m) \\ &= \tfrac{1}{2}mp(mp+1+1-\tfrac{1}{2}p-\tfrac{1}{2}p-2m) \\ &= \tfrac{1}{2}mp(m-1)(p-2)\,,\end{aligned}$$

and therefore

$$c_{\mathbb{Z}_p} = \tfrac{1}{4}m(m-1)(p-2)\,.$$

Similarly we compute

$$c_{\mathrm{D}_{\frac{1}{2}}^1} = \tfrac{1}{4}[m(p+2)+1+(-1)^{p/2}]\,,$$

$$c_{\mathrm{D}_{\frac{1}{2}}^2} = \tfrac{1}{4}[m(p-2)+1+(-1)^{p/2}]\,.$$

Moreover there are $\frac{1}{2}p-1$ two-dimensional irreducible representations W_2^j $(j=1,\ldots,\frac{1}{2}p-1)$ with multiplicities

$$c_2^j = \tfrac{1}{2}m[mp+1+(-1)^j]\,.$$

In the case where p is odd there is just one nontrivial one-dimensional representation and we compute

$$c_{\mathbb{Z}_p} = \tfrac{1}{4}m(m-1)(p-1)\,,$$

and for all the $\frac{1}{2}(p-1)$ two-dimensional irreducible representations W_2^j we obtain

$$c_2^j = \tfrac{1}{2}m(mp+1)\,.$$

From these computations it now follows that the representation W distinguishes all the subgroups if and only if $m \geq 2$ and $p \geq 3$. $\square$

We end this section by computing the distances $d(A, \mathbb{Z}_2^{(k)})$ $(k=1,\ldots,p)$ and $d(A,\mathbb{Z}_p)$ between an element $A \in W$ and the fixed-point spaces of the p reflections and the basic rotation in D_p. These formulae will be used in the numerical computations of the following section.

Again we write $A = (a_{ij})_{i,j=1,\ldots,p}$ in block form such that each a_{ij} itself is an $m \times m$ matrix. With this notation the fixed-point spaces can be written as

$$\begin{aligned}\mathrm{Fix}(\mathbb{Z}_2^{(k)}) &= \{A \in W\colon (a_{ij}) = (a_{p-i+k,p-j+k})\,, \\ &\qquad 1 \leq k \leq p\}\,, \\ \mathrm{Fix}(\mathbb{Z}_p) &= \{A \in W\colon (a_{ij}) = (a_{i+l,j+l})\,, \\ &\qquad 1 \leq l \leq p-1\}\,,\end{aligned}$$

where the values of the indices have to be evaluated modulo p. Hence the distances are given by

$$d(A,\mathbb{Z}_2^{(k)})^2 = \tfrac{1}{4}\sum_{i,j} |(a_{ij} - a_{p-i+k,p-j+k})|^2\,, \qquad k=1,\ldots,p\,,$$

$$d(A,\mathbb{Z}_p)^2 = \sum_{i,j}\left|\left(a_{ij} - \frac{1}{p}\sum_{l=0}^{p-1} a_{i+l,j+l}\right)\right|^2\,.$$

In these last expressions $|(b_{ij})|^2$ denotes the sum of the squares of the entries of the $m \times m$ matrix (b_{ij}).

9. An example of three cells

As an example of the results in the previous section we consider the following system of three coupled cells:

$$\begin{aligned} \dot{x}_j &= x_{j+1} + \delta x_j^2 x_{j+1} \,, \\ \dot{x}_{j+1} &= -x_j - (x_j^2 - \lambda)x_{j+1} \\ &\quad + \alpha(x_{j-1} - 2x_{j+1} + x_{j+3}) + \beta x_j x_{j+1} \,, \quad (9.1) \end{aligned}$$

where $j = 1, 3, 5$, $x_0 = x_6$ and $x_8 = x_2$. We fix three of the four parameters in this system setting

$$\alpha = -0.5 \,, \quad \beta = -1.8 \,, \quad \delta = -0.28 \,,$$

and consider λ as the bifurcation parameter.

The dynamical system (9.1) possesses D_3 symmetry, where the elements of D_3 act by block permutation matrices. Using the previously defined notation we have $p = 3$, $m = 2$ and $n = 6$ (see section 8).

The equivariant polynomial observable that we use to detect symmetry is $\phi(x) = x \cdot x^{\mathrm{t}}$ which leads to the computation of the *correlation matrix*

$$K_\phi^{\mathrm{E}}(x(0)) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t) \cdot x(t)^{\mathrm{t}} \, \mathrm{d}t \,.$$

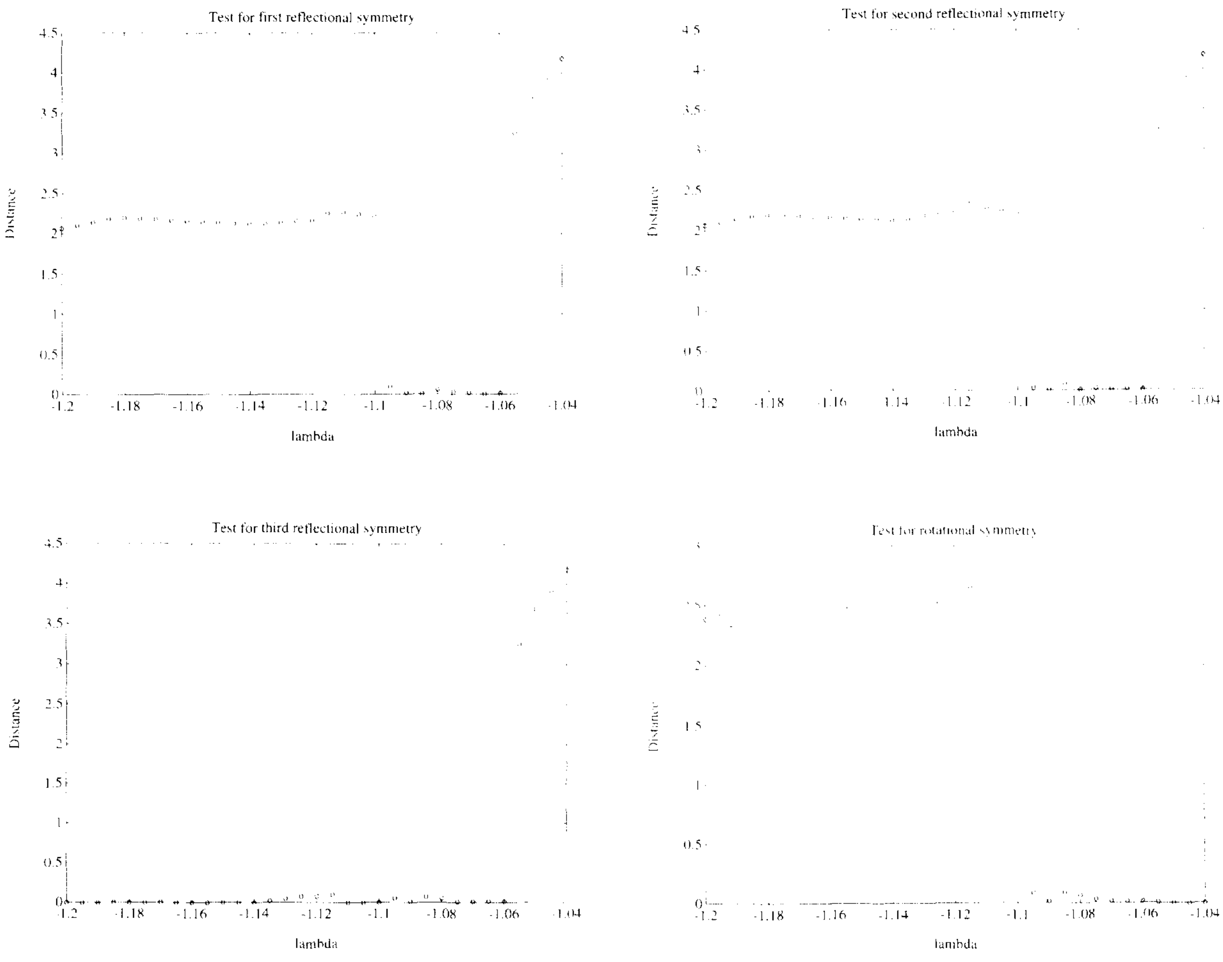


Fig. 4. The distances for $K_\phi^{\mathrm{E}}(x(0))$.

Since $m=2$ and $p=3$ we already know from theorem 8.1 that this observable is a detective.

We computed both $K_{\Phi}^{E}(x(0))$ and $K_{\Phi}(A)$ numerically varying λ from -1.20 to -1.04. In fig. 4 we show the distances between $K_{\Phi}^{E}(x(0))$ and the fixed-point spaces of the reflections and the rotations for this range of the parameter value, while in fig. 5 we show these distances for $K_{\Phi}(A)$. In six dimensions the memory requirements of the integral test are already substantial and it is no longer possible to cover a whole region of space containing the attractor by a grid as we did in section 6. So in this example we were forced to use a different method for storing data. What we did was to generate a covering of the attractor during the simulation. This resulted in the integral test running much more slowly than the ergodic test (roughly by a factor of five). If more sophisticated storage techniques were used, this difference in computation speed could probably be minimized in this dimension. However, it is clear that we are close to the limits of the direct use of the integral test in six dimensions.

With respect to symmetry types everything can be read off from either figure: For $-1.20 \leq \lambda \leq -1.10$ the attractor is essentially $\mathbb{Z}_2$ symmetric, for $-1.10 \leq \lambda \leq -1.06$ the attractor has D_3 sym-

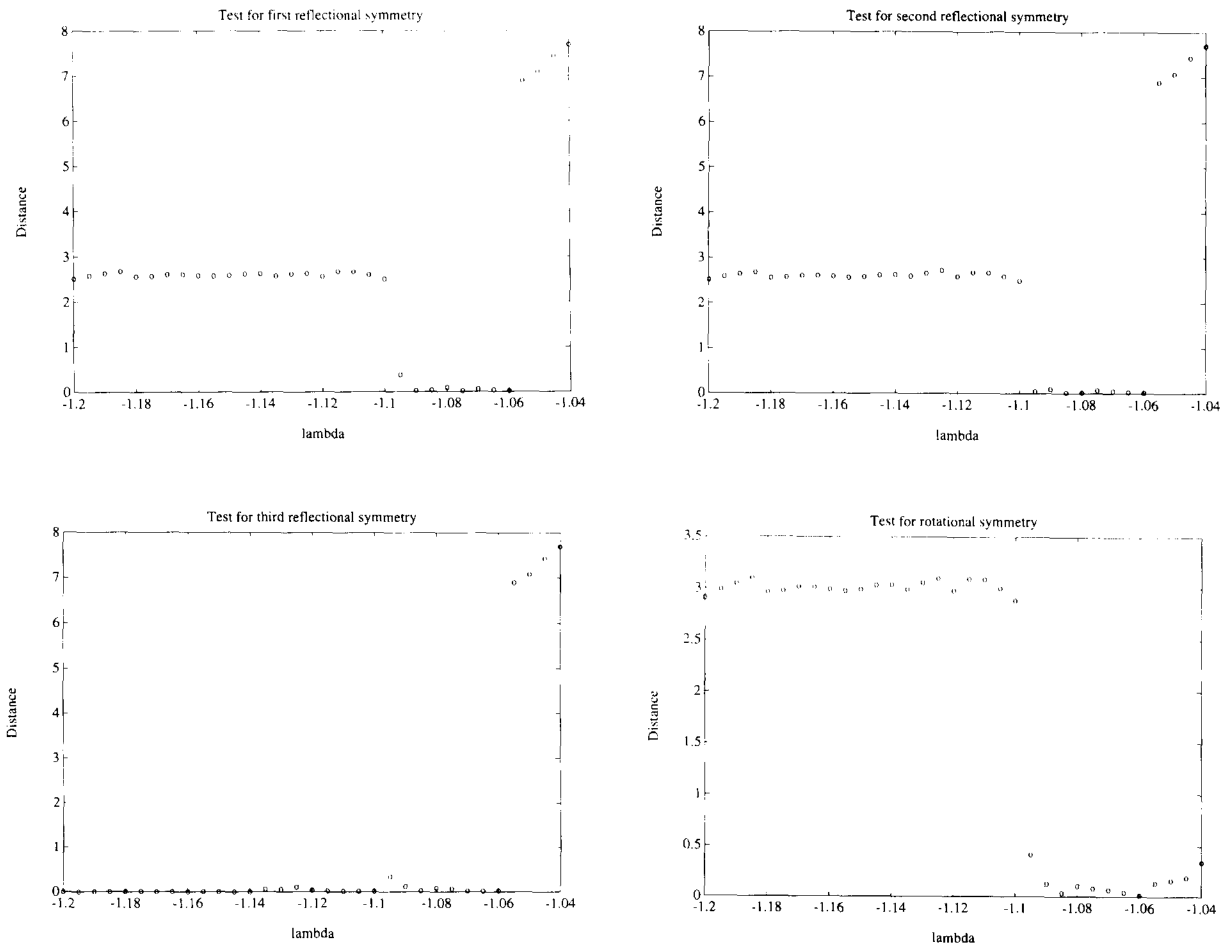


Fig. 5. The distances for $K_{\Phi}(A)$.

metry and, finally, for $\lambda \geq -1.06$ the attractor becomes $\mathbb{Z}_3$ symmetric. For this interpretation the distance itself is not relevant but the change in distance that occurs at parameter values where *symmetry creation* occurs, i.e., where conjugate attractors collide and the resulting attractor has more symmetry than the single attractors before collision.

On the other hand, at classical symmetry-breaking bifurcations, the distance that we compute should vary continuously. In fact, further inspection shows that there is a period-doubling sequence occurring for $\lambda \in (-1.14, -1.11)$ in which the $\mathbb{Z}_2$ symmetry is lost. But this can hardly be seen in fig. 4.

In the computation of $K_\phi(A)$ we again made use of the fact that the computation of the integral allows one to introduce a criterion for stopping the integration automatically. Hence the number of boxes actually used in the computation of the integral depends crucially on the dynamical complexity of the corresponding attractor. Roughly speaking, in this example the dynamical behavior becomes more and more complicated as λ is increased to approximately -1.06. For $\lambda > -1.06$ the attractor is just a discrete rotating wave. Accordingly, it can be observed (fig. 6) that the number of boxes grows rapidly when the dynamical behavior becomes more complex and that sometimes more than 50 000 boxes were needed to satisfy the criterion for stopping the integration.

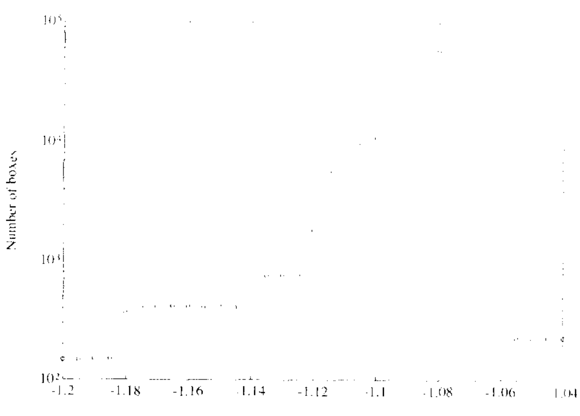


Fig. 6. The number of boxes used for the computation of $K_\phi(A)$.

Finally we observe that the computation of $K_\phi(A)$ for the D_3 symmetric ("chaotic") attractor at $\lambda = -1.095$ requires about three times the number of boxes than for the $\mathbb{Z}_2$ symmetric ("chaotic") attractor at $\lambda = -1.1$. This indicates that this transition is related to a *symmetry increasing bifurcation* (cf. [2]) in which three conjugate $\mathbb{Z}_2$ symmetric attractors collide.

10. Detectives for PDEs and experiments

In this section we discuss how we might use our results to compute the symmetry of an attractor from either experimental data or numerical computation of solutions to PDEs. In [4] we showed by example that the symmetry of an attractor for a PDE, in this case the Brusselator, could be visualized as a symmetry of the time-average of the solution. We begin this section by discussing, through the use of the ergodic theorem, why this observation is valid. We then repeat the numerical experiment for the Brusselator illustrating how detectives can simplify the observation of symmetry. At the end of the section we discuss how to use these techniques in a system with square symmetric geometry.

To indicate how the symmetry of an attractor manifests itself in physical space, we assume that $u(x, t)$ is (one component of) a solution to a PDE. We let $U(x)$ be the time-average of this solution, that is,

$$U(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T u(x, t)\, \mathrm{d}t\,. \tag{10.1}$$

We claim that if the attractor of the PDE has a symmetric ergodic measure, then

$$U(\sigma x) = U(x) \tag{10.2}$$

for all symmetries σ of the attractor. Indeed, the right hand side of (10.1) is evaluated using the ergodic theorem to be a space integral over the attractor. Using change of variables and symmetry invariance of the ergodic measure yields the desired result.

As an example we consider the Brusselator which is given by the following system of reaction diffusion equations:

$$\frac{\partial u}{\partial t} = \frac{D_1}{\lambda^2} \frac{\partial^2 u}{\partial x^2} + u^2 v - (B+1)u + A ,$$
$$\frac{\partial v}{\partial t} = \frac{D_2}{\lambda^2} \frac{\partial^2 v}{\partial x^2} - u^2 v + Bu . \tag{10.3}$$

Here u, v, A and B represent chemical concentrations and D_1, D_2 are diffusion constants. The parameter λ is a characteristic dimension of the system and we shall treat λ as the bifurcation parameter. We impose Dirichlet boundary conditions:

$$u(0,t) = u(1,t) = A ,$$
$$v(0,t) = v(1,t) = B/A .$$

Then the problem has a reflectional symmetry given by

$$\kappa(u(x,t), v(x,t)) = (u(1-x,t), v(1-x,t)) .$$

We fix four of the parameters by setting

$$A = 2 , \quad B = 5.45 , \quad D_1 = 0.008 , \quad D_2 = 0.004 .$$

By doing this we follow [8], since Holodniok et al. found complicated dynamics in (10.3) for this set of parameter values by numerical simulation.

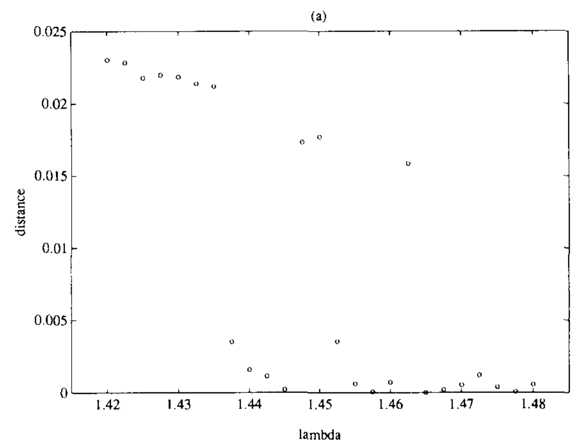
For a construction of an appropriate observable we make use of the preceding discussion of how symmetry of an attractor manifests itself in physical space (see (10.2)). Since $\mathbb{Z}_2$ has just one nontrivial one-dimensional representation our observable should be a nonzero equivariant mapping from a suitable function space into $\mathbb{R}$. In our numerical computations we have chosen the following two observables:

$$\phi_1(u,v) = u(0.3) - u(0.7) ,$$
$$\phi_2(u,v) = v(0.3) - v(0.7) ,$$

and computed the corresponding observations

$$K^{\mathrm{E}}_{\phi_j} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi_j(u,v)\, \mathrm{d}t \quad (j = 1,2) .$$

The results are presented in fig. 7. In those computations we have chosen $T = 60\,000$. A symmetry creation can be observed to occur beyond $\lambda \approx 1.45$. This was already mentioned in [4], where numerical simulations were performed for $\lambda = 1.45$ and $\lambda = 1.47$. However, the results in fig. 7 show that in between, for $\lambda = 1.4625$, the



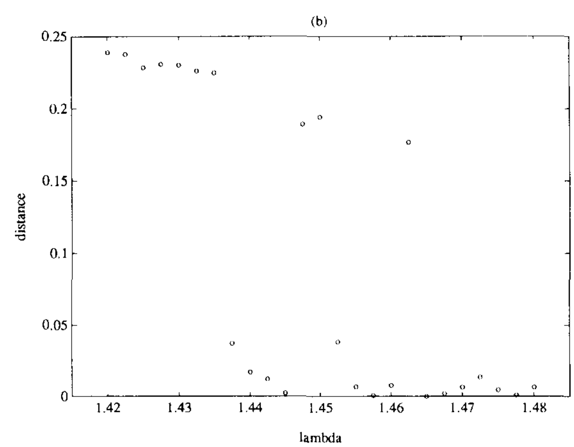


Fig. 7. The absolute value of the observations $K^{\mathrm{E}}_{\phi_1}$, (a), and $K^{\mathrm{E}}_{\phi_2}$, (b).

attractor again loses the $\mathbb{Z}_2$ symmetry. Another gain and loss of symmetry occurs for $\lambda \approx 1.44$. Finally, observe how similar the two observables behave qualitatively.

We end this section with a discussion of systems with square symmetry. For these systems we imagine taking time series at eight symmetrically placed points as illustrated in fig. 8. The observable ϕ is then the composition of a mapping from the state space of the experiment to $\mathbb{R}^8$ – the values of the state at these eight points – and a detective from $\mathbb{R}^8$ into W.

Our method for detecting symmetries is to compute the distance of the ergodic observation K_ϕ^{F} defined in section 7 to the fixed-point subspace corresponding to each cyclic subgroup of D_4. We begin by describing the symmetries of D_4 in physical space:

identity	I
rotation clockwise by 90°	r_{90}
identity	$r_{90}^2 = -I$
rotation counterclockwise by 90°	r_{90}^3
reflection across horizontal line	r_{h}
reflection across vertical line	r_{v}
reflection across northwest diagonal	r_{nw}
reflection across northeast diagonal	r_{ne} .

We use the following notation for certain subgroups:

$$\mathbb{Z}_4 = \{I_{\mathrm{I}}, r_{90}, r_{90}^2, r_{90}^3\}\,, \quad \mathrm{D}_2^s = \{r_{\mathrm{h}}, r_{\mathrm{v}}\}$$
$$\text{and} \quad \mathrm{D}_2^p = \{r_{\mathrm{nw}}, r_{\mathrm{ne}}\}\,.$$

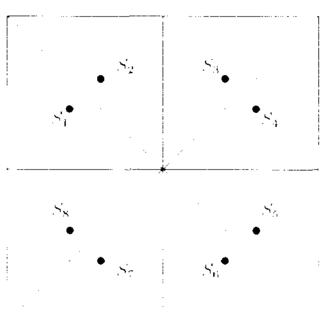


Fig. 8. Eight symmetrically placed points.

Let $S = (S_1, \ldots, S_8) \in \mathbb{R}^8$ be the time dependent values of the time series. We now explain how this data can be processed to determine the symmetries of an attractor. First recall that the group D_4 has four distinct one-dimensional and one two-dimensional irreducible representations. Using the notation of section 8 we denote them by

$$W_{\mathrm{D}_4},\ W_{\mathbb{Z}_4},\ W_{\mathrm{D}_2^p},\ W_{\mathrm{D}_2^s} \text{ and } W_2\,.$$

As previously, the subscripts on the one-dimensional irreducible representations indicate the kernels of those representations; in particular, W_{D_4} denotes the trivial representation.

We begin by writing $\mathbb{R}^8$ as a direct sum of irreducible representations. Abstractly,

$$\mathbb{R}^8 = W_{\mathrm{D}_4} \oplus W_{\mathbb{Z}_4} \oplus W_{\mathrm{D}_2^p} \oplus W_{\mathrm{D}_2^s} \oplus W_2^2\,.$$

Concretely,

$$W_{\mathrm{D}_4} = \mathbb{R}\{(1, 1, 1, 1, 1, 1, 1, 1)\}\,,$$
$$W_{\mathbb{Z}_4} = \mathbb{R}\{(1, -1, 1, -1, 1, -1, 1, -1)\}\,,$$
$$W_{\mathrm{D}_2^p} = \mathbb{R}\{(1, 1, -1, -1, 1, 1, -1, -1)\}\,,$$
$$W_{\mathrm{D}_2^s} = \mathbb{R}\{(1, -1, -1, 1, 1, -1, -1, 1)\}\,,$$
$$W_2 = \mathbb{R}\{(1, 1, 0, 0, -1, -1, 0, 0),$$
$$(0, 0, 1, 1, 0, 0, -1, -1)\}\,,$$
$$W_2 = \mathbb{R}\{(1, -1, 0, 0, -1, 1, 0, 0),$$
$$(0, 0, 1, -1, 0, 0, -1, 1)\}\,.$$

It is now a simple matter to compute the fixed-point subspaces of the various cyclic subgroups of D_4 acting on the five-dimensional subspace $W(\mathrm{D}_4) = W_{\mathbb{Z}_4} \oplus W_{\mathrm{D}_2^p} \oplus W_{\mathrm{D}_2^s} \oplus W_2$, which is the sum of all the nontrivial irreducible representations of D_4. We shall write, in coordinates

$$W_2 = x(1, 1, 0, 0, -1, -1, 0, 0)$$
$$+ y(0, 0, 1, 1, 0, 0, -1, -1)\,.$$

Table 1
Fixed-point subspaces in $W(\mathrm{D}_4)$.

$\mathrm{Fix}(\gamma)$	Fixed-point subspace	$d(w, \mathrm{Fix}(\gamma))^2$
$\mathrm{Fix}(r_{90})$	$W_{\mathbb{Z}_4} \oplus \{0\} \oplus \{0\} \oplus \{0\}$	$b^2 + c^2 + x^2 + y^2$
$\mathrm{Fix}(r_{\mathrm{v}})$	$\{0\} \oplus \{0\} \oplus W_{\mathrm{D}_2^s} \oplus \{x = y\}$	$a^2 + b^2 + \frac{1}{2}(x - y)^2$
$\mathrm{Fix}(r_{\mathrm{h}})$	$\{0\} \oplus \{0\} \oplus W_{\mathrm{D}_2^s} \oplus \{x = -y\}$	$a^2 + b^2 + \frac{1}{2}(x + y)^2$
$\mathrm{Fix}(r_{\mathrm{ne}})$	$\{0\} \oplus W_{\mathrm{D}_2^p} \oplus \{0\} \oplus \{x = 0\}$	$a^2 + c^2 + y^2$
$\mathrm{Fix}(r_{\mathrm{nw}})$	$\{0\} \oplus W_{\mathrm{D}_2^p} \oplus \{0\} \oplus \{y = 0\}$	$a^2 + c^2 + x^2$
$\mathrm{Fix}(-I)$	$W_{\mathbb{Z}_4} \oplus W_{\mathrm{D}_2^p} \oplus W_{\mathrm{D}_2^s} \oplus \{0\}$	$x^2 + y^2$

If we denote a point $w \in W(\mathrm{D}_4)$ by $w = (a, b, c, x, y)$, then it is a simple matter to write down the fixed-point subspaces in $W(\mathrm{D}_4)$ and the distances squared of w to these fixed-point subspaces. These data are given in table 1.

Finally, denote the D_4-equivariant projection by $\pi: \mathbb{R}^8 \to W(\mathrm{D}_4)$. It is easy to write π in coordinates. Let S be in $\mathbb{R}^8$, then

$$a = S \cdot (1, -1, 1, -1, 1, -1, 1, -1)\,,$$

$$b = S \cdot (1, 1, -1, -1, 1, 1, -1, -1)\,,$$

$$c = S \cdot (1, -1, -1, 1, 1, -1, 1, 1, -1, -1, 1)\,,$$

$$x = S \cdot (1, 1, 0, 0, -1, -1, 0, 0)\,,$$

$$y = S \cdot (0, 0, 1, 1, 0, 0, -1, -1)\,.$$

Thus to compute the symmetry of an attractor, one computes the ergodic observation $S = K_\phi^{\mathrm{E}}$ from a time series, then one computes the vector $w = \pi(S) \in W(\mathrm{D}_4)$, and finally one computes the distances to the various fixed-point subspaces. The exact symmetry is determined by which of these distances are (approximately) zero.

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References

[1] P. Aswin, M. Golubitsky and I. Melbourne, Symmetry groups of attractors, Arch. Rat. Mech. Anal., submitted.

[2] P. Chossat and M. Golubitsky, Symmetry increasing bifurcation of chaotic attractors, Physica D 32 (1988) 423–436.

[3] I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai, Ergodic Theory, Grundlehren Vol. 245 (Springer, New York, 1982).

[4] M. Dellnitz, M. Golubitsky and I. Melbourne, Mechanisms of symmetry creation, in: Bifurcation and Symmetry, eds. E. Allgower, K. Böhmer and M. Golubitsky, ISNM Vol. 104 (Birkhäusser, Basel, 1992) pp. 99–109.

[5] I. Melbourne, M. Dellnitz and M. Golubitsky, The structure of symmetric attractor, Arch. Rat. Mech. Anal., to appear.

[6] M. Field and M. Golubitsky, Symmetric chaos, Comput. Phys. Sept./Oct. (1990) 470–479; see also: Symmetry in Chaos (Oxford Univ. Press, Oxford, 1992).

[7] M. Golubitsky, I.N. Stewart and D.G. Schaeffer, Singularities and Groups in Bifurcation Theory: Vol. II, Appl. Math. Sci. Ser. Vol. 69 (Springer, New York, 1988).

[8] M. Holodniok, M. Kubiček and M. Marek, Desintegration of an invariant torus in a reaction–diffusion system, in: Numerical Treatment of Differential Equations, ed. K. Strehmel (Teubner, 1991) pp. 342–350.

[9] G. King and I. Stewart, Symmetric chaos, in: Nonlinear Equations in the Applied Sciences, eds. W.F. Ames and C.F. Rodgers (Academic Press, New York, 1991) pp. 257–315.

[10] W. Miller, Symmetry Groups and Their Applications (Academic Press, New York, 1972).

[11] J. Patera, R.T. Sharp and P. Winternitz, Polynomial irreducible tensors for the point groups, J. Math. Phys. 19 (1978) 2362–2376.