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Stability of Shock Waves for a Single Conservation Law

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1. INTRODUCTION

In [5] it was proved that *generically* the weak solution of the Cauchy problem for a quasilinear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \qquad u(x, 0) = \phi(x) \tag{1.1}$$

is piecewise smooth, having jump discontinuities along a finite number of smooth shock curves. (We assume that f is C^∞ and strictly convex.) In this paper we extend the result of [5] to show that generically the topological and differentiable structure of the shock set is unaffected by small perturbations of the initial data. As in [5] we consider initial data in the Schwartz space, although analogous results hold for periodic initial data. We shall call a solution u, corresponding to initial data ϕ, *stable* if there is a neighborhood $\mathcal{N}$ of ϕ in $\mathcal{S}(\mathbb{R})$ with the following property: for any $\phi_1 \in \mathcal{N}$ there is a diffeomorphism of the halfspace $Z = \mathbb{R} \times [0, \infty)$ which maps the shock set of u_1 onto that of u. The following theorem is our main result.

THEOREM. *There is an open dense set $\Omega \subset \mathcal{S}(\mathbb{R})$ such that for $\phi \in \Omega$ the solution of* (1.1) *is stable.*

We note that J. Guckenheimer [1] has obtained a similar result without a convexity hypothesis on f, but his results give only a homeomorphism of the shock sets, rather than a diffeomorphism. Moreover our proof of this theorem fits naturally into the context of Mather's theory [4] of stability of mappings. We have used this theory freely, and unlike [5], some prior knowledge of singularity theory is probably

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necessary for reading this paper. (Familiarity with [5] is also assumed.) The proof of our main theorem is contained in Section 3, where we verify the conditions for infinitesmal stability. The "Infinitesmal stability implies stability" yoga in the appropriate context (Latour [2]) is discussed in Section 2, along with certain preliminary material.

2. Preliminaries

We recall the result of Lax [3] that was the basis for the analysis in [5]. Let

$$F(x, t, u) = t[ua(u) - f(u)] + \Phi(x - a(u)t),$$

where $a(u) = f'(u)$ and $\Phi(y) = \int_0^y \phi(x)\, dx$. Lax showed that for almost every (x, t) there is a unique value of u which minimizes $F(x, t, \cdot)$ and that the function $u(x, t)$ defined (almost everywhere) to be equal to the minimizing value u is the solution of (1.1). We also recall some computations from [5]. Let

$$S = \{(x, t, u): (\partial F/\partial u)(x, t, u) = 0\}.$$

Then S is a smooth surface in $Z \times \mathbb{R}$, and

$$(\partial F/\partial x)(x, t, u) = u \quad \text{on} \quad S, \tag{2.1}$$

$$(\partial F/\partial t)(x, t, u) = -f(u) \quad \text{on} \quad S. \tag{2.2}$$

In this paragraph we attempt to motivate our approach. Let ϕ, ϕ_1 be two initial data functions with associated functions F, F_1 respectively. Suppose there is a diffeomorphism G of $Z \times \mathbb{R}$ of the form

$$G(p, u) = (g(p), \Psi(p, u)),$$

where g is a diffeomorphism of Z, such that

$$F_1(p, u) = F \circ G(p, u). \tag{2.3}$$

Then the shock set of u_1 is the image under g of the shock set of u. This follows from the observation that the shock set of a given solution consists of those points $(x, t) \in Z$ for which $F(x, t, \cdot)$ fails to possess a nondegenerate minimum, and this property is invariant under a nonsingular change of coordinates. (Since $F(p, u) \to \infty$ as $u \to \pm\infty$, (2.3) can only be satisfied for an orientation preserving map G.) Thus we shall attack the stability of the shock sets through the associated function F. However, in general it will be necessary to include in (2.3) a second diffeomorphism which operates on the range. (In (2.3) G acts on the domain.) This diffeomorphism in the range will have to depend on x, t, and we introduce the following notation to deal with this situation.

For any initial function ϕ with associated F, let $\bar{F}: Z \times \mathbb{R} \to Z \times \mathbb{R}$ be the map

$$\bar{F}(x, t, u) = (x, t, F(x, t, u)).$$

Let $C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$ be the set of smooth maps $G: Z \times \mathbb{R} \to Z \times \mathbb{R}$ with the form

$$G(p, u) = (g(p), \Psi(p, u)),$$

where $g: Z \to Z$ and $\Psi: Z \times \mathbb{R} \to \mathbb{R}$. In particular $\bar{F} \in C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$, where g is the identity. With this notation (2.3) is a special case of the equation

$$\bar{F}_1 = G_1 \circ \bar{F} \circ G_2\,, \tag{2.4}$$

where $G_1 = (g^{-1}, \Psi_1)$ and $G_2 = (g, \Psi_2)$; actually (2.3) is contained in the third component of the vector equation (2.4), and the first two components represent trivialities.

The following definitions are from Latour [2].

Definition 2.5. $\bar{F} \in C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$ is *stable* if there is an open neighborhood $\mathscr{N}$ of $\bar{F}$ (in the Whitney C^∞ topology) with the property that for any $\bar{F}_1 \in \mathscr{N}$ there exist diffeomorphisms $G_i \in C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$ such that $\bar{F}_1 = G_1 \circ \bar{F} \circ G_2^{-1}$.

Definition 2.6. $\bar{F} \in C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$ is *infinitesimally stable* if for every smooth function $\tau \in C^\infty(Z \times \mathbb{R})$ there exist smooth coefficient functions $a(x, t)$, $b(x, t)$, $c(x, t, u)$ and $\mathscr{N}(x, t, u)$ such that

$$\tau = a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial u} + \mathscr{N} \circ \bar{F}. \tag{2.7}$$

We have adapted Definition 2.6 to our context. It expresses the familiar requirement that an arbitrary vector field along $\bar{F}$ (i.e., a vector field associated to the family of maps $C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$, may be written as the sum of vector fields coming from diffeomorphisms of the domain and range. A vector field along $\bar{F}$ may be written in terms of the canonical coordinates on $Z \times \mathbb{R}$ as

$$\tau_1(x, t)\frac{\partial}{\partial x} + \tau_2(x, t)\frac{\partial}{\partial t} + \tau_3(x, t, u)\frac{\partial}{\partial u}.$$

In our situation $\bar{F} = (g, F)$, where g is the identity map on Z. Matching the first two components of a vector field along $\bar{F}$ is therefore trivial, and (2.7) requires that this matching is possible for the third component as well. The first three terms on the right in (2.7) result from a diffeomorphism of the domain and the last term from a diffeomorphism of the range.

By the main theorem of [2], to prove that a map $\bar{F}$ is stable it suffices to show merely that $\bar{F}$ is infinitesimally stable. Actually Latour's result is stated only for mappings over compact manifolds, but in fact the same result holds providing the map is proper. However the Whitney C^∞-topology is a technical complication that hinders our application of Latour's results. The Whitney topology is exceedingly fine at infinity and the map $\phi \to \bar{F}$ of $\mathscr{S}(\mathbb{R}) \to C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$ is not continuous. We shall use the asymptotic analysis in Section 4 of [5] to overcome this difficulty.

3. Proof of the Main Theorem

We begin the proof by verifying the conditions for infinitesimal stability. It turns out that in solving (2.7) we may fix (x, t) and work with functions of one variable, and we consider this case first. If $h \in C^\infty(\mathbf{R})$ and if k is a positive integer, let

$$j_k h(u) = (h'(u), h''(u), \dots, h^{(k)}(u)). \tag{3.1}$$

If $\alpha = (k_1, \dots, k_m)$ is a multi-index of positive integers, let

$$j_\alpha h(u_1, \dots, u_m) = (j_{k_1} h(u_1), \dots, j_{k_m}(u_m)). \tag{3.2}$$

If $T = \{l_1, \dots, l_n\}$ is a subset of the integers $1, \dots, m-1$, let

$$j_T h(u_1, \dots, u_m) = (h(u_{l_1}) - h(u_{l_1+1}), \dots, h(u_{l_m}) - h(u_{l_m+1})).$$

Finally, let $j_{T,\alpha} h = (j_T h, j_\alpha h)$. Thus the formula

$$j_{T,\alpha} h(u_1, \dots, u_m) = 0 \tag{3.3}$$

is a shorthand for $l + |\alpha|$ equations; here $l = \|T\|$, the cardinality of T, and $|\alpha| = k_1 + \cdots + k_m$.

LEMMA 3.4. *Let $h \in C^\infty(\mathbb{R})$ satisfy the following conditions:*

(i) *h' has only finitely many zeros.*

(ii) *(3.3) has no solution with distinct $u_1, \ldots, u_m$ if $\|T\| + |\alpha| > m + 2$.*

Then for any function $\tau \in C^\infty(\mathbb{R})$ there exist numbers $a, b \in \mathbb{R}$ and functions $c, \eta \in C^\infty(\mathbb{R})$ such that

$$\tau(u) = au + bu^2 + c(u)\, h'(u) + \eta[h(u)]. \tag{3.5}$$

Proof. Let $\alpha_1, \ldots, \alpha_N$ be the zeros of h'. If each of these zeros were simple and if $h(\alpha_i) \neq h(\alpha_j)$ for $i \neq j$, our problem would be trivial; indeed, choose η such that $\eta[h(\alpha_j)] = \tau(\alpha_j)$ for $j = 1, \ldots, N$ and let $c = (\tau - \eta \circ h)/h'$; then (3.5) is satisfied with $a = b = 0$. Of course either of these assumptions may fail. Suppose for example $h''(\alpha_1) = 0$ but $h'''(\alpha_1) \neq 0$ and the other zeros of h' are simple. Let $a = \tau'(\alpha_1)$, choose η such that $\eta[h(\alpha_j)] = \tau(\alpha_j) - a\alpha_j$, and let $c = (\tau - au - \eta \circ h)/h'$. Similarly we could use the term bu^2 to handle a triple zero of h', and a zero of higher multiplicity cannot occur, as this would violate condition (ii) with $T = \phi$, $\alpha = (4)$.

In the other direction, suppose $h(\alpha_1) = h(\alpha_2)$, but otherwise our assumptions above are valid. Let $a = [\tau(\alpha_1) - \tau(\alpha_2)]/(\alpha_1 - \alpha_2)$, choose η such that $\eta[h(\alpha_j)] = \tau(\alpha_j) - a\alpha_j$ for $j = 2, \ldots, N$, and let $c = (\tau - au - \eta \circ h)/h'$. We can handle the case $h(\alpha_1) = h(\alpha_2) = h(\alpha_3)$ similarly, and condition (ii) with $T = \{1, 2, 3\}$, $\alpha = (1, 1, 1, 1)$ precludes the next case, $h(\alpha_i) = h(\alpha_j)$ for $i, j = 1, 2, 3, 4$.

A complete listing of the ways in which these two assumptions may fail, consistent with (ii), is given in Table 1. The reader may verify that all cases may be handled analogously to the above examples. This completes the proof of the lemma.

It follows from the same proof that under conditions (i) and (ii) we may expand

$$\tau(u) = au + bf(u) + c(u)\, h'(u) + \eta[h(u)], \tag{3.6}$$

where $f(u)$ is the function in (1.1), since the hypothesis $f''(u) \neq 0$ allows us to match the necessary derivatives. Moreover, if the data in

TABLE 1

Degenerate critical points	
$h''(\alpha_1) = 0$	
$h''(\alpha_1) = 0,$	$h''(\alpha_2) = 0$
$h'''(\alpha_1) = 0$	
Equal critical values	
$h(\alpha_1) = h(\alpha_2)$	
$h(\alpha_1) = h(\alpha_2) = h(\alpha_3)$	
$h(\alpha_1) = h(\alpha_2),$	$h(\alpha_3) = h(\alpha_4)$
Mixed	
$h''(\alpha_1) = 0,$	$h(\alpha_2) = h(\alpha_3)$
$h''(\alpha_1) = 0,$	$h(\alpha_1) = h(\alpha_2)$

Lemma 3.4 depend smoothly on one or more parameters, the same may be assumed of the coefficients in (3.6). This remark justifies solving (3.2) with (x, t) fixed: the smooth local solutions may be pasted together with a partition of unity to get a global solution.

Suppose ϕ is an initial function with associated F. For given (x, t), $F(x, t, \cdot)$ is a function of one variable, and we define $j_{T\alpha}F: Z \times \mathbb{R} \to \mathbb{R}^{l+|\alpha|}$ by an obvious extension of the above notation. The following transversality theorem is a minor extension of Theorem 3.1 of [5], and may be proved by exactly the same argument.

PROPOSITION 3.7. *The system of equations*

$$j_{T,\alpha}F(x, t, u_1, \ldots, u_m) = 0$$

has no solution with distinct $u_1, \ldots, u_m$ *if* $\| T \| + | \alpha | > m + 2$, *except possibly for initial data in a subset of* $\mathscr{S}(\mathbb{R})$ *of the first category.*

We may now prove our main theorem. Let ϕ be an initial function, with associated F. Now by (2.1) there is a smooth function $c_1(x, t, u)$ such that $\partial F/\partial x = u + c_1(\partial F/\partial u)$, since $\partial F/\partial x - u$ vanishes on the smooth surface S defined by the equation $\partial F/\partial u = 0$. Similarly, by (2.2) there is a function $c_2(x, t, u)$ such that $\partial F/\partial t = -f(u) + c_2(\partial F/\partial u)$. Therefore (3.2) is solable if and only if

$$\tau = au - bf(u) + c(\partial F/\partial u) + \eta \circ \bar{F} \tag{3.10}$$

is solable. But (3.10) is solable for arbitrary τ if the conditions of Lemma 3.4 are satisfied, and by Proposition 3.7 these conditions are indeed satisfied for generic initial data. Therefore for generic initial data $\bar{F}$ is infinitesimally stable, so by [2], generically $\bar{F}$ is stable.

Choose a neighborhood $\mathscr{N}$ of $\bar{F}$ in $C_Z^\infty(Z \times \mathbb{R}, Z \times \mathbb{R})$ verifying the stability condition (Definition 2.5). Now the various topologies on spaces of smooth functions all coincide if the domain is compact. In particular,

$$\mathscr{M} = \{\phi \in \mathscr{S}(\mathbb{R}): \bar{F} \mid K = \tilde{F} \mid K \text{ for some } \tilde{F} \in \mathscr{N}\}$$

is open in $\mathscr{S}(\mathbb{R})$, providing K is compact.

Let $\tau > 0$ be given. We may find a neighborhood $\mathscr{M}_1 \subset \mathscr{M}$ of ϕ and a constant A such that for $\phi_1 \in \mathscr{M}_1$ the associated solution u_1 has no shocks in

$$\{(x, t): |x| \geqslant A \text{ and } t \leqslant \tau\}.$$

On the other hand, according to Section 4 of [5] we may assume, by further restricting $\mathscr{M}_1$ if necessary, that for $\phi_1 \in \mathscr{M}_1$ the shock set of u_1 for $t \geqslant \tau$ consists of two nonintersecting curves, when τ is some constant independent of ϕ_1. Thus the diffeomorphism of the shock set which may be constructed on $K = [-A, A] \times [0, \tau]$ using Latour's results, can be extended to a diffeomorphism on A. This completes the proof.

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