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AN INTRODUCTION TO CATASTROPHE THEORY AND ITS APPLICATIONS*

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Abstract. This article is divided into two parts. In the first we give a description of the basic theorems of elementary catastrophe theory, along with heuristic explanations of why these theorems are valid. In particular, the main ideas in Mather's proof of Thom's classification theorem are presented.

The second part contains three applications of catastrophe theory to the buckling of beams, optics, and convex conservation laws. In these sections we attempt to state the problems precisely, to show how catastrophe theory may be used in a mathematically rigorous fashion, and to state what *new* information can be obtained by the use of catastrophe theory.

CONTENTS

Introduction. During the past decade catastrophe theory, created by René Thom in his book *Structural Stability and Morphogenesis* [32], has generated substantial interest among both mathematicians and users of mathematics. Various comments about catastrophe theory have ranged from "the first theory to explain how a continuous change of parameters can cause discontinuous phenomena" or "the type of mathematics necessary to study qualitative problems in biology and the social sciences" to "some nice observations surrounded by totally unwarranted speculation." The truth, as usual, probably lies somewhere in between.

There now exist several expository articles describing catastrophe theory to the lay audience ([37], [24], [29], [27], [54], and [30]) and several expositions concerned with the technical theorems—first proved by John Mather [19]—which require relatively large investments of time and energy for most mathematicians to absorb and appreciate ([5], [36], [35], [50], [60], [55], and [21]).

Our purpose, in the first part of this article, is to give concise but complete descriptions of the basic theorems of elementary catastrophe theory along with heuristic descriptions explaining why these theorems should be true. In the second part we describe some applications which—we hope—will indicate some of the power and limitations of this theory.

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As a rule the theorems stated here are true while the motivating discussions between these theorems are to be taken with a grain of salt; they are meant to enlighten—not to prove. The exception is in § 6 where we give a proof of Mather's basic result on finite determinacy [20]. The physically motivated applications included here stand on firm mathematical ground; I have not attempted to describe examples from the biological or social sciences.

One should keep two questions in mind.

1. What mathematical assumptions are there on the theorems of catastrophe theory?
2. Assume that for a given application there is reason to believe that these mathematical assumptions are valid. What type of information can one expect to obtain by applying catastrophe theory?

We shall try to say more about these questions at the end of this article.

The mathematical ideas present in elementary catastrophe theory stem from the pioneering works of Morse, Whitney [52], and Poincaré [47]. Some of the simplest examples of catastrophes—particularly the cusp—have been known to many mathematicians in various guises for most of this century, e.g. [40], [41], [47], and [45]. The advantage of the theory as developed by Thom and Mather is to put these results into a coherent framework which allows precise statements, complete classifications, and possible generalizations of the known results. The person who has been most instrumental in the classification of higher order catastrophes—a topic not covered in this paper— is Arnol'd [2], [4], and [43]. There have been substantial contributions by others, e.g. Siersma [28], Tjurina [50], [51], and Saito [49]. Christopher Zeeman has been, aside from Thom himself, the person most active in trying to apply catastrophe theory to a variety of problems in the biological and social sciences [39]. There are now a number of researchers involved in trying to extend and test these applications and there is no doubt that some of these suggested applications have been responsible for a large part of the controversy surrounding catastrophe theory. In fact several articles have appeared recently attacking these proposed uses of catastrophe theory (see [63]–[65]).

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PART A—THE THEORY

In one sense elementary catastrophe theory is a generalization of theorems about critical points or singularities of C^∞ real-valued functions of n real variables to ones about parametrized families of such functions. By C^∞ we mean that all partial derivatives of all orders exist and are continuous. Thus a reasonable way to begin is by describing part of the classical theory of singularities of C^∞ functions or Morse theory.

Unless otherwise indicated all functions in this paper will be assumed to be C^∞.

A real-valued function of n variables will be denoted by $f\colon \mathbf{R}^n \to \mathbf{R}$. Such a function has a *singular point* at p if

$$(df)(p)=\left(\frac{\partial f}{\partial x_1}(p),\cdots,\frac{\partial f}{\partial x_n}(p)\right)=0.$$

DEFINITION 1. f has a *nondegenerate* (or *Morse*) singularity at p if the Hessian matrix

$$(d^2f)(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$$

is nonsingular.

There are three basic theorems that we shall consider about Morse functions—functions all of whose singularities are nondegenerate: two local and one global.

LOCAL THEOREM 1 (normal forms). *Let* 0 *be a nondegenerate singular point for* $f\colon \mathbf{R}^n \to \mathbf{R}$. *Then there exist coordinates* $x_1, \cdots, x_n$ *on some neighborhood of* 0 *in* $\mathbf{R}^n$ *such that*

$$f(x) = f(0) - (x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_n^2).$$

The number k is just the index of the Hessian matrix at 0 *viewed as a symmetric bilinear form.*

Comments. 1) This normal form shows that Morse singularities are isolated for it is easy to check that the only singularity of $f(x) = -(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + \cdots + x_n^2$ is at $x = 0$.

2) We now know what all functions look like near a Morse singularity—at least if we allow changes of coordinates. The qualitative shape of the function is determined by the index k.

3) A change of coordinates on $\mathbf{R}^n$ at 0 may be viewed as a mapping in the following way: suppose that one is given coordinates $y_1, \cdots, y_n$ on $\mathbf{R}^n$; then the new coordinates $x_1, \cdots, x_n$ are really functions $x_i = x_i(y)$. The mapping $H(y_1, \cdots, y_n) = (x_1, \cdots, x_n)$ is the desired one. We denote such a mapping by $H\colon \mathbf{R}^n \to \mathbf{R}^n$. We shall only consider smooth or C^∞ changes of coordinates; i.e. each x_i is a C^∞ function of $y_1, \cdots, y_n$. By a change in coordinates we also mean that the mapping H is invertible; i.e. we can solve for each y_i in terms of x. Thus we assume that the Jacobian matrix $(DH)_y = ((\partial x_i/\partial y_j)(y))$ has a determinant which is nowhere equal to zero; the inverse function theorem implies that H is invertible and that the inverse H^{-1} is also C^∞. The function $H\colon \mathbf{R}^n \to \mathbf{R}^n$ which represents a smooth change of coordinates is called a *diffeomorphism* on $\mathbf{R}^n$.

DEFINITION. Two functions $f, g\colon \mathbf{R}^n \to \mathbf{R}$ are called *right equivalent* if there exists a smooth change of coordinates, i.e., a diffeomorphism $H\colon \mathbf{R}^n \to \mathbf{R}^n$ and a constant K such that $f = g \circ H + K$.

In this language Local Theorem 1 states that if f has a Morse singularity at 0 with index k, then f is right equivalent to $g(x) = -(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + \cdots + x_n^2$. Moreover this diffeomorphism H satisfies $H(0) = 0$. We will prove this theorem in § 6 as a special case of a more general theorem.

LOCAL THEOREM 2 (local stability). *Morse singularities are stable, i.e. if f has a nondegenerate singularity at p with index k and g is a small perturbation of f* (*in the sense that g, dg,* and d^2g *are near f, df,* d^2f *on some neighborhood of p*) *then g also has a Morse singularity with index k at some point q near p. Moreover the only singularities which are stable in this sense are Morse.*

For example, $f(x) = x^3/3$ has a degenerate singularity at 0; we show that this singularity is not stable. Let $g(x) = x^3/3 - \varepsilon x$. For ε small g is a small perturbation of f. When $\varepsilon < 0$ g has no singularities, and when $\varepsilon > 0$ g has two nondegenerate singularities (at $x = \pm\sqrt{\varepsilon}$). So the singularity type does change. See Fig. 1.

Note. Local Theorem 2 states that it is easy to test the stability of a singularity. Just compute the determinant of the Hessian matrix and see whether or not it is zero.

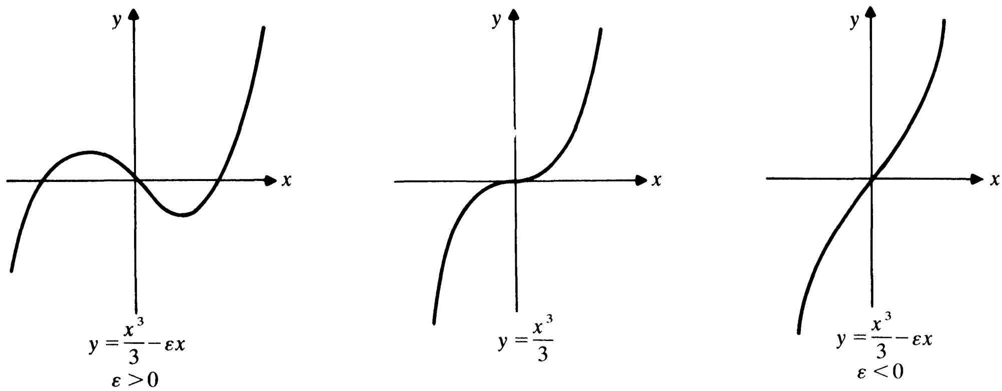

FIG. 1

Exercise. Is the singularity of $f(x, y, z)=x^2+xy-y^3/3+yz+z^2$ at 0 stable?

Let $C^\infty(\mathbf{R}^n)$ denote the space of all C^∞ functions from $\mathbf{R}^n \to \mathbf{R}$.

GLOBAL THEOREM (genericity). *The subset M of $C^\infty(\mathbf{R}^n)$ consisting of Morse functions is both open and dense.*

Notes. 1) The fact that M is open follows essentially from Local Theorem 2. What is new is the density statement.

2) To make the statement of the Global Theorem precise, we have to describe the topology used on $C^\infty(\mathbf{R}^n)$. The topology used is called the Whitney C^∞ topology. Since its definition is somewhat technical we refer the interested reader to [11]. For our purposes the salient feature is that if a sequence of functions $f_1, f_2, \cdots$ converges in this topology, then the sequence $D_a f_1, D_a f_2, \cdots$ converges uniformly on any compact subset of $\mathbf{R}^n$ where D_a is any partial differentiation operator of any order.

3) In the statement of the theorem, $\mathbf{R}^n$ may be replaced by any C^∞ n dimensional manifold (without boundary). For our purposes a *manifold of dimension n* (or an *n-manifold*) X should be thought of as a subset of some Euclidean space $\mathbf{R}^p$ such that: for each x in X there is a C^∞ map H: $\mathbf{R}^n \to \mathbf{R}^p$ such that:

(i) $H(0)=x$,

(ii) the Jacobian map $(DH)_y$ has rank n at all points y in some neighborhood V of 0,

(iii) $H(V) \subset X$ is an open neighborhood of x in X.

Examples. (a) $\mathbf{R}^n \subset \mathbf{R}^n$ is an n-manifold.

(b) S^1 = unit circle in $\mathbf{R}^2$ centered at 0 is a 1-manifold. For example let x be the point (0, 1) and V be the interval $(-1, 1)$; then $H(s)=(s, \sqrt{1-s^2})$ is a possible choice for H. Exercise: check that such a map H exists for each point on the circle S^1.

(c) S^2 = unit sphere in $\mathbf{R}^3$ and T^2 = torus (i.e. the set of points which are of distance b from the circle of radius a in the xy-plane in $\mathbf{R}^3$ with $b<a$) are both 2-manifolds.

Stated extremely loosely, these theorems say that if one were to choose a C^∞ function at "random" it would very likely be a Morse function and that locally Morse functions behave predictably and stably. Thus, if one did not know to the contrary, one might assume that in any physical situation—where measurements can only be made approximately—the only observed (differentiable) functions would be Morse functions.

Thom points out—through elementary catastrophe theory—one way in which this reasoning fails; namely, in many situations what is observed is not an individual function but rather a parametrized family of functions. (In Part B we shall analyze several of these situations.) It is then possible for a non-Morse function to appear as a single member of a "stable family." For example, as we saw before, $f(x)=x^3$ is not stable at 0; whereas we shall show that the parametrized family $F_t(x)=x^3+tx$ is stable as a parametrized family near 0. To make this notion of stability precise we need some definitions.

First we will want to analyze the behavior of a given function on some (arbitrarily small) neighborhood of a singularity, which we assume to be at 0.

DEFINITION 2. A *germ* of a function of $\mathbf{R}^n \to \mathbf{R}$ at 0 is an equivalence class of mappings in which two functions are equivalent if they are identical on some neighborhood of 0. (Note that this neighborhood depends on the choice of functions.)

Let $C_0^\infty(\mathbf{R}^n)$ be the space of germs of infinitely differentiable functions of $\mathbf{R}^n \to \mathbf{R}$ at 0. Note that $C_0^\infty(\mathbf{R}^n)$ is a ring whose operations, addition and multiplication, are induced from the standard operations of addition and multiplication of functions. We denote a germ f in $C_0^\infty(\mathbf{R}^n)$ by $f\colon (\mathbf{R}^n, 0) \to \mathbf{R}$.

A *k-parameter family of germs on $\mathbf{R}^n$ based at f* or a *k-parameter unfolding* of f is just a germ F in $C_0^\infty(\mathbf{R}^k \times \mathbf{R}^n)$ where $F(0, x)=f(x)$ for x in $\mathbf{R}^n$. We will also denote $F(t, x)$ by $F_t(x)$ for t in $\mathbf{R}^k$. In this way it is natural to think of F as a parametrized family of germs F_t in $C_0^\infty(\mathbf{R}^n)$ with $F_0=f$.

Note. In what follows we shall pass back and forth between a germ of a function and the function defining that germ.

DEFINITION 3. Let $F_t, G_s\colon (\mathbf{R}^n, 0) \to \mathbf{R}$ be two k-parameter unfoldings of f. These unfoldings are *(right) equivalent* if there is a k-parameter family of diffeomorphisms (or changes of coordinates) $H_s\colon (\mathbf{R}^n, 0) \to \mathbf{R}^n$, a smooth invertible change of parameters $t=t(s)$ and a smooth germ $K\colon (\mathbf{R}^k, 0) \to \mathbf{R}$ such that

$$H_0(x)=x \quad \text{and} \quad t(0)=0, \tag{1}$$

$$G_s(x)=F_{t(s)}(H_s(x))+K_s \tag{2a}$$

or

$$G_s=F_{t(s)} \circ H_s+K_s. \tag{2b}$$

Note. Definition 3 states that two unfoldings of the same function f are equivalent if each member of one family G is right equivalent to some member of the other family F in some smooth bijective manner.

DEFINITION 4. A k-parameter unfolding $F_t\colon (\mathbf{R}^n, 0) \to \mathbf{R}$ of f is *(locally) stable (at* $t=0$) if every nearby k-parameter unfolding G_s of f is equivalent to F_t.

By nearby—or small perturbation—we mean in some small neighborhood of F_t in the C^∞ topology of $\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$. Here is where our heuristic description of catastrophe theory begins: there is no reasonable topology on the space of C^∞ germs, only on the space of C^∞ functions. A tempting topology to use is the one given by Taylor series; i.e. two C^∞ functions are close if all of the coefficients in their Taylor expansions are close. The problem with this topology is that it is not Hausdorff, e.g. this topology does not distinguish between $\exp(-1/x^2)$ and the zero function.

Definition 4 can be made precise in the following way: for any (small) neighborhood $U_1 \times U_2$ of 0 in $\mathbf{R}^n \times \mathbf{R}^k$, there is a neighborhood V of F_t in the C^∞ topology on functions on $U_1 \times U_2$ such that if G_s is in V, then F_t and G_s are equivalent on $U_1 \times U_2$; that is, there are diffeomorphisms $H_s\colon U_1 \to U_1$ (for all s in U_2) and $t\colon U_2 \to U_2$ and a smooth function $K\colon U_2 \to \mathbf{R}$ such that $G_s(x)=F_{t(s)}(H_s(x))+K_s$. For our purposes Definition 4 includes all of the essential ideas with much less technical detail. The reader is asked to "believe" in the C^∞ topology on germs of C^∞ functions by equating the space of germs $C_0^\infty(\mathbf{R}^n \times \mathbf{R}^k)$ with C^∞ functions defined on some small fixed neighborhood $U_1 \times U_2$ of 0 and using the C^∞ topology on this space of functions $C^\infty(U_1 \times U_2)$.

The aim of elementary catastrophe theory is to determine the analogues of the three theorems of Morse theory mentioned above in the context of unfoldings of a particular function f. What Thom and Mather have done is to determine conditions on the derivatives of f which insure the existence of and which give a local normal form for a stable unfolding of f. They also show that when the parameter space or *control space* $\mathbf{R}^k$ has dimension $\leqq 5$ then there is a finite classification of stable unfoldings (in a sense to be described) as well as a global density or genericity theorem. When $k \geqq 6$ both the finite classification and the density theorems fail and for the same reasons.

Complete proofs of the theorems on Morse theory mentioned above are given in [11, Chap. II]. For a more comprehensive treatment of Morse theory see Milnor's book [22].

There is an extensive theory of singularities of mappings between manifolds of arbitrary (finite) dimension which has also been developed by Thom and Mather in recent years. A description of this theory can be found in [11], [42], or [20] and [46].

In [53] Arnol'd has developed a theory of unfoldings for matrices. This theory includes many of the ideas that occur in the theory of unfoldings for functions but in a technically simpler situation.

1. Finite codimension of singularities. For what follows we shall need the concept of a tangent space. Let X be an n-manifold in $\mathbf{R}^p$ and x a point in X. The *tangent space to X at x* is a vector subspace of $\mathbf{R}^p$ denoted by T_xX and defined as follows: let $c: \mathbf{R} \to X$ be a smooth curve such that $c(0) = x$. Then $dc/dr|_{r=0}$ is a vector tangent to the curve c at x. T_xX is the set of all such vectors.

Examples. 1) If $X = \mathbf{R}^p$, then $T_xX = \mathbf{R}^p$. To see this define for each vector v in $\mathbf{R}^p$ the curve $c_v(r) = x + rv$. This is the line through x in the direction v. Since $(dc_v/dr)|_{r=0} = v$, v is in $T_x\mathbf{R}^p$.

2) Let $X = S^2$ be the unit sphere in $\mathbf{R}^3$; i.e. $S^2 = \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}$. Let $x = (1, 0, 0)$. A curve $c: \mathbf{R} \to S^2$ with $c(0) = x$ and $c(r) = (c_1(r), c_2(r), c_3(r))$ satisfies $c_1(0) = 1$, $c_2(0) = c_3(0) = 0$, and $c_1^2(r) + c_2^2(r) - c_3^2(r) = 1$. The tangent vector $dc/dr|_{r=0} = (c_1'(0), c_2'(0), c_3'(0))$ must satisfy

$$2c_1(0)c_1'(0) + 2c_2(0)c_2'(0) + 2c_3(0)c_3'(0) = 0.$$

This implies that $c_1'(0) = 0$. Thus the tangent vectors are all parallel to the yz-plane. Intuitively this is clear.

Notes. 1) It should be reasonably clear that $\dim T_xX = \dim X$ for all x in X.

2) If X is an n-manifold in $\mathbf{R}^p$, then the *codimension of X* is just $\dim \mathbf{R}^p - \dim X = p - n$. Recall that if W is a vector subspace of the vector space V, then V/W denotes the quotient vector space and $\dim V/W = \dim V - \dim W$. Therefore $\operatorname{codim} X = \dim \mathbf{R}^p / T_xX$ for any x in X.

In the heuristics which follow, we shall replace $\mathbf{R}^p$ by the infinite dimensional vector space $C_0^\infty(\mathbf{R}^n)$. Given a germ f in $C_0^\infty(\mathbf{R}^n)$ we shall construct an (infinite dimensional) manifold $\mathcal{O}_f$ in $C_0^\infty(\mathbf{R}^n)$. We shall define $\operatorname{codim} f$ by $\operatorname{codim} \mathcal{O}_f$. Note that the computation of codimension in Note 2) above which uses the idea of quotient space can work even if both the manifold and the vector space are infinite dimensional. The original definition of codimension would not work in this case.

The manifold $\mathcal{O}_f$ promised above is the set of all germs in $C_0^\infty(\mathbf{R}^n)$ which are right equivalent to f. That is

$$\mathcal{O}_f = \{g \in C_0^\infty(\mathbf{R}^n) | \exists h: (\mathbf{R}^n, 0) \to \mathbf{R}^n \text{ a diffeomorphism and a constant } K \text{ such that } g = f \circ h + K\}.$$

DEFINITION 1.1. $\operatorname{codim} f = \operatorname{codim} \mathcal{O}_f = \dim_{\mathbf{R}} C_0^\infty(\mathbf{R}^n)/T_f\mathcal{O}_f$.

An obvious question is "How does one compute codim f?" This is equivalent to determining $T_f\mathcal{O}_f$. To compute $T_f\mathcal{O}_f$ we choose an arbitrary smooth curve $c: \mathbf{R} \to \mathcal{O}_f$ with $c(0) = f$. Then $dc/dt(0)$ is a tangent vector in $T_f\mathcal{O}_f$. Such a curve must have the form $c(t) = f \circ h_t + K_t$ from the definition of $\mathcal{O}_f$. Also we may assume that $h_0(x) = x$. What is not obvious is that K_t and $h_t(x)$ can be chosen to vary smoothly in both x and t. Fortunately this can be shown. Let $h_t(x) = (h_t^1(x), \cdots, h_t^n(x))$ in the coordinates $x_1, \cdots, x_n$ on $\mathbf{R}^n$ at 0 and use the chain rule to obtain

$$\frac{dc}{dt}(0) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \frac{\partial h_0^i}{\partial t}(x) + K_0'.$$

Algebraically $dc/dt(0)$ has the form

$$a(x) = a_1(x)\frac{\partial f}{\partial x_1}(x) + \cdots + a_n(x)\frac{\partial f}{\partial x_n}(x) + K$$

for some smooth functions $a_1(x), \cdots, a_n(x)$ and some constant K. It can be shown that for every function a of this form there is a curve c whose tangent vector at 0 is a. So

$$T_f\mathcal{O}_f = \left\{ a \in C_0^\infty(\mathbf{R}^n) | a = a_1\frac{\partial f}{\partial x_1} + \cdots + a_n\frac{\partial f}{\partial x_n} + K \right\}.$$

This set is clearly a vector subspace of $C_0^\infty(\mathbf{R}^n)$ which we shall denote by $\overline{\langle \partial f/\partial x \rangle}$. The subspace of $T_f\mathcal{O}_f$ in which $K = 0$ is denoted by $\langle \partial f/\partial x \rangle$. So we have that

$$\operatorname{codim} f = \dim_{\mathbf{R}} C_0^\infty(\mathbf{R}^n)/\overline{\left\langle \frac{\partial f}{\partial x} \right\rangle},$$

for any germ $f: (\mathbf{R}^n, 0) \to \mathbf{R}$.

Notes. (A) The subspace $\langle \partial f/\partial x \rangle$ has the property that if g is in $C_0^\infty(\mathbf{R}^n)$ and a is in $\langle \partial f/\partial x \rangle$ then ga is also in $\langle \partial f/\partial x \rangle$. A vector subspace I of $C_0^\infty(\mathbf{R}^n)$ which satisfies this property that ga is in I if a is in I is called an *ideal*. So $\langle \partial f/\partial x \rangle$ is an ideal.

(B) If g is in $C_0^\infty(\mathbf{R}^n)$ with $g(0) \neq 0$, then $1/g$ is also in $C_0^\infty(\mathbf{R}^n)$ since it is a C^∞ function defined on some (small) neighborhood of 0.

(C) Suppose that an ideal I contains a germ g with $g(0) \neq 0$. Then $I = C_0^\infty(\mathbf{R}^n)$. Proof: Since $1/g$ is in $C_0^\infty(\mathbf{R}^n)$, the function germ $g(1/g) = 1$ is in I. Now let f be any germ in $C_0^\infty(\mathbf{R}^n)$, then $f \cdot 1 = f$ is in I since I is an ideal. So $I = C_0^\infty(\mathbf{R}^n)$.

Example 1. Let $f: (\mathbf{R}^n, 0) \to \mathbf{R}$ be nonsingular. Then some $\partial f/\partial x_i(0) \neq 0$, so $\partial f/\partial x_i$ is invertible in $C_0^\infty(\mathbf{R}^n)$. Hence $\langle \partial f/\partial x \rangle = C_0^\infty(\mathbf{R}^n)$ and $\operatorname{codim} f = 0$.

(D) The ideal $\langle \partial f/\partial x \rangle$ is independent of the choice of coordinates made. Suppose $y_1, \cdots, y_n$ are some other choice of coordinates on $\mathbf{R}^n$ at 0. As observed in Comment 3) after Local Theorem 1 the y_i's are actually C^∞ functions of the x_i coordinates which we denote by $y_i = y_i(x)$. Now apply the chain rule to obtain

$$\frac{\partial f}{\partial x_i}(y) = \frac{\partial f}{\partial y_1}(y(x))\frac{\partial y_1}{\partial x_i}(x) + \cdots + \frac{\partial f}{\partial y_n}(y(x))\frac{\partial y_n}{\partial x_i}(x).$$

This calculation shows that $\langle \partial f/\partial x \rangle \subset \langle \partial f/\partial y \rangle$. The reverse inclusion follows by using the chain rule on the inverse change of coordinates.

In most elementary cases the computation of codimension involves nothing more complicated than Taylor's theorem. The useful form of the remainder term is given in this context by:

THEOREM. *Let $f: (\mathbf{R}^n, 0) \to \mathbf{R}$ be a germ and let $j^k f$ be the kth order Taylor expansion of f at* 0. *Let $p_1, \cdots, p_r$ be a basis for the homogeneous polynomials of degree $k+1$. Then there exist germs $a_1, \cdots, a_r$ in $C_0^\infty(\mathbf{R}^n)$ such that $f(x) = j^k f(x) + a_1(x)p_1(x) + \cdots + a_r(x)p_r(x)$ for x in some neighborhood of* 0.

Example. Let $p_1(x, y)=x^2+y^2, p_2(x, y)=x^2$, and $p_3(x, y)=y^2-2xy$. These homogeneous polynomials of degree 2 form a basis for such polynomials and Taylor's theorem asserts that given any germ $m(x, y)$ we may write $m(x, y)= m_0+m_1x+m_2y+a_1(x, y)(x^2+y^2)+a_2(x, y)x^2+a_3(x, y)(y^2-2xy)$. The constants are just the standard Taylor coefficients; namely, $m_0=m(0)$, $m_1=\partial m/\partial x(0)$, and $m_2= \partial m/\partial y(0)$.

Proof. It is sufficient to prove this theorem for those f which vanish to order k at 0; i.e., $j^kf(x)\equiv 0$. For general f apply the theorem to $f-j^kf$ which does vanish through order k.

The proof consists of one observation followed by an elementary induction argument. The observation is the following: Suppose that $g(0)=0$ then there are C^∞ functions $b_i(x)$ such that $g(x)=x_1b_1(x)+\cdots+x_nb_n(x)$. To see this let $q(s)= g(sx_1, \cdots, sx_n)$ and note that

$$g(x)=g(x)-g(0)=q(1)-q(0)=\int_0^1 \frac{dq}{ds}(s)\,ds$$

$$=\int_0^1 x_1\frac{\partial g}{\partial x_1}(sx)+\cdots+x_n\frac{\partial g}{\partial x_n}(sx)\,ds=x_1b_1(x)+\cdots+x_nb_n(x),$$

where $b_i(x)=\int_0^1 (\partial g/\partial x_i)(sx)\,ds$.

Next we proceed by induction on k. For $k=0$ we assume that $j^0f(0)=f(0)=0$ and let $p_1, \cdots, p_n$ be a basis for the homogeneous polynomials of degree 1. As noted above we may write $f(x)=x_1b_1(x)+\cdots+x_nb_n(x)$. Since each x_i is a homogeneous polynomial of degree 1, x_i may be written as a linear combination of the p_i's (with scalar coefficients) and $f(x)=a_1(x)p_1(x)+\cdots+a_n(x)p_n(x)$ for some smooth functions $a_i(x)$.

Now assume that the theorem is true for $k-1$ and assume that $j^kf(0)=0$. As above let $p_1, \cdots, p_r$ be a basis for the homogeneous polynomials of degree $k+1$ and let $q_1, \cdots, q_s$ be a basis for the homogeneous polynomials of degree k. Since $j^{k-1}f(0)$ is also 0, we may by induction write $f(x)=c_1(x)q_1(x)+\cdots+c_s(x)q_s(x)$. Next note that since the Taylor series of f starts with terms of degree $k+1$ and each q_i has degree k, so each c_i must start with terms of degree 1. So $c_i(0)=0$ for all i and by our first observation each c_i may be written as a linear combination of the x_j's. Since each $x_jq_i(x)$ is a homogeneous polynomial of degree $k+1$, it may be written as a linear combination of the p_i's. Hence we may write $f(x)=a_1(x)p_1(x)+\cdots+a_n(x)p_n(x)$ as desired.

Example 2. Let f: $(\mathbf{R}^n, 0)\to\mathbf{R}$ have a nondegenerate singularity at 0. By the local Morse Theorem 1, we can choose coordinates so that $f(x)=f(0)\pm x_1^2\pm\cdots\pm x_n^2$. Then $\langle\partial f/\partial x\rangle=\langle x_1, \cdots, x_n\rangle$. So $\langle\partial f/\partial x\rangle=C_0^\infty(\mathbf{R}^n)$ and codim $f=0$.

Fact. One can show easily enough that codim $f=0$ iff f is (locally) stable. So codim f is a measure of the instability of f. (Recall Local Theorem 2.)

Note. For future reference we let $\mathcal{M}$ denote the maximal ideal in $C_0^\infty(\mathbf{R}^n)$ consisting of those germs which vanish at 0. The above theorem states that $\mathcal{M}$ is precisely the ideal generated by the coordinate functions $x_1, \cdots, x_n$. Also $\mathcal{M}^k$ is the ideal of functions which vanish through order $k-1$ and is generated by any basis for the homogeneous polynomials of degree k.

Example 3. Let $f(x, y)=x^3+xy^2$. Then $\langle\partial f/\partial x\rangle=\langle 3x^2+y^2, 2xy\rangle$. We claim that any germ $l(x, y)$ can be written in the form

$$l(x, y)=l_0+l_1x+l_2y+l_3x^2+a(x, y)(3x^2+y^2)+b(x, y)2xy.$$

To accomplish this we will need several applications of Taylor's theorem. Since $p_1 = 3x^2 + y^2$, $p_2 = 2xy$, and $p_3 = x^2$ form a basis for the homogeneous polynomials of degree 2, we have that $l(x, y) = l_0 + l_1 x + l_2 y + A(x, y)(3x^2 + y^2) + B(x, y)2xy + C(x, y)x^2$. By expanding $C(x, y) = l_3 + D(x, y)x + E(x, y)y$, we see that we need only show that x^3 and x^2y are in $\langle \partial f/\partial x \rangle$. Clearly x^2y is a smooth multiple of $2xy$ while $x^3 = x(3x^2+y^2)/3 - (y/6)2xy$. So the claim is true.

Our interest is in finding the dimension of $C_0^\infty(\mathbf{R}^2)/\langle \partial f/\partial x \rangle$. Recall that two germs f and g yield the same element in this quotient vector space iff $f = g + h$ where h is in in $\langle \partial f/\partial x \rangle$.

The above claim implies that $l(x, y)$ and $l_1 x + l_2 y + l_3 x^2$ yield the same element in this quotient vector space. So $\operatorname{codim} f \leqq 3$. To see that $\operatorname{codim} f$ actually equals 3, we must show that x, y, x^2 are independent in this quotient space. To rephrase this: is it possible to find constants a, b, and c—not all zero—and germs $d(x, y)$ and $e(x, y)$ such that

$$ax + by + cx^2 = d(x, y)(3x^2 + y^2) + e(x, y)2xy? \tag{1.1}$$

The answer is no. To see this note that the right-hand side of (1.1) begins with terms of order 2 (or higher) so $a = b = 0$. Next differentiate (1.1) twice with respect to x and evaluate at 0 to obtain $2c = 6d(0)$. Next differentiate (1.1) twice with respect to y and evaluate at 0 to obtain $0 = 2d(0)$. So $c = 0$ and $\operatorname{codim} f = 3$.

Note. This example also shows us how to find a set of germs which project onto a a basis of $C_0^\infty(\mathbf{R}^n)/\langle \partial f/\partial x \rangle$. In this case x, y, x^2 gives such a basis, although x, y, y^2 would do just as well.

Example 4. Let f: $(\mathbf{R}, 0) \to \mathbf{R}$ be defined by $f(x) = x^m g(x)$ where $g(0) \neq 0$ and $m \geqq 2$. Since $\partial f/\partial x = x^{m-1}(mg(x) + xg'(x))$ and the second factor is invertible in $C_0^\infty(\mathbf{R})$—since it is nonzero at 0—we have $\langle \partial f/\partial x \rangle = \langle x^{m-1} \rangle$. Let l be in $C_0^\infty(\mathbf{R})$. By Taylor's theorem, $l(x) = l_0 + l_1 x + \cdots + l_{m-2}x^{m-2} + a(x)x^{m-1}$. Hence $C_0^\infty(\mathbf{R})/\langle \partial f/\partial x \rangle$ is isomorphic to the vector space with basis $x, x^2, \cdots, x^{m-2}$ over $\mathbf{R}$. So $\operatorname{codim} f = m - 2$.

DEFINITION 1.2. f has *finite codimension* if $\operatorname{codim} f < \infty$ and *infinite codimension* otherwise.

Example 5. $f(x) = e^{-1/x^2}$ has infinite codimension since the Taylor expansion of of $f(x)$ at 0 is identically zero. So $x, x^2, x^3, \cdots$ are all independent in $C_0^\infty(\mathbf{R})/\langle \partial f/\partial x \rangle$.

Exercise. Show that the germ x^2y in $C_0^\infty(\mathbf{R}^2)$ has infinite codimension.

2. Stability of unfoldings. One very important concept in the study of singularities is transversality. Let V be a vector space and W a manifold in V. Let F: $\mathbf{R}^k \to V$ be a C^∞ mapping and assume that $F(0) = x$ in W.

DEFINITION. F has a *transverse intersection with* W *at* 0 if every vector v in V can be written in the form $v = w + (DF)_0(r)$ where w is in T_xW, r is in R^k, and $(DF)_0$ is the Jacobian matrix of F at 0. In vector space notation we may write $V = T_xW + (DF)_0(\mathbf{R}^k)$. We denote this by $F \pitchfork W$ at 0.

Examples. (i) Let $V = \mathbf{R}^2$ and let W be the x-axis. Let F: $\mathbf{R} \to \mathbf{R}^2$ be defined by $F(t) = (t, t)$. Then $F \pitchfork W$ at 0. If $F(t) = (t, t^2)$ then F does not intersect W transversely at 0.

(ii) Let $V = \mathbf{R}^3$ and let W be the x-axis. Let F: $\mathbf{R} \to \mathbf{R}^3$ be defined by $F(t) = (t, t, 0)$. Then F does *not* intersect W transversely at 0.

Notes. (A) The reason why transversality is such a good concept comes from the following theorem first proved by Thom [57]: If $F \pitchfork W$ at 0 and G: $\mathbf{R}^k \to V$ is another C^∞ function near F then there is a point p in $\mathbf{R}^k$ near 0 with $G(p)$ in W and $G \pitchfork W$ at p. This theorem can be used to prove Local Theorem 2. See, for example, [11].

(B) Example (ii) demonstrates another useful fact about transverse intersections; namely, if $F \pitchfork W$ at 0, then $k \leqq \operatorname{codim} W$. The reason for this is simple: $\dim(T_xW + (DF)_0(\mathbf{R}^k)) \leqq \dim W + k$. So if $F \pitchfork W$ at 0, then $\dim V \leqq \dim W + k$ which implies that $\operatorname{codim} W \leqq k$.

The basic result in elementary catastrophe theory is the following:

THEOREM 2.1 (local stability for unfoldings). *Let F_t be an unfolding for the germ $f = F_0$. Then F_t is a (locally) stable unfolding of f iff $F \pitchfork \mathcal{O}_f$ at 0.*

Here again we reason by analogy with the finite dimensional case. An unfolding F_t may be thought of as a mapping $F: \mathbf{R}^k \to C_0^\infty(\mathbf{R}^n)$ defined by $F(t) = F_t$. Since $F(0) = F_0 = f$ and f is $\mathcal{O}_f$, it makes sense to ask whether or not $F \pitchfork \mathcal{O}_f$ at 0. The claim is that stability for an unfolding is equivalent to transverse intersection. We shall now proceed to give a "proof" of the sufficiency part of this theorem using the implicit function theorem. The only problem is that there is no implicit function theorem in this generality. Fortunately there is a way around these difficulties—through the notion of finite determinacy—which we shall discuss in § 6. To reiterate, our "proof" is heuristic and is given only as motivation.

The statement of the implicit function theorem that we shall use is:

THEOREM. *Let $\tilde{E}: \mathbf{R}^m \times \mathbf{R}^k \to \mathbf{R}^p$ be a C^∞ mapping with $\tilde{E}(0) = 0$. Denote the variables in $\mathbf{R}^m$ by x and the variables in $\mathbf{R}^k$ by t. If the matrix $(\partial \tilde{E}/\partial x)$ has rank p, then there is a C^∞ function $x: \mathbf{R}^k \to \mathbf{R}^m$ such that $x(0) = 0$ and $\tilde{E}(x(s), s) \equiv 0$ for all s near 0 in $\mathbf{R}^k$.*

COROLLARY. *Let $E: \mathbf{R}^m \to \mathbf{R}^p$ be a C^∞ map with $E(0) = 0$ and assume that $(DE)_0 = (\partial E/\partial x)_0$ is onto (i.e., has rank p). Let $G: \mathbf{R}^k \to \mathbf{R}^p$ be C^∞ with $G(0) = 0$. Then there is a C^∞ mapping $x: \mathbf{R}^k \to \mathbf{R}^m$ such that $E(x(s)) = G(s)$ for all s near 0 in $\mathbf{R}^k$.*

Proof. Define $\tilde{E}(x, s) = E(x) - G(s)$ and apply the implicit function theorem.

We denote by $\operatorname{Diff}_0(\mathbf{R}^n)$ the space of germs of diffeomorphisms which map $\mathbf{R}^n \to \mathbf{R}^n$. Now to proceed with the "proof" of Theorem 2.1, replace $\mathbf{R}^m$ by $\operatorname{Diff}_0(\mathbf{R}^n) \times \mathbf{R} \times \mathbf{R}^k$ and $\mathbf{R}^p$ by $C_0^\infty(\mathbf{R}^n)$ in the statement of the corollary. Given an unfolding F of f for which one wishes to determine stability, define $E: \operatorname{Diff}_0(\mathbf{R}^n) \times \mathbf{R} \times \mathbf{R}^k \to C_0^\infty(\mathbf{R}^n)$ by $E(H, K, t) = F_t \circ H + K$. Note that $E(H, K, 0) = f \circ H + K$ which is in $\mathcal{O}_f$. In fact as H and K vary with $t = 0$, E traces out all of $\mathcal{O}_f$ (by definition of $\mathcal{O}_f$). On the other hand if we take $H =$ identity map (i.e. $H(x) = x$) and $K = 0$, then $E(id, 0, t) = F_t$ so as t varies E traces out the image of the unfolding F. So we see that $(DE)_{(id,0,0)}$ is onto is equivalent to saying the $T_f\mathcal{O}_f + (DF)_0(\mathbf{R}^k) = T_fC_0(\mathbf{R}^n)$. Thus $(DE)_{(id,0,0)}$ is onto iff $F \pitchfork \mathcal{O}_f$ at 0. Next assume that the corollary to the implicit function theorem above holds for these infinite dimensional spaces and let $G: \mathbf{R}^k \to C_0^\infty(\mathbf{R}^n)$ be another unfolding of f near F. We need to show that G is equivalent to F. Here we assume that $F \pitchfork \mathcal{O}_f$ at 0. Then by the corollary there is a smooth map of $\mathbf{R}^k \to \operatorname{Diff}_0(\mathbf{R}^n) \times \mathbf{R} \times \mathbf{R}^k$ defined by $s \mapsto (H_s, K_s, s(t))$ such that $E(H_s, K_s, t(s)) = G(s)$. By definition $E(H_s, K_s, t(s)) = F_{t(s)} \circ H_s + K_s$. So $G(s) = F_{t(s)} \circ H_s + K_s$ and G is equivalent to F as unfoldings of f if we can show that the change of parameters $s \mapsto t(s)$ is invertible; this follows rather easily from the fact that G was assumed to be close to F, so t is near the identity map on $\mathbf{R}^k$.

We shall now investigate certain implications of Theorem 2.1.

COROLLARY 2.2. *There is a stable unfolding F of f iff f has finite codimension.*

Proof. Just consider note (B) above.

This corollary is analogous to Local Theorem 2 about Morse functions; namely, we now have a reasonably computable method for determining the existence of a stable unfolding for a given germ f. What is needed next is a method for finding this stable unfolding when it is known to exist. This will be analogous to Local Theorem 1 for Morse functions.

THEOREM 2.3 (normal form theorem for unfoldings). *Let f have codimension k. Let $p_1(x), \cdots, p_k(x)$ in $C_0^\infty(\mathbf{R}^n)$ project onto a basis of $C_0^\infty(\mathbf{R}^n)/\langle \partial f/\partial x\rangle$. Then*

$$F_t(x)=f(x)+t_1p_1(x)+\cdots+t_kp_k(x)$$

is a stable unfolding of t.

Proof. The fact that $p_1, \cdots, p_k$ project onto a basis just means geometrically that the k-plane spanned by $p_1, \cdots, p_k$ in $C_0^\infty(\mathbf{R}^n)$ (i.e., all germs of the form $t_1p_1(x)+\cdots+t_kp_k(x)$) is transverse to $\mathcal{O}_f$. This follows since the vector space $\langle \partial f/\partial x\rangle$ is just $T_f\mathcal{O}_f$. So by Theorem 2.1 the given unfolding F_t is stable.

Examples. 1) Let $f(x)=x^m$. As shown before $x, \cdots, x^{m-2}$ projects onto a basis of $C_0^\infty(\mathbf{R}^n)/\langle \partial f/\partial x\rangle$. So

$$F_t(x)=x^m+t_{m-2}x^{m-2}+\cdots+t_1x$$

is a stable unfolding of x^m.

2) In Example (3) after Taylor's theorem we showed that $f(x, y)=x^3+xy^2$ has codimension 3 and that x, y, x^2 project onto a basis in $C_0^\infty(\mathbf{R}^2)/\langle \partial f/\partial x\rangle$. Therefore $F_t(x)=x^3+xy^2+t_1x+t_2y+t_3x^2$ is a stable unfolding of f.

3) Let $f(x, y)=x^3+y^3$. Then $\langle \partial f/\partial x\rangle=\langle x^2, y^2\rangle$. To find a stable unfolding for f we must first compute the codimension of f. To do this we use Taylor's theorem. Note that x^2, y^2, xy form a basis for the homogeneous polynomials of degree 2. So if $m(x, y)$ is a C^∞ germ we may write $m(x, y)=m_0+m_1x+m_2y+a_1(x, y)x^2+a_2(x, y)y^2+a_3(x, y)xy$. In $C_0^\infty(\mathbf{R}^2)/\langle \partial f/\partial x\rangle$, m is equivalent to $m_1x+m_2y+a_3(x, y)xy$. Again by Taylor's theorem we may write $a_3(x, y)$ as $m_3+b_1(x, y)x+b_2(x, y)y$. Finally we have that $m(x, y)$ is equivalent to $m_1x+m_2y+m_3xy$ in $C_0^\infty(\mathbf{R}^2)/\langle \partial f/\partial x\rangle$ and $\operatorname{codim} f\leqq 3$.

Next we claim that x, y, xy project onto a basis. Suppose that $m_1x+m_2y+m_3xy=0$ in $C_0^\infty(\mathbf{R}^2)/\langle \partial f/\partial x\rangle$, that is, there are germs c and d such that

$$m_1x+m_2y+m_3xy=c(x, y)x^2+d(x, y)y^2. \tag{2.1}$$

The right hand side of (2.1) clearly never has nonzero x, y, or xy terms in its Taylor expansion. Thus $m_1=m_2=m_3=0$.

Putting this information together we see that $\operatorname{codim} f=3$ and $F_t(x, y)=x^3+y^3+t_1x+t_2y+t_3xy$ is a stable unfolding of f.

Exercise. Show that a stable unfolding for the germ $f(x, y)=x^2y+y^4$ is $F_t(x, y)=x^2y+y^4+t_1x+t_2y+t_3x^2+t_4y^2$.

For our next corollary to Theorem 2.1 we note that stable unfoldings satisfy a universality property:

THEOREM 2.4. *Let F_t be a stable unfolding of the germ f in $C_0^\infty(\mathbf{R}^n)$ with k parameters. Let G_s be any other unfolding of f with m parameters. Then there exists a smooth change of parameters $t=t(s)$, an m-parameter family of diffeomorphisms $H_s\colon (\mathbf{R}^n, 0)\to\mathbf{R}^n$, and an m-parameter set of constants K_s such that*

$$G_s(x)=F_{t(s)}\circ H_s(x)+K_s.$$

(Note: H and K depend smoothly on the parameters s.)

Proof. This theorem is proved exactly as Theorem 2.1. Here we are not assuming that G is near F so we cannot conclude that $t=t(s)$ is an invertible change of parameters. Since we do not assume that $m=k$ this should come as no surprise.

Example. Let $f(x)=x^4$. As was shown in Example 1) a stable unfolding for f is $F_t(x)=x^4+t_1x+t_2x^2$. Let $G_s(x)$ be another unfolding of f, for instance $x^4+s_1\sin(s_2x)+s_3^2x^2$ which depends on three parameters. Theorem 2.4 guarantees the existence of mappings $H_s(x)=\bar{x}(x, s)$, $K(s)$, and $t=t(s)$ all smooth in x and s such that

$x^4+s_1 \sin(s_2x)+s_3^2x^2=\bar{x}(x,s)^4+t_1(s)\bar{x}(x,s)+t_2(s)\bar{x}(x,s)^2+K(s)$. Since H_s is a diffeomorphism, we also have that $(\partial\bar{x}/\partial x)(x,s)\neq 0$ for any point (x,s) near 0.

So far in this section, we have concentrated on the problem "Given a germ f in $C_0^\infty(\mathbf{R}^n)$, find a stable unfolding F of f." The reverse question is also of interest. Given an unfolding $F_t(x)$ of the germ $f=F_0$, is F stable? From Theorem 2.1 the answer is straight-forward; F is stable if $(DF)_0(\mathbf{R}^k)+T_f\mathcal{O}_f=C_0^\infty(\mathbf{R}^n)$. We need only interpret what $(DF)_0(\mathbf{R}^k)$ is. Let $t_i(s)$ be the k-vector whose ith component is s and whose other components are zero. Then $F_{t_i(s)}$ is a curve in $C_0^\infty(\mathbf{R}^n)$ and $(d/ds)F_{t_i(s)}|_{s=0}$ is a tangent vector in $(DF)_0(\mathbf{R}^k)$. The chain rule shows that this vector is just $(\partial F/\partial t_i)(x)|_{t=0}$. Clearly these k vectors (for $1\leqq i\leqq k$) generate $(DF)_0(\mathbf{R}^k)$. So we have the following:

THEOREM (infinitesimal stability implies stability). *Given a k-parameter unfolding F_t of $f=F_0$. F is stable iff $(\partial F/\partial t_1)(x,0),\cdots,(\partial F/\partial t_k)(x,0)$ project onto a generating set for the vector space $C_0^\infty(\mathbf{R}^n)/\langle\partial f/\partial x\rangle$.*

Example. Is

$$F_t(x,y)=x^3+t_1(x+4xy^2)+\cos(t_2)y^3+\sin(t_2)xy+t_3(y+2y^3)$$

a stable unfolding of $F_0(x,y)=x^3+y^3$? To determine this we compute

$$p_1(x,y)=\frac{\partial F_t}{\partial t_1}(x,y,0)=x+4xy^2, \qquad p_2(x,y)=\frac{\partial F_t}{\partial t_2}(x,y,0)=xy$$

and

$$p_3(x,y)=\frac{\partial F_t}{\partial t_3}(x,y,0)=y+2y^3.$$

In $C_0^\infty(\mathbf{R}^2)/\langle x^2,y^2\rangle$, p_1 is equivalent to x and p_3 is equivalent to y. As was shown in Example 3) x, y, and xy project onto a set of generators of $C_0^\infty(\mathbf{R}^2)/\langle x^2,y^2\rangle$, so F_t is a stable unfolding of x^3+y^3.

A few comments about the actual proof of Theorem 2.1 are in order. Mather's original proof used two basic theorems from analysis: the Thom transversality theorem and the Malgrange preparation theorem. Proofs of these theorems can be found in [11]. These theorems were used in conjunction with a reduction of the problem to finite dimensional manifold theory. This reduction uses the notion of finite determinacy which we shall describe in § 6. Detailed descriptions of Mather's proof may be found in [5], [36], and [35]. A nice proof of Theorem 2.4 in the case of $f(x)=x^4$ (which avoids introducing advanced algebraic structures) is given in [26].

3. The classification theorem. The question we ask in this section is "How many different stable unfoldings are there with four or fewer parameters (control variables)?" A reasonable question is "Why four?" There are two answers. First, in some applications of catastrophe theory we think of the parameters as control variables for an experiment; in some cases these control variables are just space and time. Second, this is the number that Thom considered.

As we shall note later four may be replaced by five to obtain a similar answer but not by six or more.

The universality theorem, Theorem 2.4, states that two stable unfoldings of the same germ are for all purposes the same. So we may as well compute only a stable unfolding for each germ f with the fewest number of parameters, that is, an unfolding with codim f parameters. (This is the setting of Theorem 2.3.) Finally suppose that two functions f and f' are right equivalent, i.e. $f'=f\circ H+K$. If F_t is a stable unfolding of f, then it is easy to find a stable unfolding F_t' for f'; namely, let $F_t' = F_t \circ H$. For the purpose of classifying stable unfoldings we shall consider F_t and F_t' equivalent. With these observations the problem of classifying stable unfoldings with four or fewer parameters reduces to the problem of classifying all inequivalent germs in $C_0^\infty(\mathbf{R}^n)$ which have codimension $\leqq 4$. A main tool is the following:

RELATIVE MORSE LEMMA (Gromoll–Meyer [13]). *Let f be in $C_0^\infty(\mathbf{R}^n)$ and have a singularity at* 0. *Suppose the rank of $(d^2f)(0)$ is $n-l$. Then it is possible to choose coordinates $x_1, \cdots, x_l$ and $y_1, \cdots, y_{n-l}$ on $\mathbf{R}^n$ at* 0 *and a germ f_1: $(\mathbf{R}^l, 0) \to \mathbf{R}$ such that*

$$f(x_1, \cdots, x_l, y_1, \cdots, y_{n-l}) = f_1(x_1, \cdots, x_l) - (y_1^2 + \cdots + y_k^2) + (y_{k+1}^2 + \cdots + y_{n-l}^2)$$

where $(df_1)(0) = (d^2f_1)(0) = 0$.

The number k is again the index of the Hessian matrix at 0 viewed as a singular symmetric bilinear form.

Notes. 1) This implies that $\operatorname{codim} f = \operatorname{codim} f_1$ since

$$C_0^\infty(\mathbf{R}^n) \Big/ \overline{\left\langle \frac{\partial f}{\partial x} \right\rangle} \cong C^\infty(\mathbf{R}^l) \Big/ \overline{\left\langle \frac{\partial f_1}{\partial x} \right\rangle}.$$

2) In so far as our classification of germs is considered we shall assume that the germs f and f_1 are equivalent. This gives us a way to compare germs defined on different numbers of variables. In terms of the singularities associated to these germs this process makes good sense—we are lumping all singularities which differ by a nonsingular quadratic term together into one class.

3) The full power of the Relative Morse Lemma is in an infinite dimensional setting. See [13].

THEOREM 3.1 (Thom's seven elementary catastrophes). *There are precisely seven stable, universal unfoldings with $\leqq 4$ control parameters. They are the unfoldings given in Table* 1.

TABLE 1

germ	*codimension*	*Name of Catastrophe*
x^3	1	Fold
$\pm x^4$	2	Cusp or Riemann–Hugoniot
x^5	3	Swallow's tail
$\pm x^6$	4	Butterfly
$x^3 + xy^2$ or $x^3 + y^3$	3	Hyperbolic umbilic
$x^3 - xy^2$	3	Elliptic umbilic
$\pm(x^2y + y^4)$	4	Parabolic umbilic

Note. The reader may have noticed that there are in fact ten germs in this list if one distinguishes between the + and the − signs before x^4, x^6, and $x^2y + y^4$. In Thom's classification these singularities are considered to be equivalent. On the other hand these signs do make a difference if one is trying to minimize each member F_t of an unfolding rather than just find the singularities. Clearly finding minima of germs near x^4 is different from finding minima of germs near $-x^4$. In fact the stable unfolding of $-x^4$ is called the dual cusp catastrophe. More will be said in § 5. Similarly for the others.

One main step in the proof of this theorem is the use of the Relative Morse Lemma.

Claim. If $f(x, y, z)$ has a singularity at 0 with $(d^2f)(0) = 0$, then $\operatorname{codim} f \geqq 6$.

To see this, note that the partial derivatives of f begin with terms of degree 2 or higher. Let $\bar{f}_x, \bar{f}_y, \bar{f}_z$ be the terms in the Taylor expansion of f_x, f_y, f_z of degree 2.

Then

$$\begin{aligned}
\operatorname{codim} f &= \dim_{\mathbf{R}} C_0^\infty(\mathbf{R}^3) \Big/ \overline{\left\langle \frac{\partial f}{\partial x} \right\rangle} \\
&\geqq \dim_{\mathbf{R}} C_0^\infty(\mathbf{R}^3)/(\langle f_x, f_y, f_z, \text{all homogeneous polynomials of degree } 3\rangle + K) \\
&\geqq \dim_{\mathbf{R}} (\text{polynomials of degree } 2)/(\langle \bar{f}_x, \bar{f}_y, \bar{f}_z\rangle + K) \\
&\geqq 10 - 4 = 6.
\end{aligned}$$

So if $\operatorname{codim} f \leqq 5$, we can assume—using the Relative Morse Lemma—that f is a function of either 1 or 2 variables. In the previous examples we completed the case of 1 variable. More work is necessary to complete the classification when $f = f(x, y)$. See § 6.

Notes. 1) A similar classification can be carried out when $\operatorname{codim} f \leqq 6$. In codimension 5, four new singularities and thus four new stable unfoldings make their appearance (Table 2). See [4], [28], or [36].

TABLE 2

germ	*stable unfolding*
x^7	$x^7 + t_5x^5 + t_4x^4 + t_3x^3 + t_2x^2 + t_1x$
$x^2y \pm y^5$	$x^2y \pm y^5 + t_1x + t_2x^2 + t_3y + t_4y^2 + t_5y^3$
x^3y^4	$x^3 + y^4 + t_1x + t_2y + t_3y^2 + t^4xy + t_5xy^2$

2) In codimension 6 three new singularities are found. (See [4], [28], or [21].) In codimensions $\geqq 7$ an infinite number of new singularities make their appearance. We shall discuss how this happens in the next section along with an explanation of why this implies that there are an infinite number of inequivalent stable unfoldings with 6 parameters.

4. Density and moduli. In this section we shall describe the global theorem on unfoldings which was promised before.

A globally defined, C^∞, k parameter family of C^∞ functions of $\mathbf{R}^n \to \mathbf{R}$ is a C^∞ function $F: \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$ defined on all of $\mathbf{R}^n \times \mathbf{R}^k$. As usual we think of $F(x, t)$ as $F_t(x)$. Let $C^\infty(\mathbf{R}^n; \mathbf{R}^k)$ denote the space of all such unfoldings F.

DEFINITION. Let F_t be in $C^\infty(\mathbf{R}^n; \mathbf{R}^k)$ with x_0 in $\mathbf{R}^n$ and t_0 in $\mathbf{R}^k$. On a small neighborhood of (x_0, t_0) in $\mathbf{R}^n \times \mathbf{R}^k$, F defines a germ of an unfolding of F_{t_0} at x_0. F_t is *locally stable at* (x_0, t_0) if the germ F_t is a stable unfolding of F_{t_0} at x_0 in the sense described in the previous sections.

F_t is *locally stable* if it is locally stable at all points (x, t_0) in $\mathbf{R}^n \times \mathbf{R}^k$.

THEOREM 4.1. *The subset S of $C^\infty(\mathbf{R}^n; \mathbf{R}^k)$ consisting of locally stable unfoldings is both open and dense if $k \leqq 5$.*

Notes. 1) As in the case of the Global Theorem about Morse functions the fact that S is open follows essentially from the local theory (Theorem 2.1). What is new is the density statement and the fact that some restriction on the number of parameters k is necessary.

2) The topology used on $C^\infty(\mathbf{R}^n; \mathbf{R}^k)$ is again the Whitney C^∞ topology. Also one may replace both $\mathbf{R}^n$ and $\mathbf{R}^k$ by manifolds X^n and T^k respectively as long as $\dim T^k = k \leqq 5$.

At this point, it is worthwhile for the reader to review the discussion after the Global Morse Theorem in the Introduction.

One should keep in mind the goals of our analysis of unfoldings. They are to find a complete list—preferably finite—of the (locally) stable unfoldings *and* to show that the (locally) stable unfoldings are generic. The first few sections dealt with the former; now we comment on the latter. It is this genericity or density of the stable unfoldings which makes the whole project (potentially) useful. This theorem states that if we are given a parametrized family of mappings (with less than six parameters) then for all practical purposes we can assume that locally it looks like one of the members on our list.

The proof of this theorem works because there is a finite classification of locally stable unfoldings when $k \leqq 5$. (The main tool is the Thom transversality theorem. See [11].) When $k \geqq 6$ this theorem is no longer true; the reason is the appearance of moduli—a one parameter family of inequivalent singularities.

Conceptually the simplest case of moduli is the following: Let $f_m\colon (\mathbf{R}^2, 0) \to \mathbf{R}$ be the germs defined by

$$f_m(x, y) = xy(x+y)(x-my) \quad \text{for } m > 0.$$

Claim. f_{m_1} is not equivalent to f_{m_2} for $m_1 \neq m_2$.

To see this we need the concept of the cross-ratio.

Let L_1, L_2, L_3, L_4 be lines passing through the origin in the xy-plane. Let M_i $(1 \leqq i \leqq 4)$ be the slope of line L_i. Then the cross-ratio is

$$CR_L = \frac{(M_1 - M_2)(M_3 - M_4)}{(M_1 - M_3)(M_2 - M_4)}.$$

Let A be any nonsingular linear transformation of $\mathbf{R}^2 \to \mathbf{R}^2$. Four new lines $A(L_i)$ are obtained in $\mathbf{R}^2$. It is an easy calculation to check that

$$CR_{A(L)} = CR_L$$

so that the cross-ratio is invariant.

Now to return to the germs f_m. The zero set, $f_m^{-1}(0)$, of f_m consists of four lines which yield four lines in $T_0\mathbf{R}^2$ and a cross-ratio. Should f_{m_1} be equivalent to f_{m_2}, then there would exist a germ of a diffeomorphism $h\colon(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ with $f_{m_1} = f_{m_2} \circ h$. Clearly $h\colon f_{m_1}^{-1}(0) \to f_{m_2}^{-1}(0)$, so the linear map $(dh)(0)\colon T_0\mathbf{R}^2 \to T_0\mathbf{R}^2$ would take the four lines defined by f_{m_1} to the four lines defined by f_{m_2}. Thus if f_{m_1} is equivalent to f_{m_2}, then the two cross-ratios defined would have to be equal. This is obviously not true if $m_1 \neq m_2$.

What we have constructed is a l-parameter family of inequivalent germs called *moduli.* The codimension of each germ with $m \neq 0$ happens to be 8. Thus we know that the finite classification of stable unfoldings breaks down when the number of control variables is $\geqq 8$. In fact, moduli begin to appear in codimension 7. An example is given by

$$f_m(x, y, z) = x^3 + y^3 + z^3 - mxyz.$$

See [28].

Next, we give a picture of why the existence of moduli of codimension 7 insures that locally stable unfoldings are not dense in $C^\infty(\mathbf{R}^n, \mathbf{R}^k)$ when $k \geqq 6$. Let $f_m\colon (\mathbf{R}^n, 0) \to \mathbf{R}$ be an example of moduli where each f_m has codimension 7, that is codim $\mathcal{O}_{f_m}$ in $C_0^\infty(\mathbf{R}^n)$ is 7.

Let $\mathcal{O}_M = \bigcup_m \mathcal{O}_{f_m}$. These strata $\mathcal{O}_{f_m}$ fit together to give a submanifold $\mathcal{O}_M$ of codimension 6. See Fig. 2. Let F_t be an unfolding on 6 parameters such that F intersects $\mathcal{O}_M$ transversely at the point $F_0 = f$. See Fig. 3. F_t is not locally stable at $t = 0$. For if F_t

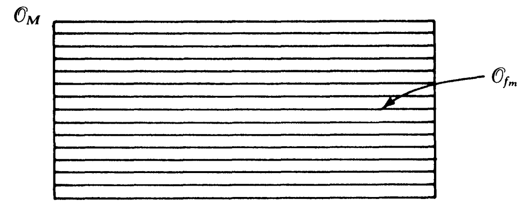

FIG. 2

were stable, then $F \pitchfork \mathcal{O}_f$ which cannot happen since dim Image $F = 6 < 7 = \operatorname{codim} \mathcal{O}_f$. (See Note (B) in § 2.) On the other hand, any unfolding near F will intersect $\mathcal{O}_M$. (See Note (A) in § 2.) So there is a neighborhood of F which consists of unfoldings which are not locally stable. Thus the density theorem is false for $k \geqq 6$.

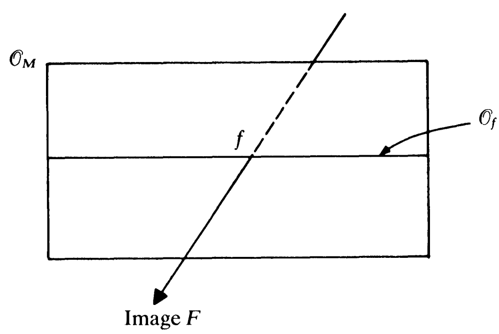

FIG. 3

Finally note that this does not contradict Corollary 2.2. For there still does exist a stable unfolding of f on 7-parameters whose image can be pictured as in Fig. 4. For this F, $F \pitchfork \mathcal{O}_f$ and is hence a stable unfolding.

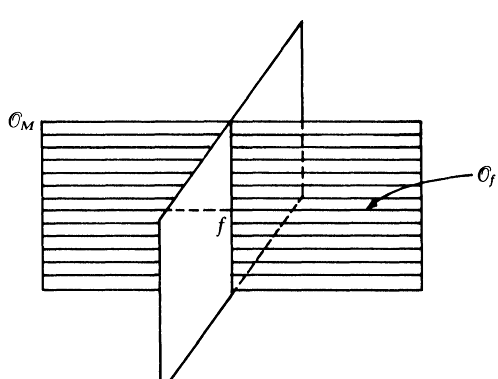

FIG. 4

5. The geometry of the cusp catastrophe. In the next part, we shall describe some applications of catastrophe theory which use the geometry of the cusp or Riemann–Hugoniot catastrophe explicitly; so we present this geometry now.

First, the names of the elementary catastrophes are derived in the following fashion: Let f: $(\mathbf{R}^n, 0) \to \mathbf{R}$ be a germ of finite codimension with stable unfolding F_t in $C^\infty(\mathbf{R}^n; \mathbf{R}^k)$. The catastrophe surface is the union of all singular points of members of the family; that is,

$$S_f = \left[(x, t) \in \mathbf{R}^n \times \mathbf{R}^k \,\middle|\, \frac{\partial F_t}{\partial x_1}(x) = \cdots = \frac{\partial F_t}{\partial x_n}(x) = 0 \right].$$

For stable unfoldings S_f is always a k-dimensional manifold in $\mathbf{R}^n \times \mathbf{R}^k$. (See [36].) Let π: $\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ be projection onto the control space, i.e., $\pi(x, t) = t$. Let $\pi_f = \pi | S_f$. π_f is called the *catastrophe map*; the names for the elementary catastrophes are derived from the geometric form of π_f—although some imagination is helpful!

For example, let $f(x) = x^3/3$. Then a stable unfolding is $F_t(x) = x^3/3 + tx$. So $S_f = ((x, -x^2) \in \mathbf{R} \times \mathbf{R})$ π_f is the projection of the parabola S_f into the t-axis. The singularity of π_f at 0 is called the *fold singularity*; the unfolding of $x^3/3$ is also called the *fold catastrophe*. See Fig. 5.

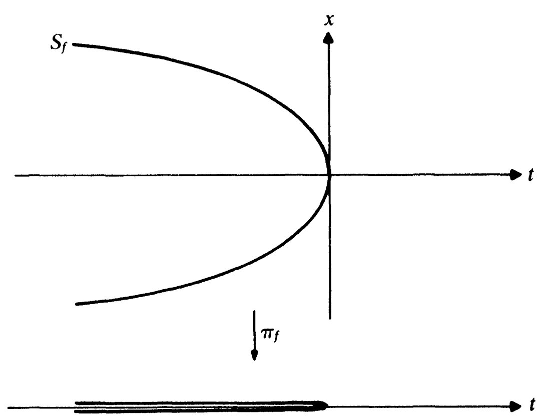

FIG. 5

In the case of the cusp catastrophe $f(x) = x^4/4$, the stable unfolding is $F_{\alpha,\beta}(x) = x^4/4 - \beta x^2/2 + \alpha x$. So $S_f = \{x^3 - \beta x + \alpha = 0\}$. To graph S_f in $\alpha\beta x$ space, note that for $\beta = 0$, $\alpha = -x^3$. When $\beta > 0$, $\alpha = \beta x - x^3$ has two singularities and when $\beta < 0$ none. See

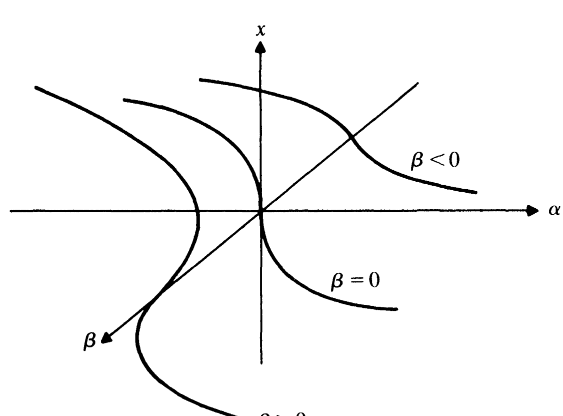

FIG. 6

Fig. 6. The surface of singularities S_f is then as in Fig. 7. The projection π_f into $\alpha\beta$ space has singularities which project to the cusp curve $\beta^3 = \frac{27}{4}\alpha^2$. Thus the unfolding of $f(x) = x^4$ is called the *cusp catastrophe*.

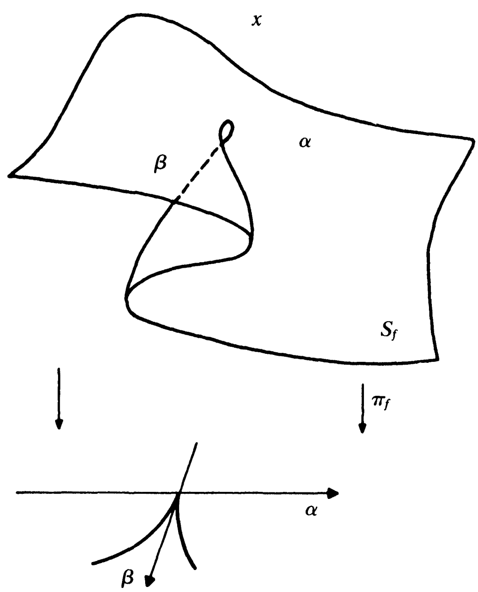

FIG. 7

We now give a more detailed description of this catastrophe. For (α, β) in the interior of the cusp $\pi_f^{-1}(\alpha, \beta)$ has three points and $F_{\alpha,\beta}(x)$ has three singularities—two local minima and one local maximum. For (α, β) outside the cusp, $\pi_f^{-1}(\alpha, \beta)$ consists of one point and $F_{\alpha,\beta}(x)$ has one singularity—a minimum. For (α, β) on the cusp, $\pi_f^{-1}(\alpha, \beta)$ consists of two points and $F_{\alpha,\beta}$ has two singularities—a minimum and an inflection.

Consider the points $A_i = (i, 3)$, $-3 \leqq i \leqq 3$ (as in Fig. 8) and the graphs $y = F_{A_i}(x)$ (as in Fig. 9).

So traveling in control space along the line $\beta = 3$, the family of functions goes through the following stages:

(−3) one relative minimum

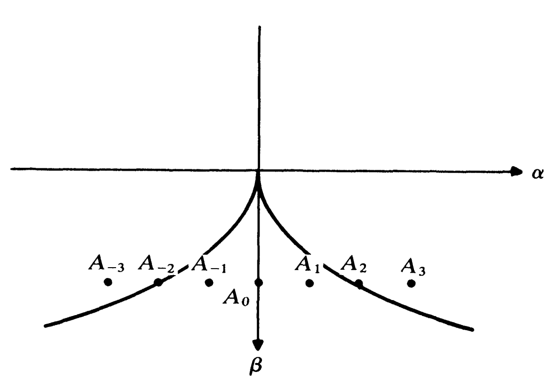

FIG. 8

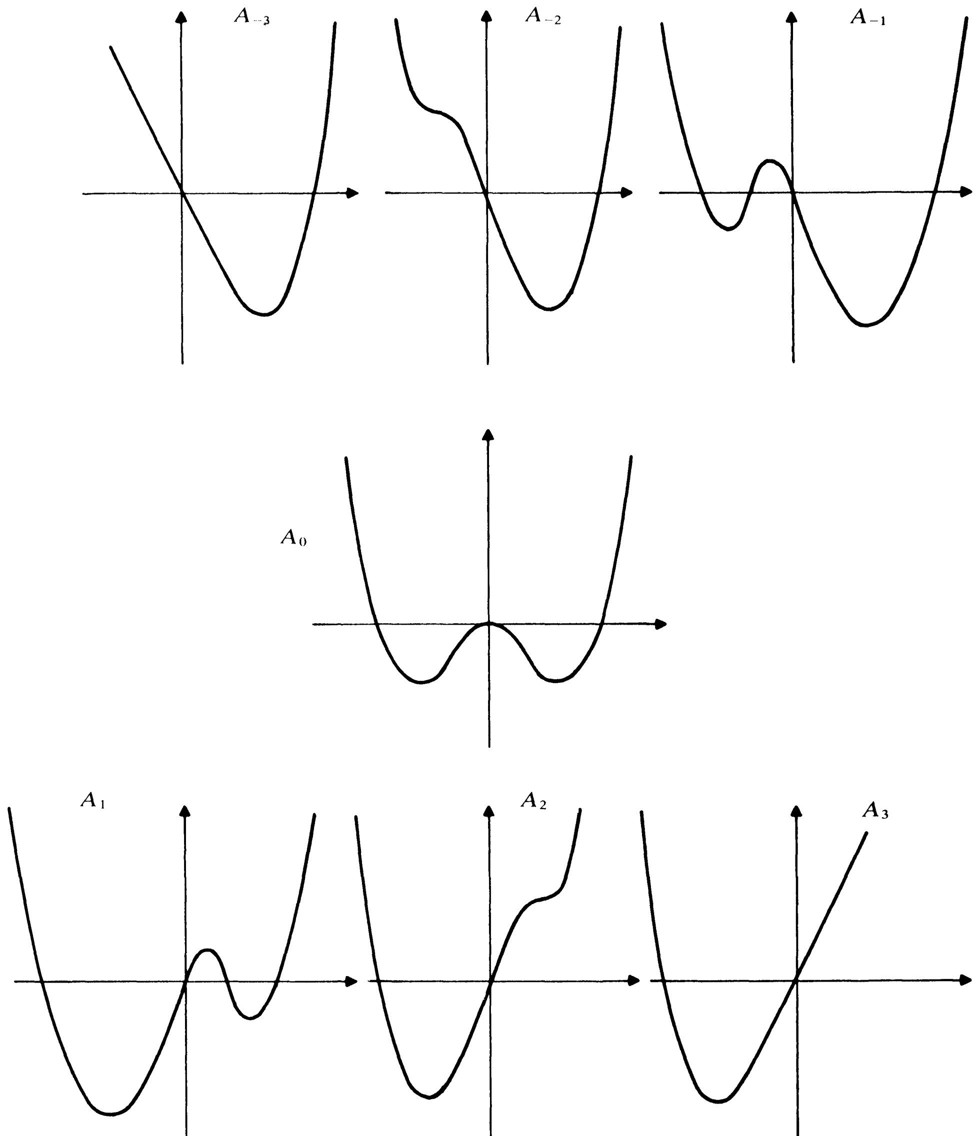

FIG. 9

(−2) formation of a new singular point
(−1) formation of a second relative minimum
(0) a position where both relative minima have equal values
(1) the new relative minimum becoming the absolute minimum
(2) the first relative minimum dying
(3) a return to one relative minimum.

Note. The unfolding of $-x^4$ is $F_t(x) = -x^4 - \beta x^2 + \alpha x$ which is the dual cusp catastrophe. If we try to find the minima of each F_t we see that for controls (α, β) outside of the cusp there are no minima while inside there is precisely one minimum. In this sense the qualitative behavior of the cusp and the dual cusp catastrophes are quite different. Similarly for the butterfly.

Finally we observe that if one is attempting to find the absolute minimum of $F_{\alpha,\beta}$, then there is a problem at the points where the relative minimum values coalesce. (See point A_0.) The *Maxwell set* of a catastrophe is the set of control points where the corresponding function has (at least) two distinct relative minima with the same minimizing value. In the case of the cusp catastrophe, the Maxwell set is the ray $\alpha = 0$, $\beta > 0$. Thus for any equivalent unfolding the Maxwell set is a smooth curve emanating from the origin.

A more complete description of minimization for catastrophes is given in [7]. The cusp picture had previously been discovered by Whitney [52] as a stable singularity of $\mathbf{R}^2 \to \mathbf{R}^2$. See [11] or [52].

6. Finite determinacy. So far our description of elementary catastrophe theory has been given in terms of infinite dimensional manifolds; e.g. $C_0^\infty(\mathbf{R}^n)$ and $\mathcal{O}_f$. For the theorems described here to be proved in the fashion outlined, one needs some form of an inverse function theorem. The way that these theorems have actually been proved is through the concept of finite determinacy.

DEFINITION 6.1. Let $f: (\mathbf{R}^n, 0) \to \mathbf{R}$ be a C^∞ germ. f is *finitely determined* if there is an integer l such that if g is also in $C_0^\infty(\mathbf{R}^n)$ and the Taylor series of g agrees with that of f to lth order at 0, then g is right equivalent to f. Such a germ is called *l-determined.*

The main result is due to Mather [20]; the proof that we give is taken from [5].

THEOREM 6.2. *A germ f in $C_0^\infty(\mathbf{R}^n)$ is finitely determined iff f has finite codimension.*

Note. The equivalency of the notions of finite codimension and finite determinacy plays a central role in the development of catastrophe theory. Because of this we shall give a complete proof that finite codimension implies finitely determined as well as the basic idea for the reverse implication.

Proof (sufficiency). Assume that the germ f has finite codimension. Let $\mathcal{M}$ denote the maximal ideal in $C_0^\infty(\mathbf{R}^n)$. (The reader may want to refer to the Note after Taylor's theorem, § 1.) We claim that f has finite codimension iff there is an s such that $\mathcal{M}^s \subset \langle \partial f/\partial x \rangle$. For suppose that $\mathcal{M}^s \not\subset \langle \partial f/\partial x \rangle$ for all s. Then choose by induction homogeneous polynomials p_s of degree s so that p_s is not in $\mathcal{M}^s + \text{span}\,(p_1, \cdots, p_{s-1})$. The set $p_1, p_2, \cdots$ is independent over $\mathbf{R}$ in $C_0^\infty(\mathbf{R}^n)/\langle \partial f/\partial x \rangle$, and $\operatorname{codim} f = \infty$. So we may assume that $\mathcal{M}^{s-1} \subset \langle \partial f/\partial x \rangle$ and show that f is s-determined.

If A and B are subspaces of $C_0^\infty(\mathbf{R}^n)$, then AB denotes the subspace consisting of all germs of the form $a_1b_1 + \cdots + a_mb_m$ where each a_i is in A and each b_i is in B. It is not hard to show that $\mathcal{M}\langle \partial f/\partial x \rangle$ consists of all germs of the form $a_1(\partial f/\partial x_1) + \cdots + a_n(\partial f/\partial x_n)$ where $a_i(0) = 0$ for all i and that $\mathcal{M}^k\mathcal{M}^s = \mathcal{M}^{k+s}$. So our assumption that f has finite codimension implies that $\mathcal{M}^s \subset \mathcal{M}\langle \partial f/\partial x \rangle$.

LEMMA 6.3. *If $\mathcal{M}^s \subset \mathcal{M}\langle \partial f/\partial x \rangle$, then f is s-determined.*

Proof. Let g be a germ whose Taylor expansion at 0 agrees with the Taylor expansion of f at 0 to order s, then $g - f$ is in $\mathcal{M}^{s+1}$. Let $f_t = f + t(g - f)$. We will show that there is a one parameter family of diffeomorphisms $\gamma_t: \mathbf{R}^n \to \mathbf{R}^n$ varying smoothly in x and t which satisfy for $0 \leqq t \leqq 1$,

(i) $\gamma_t(0) = 0$, and

(ii) $f_t \circ \gamma_t = f$ on some neighborhood of 0 in $\mathbf{R}^n$. Clearly this will prove the theorem by evaluating at $t = 1$.

The following trick first used by Moser [23] is an embryonic form of "infinitesimal stability implies stability." Differentiate (ii) with respect to t to obtain

$$\frac{\partial f_t}{\partial t}(\gamma_t(x)) + \sum_{i=1}^{n} \frac{\partial f_t}{\partial x_i}(\gamma_t(x)) \frac{\partial \gamma_t^i}{\partial t}(x) = 0, \tag{6.1}$$

where $\gamma_t = (\gamma_t^1, \cdots, \gamma_t^n)$ in local coordinates. Next evaluate at $\gamma_t^{-1}(x)$ to obtain

$$(g-f)(x) + \sum_{i=1}^{n} \frac{\partial f_t}{\partial x_i}(x) \frac{\partial \gamma_t^i}{\partial t}(\gamma_t^{-1}(x)) = 0. \tag{6.2}$$

Suppose we can find functions $v_i(x, t)$ satisfying

(a) $v_i(0, t) = 0$, and

(b) $(g-f)(x) + \sum_{i=1}^{n} \frac{\partial f_t}{\partial x_i}(x) v_i(x, t) = 0.$

Then we can solve the ODE's

$$\frac{\partial \gamma_t^i}{\partial t}(x) = v_i(\gamma_t(x), t)$$

for the desired diffeomorphism γ_t. Note (a) implies that $\gamma_t(0) = 0$. With this γ_t (6.1) can be integrated to $t = 1$ on some small neighborhood of 0 to prove the theorem. Thus we have reduced the problem to finding the v_i's.

Let $q_1, \cdots, q_N$ be a basis for the polynomials homogeneous of degree s. Since $g-f$ is in $\mathcal{M}^{s+1}$ we can write $(g-f)(x) = \sum_{i=1}^{N} a_r(x) q_r(x)$ with $a_r(0) = 0$. We will show that each q_r may be written as a linear combination of the functions $(\partial f_t/\partial x_i)(x)$. Then the v_i's needed to solve (a) and (b) are easily constructed. Since q_r is in $\mathcal{M}^s$ and $\mathcal{M}^s \subset \mathcal{M}\langle \partial f/\partial x\rangle$ we can write

$$q_r(x) = \sum_{i=1}^{n} b_{r,i}(x) \frac{\partial f}{\partial x_i}(x) = \sum_{i=1}^{n} b_{r,i}(x) \left\{ \frac{\partial f_t}{\partial x_i}(x) - t \frac{\partial}{\partial x_i}(g-f) \right\}$$

where $b_{r,i}(0) = 0$. Since $g-f$ is in $\mathcal{M}^{s+1}$, $(\partial/\partial x_i)(g-f)$ is in $\mathcal{M}^s$ and is a linear combination of the q_r's. So we have

$$q_r(x) = \sum_{i=1}^{n} b_{r,i}(x) \frac{\partial f_t}{\partial x_i}(x) + t \sum_{\rho=1}^{N} c_{r,\rho}(x) q_\rho(x),$$

where $c_{r,\rho}(0) = 0$. Writing these N equations in matrix form yields

$$K(x, t) = (I - tC(x))Q,$$

where Q is the vector of q_r's, C is the matrix $(c_{r,\rho})$, and K is the vector with coordinates $\sum_{i=1}^{n} b_{r,i}(\partial f_t/\partial x_i)$. Since $C(0) = 0$ we can invert the matrix $I - tC$ for x near 0 and obtain in coordinates that each q_r is a linear combination of the functions $(\partial f_t/\partial x_i)(x)$ and the proof is complete.

COROLLARY 6.4. *If f in $C_0^\infty(\mathbf{R}^n)$ has codimension k, then f is l-determined for some $l \leqq k+2$.* (See [36, Lem. 3.1].)

Note. This corollary proves and generalizes Local Morse Theorem 1.

Proof. Assume that f is not $(k+2)$-determined. The last lemma implies that $\mathcal{M}^{k+1} \not\subset \langle \partial f/\partial x\rangle$. We use this fact to show that $f > k$. Clearly $\mathcal{M}^s \not\subset \langle \partial f/\partial x\rangle$ for $1 \leqq s \leqq k+1$ since $\mathcal{M}^{k+1} \subset \mathcal{M}^s$ for such s. Thus there is a polynomial p_s, homogeneous of degree s, which is not in $\langle \partial f/\partial x\rangle + \underline{\operatorname{span}}\,(p_1, \cdots, p_{s-1})$ for each $s \leqq k+1$. These polynomials are independent in $C_0^\infty(\mathbf{R}^n)/\langle \partial f/\partial x\rangle$, so codim $f \geqq k+1$.

We now give a sketch of the proof that finitely determined implies finite codimension. Suppose that f is k-determined and let g be any homogeneous polynomial of degree $k+1$. For each real number t let $f_t = f + tg$. Since the Taylor series for f_t agrees with the Taylor series for f through order k, we have that

$$f_t = f \circ H_t \tag{6.3}$$

for some germ of a diffeomorphism H_t: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$. Now suppose that H_t can be chosen to vary smoothly in t and that $H_0 = \text{identity}$. Then we could differentiate (6.3) with respect to t and obtain at $t = 0$,

$$g(x) = \frac{\partial f}{\partial x_1}(x)\frac{\partial H_t^1}{\partial t}(x) + \cdots + \frac{\partial f}{\partial x_n}(x)\frac{\partial H_t^n}{\partial t}(x)$$

where $H_t = (H_t^1, \cdots, H_t^n)$ in coordinates. Thus g is in $\langle \partial f/\partial x\rangle$ and since g was arbitrary we have $\mathcal{M}^{k+1} \subset \langle \partial f/\partial x\rangle$. As was shown above this implies that f has finite codimension.

There is indeed a technical problem in showing that H_t can be chosen to vary smoothly. To accomplish this task requires some knowledge about the actions of finite dimensional Lie groups. The reduction to finite dimensions is effected through the use of finite determinacy.

We now give some examples to illustrate the strength of Lemma 6.3.

Examples. 1) The germ $f(x, y) = x^2y + y^{k+1}$ is $(k+1)$-determined when $k \geqq 2$. Using Lemma 6.3 we must show that $\mathcal{M}^{k+1} \subset \mathcal{M}\langle \partial f/\partial x\rangle = \langle x, y\rangle\langle 2xy, x^2 + (k+1)y^k\rangle = \langle x^2, xy^2, x^3, y^{k+1}\rangle \equiv A$. To demonstrate this inclusion we need to show that x^iy^{k+1-i} is in A for all i between 0 and $k+1$. This is clearly true by inspection.

2) The map germ $f(x, y) = x^3 + xy^2$ is 3-determined. (We showed previously that $\operatorname{codim} f = 3$.) Since $\langle \partial f/\partial x\rangle = \langle 3x^2 + y^2, 2xy\rangle$, $\mathcal{M}\langle \partial f/\partial x\rangle = \langle 3x^3 + xy^2,\ 3x^2y + y^3,\ 2x^2y,\ 2xy^2\rangle$ which is easily seen to be $\mathcal{M}^3$. So Lemma 6.3 implies that f is 3-determined. The same calculation shows that $x^3 - xy^2$ is also 3-determined.

We can now sketch a proof of Thom's classification theorem, Theorem 3.1. Our proof follows [5] and [35]. From the preliminaries of § 3, we may assume that $f = f(x, y)$ has codimension $\leqq 4$ and that the Taylor expansion of f at 0 begins with terms of order 3. Let $P(x, y)$ be the terms in the Taylor expansion of f at 0 homogeneous of order 3. If $P \equiv 0$, then $\operatorname{codim} f \geqq 7$. So $P \not\equiv 0$. Note that $P(x, y) = y^3Q(x/y)$ where Q is a polynomial of degree 3 in one variable. We may factor Q over the complex numbers to obtain

$$P(x, y) = (a_1x + b_1y)(a_2x + b_2y)(a_3x + b_3y).$$

Since Q has odd degree it must have at least one real root, so we assume that a_1 and b_1 are real. There are four cases:

(I) The vectors (a_i, b_i) are pairwise independent and real,
(II) $P(x, y) = (a_1x + b_1y)(a_2x + b_2y)^2$ with (a_1, b_1) and (a_2, b_2) independent,
(III) $P(x, y) = (a_1x + b_1y)^3$,
(IV) $P(x, y) = (a_1x + b_1y)(a_2x + b_2y)(\bar{a}_2x + \bar{b}_2y)$ with (a_2, b_2) not real. One then shows, by linear changes of coordinates, that $P(x, y)$ has one of the following forms:

(I) $P(x, y) = x(x - y)(x + y) = x^3 - xy^2$,
(II) $P(x, y) = x^2y$,
(III) $P(x, y) = x^3$,
(IV) $P(x, y) = x(x^2 + y^2) = x^3 + xy^2$.

Since (I) and (IV) are 3-determined, f is equivalent to P by a change of coordinates in these cases. A calculation shows that if P satisfies (III), then $\operatorname{codim} f \geqq 5$. So we are left with the hardest case (II). Since x^2y does not have finite codimension, it is not finitely determined. So there is a largest k for which f is equivalent by a change of coordinates to x^2y through order k. Suppose $f \circ \gamma$ has the same Taylor expansion at 0 as x^2y through order k. Then one constructs a change of coordinates μ such that $f \circ \mu$ has the same Taylor expansion as $x^2y \pm y^{k+1}$ through order $k+1$. Example 1) shows that $x^2y \pm y^{k+1}$ is $(k+1)$-determined. So f is equivalent to this germ. Next one checks that the codimension of $x^2y \pm y^{k+1}$ is >4 if $k>3$. So the only possibility is $x^2y \pm y^4$ which is equivalent to $\pm(x^2y+y^4)$.

PART B—THE APPLICATIONS

In this part, we shall sketch several applications of catastrophe theory. Our purpose will be to describe *how* catastrophe theory is used rather than to fully describe the end results. The reader is referred to the actual papers in which these applications appeared for the details.

7. The buckling beam (Zeeman [38]). The buckling or Euler beam problem is rather easy to understand and, perhaps because of this, is a basic problem in the classical theory of bifurcations. The problem is as follows: given a beam and a force b applied to the ends (see Fig. 10), what are the possible steady-state shapes? For b sufficiently small

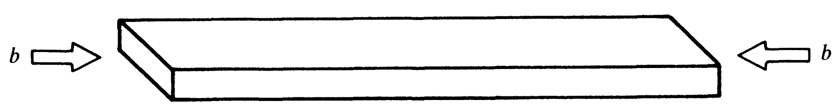

FIG. 10

the beam contracts uniformly but boes not bend. If b is increased past some critical point b_0 the beam buckles into one of two positions shown in Fig. 11.

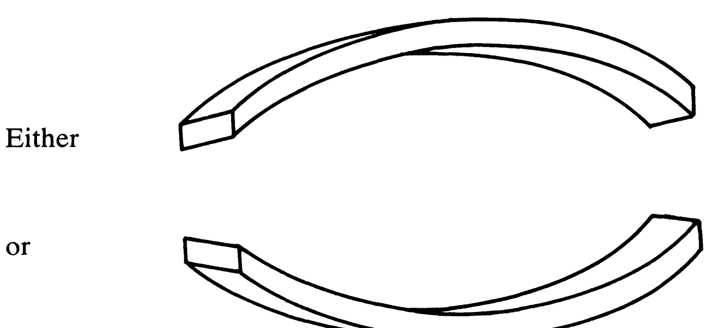

FIG. 11

If one analyzes the differential equations for the beam problem, one finds that for $b<b_0$ there is a unique solution which is the straight beam while for $b>b_0$ but not too large there are, in fact, three solutions—the unbuckled state remains as a possible solution. What distinguishes amongst these solutions is the (potential) energy function. The two buckled states have minimum energy and are the preferred physical states. See [25].

For a given b, we describe the solution to the buckled beam by a function $f(s)$ where s is the arclength along the beam and $f(s)$ is the vertical displacement of the point s to the horizontal. Let x be the maximum displacement of f. Then the classical nonlinear theory leads to the bifurcation diagram in Fig. 12 where each point on the b-axis represents a straight beam and each point on the parabolically shaped curve (actually an elliptic function) represents a buckled solution with force b and maximum deflection x.

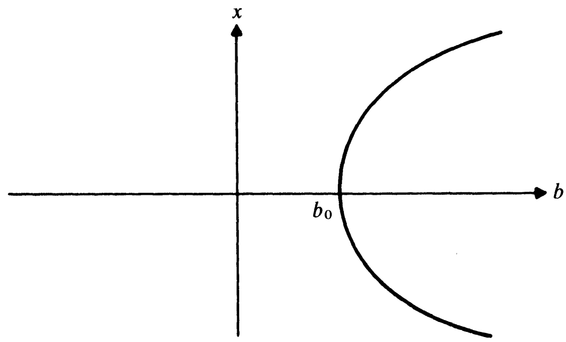

FIG. 12

The following has been known for some time: if you change the buckling beam problem by one of the following:

1) Assume that the central axis of the beam is slightly curved with the maximum amplitude given by α which is small but positive;

2) Assume that a small load α is hung from the center of the beam (see Fig. 13); then the bifurcation diagram is qualitatively shown by one of the two cases indicated by

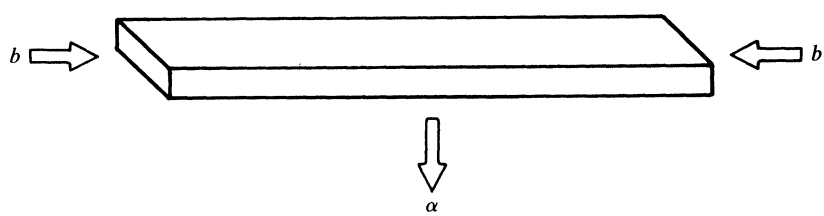

FIG. 13

the dotted lines in Fig. 14. Since real beams tend to have the imperfections suggested above, one should not be surprised that actual measurements also yield bifurcation diagrams qualitatively similar to those with the dotted lines.

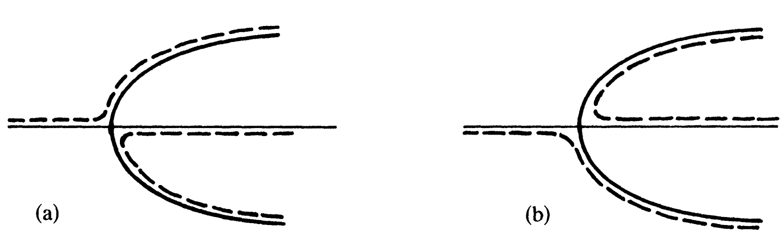

FIG. 14

The use of catastrophe theory in this problem will be to explain mathematically the qualitative differences between the idealized buckling beam problem and the "real" experiments.

Now we shall be more precise. For convenience, we assume that the beam has length π. The space $\mathscr{F}$ of possible solutions consists of those f: $[0, \pi] \to \mathbf{R}$ such that:

1) f is C^∞.

2) $f(0)=f(\pi)=0$ (since the ends are supported).

3) $f''(0)=f''(\pi)=0$ (since there is no bending moment at the ends).

The method of solution is to minimize the energy function V on the space of possible solutions $\mathcal{F}$. If f is in $\mathcal{F}$, then the energy function used in this model is

$$V(f)=\int_0^\pi \frac{1}{2}\frac{\mu f''(s)^2}{1-f'(s)^2}-b(1-\sqrt{1-f'(s)^2})\,ds$$

where μ is the modulus of elasticity per unit length. The solution to this variational problem was given by Euler in terms of elliptic functions in 1744; namely,

THEOREM. *The bifurcation point is $b_0=\mu$. For b slightly greater than μ the solution is $f(s)=x\sin(s)+\cdots$, where the constant x is just the first coefficient in the Fourier expansion of f or the* first harmonic.

This first harmonic is the key to applying catastrophe theory to this problem. The space $\mathcal{F}$ is infinite dimensional; to use catastrophe theory we must reduce this problem to a finite dimensional one. (The reader is referred to Chillingworth [8] for an alternative approach. There the reduction to finite dimensions is accomplished by use of the Relative Morse Lemma.)

The observations about x are:

1) The first harmonic x agrees to first order with the maximum vertical displacement parameter described above. Thus, using this x does not qualitatively change the bifurcation diagram.

2) Define $\bar{V}: \mathbf{R}\to\mathbf{R}$ by $\bar{V}(x)=$ energy assigned to $x\sin(s)=V(x\sin(s))$. If f is a critical point of $V: \mathcal{F}\to\mathbf{R}$, then the first harmonic x of f will be a critical point for $\bar{V}$. This can be seen by perturbing f to $f+t\sin(s)$ and differentiating with respect to t. Since f solves the variational problem V, this differentiation is zero and implies that $(d/dx)\bar{V}(x)=0$. The classical theory of the buckling beam states that the reverse is also true; if $(d/dx)\bar{V}(x)=0$, then there exists a solution to the buckling beam problem whose first harmonic is x.

The way that we shall use catastrophe theory is to reduce the problem of minimizing V to that of minimizing $\bar{V}$. Since V depends on the force b, so does $\bar{V}$. So in fact $\bar{V}$ is a 1-parameter family of functions. Choose coordinates so that $\mu=1/\pi$ and $b=\mu+\beta/\pi$. A computation yields

$$\bar{V}_\beta(x)=-\frac{\beta}{4}x^2+\frac{1-3\beta}{64}x^4+\text{higher order terms.}$$

Note that when x and β are small $d\bar{V}_\beta/dx=0$ yields the graph in Fig. 15, that is, the standard bifurcation diagram. (Compare with Fig. 12.)

The unfolding theorem tells us that to fully describe the functions in a parametrized family near $x^4+\cdots$, we need two parameters. So qualitatively (in the *mathematical* sense) there should be another parameter. Moreover the unfolding theorem guarantees that we can change coordinates so that

$$\bar{V}_{\alpha,\beta}(x)=x^4/4-\beta x^2/2+\alpha x.$$

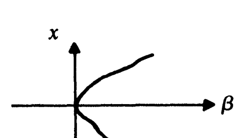

FIG. 15

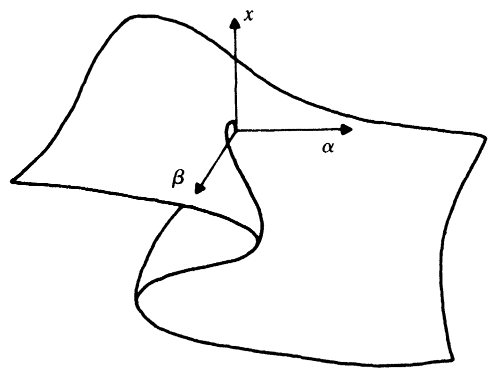

FIG. 7′

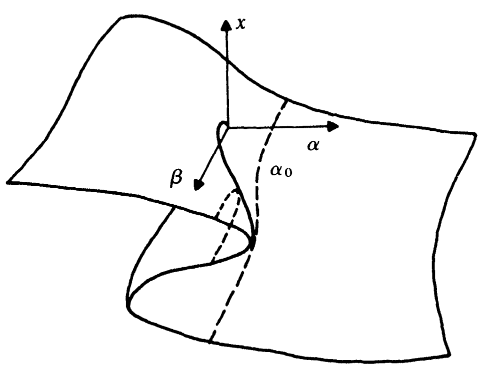

FIG. 16

This parameter α can be added physically by either of the methods mentioned at the beginning of this section; e.g. a load or imperfection. The full bifurcation set is then $(d/dx)\bar{V}_{\alpha,\beta}=0$ which was discussed in § 5, Part A. See Fig. 7′. Cutting this surface by the plane $\alpha=\alpha_0>0$ (Fig. 16) yields the bifurcation diagram given in Fig. 14(b). Cutting this surface by the plane $\alpha=\alpha_0<0$ yields the other bifurcation diagram. (See Fig. 14(a).)

Thus what catastrophe theory does in this problem is give a mathematical method for describing observed changes in the bifurcation diagram (Fig. 12) when the idealized problem is perturbed. The theory also shows that to fully describe the bifurcation set (Fig. 7′) only one new parameter α is needed.

Note added in proof. A deeper question is to describe all possible perturbations of the original bifurcation diagram (Fig. 12) as planar diagrams. Theorem 2.4 guarantees that this is equivalent to the geometric problem of choosing curves in (α, β) space and cutting the cusp surface above these curves. As we saw Figs. 12 and 14 can be obtained from the curves $\alpha=0$ and $\alpha=\alpha_0$. By use of techniques of singularity theory similar to those of catastrophe theory this problem has been solved in [66]. The full description of all possible perturbed diagrams is more complicated than has been indicated here.

Hunt and Thompson [34] have given examples of bifurcation problems which exhibit each of the elementary catastrophes along with ideas about how this sort of analysis can be useful in engineering problems. These methods stem from ideas of Koiter [16]. Other examples are found in [27].

8. Asymptotic expansions of oscillatory integrals. (Arnol'd [3] and Duistermaat [9]). One method for finding an approximate solution to the reduced wave equation for large frequencies λ is the classical WKB method. The reduced wave equation is

$$\Delta u+\lambda^2 u=0, \tag{8.1}$$

where

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$$

and u is a component of the electromagnetic field. The basic idea is to assume that the approximate solution $u=u(x,\lambda)$ has the oscillatory form $u(x,\lambda)=a(x,\lambda)\,e^{i\lambda\phi(x)}$ where ϕ is the phase and a is the amplitude. We assume that a has an asymptotic series in λ (i.e., a formal Taylor series expansion at $\lambda=\infty$), so

$$a(x,\lambda)=a_0(x)+\frac{a_1(x)}{\lambda}+\cdots+\frac{a_n(x)}{\lambda^n}+\cdots.$$

Then one can compute the asymptotic expansion of $\Delta u+\lambda^2 u$. The WKB method is simply to equate the coefficient of λ^{-n} for each n to zero. The first term yields the *eikonal equation*:

$$|\nabla\phi|^2=1, \tag{8.2}$$

while the higher order terms yield equations called *transport equations.* The distinctive feature of these equations is that the eikonal equation is an equation in the phase function ϕ alone and that once ϕ is found the higher order transport equations can be solved recursively for the $a_i(x)$'s, $(i=0,1,\cdots)$ by standard methods of ordinary differential equations.

Thus the key to finding approximate solutions to the reduced wave equation for large λ is finding a solution to the eikonal equation for ϕ. Then enough of the a_i's can be found to obtain a good approximate solution for large λ.

It turns out that solving for ϕ locally is well understood and falls in the domain of the classical Hamilton–Jacobi theory. This theory defines lines known as light rays along which the phase ϕ is determined. The problem is that the solutions ϕ may not exist globally in x-space. Mathematically the light rays may intersect yielding inconsistent information for ϕ. Such intersections are called caustics. Physically this happens for instance when light caustics form on the bottom of a wine glass.

One remedy for this situation is to generalize the class of functions u considered as possible solutions. A standard and reasonable class to consider is generalized sums of oscillatory solutions; namely, oscillatory integrals.

$$u(x,\lambda)=\int_{\alpha\in\mathbf{R}^n} a(x,\alpha,\lambda)\,e^{i\phi(x,\alpha)\lambda}\,d\alpha,$$

where for fixed x,λ the amplitude a has compact support (so that this integral makes sense). The object will be to repeat the WKB method in this more general setting.

Physically whereas the original formulation assumed that light takes a unique path from one point to another, this more general setting allows light to travel along multiple paths corresponding to various reflection and diffraction patterns. The introduction of α allows one to sum the various contributions to u over the various paths.

Now we assume that $a(x,\alpha,\lambda)$ has an asymptotic expansion in λ,

$$a(x,\alpha,\lambda)=a_0(x,\lambda)+\frac{a_1(x,\alpha)}{\lambda}+\cdots+\frac{a_n(x,\alpha)}{\lambda^n}+\cdots$$

and try to compute the asymptotic expansion of $\Delta u+\lambda^2 u$. This is where our problem starts; in order to do this we need to know how to compute an asymptotic series for an oscillatory integral.

First we describe a preliminary result. Suppose that $a_i(x,\alpha)$ all have compact support on some small neighborhood of (x_0,α_0) and that $(\partial\phi/\partial\alpha_i)(x_0,\alpha_0)\neq 0$ for some i. Then by use of integration by parts one shows that the asymptotic expansion of $u(x,\alpha,\lambda)$ vanishes to infinite order; that is,

$$u(x,\alpha,\lambda)=O(\lambda^{-N}) \text{ for arbitrarily large } N.$$

Thus, to compute the asymptotic expansion of $u(x,\alpha,\lambda)$, we can use a partition of unity argument to localize the support of all the $a_i(x,\alpha)$'s in some small neighborhood of (x_0,α_0) where $(\partial\phi/\partial\alpha_i)(x_0,\alpha_0)=0$ for all i. This process is called the method of *stationary phase.*

Next, we consider this generalized phase function $\phi(x,\alpha)$ as a germ of a 3-parameter unfolding $\phi_x(\alpha)$ for x near x_0 and α near α_0. The comments above state that ϕ_{x_0} has a singularity at α_0. Catastrophe theory is used to answer the following question: "Can one put the unfolding ϕ_x in some normal form and then use this normal form to actually compute the asymptotic expansions for the oscillatory integrals?" The answer is yes—as long as one assumes that the unfolding ϕ_x is (locally) stable. Since the number of control parameters is $3\leqq 5$, the density theorem, Theorem 4.1, states that this is a reasonable hypothesis.

Now in order to put ϕ into a normal form using catastrophe theory, we must be able to change ϕ by an arbitrary parametrized change of variables $H_x(\alpha)$. This can be accomplished by nothing more complicated that what is allowed under the standard change of variable theorem for multiple integration. (Note this changes the amplitude a but we need to compute the asymptotic expansions for arbitrary a so no problem arises.)

Thus catastrophe theory tells us that we need only compute the asymptotic expansions when $n=2$ and the phase function $\phi(x,\alpha)$ has one of the five forms:

$$\begin{aligned}
&\alpha_1^3-x_1\alpha_1\pm\alpha_2^2,\\
&\alpha_1^4-x_1\alpha_1^2-x_2\alpha_1\pm\alpha_2^2,\\
&\alpha_1^5-x_1\alpha_1^3-x_2\alpha_1^2-x_3\alpha_1\pm\alpha_2^2,\\
&\alpha_1^3+\alpha_2^3+x_1\alpha_1-x_2\alpha_2-x_3\alpha_1\alpha_2,\\
&\alpha_1^3-x_1\alpha_2^2-x_1\alpha_1-x_2\alpha_2-x_3\alpha_2^2.
\end{aligned}$$

The first form yields the classical Airy function (1838). See [1]. These asymptotic expansions are computed in the papers of Arnol'd and Duistermaat along with a much more detailed description of the theory sketched here.

These methods can be used to describe various phenomena of light near caustics as well as to analyze and solve other linear partial differential equations. Much of what has been described has been known to mathematicians and physicists for quite a while in the nature of special cases; what is gained by the infusion of catastrophe theory is:

1) a unified treatment of a theory which was up until now an unsatisfactory mix of special cases;

2) a list of all the special cases which need to be treated.

A word of warning; we have assumed at a crucial point that the phase function $\phi_x(\alpha)$ is a stable unfolding. When ϕ is unconstrained, this is a reasonable, almost necessary, hypothesis. On the other hand in many applications, one may need to assume that ϕ_x has symmetry (e.g. spherical symmetry); in these cases ϕ_x is no longer stable and the theory (as developed so far) breaks down.

Recently Poènaru [48] and Bierstone [44] have proved theorems which should help to classify stable unfoldings in the presence of the action of a compact Lie group. Wassermann ([58], [59]) has done some work in this direction.

Some interesting examples are given by M. V. Berry in [62].

9. Convex conservation laws (Schaeffer [26]). A conservation law states that a change of a physical quantity in time over a region G in space is equal to the flux of that quantity across the boundary of G. An excellent survey of this subject is given by Lax [17].

To be more precise, let

$$u(x,t) = \text{density of the physical quantity at time } t,$$

$$\mathbf{f} = \mathbf{f}\left(u, \frac{\partial u}{\partial x}, \cdots, \text{etc.}\right) = \text{flux},$$

$$\mathbf{n} = \text{unit outward normal on the boundary of } G.$$

Then the conservation law is

$$\frac{d}{dt}\int_G u(x,t)\,dx = -\int_{\partial G} \mathbf{f}\cdot\mathbf{n}\,ds,$$

where ds is surface area. If u and f are differentiable, then we may apply the divergence theorem to obtain

$$u_t + \operatorname{div}\mathbf{f} = 0. \tag{9.1}$$

We look at one of the simplest forms of a conservation law; namely, assume that

(a) x is 1-space variable,

(b) $f = f(u)$; i.e. flux depends only on density,

(c) f is C^∞ and convex; i.e. $f''(u) \geqq \varepsilon > 0$.

These conservation laws tend not to be physically interesting in and of themselves, but they are useful as model equations for more complicated situations. For example see [61, p. 76] for a model of traffic flow. Aside from this, these equations do present some mathematical interest.

Assuming the above, we see that the differential equation (9.1) is

$$u_t + \frac{d}{dx}f(u) = 0.$$

We also assume C^∞ initial data $u(x,0) = \phi(x)$.

The following example demonstrates the problem. Let $f(u) = u^2/2$. Then try to solve

$$u_t + uu_x = 0,$$

$$u(x,0) = \phi(x),$$

where $\phi(x)$ is nonnegative with compact support. See Fig. 17.

Assuming that u is a smooth solution to the given PDE one can set up an associated ODE called the *characteristic* equation which is quite useful in solving the original PDE. Let

$$\frac{dx}{dt}(t) = u(x(t),t) \quad \text{and} \quad x(0) = x_0.$$

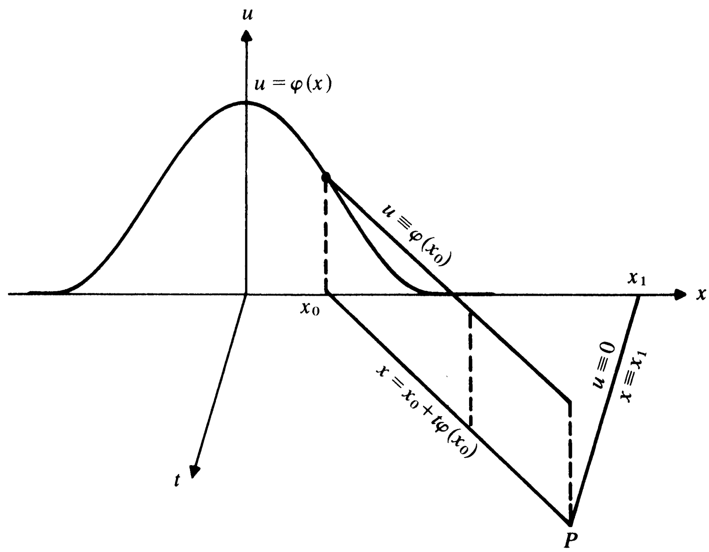

FIG. 17

To solve the characteristic equation differentiate with respect to t to obtain

$$\frac{d^2x}{dt^2} = u_x\frac{dx}{dt} + u_t = u_x u + u_t = 0.$$

So $(x(t), t)$ is a straight line called a *characteristic*. Since the slope dx/dt is constant, u is constant along characteristics; in fact, $u(x(t), t) = u(x_0, 0) = \phi(x_0)$. This yields the curve shown in Fig. 17.

As is shown in Fig. 17 u is constant and positive along the characteristic emanating from x_0 while u is identically zero along the characteristic emanating from x_1. Since these characteristics collide at P we have a contradiction and the assumption that u is smooth is false.

From the geometric point of view, what is happening is the following. Think of $u = \phi(x)$ as a wave; the characteristic equation states that this wave moves to the right with speed proportional to its height. So eventually (by Stage 3 in Fig. 18) the wave breaks over itself and there are several possible values for u at a given point (x, t). In fact the geometric surface traced out by this wave is just Fig. 19. One might hope that catastrophe theory applies.

There is an important analytic theorem which indicates how catastrophe theory could apply.

THEOREM (Lax). *Let* $F(x, t, u) = t(uf'(u) - f(u)) + \Phi(x - f'(u)t)$ *where* $\Phi(y) = \int_0^y \phi(x)\,dx$. *Let* $u(x, t)$ *be the point where the global minimum of* $F(x, y, \cdot)$ *occurs. Then* $u(x, t)$ *is the unique weak solution to* (9.1) *which also satisfies the entropy condition.* See [18].

Notes. 1) Since the solution u is in general not smooth for all time t, we must look for discontinuous or *weak* solutions to the integral form of our original conservation law. Of course at those points where the weak solution is in fact smooth it must satisfy the PDE (9.1). This theorem guarantees the existence of a weak solution as well as providing a method for computing it.

2) It should not be difficult to see that there are many possible weak solutions to the given conservation law with fixed initial data. Since physically relevant solutions

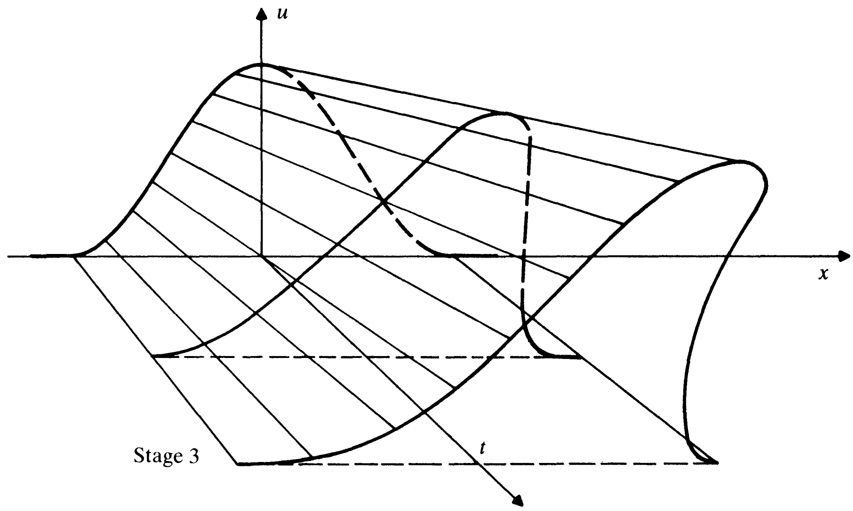

FIG. 18

tend to be unique some extra condition on the PDE is required and this is the entropy condition. To describe this conditon would require a certain amount of technical detail and since it is not really germane to our discussion we refer the interested reader to Lax [17].

This theorem states that in order to find the weak solution u, we must analyze the surface

$$\frac{d}{du}F_{x,t}(u)=0,$$

where $F_{x,t}(u)=F(x, t, u)$. (Note geometrically we are just describing the surface in Fig. 19.) This is precisely the situation of catastrophe theory. Since $F_{x,t}$ is a 2-parameter family of functions, we know that generically it can do no worse than vanish to 4th order and still be locally stable.

Let $\mathcal{S}$ = Schwartz space in $C^\infty(\mathbf{R})=C^\infty$ functions which vanish to ∞ order at ∞. A subset P of $\mathcal{S}$ is *residual* if it is the countable intersection of open dense subsets of $\mathcal{S}$.

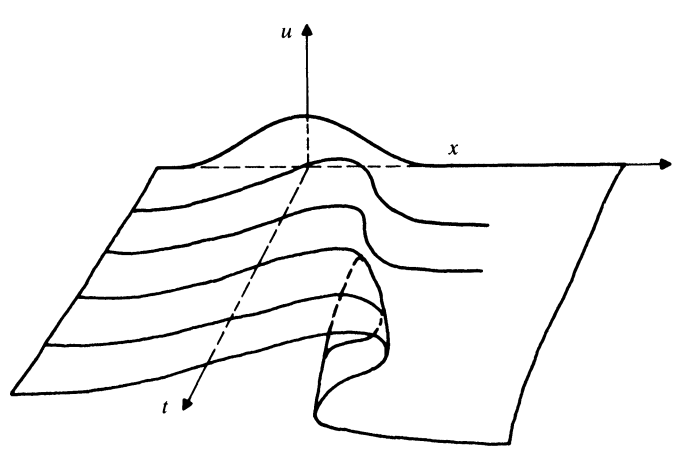

FIG. 19

THEOREM (Schaeffer). *For a residual set of initial data ϕ in $\mathcal{S}$, the corresponding weak solution u given by Lax satisfies the following regularity assumptions*:

1) *u is smooth off a finite union of smooth curves Γ in the (x, t) half plane*;
2) *Across each curve γ in Γ, u has a jump discontinuity. These curves are called* shocks.
3) *At most two shock curves intersect at a point and one shock emanates from this collision.*

The method of proof is to analyze the local behavior of the minima of $F_{x,t}(\cdot)$. The idea is as follows: let

$$\Gamma_2 = \left\{(x, t) | F_{x,t} \text{ has exactly one absolute minimum at } u_0 \text{ and } \frac{\partial^2 F}{\partial u^2}(x, t, u_0) \neq 0\right\},$$

$$\Gamma_1 = \left\{(x, t) | F_{x,t} \text{ has exactly two absolute minima at } u_0 \text{ and } u_1, \text{ and } \frac{\partial^2 F}{\partial u^2}(x, t, u_i) \neq 0\right\},$$

$$\Gamma_0^{(c)} = \left\{(x, t) | F_{x,t} \text{ has exactly three absolute minima at } u_0, u_1, \text{ and } u_2 \text{ and } \frac{\partial^2 F}{\partial u^2}(x, t, u_i) \neq 0\right\},$$

$$\Gamma_0^{(f)} = \left\{(x, t) | F_{x,t} \text{ has exactly one absolute minimum at } u_0 \text{ and } \frac{\partial^2 F}{\partial u^2}(x, t, u_0) = 0 \text{ but } \frac{\partial^4 F}{\partial u^4}(x, t, u_0) \neq 0\right\}.$$

The first step is to show—using transversality theory—that for each initial data in some residual subset in $\mathcal{S}$ the (x, t) half-plane is the union of Γ_2, Γ_1, $\Gamma_0^{(c)}$, and $\Gamma_0^{(f)}$; that is, each point (x, t) is included in precisely one of these four sets.

The second step is to analyze the local structure of the Lax solution near a point in each of the Γ_i's.

Case Γ_2: Γ_2 is an open subset of $\mathbf{R}^2$ and the Lax solution is C^∞ on Γ_2. This requires only the implicit function theorem.

Case Γ_1: Γ_1 is an at most countable collection of smooth curves across which the Lax solution has a jump discontinuity. These curves are the shock curves. Again, only the implicit function theorem is needed.

Case $\Gamma_0^{(f)}$: It is in the analysis of the Lax solution near points in this set that catastrophe theory is used. If (x_0, t_0, u_0) is in $\Gamma_0^{(f)}$, then $(\partial^3 F/\partial u^3)(x_0, t_0, u_0) = 0$ since $F_{x_0,t_0}(\cdot)$ has a minimum at u_0. So $F_{x_0,t_0}(u) = K(u-u_0)^4 +$ higher order terms, and hence factors through the cusp catastrophe (Theorem 2.4). Shock curves start at such points (which are isolated) and are just the Maxwell sets described in § 5. As was shown there, these curves emanate smoothly from (x_0, t_0).

Case $\Gamma_0^{(c)}$: This set consists of collision points of shock curves; some computations are necessary to show that two shocks collide at such points and that it is not the case that one shock bifurcates into two.

Here the application of catastrophe theory enables one to describe rather explicitly the qualitative behavior of "most" solutions even though the set of all solutions has no such qualitative description.

This theorem has been extended in [12], using more results from the theory of singularities, to conclude that residual can be replaced by open dense.

Unfortunately, the results described here rely heavily on Lax's theorem so there is little likelihood that these methods will yield any new information about more complicated conservation laws.

Other authors have obtained similar results using different techniques, in particular, Guckenheimer [14] and Jennings [15].

Summary. In this article I have charted a rather conservative course through the material and applications of elementary catastrophe theory. With regard to the choice of material I have tried to present only that part of the mathematics which is needed to understand the possible uses and the necessary assumptions of the theory. These assumptions are—of course—important. Catastrophe theory can only be applied rigorously to problems which come naturally equipped with a C^∞ parametrized family of functions; moreover these functions should not be restricted by symmetry or other requirements. The applications have been chosen to meet these criteria. The potential function for the buckling beam, the phase functions from optics, and the Lax functional for convex conservation laws all present opportunities for the rigorous application of elementary catastrophe theory.

The theorems are about two concepts—stability and genericity. The stability results are the normal form Theorem (2.3) and the universality Theorem 2.4. In each of the applications, these theorems allow qualitative results to be obtained by computing with a specific model. For convex conservation laws the model gives the structure of a solution near the formation of a shock; for the buckling beam the model gives the qualitative nature of the bifurcation diagram near the idealized problem; and for oscillatory integrals the models allow the explicit computations of asymptotic series. In each example the genericity result (Theorem 4.1) permits the investigation of "almost all" situations to be accomplished by the inspection of a finite number of cases. This is most apparent in the oscillatory integrals example.

There has been no argument about the beauty of the mathematics with which Thom and Mather have presented us. The nature of the ideas has an appeal to mathematicians which suggests its applicability. I believe that the examples described here along with similar problems which have been recently investigated give ample support to the claim that catastrophe theory is rigorously applicable. But like any other tool it does have its limitations.

Much of the discussion between proponents and opponents of catastrophe theory centers on the question of just how universal is the catastrophe theory approach to applications. The answer may well depend on how broadly one interprets the term "catastrophe theory." If one interprets this term broadly to include the general techniques and spirit of elementary catastrophe theory then there are many people who believe that deep qualitative insights are to be discovered about a variety of problems.

It is in this vein that many of the suggested applications of catastrophe theory—particularly in the biological and social sciences—should be viewed. Thom has described seven local models which are—at least in the category of unfoldings with four or fewer parameters—stable (they will not disappear under small perturbations) and complete (generic). These models give pictures which may well appear in many different situations. To the extent that these models give a realistic picture of certain standard arguments they should be incorporated into our language as linguistic models [33]. The list of—seemingly a thousand and one—different uses of the cusp catastrophe is just one example of this process in action.

More concretely, when Zeeman [37] attempts to describe the various stages of the disease Anorexia Nervosa in terms of pictures given by the butterfly catastrophe and finds that the treatment suggested by this model and the actual treatment reinforce each other, is it blind luck? This application of catastrophe theory may not be rigorous as it is difficult to describe which parametrized family one is analyzing but still it may be useful. Such applications are neither less scientific nor less realistic than the statistical approach given by linear regressions. The world may not be a butterfly but then it's probably not a plane either!

These sorts of applications of catastrophe theory are intellectually interesting and fun though they are not yet convincing. Only the future will tell whether this pictorial approach to modeling given by catastrophe theory (or some generalization) will be useful.

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