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BIFURCATION ON THE HEXAGONAL LATTICE AND THE PLANAR BÉNARD PROBLEM

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CONTENTS

One of the simplifications, used by Sattinger (1978), in studying the planar Bénard problem is to assume that the solutions are doubly periodic with respect to the hexagonal lattice in the plane. Once one makes this assumption, the generic situation is that the kernel of the linearized Boussinesq equations (linearized about the pure conduction solution) is six-dimensional, the eigenfunctions being superpositions of plane waves along three directions at mutual angles of 120°. In this situation the Liapunov–Schmidt procedure leads to a reduced bifurcation problem of the form $g(x, \lambda) = 0$ where g: $\mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}^6$ is smooth. Here λ represents the Rayleigh number. Moreover, such a g must commute with the symmetry group of the hexagonal lattice.

In the paper we study such covariant bifurcation problems from the point of view of singularity theory and group theory, thus refining the work of Sattinger (1978). In particular we are able to classify the simplest such bifurcation problems as well as all of their perturbations. We find that stable rolls and stable hexagons occur as possible solutions. In addition, we find a rich structure of non-stable equilibrium solutions including wavy rolls and false hexagons appearing in the unfoldings of even the simplest degenerate bifurcation problems.

Introduction

In this paper we use the singularity-theory methods developed by Golubitsky & Schaeffer (1979) to study the problem of pattern formation as it relates to bifurcation with respect to the hexagonal lattice. This work is motivated by the planar Bénard problem, where a variety of spatial patterns such as rolls, hexagons, wavy rolls and cross rolls have been observed in experiments, and by the work of D. H. Sattinger (1978) in which the mathematical formulation of the specific problems we study is given. We study precisely the cases considered by Sattinger, improving on his results in two distinct ways, which we explain.

We consider bifurcation problems $g(x, \lambda)$ where

$$g: \mathbb{R}^6 \times \mathbb{R} \rightarrow \mathbb{R}^6 \tag{0.1}$$

is a C^∞ map germ. Moreover, we assume that g commutes with the two-dimensional compact symmetry group Γ of the planar hexagonal lattice. In § 1 we give a precise formulation of the action of Γ on $\mathbb{R}^6$ we consider. For the moment, we describe the relation between the Bénard problem and g; the reader is warned that substantial work is needed to make this relation rigorous. Nevertheless, the form and complexity of the results we present here suggest that some such relation may well exist.

The Bénard problem describes – through the Boussinesq equations – thermal conduction and convection of a fluid contained between two parallel infinite planes. The motion is driven by a temperature gradient λ between the upper and lower planes. The mathematically simplest form of the problem is based on the observation that the pure conduction solution loses stability as the temperature gradient λ increases. Moreover, in many experimental situations, the new convection solution has the form of rolls or hexagons, the nomenclature being described at the end of § 4.

Observing that both rolls and hexagons may be described by functions in $\mathscr{F}^\Lambda$, the space of functions that are doubly periodic with respect to the hexagonal lattice Λ, Busse (1962), Sattinger (1978) and others have noted that one can understand much of the structure of steady solutions to the Boussinesq equations, $\mathscr{B}$, by restricting $\mathscr{B}$ to operate on $\mathscr{F}^\Lambda$. With this assumption one may study the solutions of $\mathscr{B}$ near the pure conduction solution by using a Liapunov–Schmidt reduction at the first eigenvalue, thus obtaining a mapping g between finite-dimensional spaces whose zero-set is in one-to-one correspondence with the solutions of $\mathscr{B}$. However, to perform such a reduction one must be able to compute – explicitly – the first eigenvalue λ_0 of the linearization $\mathscr{L}$ of $\mathscr{B}$ around the pure conduction solution, the space of eigenfunctions of $\mathscr{L}$, and the beginnings of the Taylor expansion of g. Each of these steps is difficult; however, it is possible to give a plausible argument for what the form of the answer should be and this answer depends crucially on the existence of a symmetry group for this problem. We note here that Fife (1970) has shown that $\mathscr{L}$ is Fredholm of index zero, and that Busse (1962) has completed these computations when the boundary conditions on the plane layer are free on top and rigid below.

As is well known the operator $\mathscr{B}$ commutes with the Euclidean group of rigid motions in the plane and so the problem $\mathscr{B}|\mathscr{F}^\Lambda$ commutes with the subgroup Γ of rigid motions leaving $\mathscr{F}^\Lambda$ invariant. Suppose now that the kernel of $\mathscr{L}$ contains plane waves in a single direction θ. Standard results imply that $\ker \mathscr{L}$ is invariant under the group Γ; hence $\ker \mathscr{L}$ must also contain plane waves in the directions $\theta + \frac{2}{3}\pi$ and $\theta + \frac{4}{3}\pi$. Thus $\ker \mathscr{L}$ is at least six-dimensional (note that one has two independent eigenfunctions for each direction of plane waves; namely, sine and cosine). Sattinger makes the genericity assumption that $\dim\ker\mathscr{L}$ is exactly 6. Then the Liapunov–Schmidt procedure guarantees the existence of the $g(x,\lambda)$ in (0.1) whose linear terms in x vanish. Moreover, as proved in Sattinger (1979), g must commute with the action of Γ on $\ker\mathscr{L}$.

Sattinger (1978) studies two types of bifurcation problem g. First he observes that the symmetry conditions on g imply that there is precisely one quadratic term, q, in $g(x,0)$ that can possibly be non-zero. The two cases studied are the simplest bifurcation problem for which q is non-zero and the simplest one for which q is zero. We consider here essentially the same two cases using singularity theory. (In our analysis it will be necessary to assume that the reader has some familiarity with the results of Golubitsky & Schaeffer (1979).) We note that q is zero for the idealized form of the Bénard problem considered by Busse (1962).

In his analysis, Sattinger considers only the lowest-order non-zero terms $\tilde{g}$ of g explicitly and proves that if a non-trivial solution exists for the reduced bifurcation equations $\tilde{g} = 0$ then such a solution persists for the full equations $g = 0$. (Note that when $q = 0$, Sattinger's $\tilde{g}$ includes the cubic terms of g.) An advantage of our approach is that we work directly with g. Hence we are able to show that there are solutions to $g = 0$ that are trivial in the reduced bifurcation equations $\tilde{g} = 0$, which were not observed by Sattinger. (To make this observation one must include terms of order four and five in the analysis.) This fact has also been observed by Dancer (1980) using methods similar to, but more refined than, those of Sattinger.

The reason that we are able to work with g directly is that we view the space of all gs commuting with Γ as a module over the ring of invariant functions and thus are able to give an explicit presentation for such gs. The results are given in §§2 and 3.

The way that we find specific solutions to $g = 0$ is firmly rooted in singularity theory. We show that if one assumes certain non-degeneracy conditions on g then g may be transformed by an appropriate change of coordinates (called Γ-equivalence) to a relatively simple normal form. The normal forms are derived in §§8 and 9. In §8 we transform the 'lowest-order terms' of g into this normal form and then in §9 show that the higher-order terms may also be transformed away. Once we have the normal form we use the theorem of §4 to find explicitly the solutions to $g = 0$ for the various examples considered in this paper. The computations are described in §10.

We also perform in §5 (for several types of solutions), the linearized stability analysis of g to see whether solutions are (orbitally) stable. The method of computation depends crucially on the existence of the symmetry group Γ and some of the results are presented in Sattinger (1978). Also observe that one must show that these stability assignments are invariants of Γ-equivalence. This is not *a priori* obvious. However, it is true for the most important cases we consider here, and the relevant results are also given in §5. The computation of stability assignments for the normal forms is given in §10. Again we mention the results of Dancer (1980) for an alternative approach.

The second important way in which our analysis differs from that of Sattinger is that we compute the universal unfolding of the two cases mentioned, thus enabling us to classify the possible bifurcation diagrams that can be obtained by small perturbation of the normal forms we present. It is here that our most interesting contribution lies as we can prove that these perturbations have bifurcation diagrams of great complexity involving secondary and tertiary bifurcations and turning points. The unfolding results are given in §9 and the computation of the bifurcation diagrams of both the unperturbed and the perturbed problems are given in §10. Sections 6 and 7 contain technical results that are necessary in order to complete the singularity-theory analysis of §§8 and 9.

To give the reader a flavour of the results we prove, we show, in schematic form, in figure 1 two of the more interesting bifurcation diagrams, which occur as perturbations of the case where the quadratic term q is 0. Stable solutions are indicated by heavy black lines, and secondary bifurcations and turning points by black dots. Here we use the notation T for the trivial solution, R for rolls, H1 and H2 for families of hexagons, Δ for triangles, FH for false hexagons and WR for wavy rolls. The terminology is explained in §4; note that false-hexagon solutions might in an experimental situation be confused with hexagons. The λ-axis is the horizontal one while the vertical axis is more or less the norm of the solution.

The steady-state theory predicts in the case in figure 1*a* a jump to hexagons and then a jump to rolls as λ is increased, with hysteresis effects in both jumps as λ is decreased.

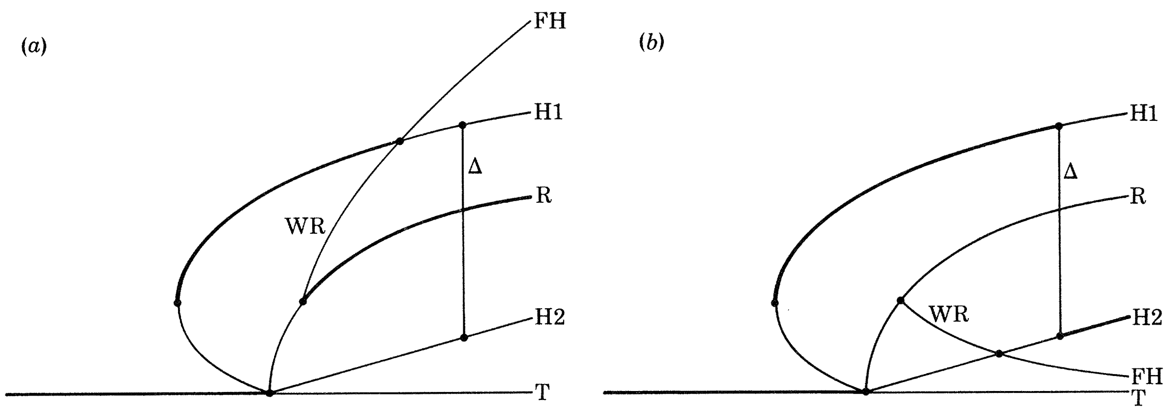

FIGURE 1. Sample results.

An interesting fact about the families H1 and H2 is that they correspond to steady flows that are approximately equal and opposite. The possibility of two such families is well known (see Busse 1979). That there is the mathematical possibility of a jump to one such solution and then a jump to the other – as is indicated in figure 1*b* – seems not to have been known previously. This fact is similar to the results of Golubitsky & Schaeffer (1982) concerning the spherical Bénard problem.

An interesting observation involves the way that the transition between the two families of hexagons may be accomplished by using triangle solutions. This observation, made to us by Jim Swift, is presented in §4.

In §11 we describe as best we can the relation between our work and the planar Bénard problem. A final point worth a mention is that our results are proved for bifurcation problems involving the hexagonal lattice and not explicitly for the Bénard problem. These results may therefore be useful in crystallography and in the study by Ermentrout & Cowan (1979) of visual hallucination patterns.

1. Group-theory preliminaries

In this paper we study bifurcation problems that commute with the action of a particular subgroup, Γ, of the Euclidean group – consisting of rigid motions on the plane – on a particular class, N, of doubly periodic functions. In this section we define the subgroup Γ, the class N, and the action of Γ on N.

Let Λ be the hexagonal lattice in the plane. More precisely, let $w_1 = (1, 0)$ and $w_2 = (\cos\frac{1}{3}\pi, \sin\frac{1}{3}\pi)$. Then

$$\Lambda = \{nw_1 + mw_2 | n, m \in Z\}.$$

Let $\mathscr{F}^{\Lambda}$ be the vector space of real-valued, smooth, Λ-periodic functions defined on $\mathbb{R}^2$. So $\psi(X) \in \mathscr{F}^{\Lambda}$ if $\psi(X + nw_1 + mw_2) = \psi(X)\ \forall n, m \in Z$.

Let σ be a rigid motion of $\mathbb{R}^2$ and let $u : \mathbb{R}^2 \to \mathbb{R}$ be a function. Then σ acts on u by

$$T_{\sigma} u(X) = u(\sigma(X)).$$

Define Γ to be the subgroup of the Euclidean group that leaves $\mathscr{F}^{\Lambda}$ invariant; that is $\gamma \in \Gamma$ if

$$T_{\gamma}(\mathscr{F}^{\Lambda}) = \mathscr{F}^{\Lambda}.$$

One can define Γ more explicitly as follows. The Euclidean group consists of the orthogonal group $O(2)$ and translations. The subgroup of $O(2)$ that preserves $\mathscr{F}^{\Lambda}$ is precisely D_6, the group of symmetries of the hexagon. Observe that every translation preserves $\mathscr{F}^{\Lambda}$. However, the action of the group of translations on $\mathscr{F}^{\Lambda}$ is not faithful, as translation by an element of the lattice Λ acts as the identity on $\mathscr{F}^{\Lambda}$. So the translations acting on $\mathscr{F}^{\Lambda}$ may be thought of as the action of the 2-torus T^2 on $\mathscr{F}^{\Lambda}$. By a slight abuse of notation, we see that $\Gamma = D_6 + T^2$ and we may assume that Γ is compact.

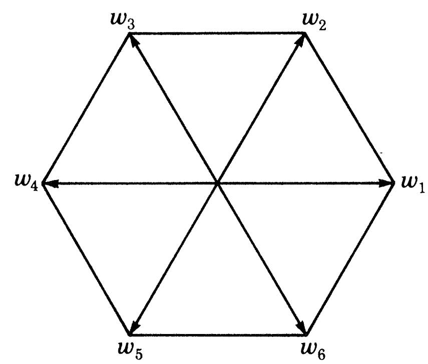

FIGURE 2. Labelling of vertices of the hexagon.

Let $w_1, \ldots, w_6$ be the vertices of the hexagon as given in figure 2. Let N be the subspace of $\mathscr{F}^{\Lambda}$ generated by the six independent plane waves (sin) $4\pi w_j \cdot X$) and $\cos(4\pi w_j \cdot X)$) in the directions w_j, $j = 1, 2, 3$, of the hexagon. (Note that the coefficient 4π is necessary so that the plane waves lie in $\mathscr{F}^{\Lambda}$.) Denote complex conjugation by an overbar and observe that

$$N = \{\psi \in \mathscr{F}^{\Lambda} | \psi = \sum_{j=1}^{6} z_j \,\mathrm{e}^{4\pi \mathrm{i} w_j \cdot X} \quad \text{where} \quad z_j \in \mathbb{C} \quad \text{and} \quad \psi = \overline{\psi}\}.$$

Here $X \cdot Y$ indicates the usual dot product on $\mathbb{R}^2$. Observe that $\psi = \overline{\psi}$ implies $z_j = \bar{z}_{j+3}$ for $j = 1, 2, 3$. It is clear that one may identify N with $\mathbb{C}^3$; that is, one may identify $\psi \in N$ with the triple (z_1, z_2, z_3). Next, we describe the action of Γ on N by its action on the coordinates (z_1, z_2, z_3).

It is not hard to show that the subspace N is invariant under the action of Γ on $\mathscr{F}^{\Lambda}$ and that the induced representation of Γ on N is irreducible. In fact, one can show that the action of T^2 on N is equivalent to

$$(s, t) \cdot (z_1, z_2, z_3) = (\mathrm{e}^{\mathrm{i}s} z_1, \mathrm{e}^{\mathrm{i}(s+t)} z_2, \mathrm{e}^{\mathrm{i}t} z_3), \quad s, t \in T^2. \tag{1.1}$$

More explicitly, let
$$a = \alpha w_1 + \beta w_2,$$
$$\psi(X) = \sum_{j=1}^{3} (z_j \mathrm{e}^{4\pi \mathrm{i} w_j \cdot X} + \bar{z}_j \mathrm{e}^{-4\pi \mathrm{i} w_j \cdot X}),$$
$$s \equiv 2\pi(2\alpha+\beta) \bmod 2\pi, \quad t \equiv 2\pi(\beta-\alpha) \bmod 2\pi.$$

If one computes $T_a\psi$ and denotes the coordinates of $T_a\psi$ by $(s,t)\cdot(z_1,z_2,z_3)$ then one obtains (1.1). The hexagonal group D_6 acts on N by permuting the z_is in exactly the same way as the group element permutes the vertices w_i of the hexagon. The 11 non-trivial elements of D_6 act on N by sending (z_1,z_2,z_3) to

$$\left.\begin{array}{lll} (a)\ (z_2,z_3,\bar{z}_1), & (b)\ (z_3,\bar{z}_1,\bar{z}_2), & (c)\ (\bar{z}_1,\bar{z}_2,\bar{z}_3), \\ (d)\ (\bar{z}_2,\bar{z}_3,z_1), & (e)\ (\bar{z}_3,z_1,z_2), & (f)\ (\bar{z}_1,z_3,z_2), \\ (g)\ (z_3,z_2,z_1), & (h)\ (z_2,z_1,\bar{z}_3), & (i)\ (z_1,\bar{z}_3,\bar{z}_2), \\ (j)\ (\bar{z}_3,\bar{z}_2,\bar{z}_1), & (k)\ (\bar{z}_2,\bar{z}_1,z_3). & \end{array}\right\} \tag{1.2}$$

Observe that D_6 is generated (as a group) by the two elements
$$(z_2,z_3,\bar{z}_1) \quad \text{and} \quad (z_3,z_2,z_1). \tag{1.3}$$

2. The ring of invariant functions

Let Γ be the group defined in §1, which preserves $\mathscr{F}^{\Lambda}$. As noted Γ acts on N, which we may identify with $\mathbb{C}^3$. Let $z = (z_1,z_2,z_3)$ be in $\mathbb{C}^3$. Let $f\colon \mathbb{C}^3 \to \mathbb{R}$ be the germ of an invariant C^∞ function; that is, f is defined on a neighbourhood of zero in $\mathbb{C}^3$, is C^∞ on that neighbourhood, and satisfies $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$. Let $\mathscr{E}^\Gamma$ denote the set of such germs and observe that $\mathscr{E}^\Gamma$ is a ring (under the usual operations of addition and multiplication of functions).

In this section we wish to describe $\mathscr{E}^\Gamma$ explicitly. Let $u_i = z_i\bar{z}_i$, $i = 1, 2, 3$, and let σ_j $(j = 1,2,3)$ be the elementary symmetric polynomials in the u_i; i.e. let
$$\sigma_1 = u_1+u_2+u_3, \quad \sigma_2 = u_1u_2+u_1u_3+u_2u_3, \quad \sigma_3 = u_1u_2u_3.$$
Let
$$q = 2\,\mathrm{Re}\,(z_1\bar{z}_2z_3) = z_1\bar{z}_2z_3+\bar{z}_1z_2\bar{z}_3.$$
Then we have

Proposition 2.1. *Let $f(z)$ be in $\mathscr{E}^\Gamma$. Then there exists a smooth map germ $g\colon \mathbb{R}^4 \to \mathbb{R}$ such that*
$$f(z) = g(\sigma, q) \tag{2.2}$$
where $\sigma = (\sigma_1,\sigma_2,\sigma_3)$. Moreover, if f is a polynomial then there is a unique polynomial g satisfying (2.2)

Note. Proposition 2.1 implies that the ring of invariant polynomials in $\mathscr{E}^\Gamma$ is itself a polynomial ring in the four variables σ, q.

Proof. A theorem of G. Schwarz (1975) states that if we prove the result for polynomials then it follows automatically for C^∞ germs. Now a polynomial mapping $\mathbb{C}^3 \to \mathbb{R}$ has the form
$$f(z) = \sum a_{\alpha\beta} z^\alpha \bar{z}^\beta \tag{2.3}$$
where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $\beta = (\beta_1,\beta_2,\beta_3)$ are multi-indices. The fact that f is real-valued means that $f(z) = \overline{f(z)}$, so
$$\bar{a}_{\alpha\beta} = a_{\beta\alpha}. \tag{2.4}$$
Note that $z \to \bar{z}$ is in Γ. Hence $f(z) = f(\bar{z})$ implies that $a_{\alpha\beta} = a_{\beta\alpha}$. It follows from (2.4) that the
$$a_{\alpha\beta} \text{ are real and symmetric.} \tag{2.5}$$

The action of T^2 in Γ was shown to be

$$(s,t)\cdot z \to (\mathrm{e}^{\mathrm{i}s}z_1, \mathrm{e}^{\mathrm{i}(s+t)}z_2, \mathrm{e}^{\mathrm{i}t}z_3).$$

So

$$f((s,t)\cdot z) = \sum a_{\alpha\beta}\,\mathrm{e}^{\mathrm{i}\omega(\alpha,\beta)}z^{\alpha}\bar{z}^{\beta}$$

where

$$\omega(\alpha,\beta) = s[(\alpha_1+\alpha_2)-(\beta_1+\beta_2)] + t[(\alpha_2+\alpha_3)-(\beta_2+\beta_3)].$$

The invariance conditions (for all s and t) imply

$$a_{\alpha\beta} = 0 \quad \text{if} \quad \alpha_1-\beta_1 \neq \beta_2-\alpha_2 \quad \text{or} \quad \alpha_3-\beta_3 \neq \beta_2-\alpha_2. \tag{2.6}$$

It follows from (2.6) and (2.4) that f has the form

$$f(z) = \sum \dot{b}_{\gamma m} u^{\gamma}(y^m+\bar{y}^m)$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a multi-index and $y = z_1\bar{z}_2 z_3$.

Next observe that $y+\bar{y} = q$ and that $y\bar{y} = u_1u_2u_3$. As $y^m+\bar{y}^m$ is symmetric in y and $\bar{y}$, it may be written as a polynomial in $y+\bar{y}$ and $y\bar{y}$. Hence f has the form

$$f(z) = \sum c_{\gamma m} u^{\gamma} q^m. \tag{2.7}$$

One may now check that q is invariant under that action of D_6. Also D_6 acts on (u_1, u_2, u_3) as S_3, the group of permutations on three letters. It follows that f is a function of q and the elementary symmetric polynomials in the u-variables $(\sigma_1, \sigma_2, \sigma_3)$. This proves the first part of proposition 2.1.

Now suppose that $g(\sigma, q) = 0$ for all $z \in \mathbb{C}^3$ and that g is a polynomial. Then $g \equiv 0$. Observe that the image of $z \to (\sigma, q)$ has a non-empty interior. (One may prove this by computing the Jacobian of this map and showing that the rank equals four at some point z.) Thus g vanishes on an open set and must be identically zero as it is a polynomial. □

3. The module of equivariant mappings

In this section we study smooth (germs of) mappings $g: \mathbb{C}^3 \to \mathbb{C}^3$ that commute with the action of Γ. Let E^{Γ} denote the set of all such mappings g and observe that E^{Γ} is a module over the ring $\mathscr{E}^{\Gamma}$. Our main result is

Proposition 3.1. *E^{Γ} is a free module over the ring $\mathscr{E}^{\Gamma}$ with basis*

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} u_1 z_1 \\ u_2 z_2 \\ u_3 z_3 \end{bmatrix}, \quad \begin{bmatrix} u_1^2 z_1 \\ u_2^2 z_2 \\ u_3^2 z_3 \end{bmatrix}, \quad \begin{bmatrix} z_2\bar{z}_3 \\ z_1 z_3 \\ \bar{z}_1 z_2 \end{bmatrix}, \quad \begin{bmatrix} u_1 z_2\bar{z}_3 \\ u_2 z_1 z_3 \\ u_3\bar{z}_1 z_2 \end{bmatrix}, \quad \begin{bmatrix} u_1^2 z_2\bar{z}_3 \\ u_2^2 z_1 z_3 \\ u_3^2\bar{z}_1 z_2 \end{bmatrix}. \tag{3.2}$$

This proposition has the following interpretation for bifurcation problems. Suppose $g: \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}^3$ commutes with Γ; that is $g(\gamma\cdot z, \lambda) = \gamma\cdot g(z,\lambda)$ for all $\gamma \in \Gamma$. Then g has the form

$$g(z,\lambda) = (H_1 z_1 + K_1 z_2\bar{z}_3, \quad H_2 z_2 + K_2 z_1 z_3, \quad H_3 z_3 + K_3 \bar{z}_1 z_2), \tag{3.3}$$

where

$$H_j = h_1 + h_3 u_j + h_5 u_j^2, \quad K_j = k_2 + k_4 u_j + k_6 u_j^2 \tag{3.4}$$

and the h_l and k_l are invariant functions. In particular $h_l = h_l(\sigma, q, \lambda)$ and $k_l = k_l(\sigma, q, \lambda)$. Note that the subscripts on the hs and ks, which may seem arbitrary, have the following logic. The subscript refers to the degree of homogeneity of the term in (3.3) corresponding to the constant term in the invariant function. So $h_5(0)$ corresponds to the term $(u_1^2 z_1, u_2^2 z_2, u_3^2 z_3)$, which is of degree 5 in the z-variables.

Proof of proposition 3.1. A generalization of Schwarz's theorem given by Poenaru (1976) state that if we can show that the mappings in (3.2) constitute a free basis for the module of polynomial mappings g in E^Γ then they form a free basis for E^Γ over $\mathscr{E}^\Gamma$.

So let $g: \mathbb{C}^3 \to \mathbb{C}^3$ be a polynomial mapping that commutes with Γ. We first must show that g is a linear combination of the generators in (3.2) with coefficients that are polynomials in $\mathscr{E}^\Gamma$. Let

$$g(z) = (g_1(z), g_2(z), g_3(z)).$$

As observed by Sattinger (1978) g is determined by g_1. Using the group elements $z \to (z_2, z_3, \bar{z}_1)$ and $z \to (z_3, z_2, z_1)$ in D_6 one has

$$\left.\begin{aligned} g_2(z_1, z_2, z_3) &\equiv g_1(z_2, z_3, \bar{z}_1), \\ g_3(z_1, z_2, z_3) &\equiv g_1(z_3, z_2, z_1). \end{aligned}\right\} \tag{3.5}$$

The remaining elements in D_6 yield restrictions on g_1 that may be summarized by:

$$g_1(\bar{z}) = \overline{g_1(z)}, \tag{3.6a}$$

$$g_1(z_1, z_2, z_3) = g_1(z_1, \bar{z}_3, \bar{z}_2). \tag{3.6b}$$

As g_1 is a polynomial we may assume that it has the form

$$g_1(z) = \sum a_{\alpha\beta} z^\alpha \bar{z}^\beta$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are multi-indices. The identity (3.6a) implies that $a_{\alpha\beta}$ is real for all α, β. We shall use the identity (3.6b) later.

Next observe that, since g commutes with the action of T^2,

$$a_{\alpha\beta} = 0 \quad \text{if} \quad \alpha_1 - \beta_1 + \alpha_2 - \beta_2 \neq 1, \quad \text{or} \quad \alpha_2 - \beta_2 + \alpha_3 - \beta_3 \neq 0. \tag{3.7}$$

Now let $\beta_2 - \alpha_2 = n$. One has, using (3.7), that if $a_{\alpha\beta} \neq 0$ then

$$z^\alpha z^\beta = \begin{cases} u_1^{\beta_1}\, u_2^{\alpha_2}\, u_3^{\beta_3}\, y^n z_1, & n \geqslant 0, \\ u_1^{\alpha_1}\, u_2^{\beta_2}\, u_3^{\alpha_3}\, \bar{y}^{-(n+1)}\, z_2 \bar{z}_3, & n < 0, \end{cases} \tag{3.8}$$

where $u_j = z_j \bar{z}_j$ and $y = z_1 \bar{z}_2 z_3$ as in §2. It follows from (3.8) that g_1 has the form

$$g_1(z) = A(u, y)\, z_1 + B(u, \bar{y})\, z_2 \bar{z}_3 \tag{3.9}$$

where $u = (u_1, u_2, u_3)$ and A and B are polynomials.

We now claim that any polynomial $C(u, y)$ may be written in the form

$$C(u, y) = D(u, q) + E(u, q)\, y \tag{3.10}$$

where $q = y + \bar{y}$. To see this write

$$C(u, y) = \tfrac{1}{2}[C(u, y) + C(u, y)] + \tfrac{1}{2}[C(u, y) - C(u, y)]. \tag{3.11}$$

The first term in (3.11) is symmetric in y and $\bar{y}$ and thus has the form $p(u, y + \bar{y}, y\bar{y})$. As $y\bar{y} = u_1 u_2 u_3$, one sees that the first term in (3.11) is a polynomial in the variables u and q. Moreover

$$C(u, y) - C(u, \bar{y}) = \sum C_n(u)\, (y^n - \bar{y}^n)$$

by Taylor's theorem and

$$y^n - \bar{y}^n = (y - \bar{y})\, (y^{n-1} + y^{n-2}\bar{y} + \ldots + \bar{y}^{n-2} y + \bar{y}^{n-1}). \tag{3.12}$$

Now the second factor in (3.12) is symmetric in y and $\bar{y}$ and hence is a polynomial in u and q. Finally $y-\bar{y} = 2y-q$ thus proving the claim.

Using the claim (3.10) we may write

$$g_1(z) = \alpha(u,q)\,z_1+\beta(u,q)\,z_2\bar{z}_3 \tag{3.13}$$

for suitably chosen real-valued polynomials α and β.

We now observe that

$$\alpha(u,q) = \alpha_1(\sigma,q)+\alpha_2(\sigma,q)\,u_1+\alpha_3(\sigma,q)\,u_2+\alpha_4(\sigma,q)\,u_1^2+\alpha_5(\sigma,q)\,u_1u_2+\alpha_6(\sigma,q)\,u_2^2$$

where again σ_1, σ_2, and σ_3 are the elementary symmetric polynomials in the variables u_1, u_2, u_3. Moreover the α_is are unique. The result is given in Golubitsky & Guillemin (1974, p. 108, exercise B). We now apply identity (3.6 b) to g_1, which implies that α and β in (3.13) are invariant under $(z_2, z_3) \to (\bar{z}_3, \bar{z}_2)$. It follows that α_3, α_5 and α_6 are identically zero in (3.14) and, with use of (3.5), that (3.2) gives a set of generators for the module E^{Γ}.

We complete the proof of proposition 3.1 by showing that the generators listed in (3.2) are free. First note that g_1 has the form $H_1z_1+K_1z_2\bar{z}_3$. Now suppose $g_1 \equiv 0$. Then we may view g_1 as a homogeneous linear system in $(z_1,\bar{z}_1)$ and $(z_2\bar{z}_3, \bar{z}_2z_3)$ with coefficients H_1 and K_1. As

$$\det\begin{bmatrix} z_1 & z_2\bar{z}_3 \\ \bar{z}_1 & \bar{z}_2z_3 \end{bmatrix} = y-\bar{y}$$

one has that $H_1 \equiv K_1 \equiv 0$ on $y \neq \bar{y}$. By continuity one has $H_1 \equiv K_1 \equiv 0$. As noted previously the α_is in (3.14) are unique. Hence $H_1 \equiv 0$ implies $h_1 \equiv h_3 \equiv h_5 \equiv 0$ with the notation given in (3.4). Similarly for K_1. Thus the generators (3.2) form a free basis as there do not exist any relations between them.

4. How to solve the bifurcation equations

Equation (3.3) shows that a bifurcation problem commuting with the group Γ has the form

$$g(z) = (H_1z_1+K_1z_2\bar{z}_3,\ H_2z_2+K_2z_1z_3,\ H_3z_3+K_3\bar{z}_1z_2) \tag{4.1}$$

where

$$\left.\begin{aligned} H_j &= h_1+u_jh_3+u_j^2h_5, \\ K_j &= k_2+u_jk_4+u_j^2k_6. \end{aligned}\right\} \tag{4.2}$$

We suppress the explicit dependence of (4.1) on λ until that dependence is needed.

In theorem 4.4 we discuss three separate though intimately related issues. First we determine explicitly what symmetries a given solution z to $g(z) = 0$ may have. By definition, the symmetries of a solution z are given by the *isotropy subgroup* of z, denoted by Σ_z, and defined by

$$\Sigma_z = \{\gamma\in\Gamma \,|\, \gamma\cdot z = z\}. \tag{4.3}$$

The calculation of Σ_z is aided by the following observation. If w and z lie on the same orbit of Γ then the corresponding isotropy subgroups are conjugate. So, to calculate the (conjugacy classes of) isotropy subgroups that are possible, one may choose specific representatives on the orbit and calculate the isotropy subgroup for that representative.

The second issue is the explanation of how to solve the bifurcation equations $g = 0$. The method we use is to restrict g to the specific orbit representatives alluded to. Finally, the third issue is to name the various orbit types in ways that are physically observable.

Let $z_j = x_j + \mathrm{i}y_j$, $j = 1, 2, 3$. We prove

THEOREM 4.4. *Each orbit of the action of Γ on $\mathbb{C}^3 = \{(z_1, z_2, z_3)\}$ intersects the half-plane $y_1 = y_3 = 0$; $y_2 \geqslant 0$. A unique representative $(x_1, x_2 + \mathrm{i}y_2, x_3)$ for each orbit is given in the following table along with the (conjugacy class of) isotropy subgroup(s) associated with that orbit. In the last column we give the equations to which the system $g(z) = 0$ reduces at the representative point on the orbit. The nomenclature for the various orbit types is explained at the end of this section.*

	nomenclature	orbit representative	isotropy subgroup	equations
I	Trivial solution	$z = 0$	Γ	—
II	Rolls	$x_2 = x_3 = y_2 = 0$ $x_1 > 0$	$S^1 + \mathbb{Z}_2 + \mathbb{Z}_2$ (1.1) $(0, t)$ (1.2f, i, j)	$H_1 = 0$
III	Hexagons	$y_2 = 0$	D_6	$H_1 + x_1 K_1 = 0$
III+	*l*-Hexagons *g*-Hexagons	$x_1 = x_2 = x_3 > 0$ $x_1 = x_2 = x_3 < 0$	(1.2)	
IV		$x_1 = x_3 > 0$ $y_2 = 0$	$\mathbb{Z}_2 + \mathbb{Z}_2$ (1.2c, g, j)	$H_1 + x_2 K_1 = 0$ $x_2[h_3 + (x_1^2 + x_2^2) h_5 + x_1^2 x_2 k_6] = k_2$
$\mathrm{IV_{WR}}$	Wavy rolls	$\lvert x_2/x_1 \rvert > 2$		
$\mathrm{IV_T}$	Transition	$\lvert x_2/x_1 \rvert = 2$		
$\mathrm{IV_{FH}}$	False hexagons	$0 < \lvert x_2/x_1 \rvert < 2$		
$\mathrm{IV_{PQ}}$	Patchwork quilt	$x_2 = 0$		
V	Triangles	$x_1 = x_3 > 0$ $u_1 = u_2$ $y_2 > 0$; let $z_2 = x_1 \mathrm{e}^{\mathrm{i}\theta}$	D_3 (1.2g) (1.2a, f)$\cdot(0, \theta)$ (1.2e, h)$\cdot(\theta, 0)$	$H_1 = 0$ $K_1 = 0$
V+	*l*-Triangles	$0 < \theta < \frac{1}{2}\pi$		
$\mathrm{V_{RT}}$	Regular triangles	$\theta = \frac{1}{2}\pi$		
V−	*g*-Triangles	$\frac{1}{2}\pi < \theta < \pi$		
VI		$x_1 = x_3 > 0$ $u_1 \neq u_2$ $y_2 > 0$	$\mathbb{Z}_2$ (z_3, z_2, z_1)	$H_1 = 0$ $K_1 = 0$ $h_3 + (u_1 + u_2) h_5 = 0$ $k_4 + (u_1 + u_2) k_6 = 0$
VII		$y_2 = 0$ $u_2 < u_1 < u_3$	$\mathbb{Z}_2$ $(\bar{z}_1, \bar{z}_2, \bar{z}_3)$	$x_1 H_1 + x_2 x_3 K_1 = 0$ $x_2 H_2 + x_1 x_3 K_2 = 0$ $x_3 H_3 + x_1 x_2 K_3 = 0$
VIII		$y_2 > 0$ $u_2 < u_1 < u_3$	$\{1\}$	$h_1 = h_3 = h_5 = 0$ $k_2 = k_4 = k_6 = 0$

Remarks. (*a*) There are two types of hexagonal solutions III given by $x > 0$ and $x < 0$. These two types correspond to different orbits of Γ with isotropy subgroup D_6 and to different, physically observed, solutions of the Bénard problems. See Busse (1978) where such solutions are denoted by l-(liquid) and g-(gas) hexagons.

(*b*) Type IV and V solutions have different physically observable characteristics depending on the exact orbit on which the solution lies. It is likely that type VI, VII, and VIII solutions have a variety of observable characteristics depending on the exact orbit type. We have not attempted to analyse these solution types here.

(*c*) An alternative choice of the representative of the orbit of type *V* is

$$z = x_1(\mathrm{e}^{\mathrm{i}\theta'}, \mathrm{e}^{-\mathrm{i}\theta'}, \mathrm{e}^{\mathrm{i}\theta'}), \quad \theta' = \tfrac{1}{3}\theta,$$

where θ is as defined in the theorem. For this z one can check readily that the isotropy subgroup is D_3, the non-trivial group elements being (1.2*b*, *d*, *g*, *i*, *k*). Note, that when $\theta = \frac{1}{2}\pi$, $z = x_1(i, -i, i)$ and one obtains regular triangles.

Proof. We begin by analysing the structure of the orbit space. Let $z = (z_1, z_2, z_3)$ be in $\mathbb{C}^3$. By an appropriate choice of $(s, t) \in T^2$ we can 'rotate' z so that z_1 and z_3 are real and positive; that is, $y_1 = y_3 = 0$ and $x_1, x_3 \geqslant 0$. Applying the group element (1.2*c*) to z allows us to assume that $y_2 \geqslant 0$.

Note that the group T^2 acts orthogonally on $\mathbb{C}^3$; in particular, the lengths of the zs are preserved. The group D_6 just permutes the vectors, perhaps adding complex conjugations. Thus, the number of u_is that are equal is an invariant of the group action. Now suppose $u_1 = u_2 = u_3 = u$. If $u = 0$, then $z = 0$ and we have the trivial solution whose isotropy subgroup is Γ. If $u \neq 0$, then there are two possibilities. First z_2 is real; so $z = (x, \pm x, x)$. One can apply the element (π, π) of T^2 to show that $(x, -x, x)$ and $-(x, x, x)$ are on the same orbit. Thus, for z_2 real one has two orbits of solutions $\pm(x, x, x)$. These points have isotropy subgroup D_6 and are the hexagons of type III. Secondly z_2 is not real ($y_2 > 0$). Then one obtains the type V orbits of triangles. One may check that the isotropy subgroup is D_3, the one listed for type V. See also remark (*c*) following the statement of theorem 4.4.

Now suppose that two of the u_is are equal and the third is different. If $u_2 = u_3 = 0$ then one obtains the type II orbit of rolls with isotropy group $S^1 + \mathbb{Z}_2 + \mathbb{Z}_2$. Now suppose that $u_1 = u_3 \neq 0$. If z_2 is real then one obtains the type IV orbits whose isotropy subgroup is $\mathbb{Z}_2 + \mathbb{Z}_2$. If z_2 is not real one finds the type VI orbits whose isotropy subgroup is $\mathbb{Z}_2$, the non-trivial group element being reflexion across the w_2-axis of the hexagon. Finally, we consider the case where all of the u_is are distinct. By permuting the u_is we may assume that $u_2 < u_1 < u_3$. Again there are two subcases. z_2 real and z_2 not real. In the first subcase we find the orbits of type VII whose isotropy subgroup is $\mathbb{Z}_2$, the non-trivial group element being given by reflexion through the centre of the hexagon. Finally the case when z_2 is not real leads to the orbits of type VIII. There are no non-trivial elements in the isotropy subgroup of orbits of type VIII.

Next we consider the last column in the table, which gives the equations that determine whether or not $g(z) = 0$ has a solution of a given orbit type. For example, the unique representative of type II solutions that we have chosen is $z = (x, 0, 0)$, One can see from (4.1) that

$$g(x, 0, 0) = (H_1 x_1, 0, 0).$$

So the solution of $g = 0$ is given by $H_1 = 0$ as x_1 is assumed to be positive. Similarly, for orbits of type III we have chosen the unique representative $z = (x, x, x)$, $x \neq 0$. From (4.1) one sees that

$$g(x, x, x) = (H_1 x + K_1 x^2,\ H_1 x + K_1 x^2,\ H_1 x + K_1 x^2).$$

Thus the equation for determining solutions of type III is the one listed in the table.

To find the equations associated with the remaining orbit types, we observe that (4.1) has a nice linear structure. In particular, if we order the six real coordinates as x_1, x_2, x_3, y_1, y_2, y_3 and assume that $y_1 = y_3 = 0$ then (4.1) becomes

$$\left[\begin{array}{ccc|ccc} x_1 & & & x_2 x_3 & & \\ & x_2 & & & x_1 x_3 & \\ & & x_3 & & & x_1 x_2 \\ \hline 0 & & & x_3 y_2 & & \\ & y_2 & & & 0 & \\ & & 0 & & & x_1 y_2 \end{array}\right] \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0. \tag{4.5}$$

Note that the determinant of the matrix in (4.5) may be computed directly to be $(x_1 y_2 x_3)^3$. For orbits of types V, VI, and VIII this determinant is non-zero; so (4.5) reduces to

$$H_j = 0, \quad K_j = 0, \quad j = 1, 2, 3. \tag{4.6}$$

For solutions of type V note that $u_1 = u_2 = u_3$ so (4.6) reduces to $H_1 = K_1 = 0$ as listed in theorem 4.4. For solutions of type VI $u_1 \neq u_2$ and $u_1 = u_3$, so equations (4.6) reduce to

$$H_1 = H_2 = K_1 = K_2 = 0.$$

An equivalent set of equations is

$$H_1 = K_1 = H_1 - H_2 = K_1 - K_2 = 0,$$

which (after division by $u_1 - u_2$, which is assumed to be non-zero) yields the equations in the theorem. Finally equations (4.6) themselves have the linear structure

$$\left[\begin{array}{c|c} V & 0 \\ \hline 0 & V \end{array}\right] \begin{bmatrix} h_1 \\ h_3 \\ h_5 \\ k_2 \\ k_4 \\ k_6 \end{bmatrix} = 0 \tag{4.7}$$

where V is the Vandermondian matrix

$$V = \begin{bmatrix} 1 & u_1 & u_1^2 \\ 1 & u_2 & u_2^2 \\ 1 & u_3 & u_3^2 \end{bmatrix}$$

whose determinant is $(u_1 - u_2)(u_1 - u_3)(u_2 - u_3)$. This determinant is non-zero for solutions of type VIII. It follows that (4.7) reduces to the equations listed in theorem 4.4.

We have now analysed all of the orbit types except IV and VII. In these two cases $y_2 = 0$ and equation (4.5) reduces to

$$\left.\begin{aligned} x_1 H_1 + x_2 x_3 K_1 &= 0, \\ x_2 H_2 + x_1 x_3 K_2 &= 0, \\ x_3 H_3 + x_1 x_2 K_3 &= 0. \end{aligned}\right\} \tag{4.8}$$

For solutions of type IV $x_1 = x_3 > 0$, so $H_1 = H_3$ and $K_1 = K_3$. Equation (4.8) then has the form

$$H_1 + x_2 K_1 = 0, \quad x_2 H_2 + x_1^2 K_2 = 0.$$

Multiply the first equation by x_2, subtract the result from the second equation and divide by $u_1 - u_2 \neq 0$ to obtain the desired result. For solutions of types VII, equations (4.8) are the desired result. □

We now discuss the nomenclature used in the statement of theorem 4.4. Recall from § 1 that solutions to $g(z) = 0$ represent a linear combination of plane waves. In particular $z = (z_1, z_2, z_3)$ corresponds to

$$\psi(X) = 2\,\mathrm{Re}\left(\sum_{j=1}^{3} z_j \mathrm{e}^{4\pi i w_j \cdot X}\right) \tag{4.9}$$

where $X = (X_1, X_2)$ and the w_js are defined in § 1.

In reductions of the planar Bénard problem using Liapunov–Schmidt, one finds (Busse 1962) that ψ represents the vertical velocity component of the steady velocity field associated with a solution of the linearized Boussinesq equations. A key ingredient in the understanding of the geometry (or symmetry) of the steady fluid flow is given by the locus of points in the plane where the velocity of this flow is horizontal; that is, points $X = (X_1, X_2)$ where

$$\psi(X) = 0. \tag{4.10}$$

Note that the set defined in (4.10) is invariant under the action of Γ.

The ψ corresponding to solutions of type II has a most elementary form, $\psi(X) = x_1 \cos(4\pi X_1)$, since $z_1 = (x_1, 0, 0)$. The solution to (4.10) in this case consists of straight lines as in figure 3. In all the figures we shall draw the solutions within the hexagon; solutions outside the hexagon may be obtained by periodicity.

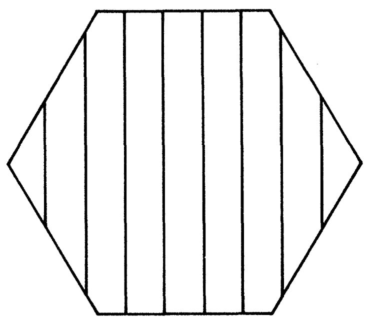

FIGURE 3. Type II solutions: rolls.

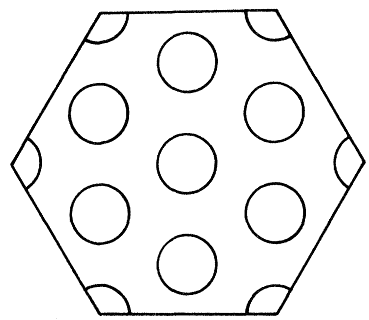

FIGURE 4. Type III solutions: hexagons.

Solutions z of type III and IV satisfy z_1, z_2, z_3 real and $x_1 = x_3$. Thus the ψ in these cases have the form

$$\phi_A(X) = \psi(X)/x_1 = \cos Y_1 + A \cos Y_2 + \cos Y_3 \tag{4.11}$$

where $Y_j = 4\pi w_j \cdot X$ and $A = x_2/x_1$. Observe that the translation $X \to X + \frac{1}{2}w_2$ transforms $\phi_A(X)$ to $-\phi_{-A}(X)$. Thus the zero set of ϕ_{-A} may be obtained from the zero set of ϕ_A by a phase shift. Note, however, that the flow for ϕ_{-A} is in the opposite direction from the flow for ϕ_A. In particular, ϕ_{-A} corresponds to a flow pointing down where ϕ_A points up. See, for example, Busse (1978) for a discussion of such a situation. We now illustrate the zero sets for ϕ_A as A varies. The pictures were obtained with computer assistance.

Solutions of type III occur when $A = 1$. Note that there are two kinds of type III solutions given by $\pm\phi_1$. As remarked, these correspond to reverse flows. The zero set of ϕ_1 is shown in figure 4. Note that the isotropy group D_6 is obvious from the figure and that the closed curves inside each hexagon form a 'six-sided' smoothed out hexagon.

Type IV solutions have more interesting behaviour. When $A \approx 1$ the zero set of ϕ_A must be approximately the zero set of ϕ_1. On the other hand when $A \gg 1$ $\phi_A(X)$ is approximately $\cos(4\pi w_2 \cdot X)$ and the zero set of ϕ_A must be approximately rolls. We call the first case *false hexagons* and the second *wavy rolls*; see figure 5. Observe that the closed curves in (b) are elongated and *not* hexagon-like. Hence the term 'false hexagons'. On the other hand these closed curves are arranged on the same hexagonal lattice as those of figure 4. To distinguish between hexagons and false hexagons in an experimental situation might be extremely difficult. There should be no problem, however, in distinguishing hexagons, pronounced wavy rolls, and rolls.

To complete our discussion for $A > 1$ we need to know at which value of A the transition from false hexagons to wavy rolls occurs. The value is $A = 2$, which follows from

LEMMA 4.12. *The zero set $\phi_A(X) = 0$ consists of non-singular curves unless $A = 0$ or $A = \pm 2$.*

The singular cases are pictured in figure 6.

Proof. We need to show that if

$$\phi_A = 0 \quad \text{and} \quad \mathrm{d}\phi_A = 0$$

then $A = 0$ or $A = \pm 2$. Note that

$$\mathrm{d}\phi_A = -(\sin Y_1\,\mathrm{d}Y_1 + A\sin Y_2\,\mathrm{d}Y_2 + \sin Y_3\,\mathrm{d}Y_3).$$

Since $Y_2 = Y_1 + Y_3$, and $\mathrm{d}Y_1$ and $\mathrm{d}Y_3$ are independent, one obtains

$$\sin Y_1 = \sin Y_3, \tag{4.13a}$$

$$\sin Y_1 = -A\sin(Y_1+Y_3). \tag{4.13b}$$

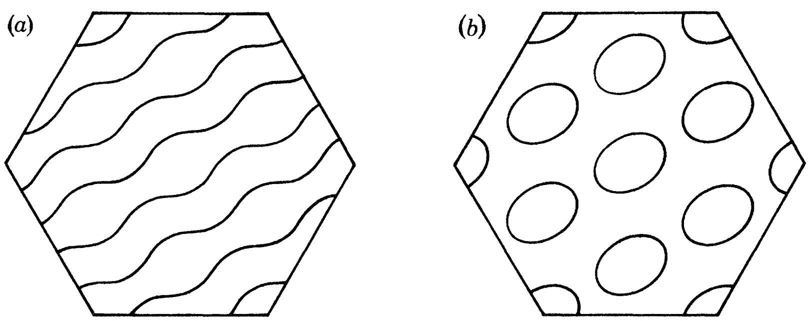

FIGURE 5. Type IV solutions for $A > 1$: (*a*) wavy rolls ($A > 2$); (*b*) false hexagons ($1 < A < 2$).

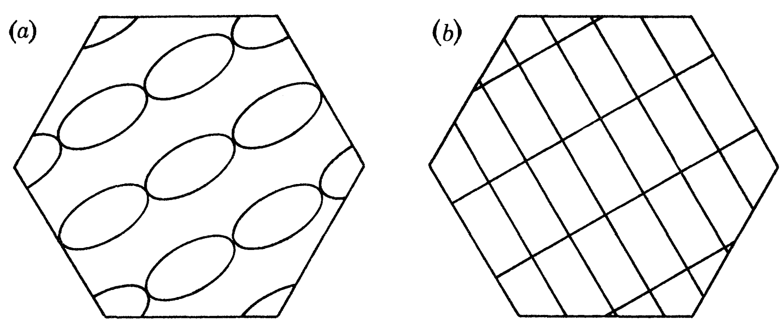

FIGURE 6. Singular zero sets of ϕ_A: (*a*) transition between wavy rolls and false hexagons ($A = 2$); (*b*) patchwork quilt ($A = 0$).

From (4.13*a*) it follows that either $\cos Y_1 = -\cos Y_3$ or $\cos Y_1 = \cos Y_3$. In the first case,

$$\phi_A = A\cos Y_2 \quad \text{and} \quad \cos Y_2 = -\cos^2 Y_1 - \sin^2 Y_1 = -1.$$

So $\phi_A = 0$ implies $A = 0$, which is one of the degenerate cases.

Now suppose $\cos Y_1 = \cos Y_3$. Then (4.13*b*) and $\phi_A = 0$ become

$$\sin Y_1 = -2A\sin Y_1\cos Y_1, \tag{4.14a}$$

$$2\cos Y_1 + A(\cos^2 Y_1 - \sin^2 Y_1) = 0. \tag{4.14b}$$

If $\sin Y_1 = 0$ in (4.14*a*), then $\cos Y_1 = \pm 1$, which implies – with use of (4.14*b*) – that $A = \pm 2$. Suppose now that $\sin Y_1 \neq 0$. Then $\cos Y_1 = -1/2A$ and it follows from (4.14*b*) that $2A^2 = -1$, a contradiction. □

Finally we note that as A approaches zero the false hexagons become more and more rectangular until at $A = 0$ one obtains the patchwork quilt of figure 6. One can show that the lines in that figure cross at right angles. The case $A = 0$ does not appear as a solution of the bifurcation problems we consider in this paper.

Our discussion of type V solutions begins with regular triangles. Here we follow remarks made to us by Jim Swift. Regular triangles occur at (real multiples of) the point $z = (\mathrm{i}, -\mathrm{i}, \mathrm{i})$. See remark (*c*) after the statement of theorem 4.4. The plane wave corresponding to z is

$$\phi(X) = \sin(4\pi w_1 \cdot X) - \sin(4\pi w_2 \cdot X) + \sin(4\pi w_3 \cdot X).$$

Note that if $w_2 \cdot X = 0$ then $w_1 \cdot X = -w_3 \cdot X$ since $w_2 = w_1 + w_3$. It follows that $\phi(X) = 0$ for all X perpendicular to w_2. Since the solutions z have triangular (D_3) symmetry it follows that ϕ also vanishes on lines that have angle $\pm\frac{1}{6}\pi$ with w_2. These three lines (and their translations on the hexagonal lattice) are the only zeros of ϕ. The graph of this zero set is given in figure 7. The actual flow will have upwelling in one triangle and downwelling in the adjacent triangles.

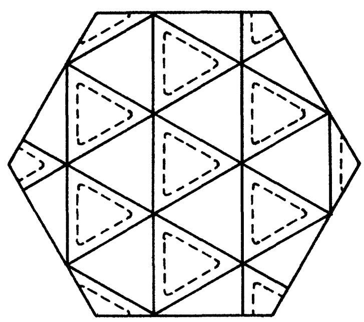

FIGURE 7. Regular triangles ($\theta = \frac{1}{2}\pi$) and triangles ($0 < \theta < \pi, \theta \neq \frac{1}{2}\pi$).

If one draws the zero set of a triangle solution near the regular triangle (say $\theta < \frac{1}{2}\pi$) where $z_2 = x_1 \mathrm{e}^{\mathrm{i}\theta}$ one obtains triangle-like curves for the zero set, as indicated by dashed curves in figure 7. Here one has upwelling inside the triangles and downwelling outside. If one takes $\theta > \frac{1}{2}\pi$ one obtains these triangle-like curves in the empty triangles in figure 7. For these solutions one has downwelling inside the triangles and upwelling outside.

Finally observe that as θ approaches 0 or π the triangle curve becomes more and more hexagonal; that is, they approach the D_6 solutions III. Letting θ travel from 0 to π, one obtains a mechanism for traversing between the two types of hexagonal solutions using steady-state triangular solutions V as intermediaries. As we shall see the bifurcation analysis admits this possibility.

5. How to compute linearized stability

The fact that $g(z)$ in (4.1) commutes with the group Γ restricts the structure of the eigenvalues of $\mathrm{d}g$, the Jacobian of g obtained by differentiation with respect to the z-variables. The restriction on the form of $\mathrm{d}g$ occurs in two distinct ways allowing one, for several solution types, to compute the eigenvalues directly from the entries of $\mathrm{d}g$.

First fix $\gamma \in \Gamma$ and observe that

$$g(\gamma \cdot z) = \gamma \cdot g(z)$$

implies – with use of the chain rule and the fact that γ acts linearly –

$$(\mathrm{d}g)_{\gamma \cdot z} \cdot \gamma = \gamma \cdot (\mathrm{d}g)_z. \tag{5.1}$$

Suppose that $\gamma \in \Sigma_z$, the isotropy subgroup corresponding to z. Then (5.1) implies that $(\mathrm{d}g)_z$ commutes with γ, thus restricting greatly the form of $\mathrm{d}g$.

Second, let γ_t be a differentiable curve in Γ with γ_0 the identity and assume that $g(z) = 0$. Since g vanishes on orbits of the action of Γ, one has

$$g(\gamma_t \cdot z) = 0. \tag{5.2}$$

Differentiating (5.2) with respect to t and evaluating at $t = 0$ yields

$$(\mathrm{d}g)_z v = 0 \tag{5.3}$$

where $v = \mathrm{d}(\gamma_t \cdot z)/\mathrm{d}t|_{t=0}$. If γ_t is a curve in Γ that crosses Σ_z with non-zero speed, then $v \neq 0$ and $(\mathrm{d}g)_z$ has v as an eigenvector.

.Using these observations, we shall also show that the signs of the eigenvalues computed in theorem 5.5 are invariants of the equivalence relation on bifurcation problems we consider in this paper (see proposition 5.24). For future reference we write (4.1) in real coordinates as

$$g(x,y) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{bmatrix} = \begin{bmatrix} H_1 x_1 + K_1(x_2 x_3 + y_2 y_3) \\ H_2 x_2 + K_2(x_1 x_3 - y_1 y_3) \\ H_3 x_3 + K_3(x_1 x_2 + y_1 y_2) \\ H_1 y_1 + K_1(x_3 y_2 - x_2 y_3) \\ H_2 y_2 + K_2(x_1 y_3 + x_3 y_1) \\ H_3 y_3 + K_3(x_1 y_2 - x_2 y_1) \end{bmatrix} \tag{5.4}$$

where H_j and K_j are defined in (4.2). Using the types of arguments indicated, we prove the following.

THEOREM 5.5. *Suppose $g(x,y) = 0$. Then the eigenvalues of $(\mathrm{d}g)_{x,y}$ are as follows. The roman numerals indicate the type of solution as defined in theorem 4.4. The multiplicity is given in parentheses following the eigenvalue.*

(I) $$h_1(0,0,\lambda)\ (6);$$

(II) $$A, D-E\,(2), \quad D+E\,(2), \quad 0$$

where $A = x_1\, \partial H_1/\partial x_1$, $D = -u_1 h_3 - u_1^2 h_5$, *and* $E = x_1 k_2$;

(III) $$A-B\,(2), \quad A+2B, \quad 3\alpha, \quad 0\ (2)$$

where $A = \partial g_1/\partial x_1$, $B = \partial g_1/\partial x_2$, *and* $\alpha = -x_1 K_1$;

(IV) $$A-C, \quad 2\alpha+\epsilon, \quad 0\ (2)$$

where $A = \partial g_1/\partial x_1$, $C = \partial g_1/\partial x_3$, $\alpha = -x_2 K_1$, *and* $\epsilon = H_2$. *The remaining two eigenvalues are the eigenvalues of the* 2×2 *matrix*

$$\begin{bmatrix} A+C & 2D \\ B & E \end{bmatrix}$$

where $B = \partial g_1/\partial x_2$, $D = \partial g_2/\partial x_1$ *and* $E = \partial g_2/\partial x_2$.

Remark. We have not been able to simplify substantially the calculation of eigenvalues for solutions of type V, VI or VII. For the bifurcation problems considered in this paper, only our inability to compute these eigenvalues for solutions of type V presents a problem.

Proof. For all solutions considered in this theorem, we may assume that $y = 0$. As a result the group element $(1.2c)$, $(\bar{z}_1, \bar{z}_2, \bar{z}_3)$, is in the isotropy subgroup. The matrix corresponding to this group element in the real coordinates is

$$Q_3 = \begin{bmatrix} I_3 & 0 \\ 0 & -I_3 \end{bmatrix}.$$

By our first observation in this section dg at a point $y = 0$ must commute with Q_3. Hence dg has the form

$$(\mathrm{d}g)_{(x,0)} = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} \tag{5.6}$$

where R and S are 3×3 matrices.

(I) Compute directly that along the trivial solutions $(\mathrm{d}g)_{(0,0)} = h_1(0,0)\,I$. Note this is a necessity as the only matrix commuting with Γ is the identity matrix I.

(II) Let z be a solution of type II. The group element $(\bar{z}_1, z_3, z_2)$ is in Σ_z and corresponds to the matrix

$$Q_6 = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 0 & 1 & 0 & & & \\ & & & -1 & 0 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & 0 & 1 & 0 \end{bmatrix}.$$

So dg has the form (5.6) and commutes with Q_6. Hence

$$(\mathrm{d}g)_{(x,0)} = \begin{bmatrix} A & B & B & & & \\ C & D & E & & 0 & \\ C & E & D & & & \\ & & & \alpha & \beta & -\beta \\ & 0 & & \gamma & \delta & \epsilon \\ & & & -\gamma & \epsilon & \delta \end{bmatrix}. \tag{5.7}$$

Now dg also commutes with the group elements of the torus, $(z_1, \mathrm{e}^{\mathrm{i}t}z_2, \mathrm{e}^{\mathrm{i}t}z_3)$, which corresponds to the matrix

$$Q_t = \begin{bmatrix} 1 & & & 0 & & \\ & c & & & -s & \\ & & c & & & -s \\ 0 & & & 1 & & \\ & s & & & c & \\ & & s & & & c \end{bmatrix}$$

where $s = \sin t$ and $c = \cos t$. Thus dg has the form

$$(\mathrm{d}g)_{(x,0)} = \begin{bmatrix} A & 0 & 0 & & & \\ 0 & D & E & & 0 & \\ 0 & E & D & & & \\ & & & \alpha & 0 & 0 \\ & 0 & & 0 & D & E \\ & & & 0 & E & D \end{bmatrix}. \tag{5.8}$$

We now use the second observation preceding the proof of the theorem. Observe that if we differentiate the curve of group elements

$$t \mapsto (\mathrm{e}^{\mathrm{i}t}z_1, \mathrm{e}^{\mathrm{i}t}z_2, z_3) \tag{5.9}$$

with respect to t and evaluate at $t = 0$ we obtain $(\mathrm{i}z_1, \mathrm{i}z_2, 0)$. Evaluating this vector at $z = (x_1, 0, 0)$ and writing the result in real coordinates yields the following eigenvector v for dg:

$$v = (0, 0, 0, x_1, 0, 0). \tag{5.10}$$

Hence $\alpha = 0$ in (5.8). One sees immediately that A and 0 are eigenvalues of dg.

We end this computation with the observation that the eigenvalues of the matrix $\begin{bmatrix} D & E \\ E & D \end{bmatrix}$ are $D-E$ and $D+E$ as the sum of these numbers gives the trace and the product gives the determinant. Both of these eigenvalues have multiplicity two as eigenvalues of $\mathrm{d}g$. Now one computes $A = \partial g_1/\partial x_1$, $D = \partial g_2/\partial x_2$, and $E = \partial g_2/\partial x_3$ explicitly using (5.4) and the fact that $H_1 = x_2 = x_3 = y = 0$ along solutions of type II.

(III) Let z be a solution of type III. In real coordinates z is $(x, x, x, 0, 0, 0)$. The group elements used in computing form (5.7) for $\mathrm{d}g$ are in D_6, which is the isotropy subgroup of solutions of type III. So we may assume that $\mathrm{d}g$ has the form (5.7). In addition, $\mathrm{d}g$ must commute with the group element $(z_2, z_3, \bar{z}_1)$ whose matrix form is

$$Q_1 = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 1 & 0 & 0 & & & \\ & & & 0 & 1 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & -1 & 0 & 0 \end{bmatrix}.$$

So $\mathrm{d}g$ has the form

$$(\mathrm{d}g)_z = \begin{bmatrix} A & B & B & & & \\ B & A & B & & 0 & \\ B & B & A & & & \\ & & & \alpha & \beta & -\beta \\ & 0 & & \beta & \alpha & \beta \\ & & & -\beta & \beta & \alpha \end{bmatrix}. \tag{5.11}$$

Next we search for vectors in $\ker \mathrm{d}g$. One may compute quickly using the curve of group elements (5.9) and the curve

$$t \mapsto (z_1, \mathrm{e}^{\mathrm{i}t} z_2, \mathrm{e}^{\mathrm{i}t} z_3)$$

that the following v_1 and v_2 are eigenvectors for $\mathrm{d}g$:

$$v_1 = (0, 0, 0, x, x, 0,) \quad v_2 = (0, 0, 0, 0, x, x). \tag{5.12}$$

Thus $\alpha = -\beta$. So the lower right-hand matrix in (5.13) has rank one with two zero eigenvalues and third eigenvalue 3α. Note that $\alpha = \partial g_4/\partial y_1 = H_1 = -x_1 K_1$ along solutions of type III.

Finally, we claim that the eigenvalues of the upper left-hand matrix are $A+2B$ and the double eigenvalue $A-B$. This follows from

$$\begin{bmatrix} A & B & B \\ B & A & B \\ B & B & A \end{bmatrix} = B \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + AI_3 \equiv BL + AI_3$$

and the fact that the eigenvalues of L are -1, -1 and 2.

(IV) The isotropy subgroup for solutions of type IV is generated by the group elements (1.2c) and (1.2g). From the first element we assume that $\mathrm{d}g$ has the form (5.6). The second group element has the matrix form

$$Q_7 = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$$

where

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Both matrices R and S in (5.6) commute with J. The result is that

$$(\mathrm{d}g)_{(x,0)} = \begin{bmatrix} A & B & C & & & \\ D & E & D & & 0 & \\ C & B & A & & & \\ & & & \alpha & \beta & \gamma \\ & 0 & & \delta & \epsilon & \delta \\ & & & \gamma & \beta & \alpha \end{bmatrix}. \tag{5.13}$$

A short calculation similar to the one in (III) shows that

$$v_1 = (0,0,0,x_1,x_2,0), \quad v_2 = (0,0,0,0,x_2,x_1) \tag{5.14}$$

are vectors in $\ker \mathrm{d}g$. As a result the lower right-hand matrix again has rank 1 with the only non-zero eigenvalue given by its trace $2\alpha+\epsilon$. Note that $\alpha = \partial g_4/\partial y_1 = H_1 = -x_2 K_1$ and $\epsilon = \partial g_5/\partial y_2 = H_2$.

To find the eigenvalues for the upper left-hand matrix requires more work. Note, however, that the vector $(1,0,-1)$ is an eigenvector with eigenvalue $A-C$ while the vectors $(1,0,1)$ and $(0,1,0)$ span an invariant subspace for this matrix. The matrix restricted to this subspace gives the 2×2 matrix

$$\begin{bmatrix} A+C & 2D \\ B & E \end{bmatrix}. \tag{5.15}$$

This completes our discussion of the eigenvalues corresponding to solutions of type IV. □

We recall that two bifurcation problems $g(x,\lambda)$ and $h(x,\lambda)$ are Γ-equivalent if

$$g(x,\lambda) = T(x,\lambda)\, H(X(x,\lambda), \Lambda(\lambda))$$

where $T(x,\lambda)$ is a non-singular $n\times n$ matrix satisfying

$$T(\gamma\cdot x,\lambda)\cdot\gamma = \gamma\cdot T(x,\lambda) \quad \forall\gamma\in\Gamma. \tag{5.16}$$

We assume in addition that $T(0,0)$ and $(\mathrm{d}_x X)(0,0)$ equal cI where $c>0$. With this assumption it was shown in Golubitsky & Schaeffer (1982) that the linearized orbital stability for a solution to $g=0$ is an invariant of Γ-equivalence if the signatures of the eigenvalues of $\mathrm{d}g$ is the same as those of $T\mathrm{d}g$ for every such T.

We now explain what we mean by the term *linearized orbital stability* for a solution z_0 to $g(z)=0$. Consider the system of ordinary differential equations

$$\dot z = g(z).$$

If $g(z_0)=0$ then z_0 is a steady-state solution for this system. It is well known that if all of the eigenvalues of $(\mathrm{d}g)_{z_0}$ lie in the correct half-plane for stability (in this case the left half-plane) then solutions to this system, $z(t)$, with initial conditions close enough to z_0 will tend to z_0 as $t\to\infty$. This is usually referred to as *linearized stability*. However, if g commutes with a continuous group then it is (virtually) impossible for a steady-state solution to be linearized stable. The reason for this lies at the heart of our analysis. If $g(z_0)=0$ then $g(\gamma\cdot z_0)=0$ for every γ in Γ, that is g vanishes on the whole orbit $\Gamma\cdot z_0$. If $\dim\Gamma\cdot z_0>0$ then g vanishes on a surface containing z_0 and $\mathrm{d}g$ must vanish on the tangent space to that surface. The calculation (5.3) shows that

$$\dim\ker\mathrm{d}g = \dim\Gamma\cdot z_0 = \dim(\Gamma/\Sigma_{z_0}) = 2-\dim\Sigma_{z_0}$$

where Σ_{z_0} is the isotropy subgroup of z_0. Thus linearized stability is impossible except at the trivial solution $z_0=0$.

However, one can show that if all of the eigenvalues of $\mathrm{d}g$ that are not forced by the group action to be zero are in the correct half-plane for stability then any solution to this system, $z(t)$, that has initial conditions sufficiently close to z_0 will tend as $t \to \infty$ to the orbit $\Gamma \cdot z_0$. In fact, slightly more is true since the group is compact and the rate of convergence to the orbit is exponential. One has that $z(t)$ actually tends to a fixed point $\gamma \cdot z_0$ on the orbit with, of course, γ close to the identity in Γ. This situation we call *linearized orbital stability*.

Finally, we note that the choice of which half-plane corresponds to stable eigenvalues and which to unstable eigenvalues is to some extent a matter of convention. It depends on whether one writes the system as $\dot{z} = g(z)$ or $\dot{z} + g(z) = 0$. In this sense there is not much difference from the point of view of singularity theory between $g(z)$ and $-g(z)$. However, if one makes the two equivalent then one has to remember to keep track of which half-plane corresponds to a stable eigenvalue.

PROPOSITION 5.24. *The linearized orbital stability of solutions of type* I, II, *and* III *is an invariant of Γ-equivalence if* $\det T > 0$. *In addition the signs of the eigenvalue $A - C$ and the determinant of* (5.15) *for solutions of type* IV *are invariants of this type of Γ-equivalence.*

Proof. The basic observation is the same as in the proof of the main theorem: namely, if γ is in the isotropy subgroup of the solution z then it follows from (5.16) that T commutes with γ.

(I) It follows that T is a positive multiple of the identity. So the result holds trivially for solutions of type I.

(II) The preceding observation implies that along solutions of type II, T has the form (5.8). In particular, let

$$T = \begin{bmatrix} a & 0 & 0 & & & \\ 0 & d & e & & 0 & \\ 0 & e & d & & & \\ & & & f & 0 & 0 \\ & 0 & & 0 & d & e \\ & & & 0 & e & d \end{bmatrix}. \tag{5.17}$$

Multiplying by T in (5.8) yields the new coordinates of $\mathrm{d}g$. Let A', D' and E' denote the non-zero entries of $T\mathrm{d}g$ (recall that $\alpha = 0$). Then

$$A' = aA, \quad D' = dD + eE, \quad \text{and} \quad E' = eD + dE. \tag{5.18}$$

So $D' - E' = (d-e)(D-E)$ and $D' + E' = (d+e)(D+E)$. As we may assume that a and d are positive and that e is as small as desired – by restricting the solution to a small neighbourhood of zero – we have proved the result for solutions of type II.

(III) We use the same kind of argument as the one made for case II. Note that here both T and $\mathrm{d}g$ have the form (5.11).

(IV) The first eigenvalue is an invariant in this case as $(1, 0, -1, 0, 0, 0)$ is an eigenvector for both $\mathrm{d}g$ and T. Finally, the determinant of (5.15) is an invariant of Γ-equivalence since T has the form (5.13) and is near the identity. Note that the vectors $(1, 0, 1, 0, 0, 0)$ and $(0, 1, 0, 0, 0, 0)$ span an invariant subspace for T as well as for $\mathrm{d}g$. □

6. Equivariant matrices

The basic notion in the study of bifurcation problems commuting with a group Γ from the singularity theory point of view is the definition of when two bifurcation problems are equivalent.

Recall again that $g(x, \lambda)$ and $h(x, \lambda)$ are *Γ-equivalent* if

$$h(x,\lambda) = T(x,\lambda)\, g(X(x,\lambda), \Lambda(\lambda))$$

with certain conditions holding on Λ, X, and T. In particular, T is a square matrix satisfying

$$T(\gamma x, \lambda)\,\gamma = \gamma T(x,\lambda). \tag{6.1}$$

We call a matrix T satisfying (6.1) an *equivariant matrix*. Note that λ is just a parameter in (6.1).

Let $\boldsymbol{E}^{\Gamma}$ denote the space of all mappings

$$T: \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3 \tag{6.2}$$

that are C^{∞}, linear in the second variable and satisfy

$$T(\gamma z, \gamma w) = \gamma T(z, w) \quad \forall \gamma \in \Gamma. \tag{6.3}$$

There is an obvious identification of the Ts in (6.1) with those in (6.2) satisfying (6.3).

Observe that $\boldsymbol{E}^{\Gamma}$ is a module over the ring of invariant functions $\mathscr{E}^{\Gamma}$. We make two calculations in this section. The first is to find an explicit set of generators for the module $\boldsymbol{E}^{\Gamma}$, the answer being given in proposition 6.6. The second calculation involves writing $T(z, g(z))$, where $g(z)$ is equivariant, in terms of the generators for the module of gs, E^{Γ} (see proposition 3.1). Here we use explicitly the results of the first calculation.

The first observation we make is that T is determined by its first component. Let $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3)$. Then write $T = (t_1, t_2, t_3)$. The action of the group elements (1.2g, h) implies

$$\left.\begin{aligned} t_2(z,w) &= t_1(z_2, z_1, \bar{z}_3, w_2, w_1, \bar{w}_3),\\ t_3(z,w) &= t_1(z_3, z_2, z_1, w_3, w_2, w_1).\end{aligned}\right\} \tag{6.4}$$

The remaining equivariance conditions on t_1 may now be summarized:

$$\left.\begin{aligned} t_1(z,w) &= \bar{t}_1(\bar{z}, \bar{w}), && (1.2c),\\ t_1(z,w) &= t_1(z_1, \bar{z}_3, \bar{z}_2, w_1, \bar{w}_3, \bar{w}_2), && (1.2i),\\ t_1(z,w) &= \mathrm{e}^{-\mathrm{i}s} t_1(\mathrm{e}^{\mathrm{i}s} z_1, \mathrm{e}^{\mathrm{i}s} z_2, z_3, \mathrm{e}^{\mathrm{i}s} w_1, \mathrm{e}^{\mathrm{i}s} w_2, w_3), && T^2,\\ t_1(z,w) &= t_1(z_1, \mathrm{e}^{\mathrm{i}t} z_2, \mathrm{e}^{\mathrm{i}t} z_3, w_1, \mathrm{e}^{\mathrm{i}t} w_2, \mathrm{e}^{\mathrm{i}t} w_3), && T^2,\end{aligned}\right\} \tag{6.5}$$

The group element inducing the equivariance condition is listed after each relation.

Proposition 6.6. *The module* $\boldsymbol{E}^{\Gamma}$ *is generated by the following* 40 *generators, listed by their first components, according to the degree of homogeneity in the z-variables.*

The following table has the following interpretation. Each of the basic generators should be multiplied by the terms on the right-hand side of the table. For example, T_4, $u_1 T_4$, and $u_1^2 T_4$ are all generators. Note that multiplication by u_1 and u_1^2 increases the degree of homogeneity by two and four respectively. Recall that $y = z_1 \bar{z}_2 z_3$.

Proof. As in proposition 3.1 it sufficies to prove this proposition when T is a polynomial. As T is linear in w we may write

$$T_1(z,w) = \sum_{j=1}^{3} [a_j(z)\, w_j + b_j(z)\, \bar{w}_j]. \tag{6.7}$$

49-2

degree	basic generators	multiply by
0	$T_1 = w_1$	$1, u_1, u_1^2$
1	$T_2 = \bar{z}_3 w_2 + z_2 \bar{w}_3$	$1, u_1, u_1^2$
2	$T_3 = z_1^2 \bar{w}_1,\ T_4 = z_1(\bar{z}_2 w_2 + z_3 \bar{w}_3)$ $T_5 = z_1(z_2 \bar{w}_2 + \bar{z}_3 w_3)$	$1, u_1, u_1^2$
3	$T_6 = y w_1,\ T_7 = z_1 z_2 \bar{z}_3 \bar{w}_1$ $T_8 = z_1^2(z_3 \bar{w}_2 + \bar{z}_2 w_3)$	$1, u_1, u_1^2$
3	$T_9 = u_2 \bar{z}_3 w_2 + u_3 z_2 \bar{w}_3$	$1, u_1$
3	$T_{10} = z_2^2 \bar{z}_3 \bar{w}_2 + z_2 \bar{z}_3^2 w_3$	1
4	$T_{11} = z_1(u_2 \bar{z}_2 w_2 + u_3 z_3 \bar{w}_3)$ $T_{12} = z_1(u_3 z_2 \bar{w}_2 + u_2 \bar{z}_3 w_3)$	$1, u_1$
4	$T_{13} = z_2^2 \bar{z}_3^2 \bar{w}_1$	1
5	$T_{14} = z_1^2(u_3 z_3 \bar{w}_2 + u_2 \bar{z}_2 w_3)$	$1, u_1$
5	$T_{15} = u_2^2 \bar{z}_3 w_2 + u_3^2 z_2 \bar{w}_3$ $T_{16} = u_3 z_2^2 \bar{z}_3 \bar{w}_2 + u_2 z_2 \bar{z}_3^2 w_3$	1
6	$T_{17} = z_1(u_2^2 \bar{z}_2 w_2 + u_3^2 z_3 \bar{w}_3)$ $T_{18} = z_1(u_3^2 z_3 \bar{w}_2 + u_2^2 \bar{z}_3 w_3)$	1
7	$T_{19} = z_1^2(u_3^2 z_3 \bar{w}_2 + u_2^2 \bar{z}_2 w_3)$ $T_{20} = u_3^2 \bar{z}_3 z_2^2 \bar{w}_2 + u_2^2 z_2 \bar{z}_3^2 w_3$	1

We now translate the equivariance conditions (6.5) into the form of T in (6.7) yielding

$$a_j(z) = \bar{a}_j(\bar{z}),\ b_j(z) = \bar{b}_j(\bar{z}), \quad j = 1, 2, 3, \tag{6.8a}$$

$$\left.\begin{aligned} a_1(z) &= a_1(z_1, \bar{z}_3, \bar{z}_2),\ b_1(z) = b_1(z_1, \bar{z}_3, \bar{z}_2), \\ b_2(z) &= a_3(z_1, \bar{z}_3, \bar{z}_2),\ b_3(z) = a_2(z_1, \bar{z}_3, \bar{z}_2). \end{aligned}\right\} \tag{6.8b}$$

$$\left.\begin{aligned} a_j(z) &= a_j(\mathrm{e}^{\mathrm{i}s} z_1, \mathrm{e}^{\mathrm{i}s} z_2, z_3), \quad j = 1, 2, \\ a_3(z) &= \mathrm{e}^{-\mathrm{i}s} a_3(\mathrm{e}^{\mathrm{i}s} z_1, \mathrm{e}^{\mathrm{i}s} z_2, z_3), \\ b_1(z) &= \mathrm{e}^{-2\mathrm{i}s} b_1(\mathrm{e}^{\mathrm{i}s} z_1, e^{\mathrm{i}s} z_2, z_3) \quad \forall s \in \mathbb{R}, \end{aligned}\right\} \tag{6.8c}$$

$$\left.\begin{aligned} a_1(z) &= a_1(z_1, \mathrm{e}^{\mathrm{i}t} z_2, \mathrm{e}^{\mathrm{i}t} z_3), \\ a_j(z) &= \mathrm{e}^{\mathrm{i}t} a_j(z_1, \mathrm{e}^{\mathrm{i}t} z_2, \mathrm{e}^{\mathrm{i}t} z_3), \quad j = 2, 3, \\ b_1(z) &= b_1(z_1, \mathrm{e}^{\mathrm{i}t} z_2, \mathrm{e}^{\mathrm{i}t} z_3) \quad \forall t \in \mathbb{R}. \end{aligned}\right\} \tag{6.8d}$$

From (6.8b) one can compute b_2 and b_3 from a_2 and a_3. As a result, we have not given explicitly the restrictions on b_2 and b_3, which can be obtained from (6.8c, d). To prove the proposition we need to find explicit forms for the a_js and b_1.

Recall the notation of previous sections: $u = (u_1, u_2, u_3)$, $u_j = z_j \bar{z}_j$ and $y = z_1 \bar{z}_2 z_3$.

Lemma 6.9. *If* (6.8) *holds, the* a_js *and* b_1 *have the form*

$$\begin{aligned} a_1(z) &= a_{11}(u, y) + a_{12}(u, \bar{y}), \\ a_2(z) &= a_{21}(u, y)\, z_1 \bar{z}_2 + a_{22}(u, \bar{y}) \bar{z}_3, \\ a_3(z) &= a_{31}(u, y)\, z_1^2 \bar{z}_2 + a_{32}(u)\, z_1 \bar{z}_3 + a_{33}(u, \bar{y})\, z_2 \bar{z}_3^2, \\ b_1(z) &= b_{11}(u, y)\, z_1^2 + b_{12}(u)\, z_1 z_2 \bar{z}_3 + b_{13}(u, \bar{y})\, z_2^2 \bar{z}_3^2. \end{aligned}$$

where the a_{jk}s *and* b_{jk}s *are suitably chosen polynomials.*

Proof. The proofs of the four cases are similar. We shall describe the results for a_3 in detail, leaving the remaining cases for the reader.

Let
$$a_3(z) = \sum A_{\alpha\beta} z^{\alpha}\bar{z}^{\beta},$$
with use of multi-index notation $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$. Constraint (6.8$a$) states that $A_{\alpha\beta}$ is real for all α, β. Constraints (6.8c, d) imply that $A_{\alpha\beta} = 0$ unless

$$\left.\begin{aligned} \alpha_1+\alpha_2-\beta_1-\beta_2-1 &= 0,\\ \alpha_2+\alpha_3-\beta_2-\beta_3+1 &= 0,\end{aligned}\right\} \tag{6.10}$$

respectively. Let $n = \beta_2-\alpha_2$. Then $\alpha_1-\beta_1 = n+1$ and $\alpha_3-\beta_3 = n-1$. There are three possibilities for $z^{\alpha}\bar{z}^{\beta}$ depending on the sign of n:

$$\left.\begin{aligned} z^{\alpha}\bar{z}^{\beta} &= u_1^{\beta_1}u_2^{\alpha_2}u_3^{\beta_3}y^{n-1}z_1^2\bar{z}_2, && n>0,\\ z^{\alpha}\bar{z}^{\beta} &= u_1^{\beta_1}u_2^{\alpha_2}u_3^{\alpha_3}z_1\bar{z}_3, && n=0,\\ z^{\alpha}\bar{z}^{\beta} &= u_1^{\alpha_1}u_2^{\beta_2}u_3^{\alpha_3}(\bar{y})^{-n-1}z_2\bar{z}_3^2, && n<0.\end{aligned}\right\} \tag{6.11}$$

The form for a_3 in lemma 6.9 follows directly from (6.11). □

In proposition 3.1, see (3.10) and (3.14), we showed that polynomials $f(u,y)$ and $h(u,q)$, where $q = y+\bar{y}$, have the form
$$f(u,y) = f_1(u,q)+f_2(u,q)\,y, \tag{6.12a}$$
$$h(u,q) = h_1+h_2u_1+h_3u_2+h_4u_1^2+h_5u_1u_2+h_6u_2^2, \tag{6.12b}$$
where each h_j is invariant, i.e. $h_j = h_j(\sigma, q)$,
$$g(u,\bar{y}) = g_1(u,q)+g_2(u,q)\bar{y} = g_3(u,q)+g_4(u,q)\,y. \tag{6.12c}$$

Note that the first equality in (6.12c) is analogous to (6.12a) and the second equality follows from $y+\bar{y} = q$.

From (6.12) and lemma 6.9 one can show that the $a_j(z)$s and $b_1(z)$ are linear combinations of the following generators with coefficients in $\mathscr{E}^{\Gamma}$, the ring of invariant functions:

$$\left.\begin{aligned} &(a_1) \quad U\cdot\{1,y\}\\ &(a_2) \quad U\cdot\{1,y\}\cdot\{z_1\bar{z}_2, \bar{z}_3\}\\ &(a_3) \quad U\cdot\{z_1^2\bar{z}_2, yz_1^2\bar{z}_2, z_1\bar{z}_3, z_2\bar{z}_3^2, yz_2\bar{z}_3^2\}\\ &(b_1) \quad U\cdot\{z_1^2, yz_1^2, z_1z_2\bar{z}_3, z_2^2\bar{z}_3^2, yz_2^2\bar{z}_3^2\}.\end{aligned}\right\} \tag{6.13}$$

where $U = \{1, u_1, u_2, u_1^2, u_1u_2, u_2^2\}$. The dot between sets in (6.13) indicates that one should take the set whose elements are all the products of two elements, one from each set.

Now many of the generators listed in (6.13) are redundant. One can eliminate any generator involving y for a_2, a_3, and b_1 using the relations

$$\begin{aligned} yz_1\bar{z}_2 &= qz_1\bar{z}_2-u_1u_2\bar{z}_3,\\ y\bar{z}_3 &= u_3z_1\bar{z}_2 = (\sigma_1-u_1-u_2)\,z_1\bar{z}_2,\\ yz_1^2\bar{z}_2 &= qz_1^2\bar{z}_2-u_1u_2z_1\bar{z}_3,\\ yz_2\bar{z}_3^2 &= u_2u_3z_1\bar{z}_3,\\ yz_1^2 &= qz_1^2-u_1z_1z_2\bar{z}_3,\\ yz_2^2\bar{z}_3^2 &= u_2u_3z_1z_2\bar{z}_3.\end{aligned}$$

We have shown that if the a_js and b_1 satisfy the symmetry conditions (6.8) then they have the special form indicated in the statement of the lemma. We have not shown that every polynomial having this special form satisfies (6.8). In fact (6.8 b), which we have not used so far, adds restrictions to a_1 and b_1. For example, we have shown that

$$a_1(z) = k_1 + k_2 u_1 + k_3 u_2 + k_4 u_1^2 + k_5 u_1 u_2 + k_6 u_2^2 \tag{6.14}$$

where $k_j = h_{j1} + h_{j2} y$ and each h_{jl} is an invariant function. Now compute

$$a_1(z_1, \bar{z}_3, \bar{z}_2) = k_1 + k_2 u_1 + k_3 u_3 + k_4 u_1^2 + k_5 u_1 u_3 + k_6 u_3^2 \tag{6.15}$$

where $k_j(z_1, \bar{z}_3, \bar{z}_2) = h_{j1} + h_{j2} y$ is, in fact, invariant under the action of this group element. From (6.14) and (6.15) it follows that

$$(u_3 - u_2)\,[k_3 + k_5 u_1 + k_6(u_2 + u_3)] \equiv 0. \tag{6.16}$$

As $u_3 \neq u_2$ one may solve for k_3 in (6.16) and substitute for k_3 in (6.14) to obtain

$$a_1 = k_1 + k_2 u_1 + k_4 u_1^2 - k_6 u_2 u_3. \tag{6.17}$$

Observe that $u_2 u_3 = \sigma_2 = u_1 \sigma_1 + u_1^2$ so we can write a_1 in the form

$$a_1(z) = k_1 + k_2 u_1 + k_4 u_1^2. \tag{6.18}$$

A similar proof shows one need not include any multiple of u_2 in the expansion of b_1 either.

We have shown that the following is a list of generators for the module E^Γ:

$$\left.\begin{array}{ll}
(a_1) & \{1, u_1, u_1^2\}\cdot\{1, y\}, \\
(a_2) & U\cdot\{z_1\bar{z}_2, \bar{z}_3\}, \\
(a_3) & U\cdot\{z_1^2\bar{z}_2, z_1\bar{z}_3, z_2\bar{z}_3^2\}, \\
(b_1) & \{1, u_1, u_1^2\}\cdot\{z_1^2, z_1 z_2 \bar{z}_3, z_2^2\bar{z}_3^2\}.
\end{array}\right\} \tag{6.19}$$

We have in (6.19) 45 generators; we claim that five more can be shown to be redundant. To see this consider

$$u_1 z_2 \bar{z}_3^2 = q z_1 \bar{z}_3 - u_3 z_1^2 z_2 \quad \text{from} \quad (a_3), \tag{6.20a}$$

$$u_1 z_2^2 \bar{z}_3^2 = q z_1 z_2 \bar{z}_3 - u_2 u_3 z_1^2 \quad \text{from} \quad (b_1). \tag{6.20b}$$

Recalling that one solves for b_2 and b_3 from a_3 and a_2 by using (6.8 b), one obtains the generators listed in proposition 6.6. □

We now compute $T(z, g(z, \lambda))$, modulo higher-order terms where g has the equivariant form (3.3):

$$g(z, \lambda) = \begin{bmatrix} H_1 z_1 + K_1 z_2 \bar{z}_3 \\ H_2 z_2 + K_2 z_1 z_3 \\ H_3 z_3 + K_3 \bar{z}_1 z_2 \end{bmatrix}$$

where $H_j = h_1 + h_3 u_j + h_5 u_j^2$ and $K_j = k_2 + k_4 u_j + k_6 u_j^2$. Let $\mathscr{E}$ denote the ring of C^∞-functions in the variables σ_1, σ_2, σ_3, q and λ. In §3 we identified E^Γ with the $\mathscr{E}$ module $E = (\mathscr{E})^6$ by the isomorphism $g \to (h_1, h_3, h_5, k_2, k_4, k_6)$. With such an identification, the computations of $T(z, g(z, \lambda))$ are summarized in proposition 6.21.

In this paper we shall only need the computation of the 'lower-order terms' of the $T_j(z, g(z, \lambda))$s. Let $\mathscr{M}$ denote the maximal ideal in $\mathscr{E}$ generated by the coordinate functions σ_1, σ_2, σ_3, q, λ. Let $\mathscr{N}$ be the ideal generated by σ_1^2, σ_2, σ_3, q, λ. Let $\mathscr{P}$ be the submodule of E given by

$$\mathscr{P} = (\mathscr{M}\cdot\mathscr{N}, \mathscr{N}, \mathscr{M}, \mathscr{M}^2, \mathscr{M}, \mathscr{M}).$$

In this paper we need only compute the generators T_j modulo terms in $\mathscr{M}\cdot\mathscr{P}$. This we now do.

Proposition 6.21. *Modulo the submodule $\mathscr{M}\cdot\mathscr{P}$ one computes $T_j(z, g(z,\lambda))$ as (recall that $h_1(0) = 0$)*

$$T_1 = (h_1, h_3, h_5, k_2, k_4, k_6),$$

$$T_2 = (\sigma_1 k_2 + 2\sigma_2 k_4 + (\sigma_1\sigma_2 - \sigma_3) k_6,\ -k_2 - 2\sigma_1 k_4 - \sigma_1^2 k_6,\ 2k_4 + \sigma_1 k_6,\ 2h_1 + \sigma_1 h_3 + (\sigma_1^2 - 2\sigma_2) h_5,\ -h_3,\ -h_5),$$

$$T_3 \equiv (\sigma_3 h_5 + qk_2,\ h_1 - \sigma_2 h_5 + qk_4,\ h_3 + \sigma_1 h_5 + qk_6,\ -\sigma_3 k_6,\ -k_2 + \sigma_2 k_6,\ -k_4 - \sigma_1 k_6),$$

$$T_4 \equiv (\sigma_1 h_1 + (\sigma_1^2 - 2\sigma_2) h_3 + (\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3) h_5 + 2qk_2 + \sigma_1 qk_4 - 2\sigma_2 qk_6,\ -h_1 + \sigma_2 h_5 - qk_4,\ -h_3 - \sigma_1 h_5 - qk_6,\ \sigma_3 k_6,\ -2k_2 - \sigma_1 k_4 + \sigma_2 k_6,\ k_4 + \sigma_1 k_6),$$

$$T_5 \equiv (\sigma_1 h_1 + (\sigma_1^2 - 2\sigma_2) h_3 + (\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3) h_5,\ -h_1 + \sigma_2 h_5,\ -h_3 - \sigma_1 h_5,\ -\sigma_3 k_6,\ 2k_2 + \sigma_1 k_4 + (\sigma_1^2 - \sigma_2) k_6,\ -k_4 - \sigma_1 k_6),$$

$$T_6 \equiv (qh_1 + \sigma_2 k_2 + \sigma_3 k_4,\ qh_3 - \sigma_1 k_2 + \sigma_3 k_6,\ qh_5 + k_2,\ -\sigma_3 h_5,\ -h_1 + \sigma_2 h_5,\ -h_3 - \sigma_1 h_5),$$

$$T_7 \equiv (\sigma_2 k_2 + \sigma_3 k_4,\ -\sigma_1 k_2 + \sigma_3 k_6,\ k_2,\ \sigma_3 h_5,\ h_1 - \sigma_2 h_5,\ h_3 + \sigma_1 h_5),$$

$$T_8 \equiv (2qh_1 + \sigma_1 qh_3 - 2\sigma_2 qh_5 + 2\sigma_3 k_4 + \sigma_1\sigma_2 k_6,\ -qh_3 + \sigma_1 k_2 - \sigma_3 k_6,\ -qh_5 - k_2,\ \sigma_3 h_5,\ -2h_1 - \sigma_1 h_3 + \sigma_2 h_5,\ h_3 + \sigma_1 h_5),$$

$$T_9 \equiv (2\sigma_2 k_2 + (\sigma_1\sigma_2 - \sigma_3) k_4 - 2\sigma_2^2 k_6,\ -2\sigma_1 k_2 - \sigma_1^2 k_4 - \sigma_3 k_6,\ 2k_2 + \sigma_1 k_4 - 2\sigma_2 k_6,\ \sigma_1 h_1 + (\sigma_1^2 - 2\sigma_2) h_3 + (-3\sigma_1\sigma_2 + 2\sigma_3) h_5,\ -h_1 + \sigma_2 h_5,\ -h_3 - \sigma_1 h_5),$$

$$T_{10} \equiv (-2\sigma_2 k_2 + (\sigma_3 - \sigma_1\sigma_2) k_4 + 2\sigma_2^2 k_6,\ 2\sigma_1 k_2 + \sigma_1^2 k_4 + \sigma_3 k_6,\ -2k_2 - \sigma_1 k_4 + 2\sigma_2 k_6,\ \sigma_1 h_1 + (\sigma_1^2 - 2\sigma_2) h_3 + (-3\sigma_1\sigma_2 + 2\sigma_3) h_5 + 2qk_2 + \sigma_1 qk_4 - 2\sigma_2 qk_6,\ -h_1 + \sigma_2 h_5 - qk_4,\ -h_3 - \sigma_1 h_5 - qk_6),$$

$$T_{11} \equiv ((\sigma_1^2 - 2\sigma_2) h_1 + (\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3) h_3 + (3\sigma_1\sigma_3 + 2\sigma_2^2) h_5 + \sigma_1 qk_2 - 2\sigma_2 qk_4 + 2\sigma_3 qk_6,\ \sigma_2 h_3 - \sigma_3 h_5 - qk_2,\ -h_1 - \sigma_1 h_3 + \sigma_2 h_5 - qk_4,\ \sigma_3 k_4 + \sigma_1\sigma_3 k_6,\ -\sigma_1 k_2 + \sigma_2 k_4 - 2\sigma_3 k_6,\ k_2 + \sigma_1 k_4 - \sigma_2 k_6),$$

$$T_{12} \equiv (2\sigma_2 h_1 + (\sigma_1\sigma_2 - \sigma_3) h_3 - 2\sigma_2^2 h_5,\ -2\sigma_1 h_1 - \sigma_1^2 h_3 - \sigma_3 h_5,\ 2h_1 + \sigma_1 h_3 - 2\sigma_2 h_5,\ 2\sigma_3 k_4 + \sigma_1\sigma_3 k_6,\ \sigma_1 k_2 - \sigma_3 k_6,\ -k_2),$$

$$T_{13} \equiv (-\sigma_2 h_1 - \sigma_3 h_3,\ \sigma_1 h_1 - \sigma_3 h_5,\ -h_1,\ qh_1 + \sigma_2 k_2 + \sigma_3 k_4,\ qh_3 - \sigma_1 k_2 + \sigma_3 k_6,\ qh_5 + k_2),$$

$$T_{14} \equiv (2\sigma_2 qh_3 - \sigma_3 qh_5 - \sigma_3 k_2 + \sigma_1\sigma_3 k_4 + 2\sigma_2\sigma_3 k_6,\ -\sigma_2 k_2 - \sigma_3 k_4,\ 2qh_3 - \sigma_1 k_2 + 2\sigma_3 k_6,\ -2\sigma_3 h_3 - \sigma_1\sigma_3 h_5,\ \sigma_3 h_5,\ h_1),$$

$$T_{15} \equiv ((\sigma_1\sigma_2 - \sigma_3) k_2 - 2\sigma_2^2 k_4 + 3\sigma_2\sigma_3 k_6,\ -\sigma_1^2 k_2 - \sigma_3 k_4,\ \sigma_1 k_2 - 2\sigma_2 k_4 + 2\sigma_3 k_6,\ -2\sigma_2 h_1 + (-3\sigma_1\sigma_2 + 2\sigma_3) h_3 + (3\sigma_1\sigma_3 + 2\sigma_2^2) h_5,\ \sigma_2 h_3 - \sigma_3 h_5,\ -h_1 - \sigma_1 h_3 + \sigma_2 h_5),$$

$$T_{16} \equiv ((\sigma_3 - \sigma_1\sigma_2) k_2 + 2(\sigma_1\sigma_3 - \sigma_2^2) k_4 + \sigma_2\sigma_3 k_6,\ \sigma_1^2 k_2 - 2\sigma_3 k_4,\ -\sigma_1 k_2 - 2\sigma_2 k_4 + \sigma_3 k_6,\ 2\sigma_2 h_1 + (\sigma_1\sigma_2 - \sigma_3) h_3 - 2\sigma_2^2 h_5 + \sigma_1 qk_2 + 2\sigma_2 qk_4 - \sigma_3 qk_6,\ -\sigma_3 h_5 - qk_2,\ 2h_1 + \sigma_1 h_3 - 2\sigma_2 h_5 + 2qk_4),$$

$$T_{17} \equiv (2\sigma_3 h_1 + (3\sigma_1\sigma_3 + 2\sigma_2^2) h_3 - 5\sigma_2\sigma_3 h_5 - 2\sigma_2 qk_2 + 2\sigma_3 qk_4,\ -\sigma_3 h_3,\ \sigma_2 h_3 - \sigma_3 h_5 - qk_2,\ \sigma_3 k_2 + \sigma_1\sigma_3 k_4 - \sigma_2\sigma_3 k_6,\ \sigma_2 k_2 - 2\sigma_3 k_4,\ \sigma_1 k_2 - \sigma_2 k_4 + \sigma_3 k_6),$$

$$T_{18} \equiv (-\sigma_3 h_1 + 2(\sigma_2^2 - \sigma_1\sigma_3) h_3 - \sigma_2\sigma_3 h_5,\ 2\sigma_3 h_3,\ 2\sigma_2 h_3 - \sigma_3 h_5,\ -\sigma_3 k_2 + \sigma_1\sigma_3 k_4 + \sigma_2\sigma_3 k_6,\ -\sigma_2 k_2 - \sigma_3 k_4,\ -\sigma_1 k_2 + \sigma_3 k_6),$$

$$T_{19} \equiv (-\sigma_3 qh_3 - \sigma_1\sigma_3 k_2 - 2\sigma_2\sigma_3 k_4 - \sigma_3^2 k_6,\ 2\sigma_3 k_2,\ \sigma_2 k_2 - \sigma_3 k_4,\ \sigma_3 h_1 - \sigma_1\sigma_3 h_3 - 2\sigma_2\sigma_3 h_5,\ \sigma_3 h_3,\ -2\sigma_3 h_5),$$

$$T_{20} \equiv (2\sigma_2^2 k_2 + \sigma_2\sigma_3 k_4 - \sigma_3^2 k_6,\ \sigma_3 k_2,\ 2\sigma_2 k_2 + \sigma_3 k_4,\ -\sigma_3 h_1 + 2(\sigma_2^2 - \sigma_1\sigma_3) h_3 - \sigma_2\sigma_3 h_5 - 2\sigma_2 qk_2 - \sigma_3 qk_4,\ 2\sigma_3 h_3,\ 2\sigma_2 h_3 - \sigma_3 h_5 - qk_2).$$

The generators u_1T_j and $u_1^2T_j$ can be computed with use of the following formula:

$$u_1T_j \equiv (\sigma_3 l_5,\ l_1-\sigma_2 l_5,\ l_3+\sigma_1 l_5,\ \sigma_3 m_6,\ m_2-\sigma_2 m_6,\ m_4+\sigma_1 m_6)$$

where $T_j = (l_1, l_3, l_5, m_2, m_4, m_6)$.

Proof. The proof is a long but straightforward calculation, which is outlined here. One substitutes the proper components of g into the generator T_j obtaining the first component f_1 of $f = T(z,g)$. Then one puts f_1 into the form $(l_1+l_3u_1+l_5u_1^2)z_1+(m_2+m_4u_1+m_6u_1^2)z_2\bar{z}_3$ truncating modulo the submodule $\mathscr{M}\cdot\mathscr{P}$. We have given the results of these calculations in the proposition in the form

$$(l_1, l_3, l_5, m_2, m_4, m_6).$$

The following identities are useful in these computations:

$$u_1^3 = \sigma_3-\sigma_2u_1+\sigma_1u_1^2, \tag{6.22a}$$

$$u_1^4 = \sigma_1\sigma_3+(\sigma_3-\sigma_1\sigma_2)u_1+(\sigma_1^2-\sigma_2)u_1^2, \tag{6.22b}$$

$$u_1^5 = (\sigma_1^2-\sigma_2)\sigma_3+(\sigma_2^2-\sigma_1^2\sigma_2+\sigma_1\sigma_3)u_1+(\sigma_1^3-2\sigma_1\sigma_2+\sigma_3)u_1^2, \tag{6.22c}$$

$$u_2u_3 = \sigma_2-\sigma_1u_1+u_1^2, \tag{6.22d}$$

$$u_2^2u_3^2 = \sigma_2^2-\sigma_1\sigma_3+(\sigma_3-\sigma_1\sigma_2)u_1+\sigma_2u_1^2, \tag{6.22e}$$

$$u_2^3u_3^3 = (\sigma_3^2-2\sigma_1\sigma_2\sigma_3+\sigma_2^3)+(\sigma_2\sigma_3-\sigma_1\sigma_2^2+\sigma_1^2\sigma_3)u_1+(\sigma_2^2-\sigma_1\sigma_3)u_1^2, \tag{6.22f}$$

$$u_2+u_3 = \sigma_1-u_1, \tag{6.22g}$$

$$u_2^2+u_3^2 = \sigma_1^2-2\sigma_2-u_1^2, \tag{6.22h}$$

$$u_2^3+u_3^3 = \sigma_1^3-3\sigma_1\sigma_2+2\sigma_3+\sigma_2u_1-\sigma_1u_1^2, \tag{6.22i}$$

$$u_2^4+u_3^4 = \sigma_1^4+3\sigma_1\sigma_3-4\sigma_1^2\sigma_2+2\sigma_2^2+(\sigma_1\sigma_2-\sigma_3)u_1+(\sigma_2-\sigma_1^2)u_1^2, \tag{6.22j}$$

$$\begin{aligned} u_2^5+u_3^5 = {} & \sigma_1^5+4\sigma_1^2\sigma_3-5\sigma_1^3\sigma_2+5\sigma_1\sigma_2^2-5\sigma_2\sigma_3+(\sigma_1^2\sigma_2-\sigma_1\sigma_3-\sigma_2^2)u_1 \\ & -(\sigma_1^3-2\sigma_1\sigma_2+\sigma_3)u_1^2, \end{aligned} \tag{6.22k}$$

$$(u_2^2+u_3^2)u_1 = -\sigma_3+(\sigma_1^2-\sigma_2)u_1-\sigma_1u_1^2, \tag{6.22l}$$

$$(u_2^3+u_3^3)u_1 = -\sigma_1\sigma_3+(\sigma_1^3-2\sigma_1\sigma_2+2\sigma_3)u_1+(\sigma_2-\sigma_1^2)u_1^2, \tag{6.22m}$$

$$(u_2^4+u_3^4)u_1 = (\sigma_2-\sigma_1^2)\sigma_3+(\sigma_1^4+3\sigma_1\sigma_3-3\sigma_1^2\sigma_2+\sigma_2^2)u_1-(\sigma_1^3-2\sigma_1\sigma_2+\sigma_3)u_1^2. \tag{6.22n}$$

We illustrate the computation with $T_{12} = z_1(u_3z_2\bar{w}_2+u_2\bar{z}_3w_3)$. The first component f_1 of $f = T_{12}(z,g)$ is (see proposition 6.6)

$$\begin{aligned} f_1 &= z_1[u_3z_2(H_2\bar{z}_2+K_2\bar{z}_1\bar{z}_3)+u_2\bar{z}_3(H_3z_3+K_3\bar{z}_1z_2)] \\ &= u_2u_3(H_2+H_3)z_1+u_1(u_3K_2+u_2K_3)z_2\bar{z}_3 \\ &= [2u_2u_3h_1+u_2u_3(u_2+u_3)h_3+u_2u_3(u_2^2+u_3^2)h_5]z_1 \\ &\qquad +[2u_1(u_2+u_3)k_2+2u_1u_2u_3k_4+u_1u_2u_3(u_2+u_3)k_6]z_2\bar{z}_3 \\ &= \{2(\sigma_2-\sigma_1u_1+u_1^2)h_1+(\sigma_1\sigma_2-\sigma_1^2u_1+\sigma_1u_1^2-\sigma_3)h_3 \\ &\qquad +[(\sigma_1^2-2\sigma_2)\sigma_2-(\sigma_1^2-2\sigma_2)\sigma_1u_1+(\sigma_1^2-2\sigma_2)u_1^2-\sigma_3u_1]h_5\}z_1 \\ &\qquad +[(\sigma_1u_1-u_1^2)k_2+2\sigma_3k_4+(\sigma_1\sigma_3-\sigma_3u_1)k_6]z_2\bar{z}_3 \\ &= [2\sigma_2h_1+(\sigma_1\sigma_2-\sigma_3)h_3-2\sigma_2^2h_5-(2\sigma_1h_1+\sigma_1^2h_3+\sigma_3h_5)u_1+(2h_1+\sigma_1h_3-2\sigma_2h_5)u_1^2]z_1 \\ &\qquad +[2\sigma_3k_4+\sigma_1\sigma_3k_6+(\sigma_1k_2-\sigma_3k_6)u_1-k_2u_1^2]z_2\bar{z}_3. \end{aligned}$$

Therefore,

$$T_{12}(z,g) = (2\sigma_2 h_1 + (\sigma_1\sigma_2 - \sigma_3)\, h_3 - 2\sigma_2^2 h_5,$$
$$-2\sigma_1 h_1 - \sigma_1 h_3 - \sigma_3 h_5,\ 2h_1 + \sigma_1 h_3 - 2\sigma_2 h_5,\ 2\sigma_3 k_4 + \sigma_1\sigma_3 k_6,\ \sigma_1 k_2 - \sigma_3 k_6,\ -k_2).$$

The computation of the other 19 generators T_j proceeds similarly.

To end the proof we show how to multiply by u_1. Suppose we want to compute $u_1 T_j(z,g)$ with $T_j(z,g) = (l_1, l_3, l_5, m_2, m_4, m_6)$. The first component of $f = T_j(z,g) \in \mathbb{C}^3$ is given by

$$f_1 = (l_1 + l_3 u_1 + l_5 u_1^2)\, z_1 + (m_2 + m_4 u_1 + m_6 u_1^2)\, z_2 \bar{z}_3.$$

It follows that the first component of $u_1 T_j(z,g) \in \mathbb{C}^3$ is given by

$$u_1 f_1 = l_1 u_1 z_1 + l_3 u_1^2 z_1 + l_5 u_1^3 z_1 + m_2 u_1 z_2 \bar{z}_3 + m_4 u_1^2 z_2 \bar{z}_3 + m_6 u_1^3 z_2 \bar{z}_3.$$

Using (6.22 a) we obtain:

$$u_1 f_1 = \sigma_3 l_5 z_1 + (l_1 - \sigma_2 l_5)\, u_1 z_1 + (l_3 + \sigma_1 l_5)\, u_1^2 z_1$$
$$+ \sigma_3 m_2 z_2 \bar{z}_3 + (m_2 - \sigma_2 m_6)\, u_1 z_2 \bar{z}_3 + (m_4 + \sigma_1 m_6)\, u_1^2 z_2 \bar{z}_3.$$

Hence $$u_1 T_j(z,g) \equiv (\sigma_3 l_5,\ l_1 - \sigma_2 l_5,\ l_3 + \sigma_1 l_5,\ \sigma_3 m_6,\ m_2 - \sigma_2 m_6,\ m_4 + \sigma_1 m_6)$$

and the proof is complete. □

7. Generators of the tangent space

To find the normal forms for and to compute the universal unfolding of our bifurcation problem (see § 9), we need the generators of the tangent space $\tilde{T}g$. These generators are defined in complex notation as (see Golubitsky & Schaeffer 1982):

$$\tilde{T}g = \{T(z,\lambda)\cdot g + (\delta g)\cdot f \mid T \in \boldsymbol{E}^{\Gamma}, f \in E^{\Gamma}\} \tag{7.1}$$

where $$(\delta g)\cdot f = (\mathrm{d}_z g)\cdot f + (\mathrm{d}_{\bar{z}} g)\cdot \bar{f} \tag{7.2}$$

and g is the bifurcation problem (3.3). In the right-hand side of (7.2) we use the notation $z\cdot w = z_1 w_1 + z_2 w_2 + z_3 w_3$. In § 6 we discussed the identification of E^{Γ} with $E = \mathscr{E}^6$. Using this identification we consider $\tilde{T}g$ to be a submodule of E. We note here that for the calculations in this paper we need only compute the generators for $\tilde{T}g$ modulo $\mathscr{M}\cdot\mathscr{P}$ where $\mathscr{P}$ is the submodule of E defined in § 6.

Proposition 7.3. *The tangent space $\tilde{T}g$ is generated by* 46 *elements: those listed in proposition* 6.21 *and, modulo $\mathscr{M}\cdot\mathscr{P}$, the following six generators:*

$$D_1 \equiv (h_1 + 2\sigma_1 h_{1,\sigma_1} + 4\sigma_2 h_{1,\sigma_2} + 6\sigma_3 h_{1,\sigma_3} + 3q h_{1,q},\ 3h_3 + 2\sigma_1 h_{3,\sigma_1} + 4\sigma_2 h_{3,\sigma_2} + 6\sigma_3 h_{3,\sigma_3} + 3q h_{3,q},$$
$$5h_5 + 2\sigma_1 h_{5,\sigma_1} + 4\sigma_2 h_{5,\sigma_2} + 6\sigma_3 h_{5,\sigma_3} + 3q h_{5,q},\ 2k_2 + 2\sigma_1 k_{2,\sigma_1} + 4\sigma_2 k_{2,\sigma_2} + 6\sigma_3 k_{2,\sigma_3} + 3q k_{2,q},$$
$$4k_4 + 2\sigma_1 k_{4,\sigma_1} + 4\sigma_2 k_{4,\sigma_2} + 6\sigma_3 k_{4,\sigma_3} + 3q k_{4,q},\ 6k_6 + 2\sigma_1 k_{6,\sigma_1} + 4\sigma_2 k_{6,\sigma_2} + 6\sigma_3 k_{6,\sigma_3} + 3q k_{6,q}),$$

$$D_2 \equiv (5\sigma_3 h_5 + 2(\sigma_1^2 - 2\sigma_2)\, h_{1,\sigma_1} + 2(\sigma_1\sigma_2 - 3\sigma_3)\, h_{1,\sigma_2} + 2\sigma_1\sigma_3 h_{1,\sigma_3} + \sigma_1 q h_{1,q},$$
$$h_1 - 5\sigma_2 h_5 + 2\,(\sigma_1^2 - 2\sigma_2)\, h_{3,\sigma_1} - 6\sigma_3 h_{3,\sigma_2},\ 3h_3 + 5\sigma_1 h_5 - 4\sigma_2 h_{5,\sigma_1} - 6\sigma_3 h_{5,\sigma_2},$$
$$\sigma_1 k_2 + 3\sigma_3 k_6 + 2(\sigma_1^2 - 2\sigma_2)\, k_{2,\sigma_1} + 2(\sigma_1\sigma_2 - 3\sigma_3)\, k_{2,\sigma_2} + 2\sigma_1\sigma_3 k_{2,\sigma_3} + \sigma_1 q k_{2,q},$$
$$-k_2 + \sigma_1 k_4 - 3\sigma_2 k_6 - 4\sigma_2 k_{4,\sigma_1} - 6\sigma_3 k_{4,\sigma_2},\ k_4 + 4\sigma_1 k_6 - 4\sigma_2 k_{6,\sigma_1} - 6\sigma_3 k_{6,\sigma_2}),$$

$$
\begin{aligned}
D_3 \equiv\ & (3\sigma_3 h_3 + 5\sigma_1\sigma_3 h_5 + 2(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3)\,h_{1,\sigma_1} - 2(2\sigma_2^2 + \sigma_1\sigma_3)\,h_{1,\sigma_2} - 4\sigma_2\sigma_3 h_{1,\sigma_3} - 2\sigma_2 q h_{1,q},\\
& -3\sigma_2 h_3 + 5\sigma_3 h_5 + 6\sigma_3 h_{3,\sigma_1},\ h_1 + 3\sigma_1 h_3 - 5\sigma_2 h_5 + 6\sigma_3 h_{5,\sigma_1},\\
& (\sigma_1^2 - 2\sigma_2)\,k_2 + \sigma_3 k_4 + 3\sigma_1\sigma_3 k_6 + 6(\sigma_3 - \sigma_1\sigma_2)\,k_{2,\sigma_1} - 2(2\sigma_2^2 + \sigma_1\sigma_3)\,k_{2,\sigma_2}\\
& -4\sigma_2\sigma_3 k_{2,\sigma_3} - 2\sigma_2 q k_{2,q},\ -3\sigma_2 k_4 + 3\sigma_3 k_6 + 6\sigma_3 k_{4,\sigma_1},\\
& -k_2 + \sigma_1 k_4 - 5\sigma_2 k_6 + 6\sigma_3 k_{6,\sigma_1}),\\
D_4 \equiv\ & (qh_3 + \sigma_1 k_2 - \sigma_3 k_6 + 3qh_{1,\sigma_1} + 2\sigma_1 q h_{1,\sigma_2} + \sigma_2 q h_{1,\sigma_3} + 2\sigma_2 h_{1,q},\\
& 2qh_5 - k_2 + \sigma_1 k_4 + \sigma_2 k_6 + 3qh_{3,\sigma_1} + 2\sigma_2 h_{3,q},\ -k_4 + 3qh_{5,\sigma_1} + 2\sigma_2 h_{5,q},\\
& h_1 + qk_4 + 3qk_{2,\sigma_1} + 2\sigma_1 q k_{2,\sigma_2} + \sigma_2 q k_{2,\sigma_3} + 2\sigma_2 k_{2,q},\\
& h_3 + 2qk_6 + 3qk_{4,\sigma_1} + 2\sigma_2 k_{4,q},\ h_5 + 3qk_{6,\sigma_1} + 2\sigma_2 k_{6,q}),\\
D_5 \equiv\ & (2\sigma_2 k_2 + 2\sigma_3 k_4 + \sigma_1 q h_{1,\sigma_1} + 2\sigma_2 q h_{1,\sigma_2} + 3\sigma_3 q h_{1,\sigma_3} + 6\sigma_3 h_{1,q},\\
& qh_3 - 2\sigma_1 k_2 + 2\sigma_3 k_6 + 6\sigma_3 h_{3,q},\ 2qh_5 + 2k_2 + 6\sigma_3 h_{5,q},\\
& \sigma_3 h_5 + \sigma_1 q k_{2,\sigma_1} + 2\sigma_2 q k_{2,\sigma_2} + 3\sigma_3 q k_{2,\sigma_3} + 6\sigma_3 k_{2,q},\\
& h_1 - \sigma_2 h_5 + qk_4 + 6\sigma_3 k_{4,q},\ h_3 + \sigma_1 h_5 + 2qk_6 + 6\sigma_3 k_{6,q}),\\
D_6 \equiv\ & (2\sigma_3 q h_5 + (\sigma_1\sigma_2 - \sigma_3)\,k_2 + \sigma_1\sigma_3 k_4 - 2\sigma_2 q h_{1,\sigma_1} - 3\sigma_3 q h_{1,\sigma_2} + 2\sigma_1\sigma_3 h_{1,q},\\
& -\sigma_1^2 k_2 - \sigma_3 k_4,\ qh_3 + \sigma_1 k_2 - \sigma_3 k_6,\\
& \sigma_3 h_3 + \sigma_1\sigma_3 h_5 + 2\sigma_3 q k_6 - 2\sigma_2 q k_{2,\sigma_1} - 3\sigma_3 q k_{2,\sigma_2} + 2\sigma_1\sigma_3 k_{2,q},\\
& -\sigma_2 h_3 + \sigma_3 h_5,\ h_1 + \sigma_1 h_3 - \sigma_2 h_5 + qk_4)
\end{aligned}
$$

where a subscript after a comma represents partial differentiation.

Proof. The generators are defined as

$$\tilde{\Gamma} g = \{T(z,\lambda)\cdot g \mid T \in \boldsymbol{E}^{\Gamma}\} + \{(\delta g)\cdot f \mid f \in \boldsymbol{E}^{\Gamma}\}.$$

The generators of the first submodule are listed in proposition 6.21. Here we have to compute the generators of the second submodule, which is generated by $(\delta g)\cdot f_1, \ldots, (\delta g)\cdot f_6$, where $f_1, \ldots, f_6$ are the generators (3.2) of E^{Γ}.

Again it suffices to compute the first component of $(\delta g)\cdot f_j$, that is

$$g_{1,z_1} f_{j1} + g_{1,z_2} f_{j2} + g_{1,z_3} f_{j3} + g_{1,\bar{z}_1} \bar{f}_{j1} + g_{1,\bar{z}_2} \bar{f}_{j2} + g_{1,\bar{z}_3} \bar{f}_{j3}. \tag{7.4}$$

where g_1 is the first component of $g \in \mathbb{C}^3$, and f_{jk} is the kth component of $f_j \in \mathbb{C}^3$.

Recall that

$$g_1 = H_1 z_1 + K_1 z_2 \bar{z}_3$$

where

$$H_1 = h_1 + h_3 u_1 + h_5 u_1^2 \quad \text{and} \quad K_1 = k_2 + k_4 u_1 + k_6 u_1^2.$$

One may compute directly that

$$
\left.
\begin{aligned}
g_{1,z_1} &= H_1 + H_{1,z_1} z_1 + K_{1,z_1} z_2 \bar{z}_3,\\
g_{1,z_2} &= H_{1,z_2} z_1 + K_{1,z_2} z_2 \bar{z}_3 + K_1 \bar{z}_3,\\
g_{1,z_3} &= H_{1,z_3} z_1 + K_{1,z_3} z_2 \bar{z}_3,\\
g_{1,\bar{z}_1} &= H_{1,\bar{z}_1} z_1 + K_{1,\bar{z}_1} z_2 \bar{z}_3,\\
g_{1,\bar{z}_2} &= H_{1,\bar{z}_2} z_1 + K_{1,\bar{z}_2} z_2 \bar{z}_3,\\
g_{1,\bar{z}_3} &= H_{1,\bar{z}_3} z_1 + K_{1,\bar{z}_3} z_2 \bar{z}_3 + K_1 z_2.
\end{aligned}
\right\} \tag{7.5}
$$

$$\left.\begin{aligned}
H_{1,z_1} &= (h_3+2h_5u_1)\bar{z}_1+H_{1,\sigma_1}\bar{z}_1+H_{1,\sigma_2}(u_2+u_3)\bar{z}_1+H_{1,\sigma_3}u_2u_3\bar{z}_1+H_{1,q}\bar{z}_2z_3,\\
H_{1,z_2} &= H_{1,\sigma_1}\bar{z}_2+H_{1,\sigma_2}(u_1+u_3)\bar{z}_2+H_{1,\sigma_3}u_1u_3\bar{z}_2+H_{1,q}z_1z_3,\\
H_{1,z_3} &= H_{1,\sigma_1}\bar{z}_3+H_{1,\sigma_2}(u_1+u_2)\bar{z}_3+H_{1,\sigma_3}u_1u_2\bar{z}_3+H_{1,q}z_1\bar{z}_2,\\
H_{1,\bar{z}_j} &= \bar{H}_{1,z_j}, j=1,2,3,\\
K_{1,z_1} &= (k_4+2k_6u_1)\bar{z}_1+K_{1,\sigma_1}\bar{z}_1+K_{1,\sigma_2}(u_2+u_3)\bar{z}_1+K_{1,\sigma_3}u_2u_3\bar{z}_1+K_{1,q}\bar{z}_2z_3\\
K_{1,z_2} &= K_{1,\sigma_1}\bar{z}_2+K_{1,\sigma_2}(u_1+u_3)\bar{z}_2+K_{1,\sigma_3}u_1u_3\bar{z}_2+K_{1,q}z_1z_3,\\
K_{1,z_3} &= K_{1,\sigma_1}\bar{z}_3+K_{1,\sigma_2}(u_1+u_2)\bar{z}_3+K_{1,\sigma_3}u_1u_2\bar{z}_3+K_{1,q}z_1\bar{z}_2,\\
K_{1,\bar{z}_j} &= \overline{K_{1,j}}, j=1,2,3,
\end{aligned}\right\} \tag{7.6}$$

$$\left.\begin{aligned}
H_{1,\sigma j} &= h_{1,\sigma j}+h_{3,\sigma j}u_1+h_{5,\sigma j}u_1^2, j=1,2,3,\\
H_{1,q} &= h_{1,q}+h_{3,q}u_1+h_{5,q}u_1^2,\\
K_{1,\sigma j} &= k_{2,\sigma j}+k_{4,\sigma j}u_1+k_{6,\sigma j}u_1^2, j=1,2,3,\\
K_{1,q} &= k_{2,q}+k_{4,q}u_1+k_{6,q}u_1^2,
\end{aligned}\right\} \tag{7.7}$$

$$\left.\begin{aligned}
f_{11} &= z_1, f_{12}=z_2, f_{13}=z_3,\\
f_{21} &= u_1z_1, f_{22}=u_2z_2, f_{23}=u_3z_3,\\
f_{31} &= u_1^2z_1, f_{32}=u_2^2z_3, f_{33}=u_3^2z_3,\\
f_{41} &= z_2\bar{z}_3, f_{42}=z_1z_3, f_{43}=\bar{z}_1z_2,\\
f_{51} &= u_1z_2\bar{z}_3, f_{52}=u_2z_1z_3, f_{53}=u_3\bar{z}_1z_2,\\
f_{61} &= u_1^2z_2\bar{z}_3, f_{62}=u_2^2z_1z_3, f_{63}=u_3^2\bar{z}_1z_2.
\end{aligned}\right\} \tag{7.8}$$

By substituting (7.5)–(7.8) into (7.4), using (6.22) and computing modulo $\mathscr{M}\cdot\mathscr{P}$ one can compute the generators D_1–D_6.

For example, we compute D_4:

$$\begin{aligned}
&g_{1,z_1}f_{41}+g_{1,z_2}f_{42}+g_{1,z_3}f_{43}+g_{1,\bar{z}_1}\bar{f}_{41}+g_{1,\bar{z}_2}\bar{f}_{42}+g_{1,\bar{z}_3}\bar{f}_{43}\\
&\quad= H_1z_2\bar{z}_3+K_1(u_2+u_3)z_1+2\operatorname{Re}(H_{1,z_1}z_2\bar{z}_3+H_{1,z_2}z_1z_3+H_{1,z_3}\bar{z}_1z_2)z_1\\
&\qquad+2\operatorname{Re}(K_{1,z_1}\bar{z}_2z_3+K_{1,z_2}z_1z_3+K_{1,z_3}\bar{z}_1z_2)z_2\bar{z}_3\\
&\quad= [qh_3+\sigma_1k_2-\sigma_3k_6+(2qh_5-k_2+\sigma_1k_4+\sigma_2k_6)u_1-k_4u_1^2]z_1\\
&\qquad+[h_1+qk_4+(h_3+2qk_6)u_1+h_5u_1^2]z_2\bar{z}_3\\
&\qquad+[3qH_{1,\sigma_1}+2\sigma_1qH_{1,\sigma_2}+\sigma_2qH_{1,\sigma_3}+2\sigma_2H_{1,q}]z_1\\
&\qquad+[3qK_{1,\sigma_1}+2\sigma_1qK_{1,\sigma_2}+\sigma_2qK_{1,\sigma_3}+2\sigma_2K_{1,q}]z_2\bar{z}_3\\
&\quad= [qh_3+\sigma_1k_2-\sigma_3k_6+3qh_{1,\sigma_1}+2\sigma_1qh_{1,\sigma_2}+\sigma_2qh_{1,\sigma_3}+2\sigma_2h_{1,q}\\
&\qquad+(2qh_5-k_2+\sigma_1k_4+\sigma_2k_6+3qh_{3,\sigma_1}+2\sigma_2h_{3,q})u_1\\
&\qquad+(-k_4+3qh_{5,\sigma_1}+2\sigma_2h_{5,q})u_1^2]z_1\\
&\qquad+[h_1+qk_4+3qk_{2,\sigma_1}+2\sigma_1qk_{2,\sigma_2}+\sigma_2qk_{2,\sigma_3}+2\sigma_2k_{2,q}\\
&\qquad+(h_3+2qk_6+3qk_{4,\sigma,1}+2\sigma_2k_{4,q})u_1+(h_5+3qk_{6,\sigma_1}+2\sigma_2k_{6,q})u_1^2]z_2\bar{z}_3
\end{aligned}$$

where the last equality is modulo $\mathscr{M}\cdot\mathscr{P}$. □

8. Change of coordinates and normal forms

One purpose of this paper is to show that under certain conditions bifurcation problems $g(z, \lambda)$ have (relatively) simple normal forms, which we denote by $n(z, \lambda)$. More precisely, we wish to find conditions on g such that g is Γ-equivalent to n: that is

$$n(z, \lambda) = T(z, \lambda) \cdot g(Z(z, \lambda), \Lambda(\lambda)) \tag{8.1}$$

where T is a 6×6 matrix satisfying

$$T(\gamma \cdot z, \lambda)\gamma = \gamma \cdot T(z, \lambda)$$

and Z satisfies

$$Z(\gamma \cdot z, \lambda) = \gamma \cdot Z(z, \lambda).$$

The proof of such a normal-form theorem proceeds in two stages. First, one computes explicitly a Γ-equivalence of g that shows that g is Γ-equivalent to n + higher-order terms. Of course, one must decide in advance which are the higher-order terms and this depends on the particular form of g. Second, one uses standard theorems from singularity theory to show that n + higher-order terms is Γ-equivalent to n. We give the first stage of this process in this section and the second stage in the next section.

We have identified bifurcation problems g that commute with the action of Γ to 6-tuples of invariant functions $(h_1, h_3, h_5, k_2, k_4, k_6)$. As noted in §5 the Jacobian matrix $\mathrm{d}_z g$ is just $h_1(0, \lambda) I$ when $z = 0$. We assume that $(z, \lambda) = (0, 0)$ is a point of bifurcation and that the trivial solution $z = 0$ changes stability at $(0, 0)$. Thus

$$h_1(0, 0) = 0 \quad \text{and} \quad h_{1,\lambda}(0, 0) \neq 0. \tag{8.2}$$

Normally in the Γ-equivalence (8.1), see Golubitsky & Schaeffer (1979), we demand that $\Lambda'(0) > 0$. We relax that assumption here with the understanding that by doing so the orientation of λ in the normal form can be reversed to obtain another example; that is, the bifurcation diagrams we draw in §10 may be read either from left to right or from right to left. On the other hand, we showed in §5 that for the Γ-equivalence (8.1) to maintain the linearized stability assignment given to a solution one must assume that $T(0,0)$ and $(\mathrm{d}_z Z)(0,0)$ – which are both multiples of the identity matrix, say $A_1 I$ and ϵI respectively – satisfy A_1, $\epsilon > 0$. We relax this assumption in (8.1) with the understanding that we have postponed deciding whether the sign of the stable eigenvalue in the linearized stability analysis is + or −.

Algebraically it makes sense to define the higher-order terms of g to be a submodule of E. Letting $\mathscr{M}$ be the maximal ideal in $\mathscr{E}$ and $\mathscr{N}$ be the ideal in $\mathscr{E}$ generated by $\sigma_1^2, \sigma_2, \sigma_3, q, \lambda$ as defined by §6, we define

$$\left.\begin{aligned} \mathscr{P}_1 &= (\mathscr{M}^2, \mathscr{M}, \mathscr{M}, \mathscr{M}, \mathscr{M}, \mathscr{M}), \\ \mathscr{P}_2 &= (\mathscr{M}\mathscr{N}, \mathscr{N}, \mathscr{M}, \mathscr{M}, \mathscr{M}, \mathscr{M}), \\ \mathscr{P} &= (\mathscr{M}\mathscr{N}, \mathscr{N}, \mathscr{M}, \mathscr{M}^2, \mathscr{M}, \mathscr{M}). \end{aligned}\right\} \tag{8.3}$$

Note that

$$\mathscr{P}_1 \supset \mathscr{P}_2 \supset \mathscr{P} \supset \mathscr{M} \cdot \mathscr{P}$$

where $\mathscr{P}$ is defined in §6. For example, we can write g modulo $\mathscr{P}$ as

$$h_1 \equiv \alpha\lambda + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + a_4 q + a_5\sigma_1^2, \tag{8.4a}$$

$$h_3 \equiv b_0 + b_1\sigma_1, \tag{8.4b}$$

$$h_5 \equiv c_0, \tag{8.4c}$$

$$k_2 \equiv \beta\lambda + d_0 + d_1\sigma_1 + d_2\sigma_2 + d_3\sigma_3 + d_4 q, \tag{8.4d}$$

$$k_4 \equiv e_0, \tag{8.4e}$$

$$k_6 \equiv f_0. \tag{8.4f}$$

We now state the main result of this section, using the notation that the normal form $n(z,\lambda)$ is identified in $\mathscr{E}^6$ with $(v_1, v_3, v_5, w_2, w_4, w_6)$ where the v_js and w_js are invariant functions.

Proposition 8.5. *As stated in* (8.2) *we assume* $\alpha \neq 0$.

(i) *Assume*
$$d_0 \neq 0, \quad a_1 + b_0 \neq 0. \tag{8.6}$$

Then $g(z,\lambda)$ *is* Γ-*equivalent modulo* $\mathscr{P}_1$ *to* $n_1(z,\lambda)$ *where*
$$n_1 = (-\lambda, 1, 0, 1, 0, 0). \tag{8.7}$$

(ii) *Assume*
$$d_0 \neq 0, \quad a_1 + b_0 = 0, \quad a_2 + a_5 + b_1 \neq 0. \tag{8.8}$$

Then $g(z,\lambda)$ *is* Γ-*equivalent modulo* $\mathscr{P}_2$ *to* $n_2(z,\lambda)$ *where*
$$n_2 = (-\lambda + \sigma_1^2, 0, 0, 1, 0, 0). \tag{8.9}$$

(iii) *Assume*
$$d_0 = 0, \quad b_0 \neq 0, \quad 3a_1 + b_0 \neq 0, \quad e_0 \neq \frac{b_0}{2}\left(\frac{3a_4}{3a_1+b_0} - \frac{\beta}{\alpha}\right). \tag{8.10}$$

Then $g(z,\lambda)$ *is* Γ-*equivalent modulo* $\mathscr{P}$ *to* $n_3(z,\lambda)$ *where*
$$n_3 = (-\lambda + a\sigma_1 + d\sigma_1^2, 1, 0, b\sigma_1 + cq, 1, 0). \tag{8.11}$$

In the normal form the condition $3a_1 + b_0 \neq 0$ *becomes*
$$3a + 1 \neq 0. \tag{8.12}$$

The constants a, b, *and* c *are defined by*
$$\left.\begin{aligned} a &= a_1/b_0, \\ b &= \epsilon^2[A_1(\epsilon^2 d_1 + \zeta a_1) + A_2\epsilon(2a_1 + b_0)], \\ c &= A_1\epsilon^5[(3a_1+b_0)\,d_4 - (3d_1 + e_0)\,a_4]/(3a_1+b_0) \end{aligned}\right\} \tag{8.13}$$

where
$$\left.\begin{aligned} \epsilon &= 2\alpha b_0(3a_1+b_0)/[(2e_0\alpha + b_0\beta)\,(3a_1+b_0) - 3\alpha a_4 b_0], \\ A_1 &= [(2e_0\alpha + b_0\beta)\,(3a_1+b_0) - 3\alpha a_4 b_0]^3/[2\alpha b_0^2(3a_1+b_0)], \\ \zeta &= \epsilon^2 a_4/(3a_1+b_0), \\ A_2 &= -A_1(\epsilon^2\beta + \zeta\alpha)/(2\epsilon\alpha). \end{aligned}\right\} \tag{8.14}$$

Moreover, any further Γ-*equivalence that maintains the form* (8.11) *leaves the values of* a, b, c, *and* d *unchanged. Therefore, one cannot scale the parameters* a, b, c, *and* d *further.*

Remarks. (*a*) We have not given an explicit formula for d in terms of the coefficients of g in (8.4) for two reasons. First it is a long and horrible expression and second we do not use the particular value of d for the results in this paper.

(*b*) In the Introduction we described in general terms the two bifurcation problems we wished to study. The first was the simplest singularity when the quadratic term $d_0 \neq 0$. This corresponds to (i) in proposition 8.5. Note that to study this example one has to assume in addition that $a_1 + b_0 \neq 0$. The second case was the simplest singularity when $d_0 = 0$ and that is given in (iii). For completeness, we considered the case when $d_0 \neq 0$ and $a_1 + b_0 = 0$, which is given in (ii).

We shall compute the general Γ-equivalence modulo $\mathscr{P}$ in two steps by considering separately the changes of coordinates in the domain and range. Let
$$g'(x,\lambda) = g(Z(z,\lambda), \Lambda(\lambda))$$

be the change in the domain and

$$g''(z, \lambda) = T(z, \lambda)\, g'(z, \lambda)$$

be the change in the range. In (8.4) we expanded g modulo $\mathscr{P}_3$ explicitly. We can, of course, do the same for g' and g''; we indicate the corresponding coefficients by adding a prime and a double prime respectively to the coefficients found in (8.4). For example, g' is identified with the 6-tuple of invariant functions

$$g' = (l_1, l_3, l_5, m_2, m_4, m_6) \tag{8.15}$$

and $l_5 \equiv c_0'$ modulo $\mathscr{P}$. Before proving proposition 8.5 we give the results of explicit computation of g' from g, and g'' from g', deferring the outline of these computations until the end of this section.

We identify the change of coordinates $Z(z, \lambda)$ with the 6-tuple of invariant functions

$$Z \sim (r_1, r_3, r_5, s_2, s_4, s_6) \tag{8.16}$$

and use the notation

$$\left.\begin{aligned} &\epsilon = r_1(0) \neq 0,\ \eta = r_{1,\sigma_1}(0),\ \theta = r_3(0),\ \zeta = s_2(0), \\ &\phi = s_{2,\sigma_1}(0),\ \mu = s_4(0),\ \gamma = \Lambda'(0) \neq 0. \end{aligned}\right\} \tag{8.17}$$

LEMMA 8.18. *The computation of g' from g modulo $\mathscr{P}$ yields the following. Note that we have only computed those coefficients that we shall need explicitly.*

$$\alpha' = \gamma\epsilon\alpha, \tag{8.19a}$$

$$a_1' = \epsilon^3 a_1 + \epsilon\zeta d_0, \tag{8.19b}$$

$$a_2' = \epsilon(\zeta^2 - 4\epsilon\theta)\, a_1 + \epsilon^5 a_2 + 2\epsilon^3\zeta a_4 + \epsilon\zeta^2 b_0 + 2(\epsilon\mu - \theta\zeta)\, d_0, \tag{8.19c}$$

$$a_4' = 3\epsilon^2\zeta a_1 + \epsilon^4 a_4 + \epsilon^2\zeta b_0 + \zeta^2 d_0, \tag{8.19d}$$

$$a_5' = \epsilon^2(2\theta + 3\eta)\, a_1 + \epsilon^5 a_5 + (\theta\zeta + \eta\zeta + \epsilon\phi)\, d_0 + \epsilon^3\zeta d_1, \tag{8.19e}$$

$$b_0' = \epsilon^3 b_0 - \epsilon\zeta d_0, \tag{8.19f}$$

$$b_1' = \epsilon^2\theta a_1 + \epsilon(3\epsilon\eta - \zeta^2)\, b_0 + \epsilon^5 b_1 - (2\epsilon\mu + \eta\zeta + \epsilon\phi)\, d_0 - \epsilon^3\zeta d_1 + \epsilon^3\zeta e_0, \tag{8.19g}$$

$$c_0' = \epsilon(3\epsilon\theta + \zeta^2)\, b_0 + \epsilon^5 c_0 + (2\epsilon\mu - \theta\zeta)\, d_0 - \epsilon^3\zeta e_0, \tag{8.19h}$$

$$\beta' = \gamma\zeta\alpha + \gamma\epsilon^2\beta, \tag{8.19i}$$

$$d_0' = \epsilon^2 d_0, \tag{8.19j}$$

$$d_1' = \epsilon^2\zeta a_1 + \epsilon(\theta + 2\eta)\, d_0 + \epsilon^4 d_1, \tag{8.19k}$$

$$d_4' = 3\epsilon\zeta^2 a_1 + \epsilon^3\zeta a_4 + \epsilon\zeta^2 b_0 + 3\epsilon^3\zeta d_1 + \epsilon^5 d_4 + \epsilon^3\zeta e_0, \tag{8.19l}$$

$$e_0' = \epsilon^2\zeta b_0 - (\epsilon\theta + \zeta^2)\, d_0 + \epsilon^4 e_0. \tag{8.19m}$$

We now compute g'' from g'. Recall the notation for $g' \in (\mathscr{E})^6$ given in (8.15).

LEMMA 8.20. *The general formula for g'' modulo $\mathscr{P}$ is given by*

$$g'' = Tg' = \sum_{j=1}^{22} A_j C_j \tag{8.21}$$

where $A_j \in \mathbb{R}$ are arbitrary except that $A_1 \neq 0$, and

$$\left.\begin{aligned}
C_1 &= (l_1, l_3, l_5, m_2, m_4, m_6),\\
C_2 &= (\sigma_1 m_2 + 2\sigma_2 m_4 - \sigma_3 m_6,\ -m_2 - 2\sigma_1 m_4,\ 2m_4,\ 2l_1 + \sigma_1 l_3 - 2\sigma_2 l_5,\ -l_3,\ -l_5),\\
C_3 &= (\sigma_3 l_5, l_1, l_3, 0, m_2, 0),\\
C_4 &= (\sigma_1 l_1 + (\sigma_1^2 - 2\sigma_2)\, l_3 + 3\sigma_3 l_5, 0, 0, 0, 3m_2, 0),\\
C_5 &= (0, 0, 0, \sigma_3 m_6, 0, m_4), & C_6 &= (\sigma_3 m_4, 0, 0, 0, 0, 0),\\
C_7 &= (0, 0, 0, \sigma_3 l_5, 0, l_3), & C_8 &= (0, 0, 0, -2\sigma_2 l_3 + 3\sigma_3 l_5, 0, 0),\\
C_9 &= (\sigma_3 l_3, 0, 0, 0, 0, 0), & C_{10} &= (0, 0, 0, \sigma_3 l_3, 0, 0),\\
C_{11} &= (0, 0, 0, \sigma_3 m_4, 0, 0), & C_{12} &= (\sigma_1 l_1, \sigma_1 l_3, 0, \sigma_1 m_2, 0, 0),\\
C_{13} &= (-q m_2, 0, 0, 0, 2m_2, 0), & C_{14} &= (0, \sigma_1 m_2, -m_2, 0, 0),\\
C_{15} &= (\sigma_1 m_2, 0, 0, 0, 0, 0), & C_{16} &= (\sigma_3 m_2, 0, 0, 0, 0, 0),\\
C_{17} &= (0, 0, 0, \sigma_2 m_2, 0, 0), & C_{18} &= (0, 0, 0, \sigma_3 m_2, 0, 0),\\
C_{19} &= (0, 0, 0, 0, 0, m_2), & C_{20} &= (0, 0, 0, 0, q m_2, 0, 0),\\
C_{21} &= (\sigma_1^2 m_2, -\sigma_1 m_2, 0, 0, 0, 0), & C_{22} &= (0, 0, 0, \lambda m_2, 0, 0).
\end{aligned}\right\} \tag{8.22}$$

We can now use this lemma to compute the coefficients of g'' in terms of those of g' using the notation corresponding to (8.4):

$$\alpha'' = A_1 \alpha', \tag{8.23a}$$

$$a_1'' = A_1 a_1' + A_2 d_0', \tag{8.23b}$$

$$a_2'' = A_1 a_2' + 2A_2 e_0' - 2A_4 b_0' + A_{15} d_0', \tag{8.23c}$$

$$a_3'' = A_1 a_3' - A_2 f_0' + (A_3 + 2A_4)\, c_0' + A_6 e_0' + A_9 b_0' + A_{16} d_0', \tag{8.23d}$$

$$a_4'' = A_1 a_4' - A_{13} d_0', \tag{8.23e}$$

$$a_5'' = A_1 a_5' + A_2 d_1' + A_4 (a_1' + b_0') + A_{12} a_1' + A_{21} d_0', \tag{8.23f}$$

$$b_0'' = A_1 b_0' - A_2 d_0', \tag{8.23g}$$

$$b_1'' = A_1 b_1' + A_{12} b_0' - A_2 (d_1' + 2e_0') + A_3 a_1' + (A_{14} - A_{21})\, d_0', \tag{8.23h}$$

$$c_0'' = A_1 c_0' + 2A_2 e_0' + A_3 b_0' - A_{14} d_0', \tag{8.23i}$$

$$\beta'' = A_1 \beta' + 2A_2 \alpha' + A_{22} d_0', \tag{8.23j}$$

$$d_0'' = A_1 d_0', \tag{8.23k}$$

$$d_1'' = A_1 d_1' + A_2 (2a_1' + b_0') + A_{12} d_0', \tag{8.23l}$$

$$d_2'' = A_1 d_2' + 2A_2 (a_2' - c_0') - 2A_8 b_0' + A_{17} d_0', \tag{8.23m}$$

$$d_3'' = A_1 d_3' + 2A_2 a_3' + A_5 f_0' + (A_7 + 3A_8)\, c_0' + A_{11} e_0' + A_{10} b_0' + A_{18} d_0', \tag{8.23n}$$

$$d_4'' = A_1 d_4' + 2A_2 a_4' + A_{20} d_0', \tag{8.23o}$$

$$e_0'' = A_1 e_0' - A_2 b_0' + (A_3 + 3A_4 + 2A_{13})\, d_0', \tag{8.23p}$$

$$f_0'' = A_1 f_0' - A_2 c_0' + A_5 e_0' + A_7 b_0' + A_{19} d_0'. \tag{8.23q}$$

We shall sketch the proofs of lemmas 8.18 and 8.20 and the calculations (8.23) at the end of this section.

Proof of proposition 8.5. Recall that A_1, γ, and ϵ are non-zero constants.

(i) We need only compute g'' modulo $\mathscr{P}_1$ in this case. The coefficients of g in (8.4) that can be non-zero modulo $\mathscr{P}_1$ are

$$\alpha, a_1, a_2, a_3, a_4, b_0, c_0, d_0, e_0, f_0. \tag{8.24}$$

Note that $d_0 \neq 0$ implies $d_0' \neq 0$. Thus one can choose A_2, A_{15}, A_{16}, A_{13}, A_{14}, A_3, and A_{19} respectively so that

$$a_1'' = a_2'' = a_3'' = a_4'' = c_0'' = e_0'' = f_0'' = 0. \tag{8.25}$$

Note that

$$A_2 = -A_1\alpha'/d_0'.$$

With use of (8.23) for the first equality and (8.19) for the second it follows that

$$\left.\begin{aligned} \alpha'' &= A_1\alpha' = A_1\gamma\epsilon\alpha, \\ b_0'' &= A_1(a_1'+b_0') = A_1\epsilon^3(a_1+b_0), \\ d_0'' &= A_1 d_0' = A_1\epsilon^2 d_0. \end{aligned}\right\} \tag{8.26}$$

As $a_1+b_0 \neq 0$ one may choose γ so that $\alpha'' = -1$, ϵ so that $b_0'' = 1$, and A_1 so that $d_0'' = 1$.

(ii) For this case we need only compute g'' modulo $\mathscr{P}_2$. The coefficients of g in (8.4) that can be non-zero modulo $\mathscr{P}_2$ are a_5, b_1, and those listed in (8.24). As before, one can show that the coefficients listed in (8.25) are zero since we still assume $d_0 \neq 0$. Note that the assumption that $a_1+b_0 = a_1'+b_0' = 0$ implies that now $b_0'' = 0$. In addition, one can choose A_{21} so that $b_1'' = 0$. It follows that:

$$\left.\begin{aligned} \alpha'' &= A_1\alpha' = A_1\gamma\epsilon\alpha, \\ a_5'' &= A_1(a_5'+b_1'+c_0') = A_1\epsilon^5(a_5+b_1+c_0), \\ d_0'' &= A_1 d_0' = A_1\epsilon^2 d_0. \end{aligned}\right\} \tag{8.27}$$

Since $a_5+b_1+c_0 \neq 0$, one can choose γ, ϵ, and A_1 so that $\alpha'' = -1$, $a_5'' = 1$, and $d_0'' = 1$.

(iii) The assumptions that $b_0 \neq 0$ and $d_0 = 0$ suffice to show, using (8.19*f*), that $b_0' \neq 0$. As a result, one can choose A_4, A_9, A_{12}, A_3, A_8, A_{10}, A_7 appropriately so that

$$a_2'' = a_3'' = b_1'' = c_0'' = d_2'' = d_3'' = f_0'' = 0.$$

Here one uses the corresponding equations in (8.23). Next observe, using (8.19*a*), that $\alpha \neq 0$ implies $\alpha' \neq 0$. So one can choose A_2 in (8.23*j*) so that $\beta'' = 0$. One can also change coordinates so that $a_4'' = 0$. To do this, note that $d_0 = 0$ implied $d_0' = 0$. Then (8.23*e*) and (8.19*d*) imply

$$a_4'' = A_1\epsilon^2[(3a_1+b_0)\zeta+\epsilon^2 a_4]. \tag{8.28}$$

Now the assumption that $3a_1+b_0 \neq 0$ allows one to choose ζ so that $a_4'' = 0$.

In the statement of the proposition we claimed that there were three coefficients that could be scaled to unity. In particular, we claim that it is possible to choose $\alpha'' = -1$, $b_0'' = 1$, and $e_0'' = 1$. Using (8.19) and (8.23) one shows

$$\alpha'' = A_1\gamma\epsilon\alpha, \tag{8.29a}$$

$$b_0'' = A_1\epsilon^3 b_0, \tag{8.29b}$$

$$e_0'' = A_1\epsilon^4[e_0-\beta b_0/2\alpha-3a_4 b_0/2(3a_1+b_0)]. \tag{8.29c}$$

In deriving (8.29*c*), one uses the facts

$$A_2 = -A_1\beta'/(2\alpha'), \tag{8.30a}$$

$$\zeta = -\epsilon^2 a_4/(3a_1+b_0). \tag{8.30b}$$

One can now choose γ so that $\alpha'' = -1$ and A_1 so that $b_0'' = 1$. Since the third factor in (8.29*c*) is assumed to be non-zero one can choose ϵ so that $e_0'' = 1$.

We have now found a change of coordinates that puts g into the normal form n_3. Note that we have used the following notation in (8.11):

$$a = a_1'', \; d = a_5'', \; b = d_1'', \; c = d_4''.$$

Using (8.19) and (8.23) one can compute the values of a, b, c, and d explicitly from the original coefficients of g. We have recorded the results for a, b, and c in (8.13) and (8.14).

Finally, one shows that an arbitrary Γ-equivalence (modulo $\mathscr{P}$) that maintains the normal form (8.11) leaves the values of a, b, c, and d unchanged. This is a long but straightforward calculation involving all the formulae in (8.19) and (8.23). □

We now sketch the proofs of lemmas 8.18 and 8.20. These are both long but straightforward calculations.

Proof of lemma 8.18. Our aim is to compute the general change of coordinates of g given by $g(Z, \Lambda)$ modulo $\mathscr{P}$. We use capital letters to indicate the various quantities in the variables Z, Λ. For example $U_j = Z_j \bar{Z}_j$ and $\Sigma_1 = U_1 + U_2 + U_3$. One can then write the first coordinate of $g(Z, \Lambda)$ modulo $\mathscr{P}$ as

$$\begin{aligned}(\alpha\Lambda + a_1\Sigma_1 + a_2\Sigma_2 + a_3\Sigma_3 + a_4 Q + a_5\Sigma_1^2) Z_1 + (b_0 + b_1\Sigma_1) U_1 Z_1 + c_0 U_1^2 Z_1 \\ + (\beta\Lambda + d_0 + d_1\Sigma_1 + d_2\Sigma_2 + d_3\Sigma_3 + d_4 Q) Z_2 \bar{Z}_3 + e_0 U_1 Z_2 \bar{Z}_3 + f_0 U_1^2 Z_2 \bar{Z}_3. \quad (8.31)\end{aligned}$$

Observe that in (8.19) we do not compute the coefficients a_3', d_2', d_3', and f_0'. We showed in the proof of proposition 8.5 that the explicit calculation of these coefficients was not necessary. As a result, we do not have to keep track of any coefficients involving σ_3. Moreover, we showed in (8.4) that when computing modulo $\mathscr{P}$ we need not keep track of any coefficients involving $\mathscr{M}^2$, $\mathscr{M}u_1$, or $\mathscr{M}u_1^2$ except for σ_1^2 and $\sigma_1 u_1$. These observations simplify the calculations substantially. The reader should find the identities (6.22) useful. We find

$$\left.\begin{aligned} U_1 &= \zeta^2\sigma_2 + \epsilon\zeta q + [\epsilon^2(2\epsilon\eta - \zeta^2)\,\sigma_1]\,u_1 + (2\epsilon\theta + \zeta^2)\,u_1^2, \\ U_1^2 &\equiv \epsilon^4 u_1^2, \\ U_2 + U_3 &= \epsilon^2\sigma_1 - 4\epsilon\theta\sigma_2 + 2\epsilon\zeta q + 2\epsilon(\theta+\eta)\,\sigma_1^2 + [(\zeta^2 - 2\epsilon\eta)\,\sigma_1 - \epsilon^2]\,u_1 - (\zeta^2 + 2\epsilon\theta)\,u_1^2, \\ \Sigma_1 &= \epsilon^2\sigma_1 + (\zeta^2 - 4\epsilon\theta)\,\sigma_2 + 3\epsilon\zeta q + 2\epsilon(\theta+\eta)\,\sigma_1^2, \\ \Sigma_2 &= U_1(U_2+U_3) + U_2 U_3 \equiv \epsilon^4\sigma_2, \\ \Sigma_3 &= U_1 U_2 U_3 \equiv 0, \\ Q &= 2\,\mathrm{Re}\,(Z_1\bar{Z}_2 Z_3) \equiv 2\epsilon^2\zeta\sigma_2 + \epsilon^3 q, \\ \Sigma_1^2 &= \epsilon^4\sigma_1^2, \\ \Lambda &\equiv \gamma\lambda. \end{aligned}\right\} \quad (8.32)$$

Next, we calculate modulo $\mathscr{P}$. As discussed we also ignore any terms of the form $\sigma_3 z_1$, $\sigma_2 z_2 \bar{z}_3$, $\sigma_3 z_2 \bar{z}_3$ and $u_1^2 z_2 \bar{z}_3$. These terms correspond to the coefficients a_3', d_2', d_3' and f_0' respectively. We obtain

$$\left.\begin{aligned} U_1 Z_1 &\equiv \{\epsilon\zeta^2\sigma_2 + \epsilon^2\zeta q + \epsilon[\epsilon^2 + (3\epsilon\eta - \zeta^2)\,\sigma_1]\,u_1 + \epsilon(3\epsilon\theta + \zeta^2)\,u_1^2\} z_1 + (\epsilon\zeta^2 q + \epsilon^2\zeta u_1)\,z_2\bar{z}_3, \\ U_1^2 Z_1 &\equiv \epsilon^5 u_1^2 z_1, \\ Z_2\bar{Z}_3 &\equiv [\epsilon\zeta\sigma_1 + 2(\epsilon\mu - \theta\zeta)\,\sigma_2 + \zeta^2 q + (\theta\zeta + \eta\zeta + \epsilon\phi)\,\sigma_1^2 - \epsilon\zeta u_1 - (2\epsilon\mu + \eta\zeta + \epsilon\phi)\,\sigma_1 u_1 \\ &\qquad + (2\epsilon\mu - \theta\zeta)\,u_1^2]\,z_1 + [\epsilon^2 + \epsilon(\theta + 2\eta)\,\sigma_1 - (\epsilon\theta + \zeta^2)\,u_1]\,z_2\bar{z}_3, \\ U_1 Z_2\bar{Z}_3 &\equiv \epsilon^3\zeta(\sigma_1 u_1 - u_1^2)\,z_1 + \epsilon^3(\zeta q + \epsilon u_1)\,z_2\bar{z}_3, \\ U_1^2 z_2\bar{z}_3 &\equiv 0. \end{aligned}\right\} \quad (8.33)$$

Finally one substitutes the results of (8.32) and (8.33) into (8.31) to obtain the formulae (8.19). □

Proof of lemma 8.20. The general form of a Γ-equivalence $T(z, g(z, \lambda))$ is given, modulo the submodule $\mathscr{M}\cdot\mathscr{P}$, in proposition 6.2 as

$$\sum_{j=1}^{20} s_j T_j(z, g(z)), \tag{8.34}$$

where each s_j has the form $a_j + b_j u_1 + c_j u_1^2$ and the a_js, b_js, and c_js are invariant functions. In this lemma we wish to find an expansion of $T(z, g(z,\lambda))$ modulo the submodule $\mathscr{P}$ as

$$\sum_{j=1}^{22} A_j C_j,$$

where the A_js are real numbers.

Observe that we first identify g with the 6-tuple of invariant functions $(l_1, l_3, l_5, m_2, m_4, m_6)$ rather than $(h_1, h_3, h_5, k_2, k_4, k_6)$ as in proposition 6.21. The first step in the proof of this lemma is to write out all of the terms in (8.34) modulo $\mathscr{P}$ and then observe that the only terms that can possibly be independent are T_j $(j = 1, \dots, 12)$. T_{14}, T_{15}, T_{17}, $u_1 T_1$, $u_1^2 T_1$, λT_1, $\sigma_1 T_1$, $\sigma_2 T_1$ qT_1 and $\sigma_1 T_2$. An equivalent set of vector-space generators is given by C_j $(j = 1, \dots, 22)$. We do not claim that the C_js are the most natural choice; they are the ones we have used. The relations between the C_js and the T_js are given by

$$\left.\begin{aligned}
C_1 &= T_1, & C_2 &= T_2,\\
C_3 &= \tfrac{1}{6}(T_5 - T_4 + 2T_3 + 4u_1 T_1), & C_4 &= T_5 + u_1 T_1,\\
C_5 &= \tfrac{1}{6}(T_4 - T_5 - 2T_3 + 2u_1 T_1), & C_6 &= \tfrac{1}{6}(2T_6 + 2T_7 - T_9 + T_{10} - 2qT_1),\\
C_7 &= \tfrac{1}{2}(T_7 - T_6), & C_8 &= \tfrac{1}{2}(T_7 - T_6 + T_9 + T_{10} - 2qT_1),\\
C_9 &= T_{11} - u_1^2 T_1, & C_{10} &= \tfrac{1}{4}(T_{15} - T_{14}),\\
C_{11} &= \tfrac{1}{3}(T_{12} + u_1^2 T_1), & C_{12} &= \sigma_1 T_1,\\
C_{13} &= \tfrac{1}{3}(T_5 - T_4 + u_1 T_1 - T_3), & C_{14} &= \tfrac{1}{6}(-T_6 - 7T_7 + 6T_8 + 2T_9 - 2T_{10} + 4qT_1),\\
C_{15} &= \tfrac{1}{2}(2T_8 - 2T_7 + T_9 - T_{10} + 2qT_1), & C_{16} &= -\tfrac{1}{2}(T_{14} + T_{15}),\\
C_{17} &= \sigma_2 T_1, & C_{18} &= T_{17},\\
C_{19} &= \tfrac{1}{3}(5u_1^2 T_1 - 3T_{11} - T_{12}), & C_{20} &= qT_1,\\
C_{21} &= \sigma_1 T_2, & C_{22} &= \lambda T_1.
\end{aligned}\right\} \tag{8.35}$$

□

9. Determinacy and unfolding

We now show that the normal forms n_1, n_2 and n_3 of proposition 8.5 are $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}$-determined where $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}$ are the submodules of $\mathscr{E}^6$ defined in (8.3). More precisely, we show that any map germ $n_j + p$ where $p \in \mathscr{P}_j$ is Γ-equivalent to n_j. Moreover, we compute the codimension and a universal unfolding for each of the normal forms n_j.

Our main result is the following:

THEOREM 9.1. *Let* $g(z,\lambda) = (h_1, h_3, h_5, k_2, k_4, k_6)$ *be the bifurcation problem defined in* (8.4) *with* $\alpha \neq 0$.

(i) *If g satisfies* (8.6), *then g is Γ-equivalent to*

$$n_1(z,\lambda) = (-\lambda, 1, 0, 1, 0, 0). \tag{9.2}$$

Moreover, $\operatorname{codim}_\Gamma n_1 = 0$; *hence all small perturbations of n_1 are Γ-equivalent to n_1.*

(ii) *If g satisfies* (8.8) *then g is Γ-equivalent to*

$$n_2(z,\lambda) = (-\lambda+\sigma_1^2, 0, 0, 1, 0, 0). \tag{9.3}$$

Moreover $\operatorname{codim}_\Gamma n_2 = 1$ *and a universal unfolding is*

$$f_2(z,\lambda,a) = (-\lambda+\sigma_1^2, a, 0, 1, 0, 0). \tag{9.4}$$

where a is near 0.

(iii) *If g satisfies* (8.10) *and*

$$\left.\begin{aligned} a_1+b_0 \neq 0,\ 2a_1+b_0 \neq 0,\ (3d_1+e_0)\,\alpha \neq (3a_1+b_0)\,\beta,\\ (3a_1+b_0)\,d_4 \neq (3d_1+e_0)\,a_4 \end{aligned}\right\} \tag{9.5}$$

then g is Γ-equivalent to

$$n_3(z,\lambda) = (-\lambda+a\sigma_1+d\sigma_1^2, 1, 0, b\sigma_1+cq, 1, 0). \tag{9.6}$$

where the modal parameters a, b, c are defined in (8.13). (*We did not compute d explicitly as it appears to be a topologically trivial parameter.*) *The modal parameters satisfy*

$$a \neq -1, -\tfrac{1}{2}, -\tfrac{1}{3};\ b \neq -\tfrac{1}{3},\ c \neq 0. \tag{9.7}$$

Moreover $\operatorname{codim}_\Gamma n_3 = 5$ (*though the topological codimension is* 1) *and a universal unfolding is*

$$f_3(z,\lambda,\tilde a,\tilde b,\tilde c,\tilde d,e) = (-\lambda+\tilde a\sigma_1+\tilde d\sigma_1^2, 1, 0, -e+\tilde b\sigma_1+\tilde c q, 1, 0) \tag{9.8}$$

where $\tilde a$, $\tilde b$, $\tilde c$, $\tilde d$ are near a, b, c, d respectively and e is near zero.

The determinacy results proceed in two steps. First one uses proposition 8.5 to show that g is Γ-equivalent to n_j+p where $p\in\mathscr{P}_j$. Next one shows that $\tilde\Gamma(n_j+p) = \tilde\Gamma(n_j)$ for all $p\in\mathscr{P}_j$ where $\tilde\Gamma(g)$ is the submodule defined in (7.1). Finally one applies theorem 1.13 of Golubitsky & Schaeffer (1979), which states that if $\tilde\Gamma(n_j+p) = \tilde\Gamma(n_j)$ for all $p\in\mathscr{P}_j$ then n_j+p is Γ-equivalent to n_j for all $p\in\mathscr{P}_j$. Once $\tilde\Gamma(n_j)$ has been computed, one can compute a universal unfolding straightforwardly. Let

$$\Gamma(n_j) = \tilde\Gamma(n_j) + \mathscr{E}_\lambda\{n_{j,\lambda}\} \tag{9.9}$$

where $\mathscr{E}_\lambda$ is the ring of germs of C^∞ functions in the variable λ. Let Q be a vector subspace of E satisfying

$$\Gamma(n_j) \oplus Q = E.$$

Then $\operatorname{codim}_\Gamma n_j = \dim Q$. Moreover, if $\{q_1, \ldots, q_l\}$ is a basis of Q then

$$f(z,\lambda,\alpha) = g(z,\lambda) + \sum_{j=1}^{l} \alpha_j q_j(z,\lambda)$$

is a universal unfolding of g. (The reader is referred to §1 of Golubitsky & Schaeffer (1979) for more detail.)

The computation of $\tilde\Gamma(n_j)$ uses

NAKAYAMA'S LEMMA. *Let A and B be submodules of E where A is finitely generated. If $A \subset B+\mathscr{M}\cdot A$, then $A\subset B$.*

Here $\mathscr{M}$ is the maximal ideal in $\mathscr{E}$.

Proof of theorem 9.1

(i) We must show that $\tilde\Gamma(n_1+p) = \tilde\Gamma(n_1)$ for all $p\in\mathscr{P}_1$. Let $\mathscr{L}_1$ be the ideal in $\mathscr{E}$ generated by λ^2, $\sigma_1\lambda$, σ_1^2, σ_2, σ_3, and q. Define the submodule $\mathscr{Q}_1$ of E by

$$\mathscr{Q}_1 = (\mathscr{L}_1, \mathscr{M}, \mathscr{E}, \mathscr{M}, \mathscr{E}, \mathscr{E}). \tag{9.10}$$

We claim

$$\tilde\Gamma(n_1+p) = \mathscr{Q}_1 \oplus \mathbb{R}\{(-\lambda,1,0,1,0,0),\ (\sigma_1,-1,0,0,0,0),\ (-\lambda,3,0,2,0,0)\} \tag{9.11}$$

for all $p \in \mathscr{P}_1$. It follows from (9.11) that $\tilde{\Gamma}(n_1+p)$ is independent of p, so $\tilde{\Gamma}(n_1+p) = \tilde{\Gamma}(n_1)$. Thus n_1+p is Γ-equivalent to n_1. It is easy to see that $\dim E/\mathscr{Q}_1 = 5$. Moreover, the vectors $n_{1,\lambda}$, $\lambda n_{1,\lambda}$ satisfy

$$n_{1,\lambda} \equiv (-1,0,0,0,0,0) \bmod \mathscr{Q}_1, \quad \lambda n_{1,\lambda} \equiv (-\lambda,0,0,0,0,0) \bmod \mathscr{Q}_1. \tag{9.12}$$

Using (9.11) and (9.12) one sees that $\Gamma(n_1) = E$ (see (9.9)). Hence $\operatorname{codim}_\Gamma n_1 = 0$.

We sketch the proof that (9.11) holds. We first show that $\mathscr{Q}_1 \subset \tilde{\Gamma}(n_1+p) + \mathscr{M}\cdot\mathscr{Q}_1$ and hence by Nakayama's lemma that $\mathscr{Q}_1 \subset \tilde{\Gamma}(n_1+p)$. Note that the module $\mathscr{Q}_1$ is generated by 19 generators: six in the first coordinate, five each in the second and fourth coordinates, and one each in the remaining coordinates (see (9.10)). Using the notation of propositions 6.21 and 7.3 and observing that $\mathscr{Q}_1 \subset P$ one computes the 19 generators modulo $\mathscr{M}\cdot\mathscr{Q}_1$:

$$\{\sigma_1,\lambda\}T_2,\ \{\sigma_1,\sigma_2,\sigma_3,q,\lambda\}\cdot\{T_1,D_1\},\ u_1T_1,\ u_1^2T_1,\ u_1T_2,\ u_1^2T_2,\ T_3,\ T_5,\ T_7. \tag{9.13}$$

(We again use the notation that the product of two sets is the set of pairwise products with one factor from each set.) It is then a relatively easy computation to show that (9.13) is a set of generators for the module $\mathscr{Q}_1$ (modulo $\mathscr{M}\cdot\mathscr{Q}_1$). Next one shows that T_1, T_2, and D_1 form a (vector space) basis for $\tilde{\Gamma}(n_1+p)$ modulo $\mathscr{Q}_1$, thus proving (9.11).

(ii) We must show that $\tilde{\Gamma}(n_2+p) = \tilde{\Gamma}(n_2)$ for all $p \in \mathscr{P}_2$. Let $\mathscr{L}_2$ be the ideal generated by $\lambda\sigma_1$, λ^2, σ_1^3, σ_2, σ_3 and q and recall that $\mathscr{N}$ is the ideal generated by σ_1^2, σ_2, σ_3, q and λ. Define the submodule $\mathscr{Q}_2$ of E by

$$\mathscr{Q}_2 = (\mathscr{L}_2, \mathscr{N}, \mathscr{M}, \mathscr{M}, \mathscr{E}, \mathscr{E}). \tag{9.14}$$

We claim

$$\tilde{\Gamma}(n_2+p) = \mathscr{Q}_2 \oplus \mathbb{R}\{(-\lambda+\sigma_1^2,0,0,1,0,0),\ (\sigma_1,-1,0,0,0,0),\ (\sigma_1^2,-\sigma_1,0,0,0,0),\ (0,-\sigma_1,1,0,0,0),\ (\lambda+3\sigma_1^2,0,0,0,0,0)\} \tag{9.15}$$

for all $p \in \mathscr{P}_2$. It follows from (9.15) that $\tilde{\Gamma}(n_2+p)$ is independent of p, so $\tilde{\Gamma}(n_2+p) = \tilde{\Gamma}(n_2)$. Thus n_2+p is Γ-equivalent to n_2. It is easy to see that $\dim E/\mathscr{Q}_2 = 8$. Now the vectors $n_{2,\lambda}$, $\lambda n_{2,\lambda}$ satisfy

$$n_{2,\lambda} \equiv (-1,0,0,0,0,0) \bmod \mathscr{Q}_2, \quad \lambda n_{2,\lambda} \equiv (-\lambda,0,0,0,0,0) \bmod \mathscr{Q}_2. \tag{9.16}$$

Using (9.15) and (9.16) one sees that

$$E = \Gamma(n_2) \oplus \mathbb{R}\{(0,1,0,0,0,0)\}.$$

Thus $\operatorname{codim}_\Gamma n_2 = 1$ and a universal unfolding of n_2 is given by f_2 in (9.4).

We now sketch the proof that (9.15) holds. We begin by showing that $\mathscr{Q}_2 \subset \Gamma(n_2+p) + \mathscr{M}\cdot\mathscr{Q}_2$ and hence by Nakayama's lemma that $\mathscr{Q}_2 \subset \tilde{\Gamma}(n_2+p)$. Note that the module $\mathscr{Q}_2$ has 23 generators: six in the first coordinate, five each in the second, third and fourth coordinates, and one each in the last two coordinates. Using propositions 6.21 and 7.3 along with the observation that $\mathscr{Q}_2 \subset \mathscr{P}$ one computes the 23 generators modulo $\mathscr{M}\cdot\mathscr{Q}_2$

$$\{\lambda,\sigma_2,\sigma_3,q\}\{T_1,T_2,T_8\},\ \{\sigma_1,u_1,u_1^2\}T_1,\ \{\lambda,\sigma_1\}D_1,\ \sigma_1^2T_2,\ T_4,\ T_5,\ T_6+T_8,\ \{\sigma_1,u_1\}T_6. \tag{9.17}$$

One then shows that (9.17) is a set of generators for the module $\mathscr{Q}_2$ (modulo $\mathscr{M}\cdot\mathscr{Q}_2$). Finally one shows that there are five independent terms in $\tilde{\Gamma}(n_2+p)$ modulo $\mathscr{Q}_2$:

$$T_1,\ T_2,\ \sigma_1T_2,\ T_6,\ D_1.$$

These terms span the same vector subspace as the right-hand summand in (9.15).

(iii) We must show that $\tilde{\Gamma}(n_3+p) = \tilde{\Gamma}(n_3)$ for all $p \in \mathscr{P}$. Let $\mathscr{L}_3$ be the ideal in $\mathscr{E}$ generated by σ_3, quadratic terms involving σ_2, q and λ, and σ_1^3. More precisely $\mathscr{L}_3$ is generated by the 11 generators

$$\sigma_1,\ \sigma_1\sigma_2,\ \sigma_1 q,\ \sigma_1\lambda,\ \sigma_2^2,\ \sigma_2 q,\ \sigma_2\lambda,\ q^2,\ q\lambda,\ \lambda^2,\ \sigma_3.$$

Let $\mathscr{L}_4$ be the ideal in $\mathscr{E}$ generated by σ_2, σ_3, and the quadratic terms in σ_1, q and λ. So $\mathscr{L}_3$ is generated by the eight generators

$$\sigma_1^2,\ \sigma_1 q,\ \sigma_1\lambda,\ q^2,\ q\lambda,\ \lambda^2,\ \sigma_2,\ \sigma_3.$$

Recall that $\mathscr{N}$ is the ideal generated by λ, σ_1^2, σ_2, σ_3, and q. Define the submodule $\mathscr{Q}_3$ of E by

$$\mathscr{Q}_3 = (\mathscr{L}_3, \mathscr{N}, \mathscr{M}, \mathscr{L}_4, \mathscr{M}, \mathscr{E}). \tag{9.18}$$

A simple count shows that there are 35 generators for the submodule $\mathscr{Q}_3$. Also, $\dim E/\mathscr{Q}_3 = 14$. We claim

$$\tilde{\Gamma}(n_3+p) = \mathscr{Q}_3 \oplus \mathbb{R}\{T_1, \sigma_1 T_1, T_2, T_5, u_1 T_1, D_1, D_4\} \tag{9.19}$$

using the notation of propositions 6.21 and 7.3. Moreover, modulo $\mathscr{Q}_3$, one has

$$\left.\begin{aligned}
T_1 &\equiv (-\lambda a\sigma_1 + d\sigma_1^2,\ 1,\ 0,\ b\sigma_1 + cq,\ 1,\ 0),\\
\sigma_1 T_1 &\equiv (a\sigma_1^2,\ \sigma_1,\ 0,\ 0,\ 0,\ 0),\\
T_2 &\equiv (b\sigma_1^2 + 2\sigma_2,\ -(b+2)\,\sigma_1,\ 2,\ -2\lambda + (2a+1)\,\sigma_1,\ -1,\ 0),\\
T_5 &\equiv ((a+1)\,\sigma_1^2 - 2\sigma_2,\ -a\sigma_1,\ -1,\ 0,\ 0,\ 0),\\
u_1 T_1 &\equiv (0,\ a\sigma_1,\ 1,\ 0,\ 0,\ 0),\\
D_1 &\equiv (-\lambda + 3a\sigma_1 + 5d\sigma_1^2,\ 3,\ 0,\ 4b\sigma_1 + 5cq,\ 4,\ 0),\\
D_4 &\equiv ((3a+1)\,q + b\sigma_1^2,\ (1-b)\,\sigma_1,\ -1,\ -\lambda + a\sigma_1 + (3b+1)\,q,\ 1,\ 0).
\end{aligned}\right\} \tag{9.20}$$

From (9.20) one sees the right-hand side of (9.19) is independent of p and hence $\tilde{\Gamma}(n_3+p) = \tilde{\Gamma}(n_3)$ for all p and n_3+p is Γ-equivalent to n_3. Now the vectors $n_{3,\lambda}$, $\lambda n_{3,\lambda}$ satisfy

$$\left.\begin{aligned}
n_{3,\lambda} &\equiv (-1,\ 0,\ 0,\ 0,\ 0,\ 0) \bmod \mathscr{Q}_3,\\
\lambda n_{3,\lambda} &\equiv (-\lambda,\ 0,\ 0,\ 0,\ 0,\ 0) \bmod \mathscr{Q}_3.
\end{aligned}\right\} \tag{9.21}$$

Hence $\Gamma(n_3) = \tilde{\Gamma}(n_3) \oplus \mathbb{R}\{n_{3,\lambda}, \lambda n_{3,\lambda}\}$. One sees that $\operatorname{codim}_\Gamma n_3 = 5$. It is a straightforward calculation to show that f_3 as defined in (9.8) is a universal unfolding of n_3.

We now give a sketch of the calculations needed to prove (9.19). The first step is to show that $\mathscr{Q}_3 \subset \tilde{\Gamma}(n_3+p) + \mathscr{M}\cdot\mathscr{Q}_3$ and hence by Nakayama's lemma that $\mathscr{Q}_3 \subset \tilde{\Gamma}(n_3+p)$. To do this compute the following 35 generators of $\tilde{\Gamma}(n_3+p)$ modulo $\mathscr{Q}_3$:

$$\begin{aligned}
&\{\sigma_1^2, \sigma_2, \lambda, q\}T_1,\ \{\lambda, \sigma_1, \sigma_2, \sigma_3, q\}u_1 T_1,\ u_1^2 T_1,\ T_3 - u_1 T_1,\ \{\lambda, \sigma_1\}T_2,\ T_4 - T_5,\ \{q, \sigma_2, \lambda, \sigma_1\}T_5,\ T_6,\\
&\qquad \{1, u_1, u_1^2\}T_7,\ \{1, u_1\}T_8,\ \{1, u_1\}T_9,\ T_{10},\ T_{11},\ T_{13},\ \{\lambda, q, \sigma_1\}D_1,\ \{q, \sigma_1\}D_4,\ D_5.
\end{aligned} \tag{9.22}$$

This computation leads to a 35×35 matrix which, if we assume the non-degeneracy conditions (9.7), is non-singular. This calculation is extraordinarily tedious; however, since the matrix is relatively sparse, the calculation can be done by hand. Finally, one shows that modulo $\mathscr{Q}_3$ the elements in (9.20) constitute a (vector space) basis for $\tilde{\Gamma}(n_3+p)$ thus yielding (9.19). □

10. Analysis of the normal forms

In this section we use the results of the previous sections to draw schematic bifurcation diagrams for the three normal forms considered in this paper. The examples are $A(z,\lambda)$, $B(z,\lambda)$, and $C(z,\lambda)$ which are defined respectively by

$$h_1 = -\lambda,\ h_3 = 1,\ h_5 = 0,\ k_2 = 1,\ k_4 = 0,\ k_6 = 0,$$

$$h_1 = -\lambda+\sigma_1^2,\ h_3 = a,\ h_5 = 0,\ k_2 = 1,\ k_4 = 0,\ k_6 = 0,$$

$$h_1 = -\lambda+a\sigma_1+d\sigma_1^2,\ h_3 = 1,\ h_5 = 0,\ k_2 = b\sigma_1+cq-e,\ k_4 = 1,\ k_6 = 0$$

where the following non-degeneracy conditions hold:

$$a+1 \neq 0,\ 2a+1 \neq 0,\ 3a+1 \neq 0,\ 3b+1 \neq 0 \quad \text{and} \quad c \neq 0. \tag{10.1}$$

We have chosen the coefficient of λ to be -1 for all the examples. To do this we needed to use the coordinate change $\lambda \to -\lambda$. With this in mind, one can read all of the bifurcation diagrams 'backwards' obtaining a new set of examples. In addition, to arrive at the normal forms above, we had to consider $g(z,\lambda)$ equivalent to $-g(z,\lambda)$. Clearly, this makes no difference when considering the zero set of g. It does, however, change the linearized stability assignments associated with a given solution (see § 5). In the subsequent bifurcation diagrams we have given many of the linearized stability assignments as, for example, $3+1-$. This notation indicates that $\mathrm{d}g$ on the given solution has three positive eigenvalues, one negative eigenvalue, and two zero eigenvalues (as the number of eigenvalues must be six). The equivalence of g with $-g$ may be interpreted as a choice of whether the sign of the stable eigenvalue is to be considered as $+$ or $-$. We also use the change of coordinates $z \to -z$, which has the effect of interchanging the solutions III^+, IV^+ and V^+ with the solutions III^-, IV^- and V^- respectively.

Analysis of example A (*the codimension* 0 *case*).

Using theorem 4.4 one finds a unique representative for each orbit of solutions to $A = 0$. The results are

$$\left.\begin{aligned} &(\mathrm{II}) \quad \lambda = x^2,\ x = x_1 > 0,\\ &(\mathrm{III}) \quad \lambda = x+x^2,\ x = x_1 = x_2 = x_3 \neq 0. \end{aligned}\right\} \tag{10.2}$$

There are no solutions of type IV–VIII located near the origin. The stability assignments for the solutions listed in (6.2) can be computed by using theorem 5.7. The eigenvalues for $\mathrm{d}A$ are given by:

$$\left.\begin{aligned} &(\mathrm{II}) \quad 2x^2,\ -x+x^2\ \text{(twice)},\ x+x^2\ \text{(twice)},\ 0,\\ &(\mathrm{III}) \quad -2(x+x^2)\ \text{(twice)},\ x+2x^2\ \text{(twice)},\ -3x,\ 0\ \text{(twice)}. \end{aligned}\right\} \tag{10.3}$$

The bifurcation diagram for example A is given in figure 8. Observe that no physical problem could be described completely by using a local analysis ending with example A. The reason is that every non-trivial solution branch in this example represents an unstable solution.

Analysis of example B (*Γ-codimension* 1).

In this example a is an unfolding parameter, which is assumed to be near zero. We shall analyse the three cases $a = 0$, $a > 0$, $a < 0$.

Using theorem 4.4 one solves $B = 0$ explicitly as

$$\left.\begin{aligned} &(\mathrm{II}) \quad \lambda = ax^2 + x^4,\ x > 0, \\ &(\mathrm{III}) \quad \lambda = x + ax^2 + 9x^4,\ x \neq 0. \end{aligned}\right\} \tag{10.4}$$

There are no solutions at type IV–VIII located near the origin. The eigenvalues for $\mathrm{d}B$ along the solutions listed in (10.4) are given by

$$\left.\begin{aligned} &(\mathrm{II}) \quad 2ax^2 + 4x^4,\ -x - ax^2 \text{ (twice)},\ x - ax^2 \text{ (twice)}, \\ &(\mathrm{III}) \quad -2x + 2ax^2 \text{ (twice)},\ x + 2ax^2 + 36x^3 \text{ (twice)},\ -3x,\ 0 \text{ (twice)}. \end{aligned}\right\} \tag{10.5}$$

The bifurcation diagram for example B with $a < 0$ is given in figure 9. The bifurcation diagram for $a > 0$ is the same as figure 8. Again the local analysis does not yield a physically interesting problem.

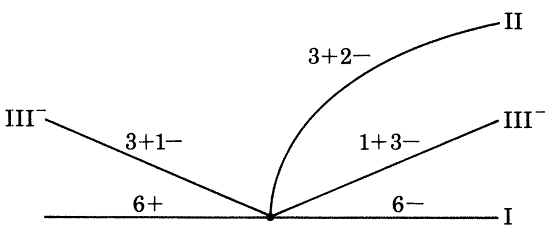

FIGURE 8. The bifurcation diagram for example A.

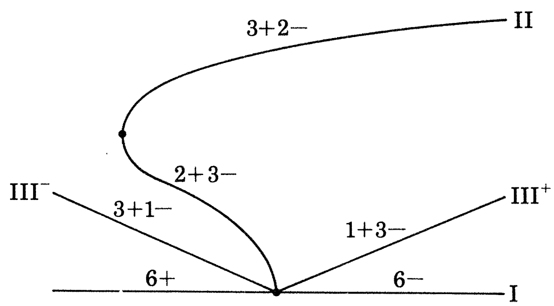

FIGURE 9. The perturbed bifurcation diagram for example B; $a < 0$.

Analysis of example C (*Γ-codimension* 5, *topological codimension* 1).

For this normal form the parameters a, b, c, and d are modal parameters while the parameter e is a true unfolding parameter. We shall present the bifurcation diagrams associated with each region of the modal parameter space defined by the non-degeneracy conditions (10.1). We first present the diagrams at the organizing centre ($e = 0$) and then the perturbed diagrams when $e > 0$ and $e < 0$.

Using theorem 4.4 one may compute the solutions to $C = 0$ explicitly as:

$$\left.\begin{aligned} &(\mathrm{II}) \quad \lambda = (a+1)x^2 + dx^4, && x > 0, \\ &(\mathrm{III}) \quad \lambda = -ex + (3a+1)x^2 + (3b+1)x^3 + (2c+9d)x^4, && x \neq 0, \\ &(\mathrm{IV}) \quad \lambda = (2a+1)x_1^2 + (a+1)x_2^2 + x_1^2 x_2 + d(2x_1^2 + x_2^2)^2, && x_1 > 0, \\ &\text{where } bx_2^2 + (2cx_1^2 - 1)x_2 + 2bx_1^2 - e = 0, \\ &(\mathrm{V}) \quad \lambda = (3a+1)x_1^2 + 9dx_1^4, && x_1 > 0, \end{aligned}\right\} \tag{10.6}$$

where $x_1^2 = e/(3b+1+2cx_2)$, $|x_2| < x_1$. There are no solutions of type VI–VIII located near the origin for this example. The eigenvalues for $\mathrm{d}C$ along the solutions listed in (10.6) can be computed by using theorem 5.7 and are given by

$$\left.\begin{aligned} &(\mathrm{II}) \quad 2(a+1)x^2 + 4dx^3,\ ex - x^2 - bx^3 \text{ (twice)},\ -ex - x^2 + bx^3 \text{ (twice)},\ 0, \\ &(\mathrm{III}) \quad 2(ex + x^2 - 3bx^3 - 2cx^4) \text{ (twice)},\ 3[ex - (3b+1)x^3 - 2cx^4], \\ &\qquad\quad -ex + 2(3a+1)x^2 + 3(3b+1)x^3 + 4(9d+2c)x^4,\ 0 \text{ (twice)}, \\ &(\mathrm{IV}) \quad 2(x_1^2 - x_2^2),\ -x_1^2 - 2x_2^2 - 3x_1^2 x_2,\ 0 \text{ (twice)}. \end{aligned}\right\} \tag{10.7}$$

One may now draw the bifurcation diagrams for the unperturbed problem ($e = 0$) (see figure 10). The axes on this figure – and the subsequent figures – are λ and the norm of x. (Using the norm leads to what seems to be a simpler representation of the solution set. However, the reader should be warned that these diagrams are schematic: they convey only the way the various branches connect, though they do present faithfully the ordering of the various intersections. Please recall this remark when using figures 11–14.) We make several observations about figure 10.

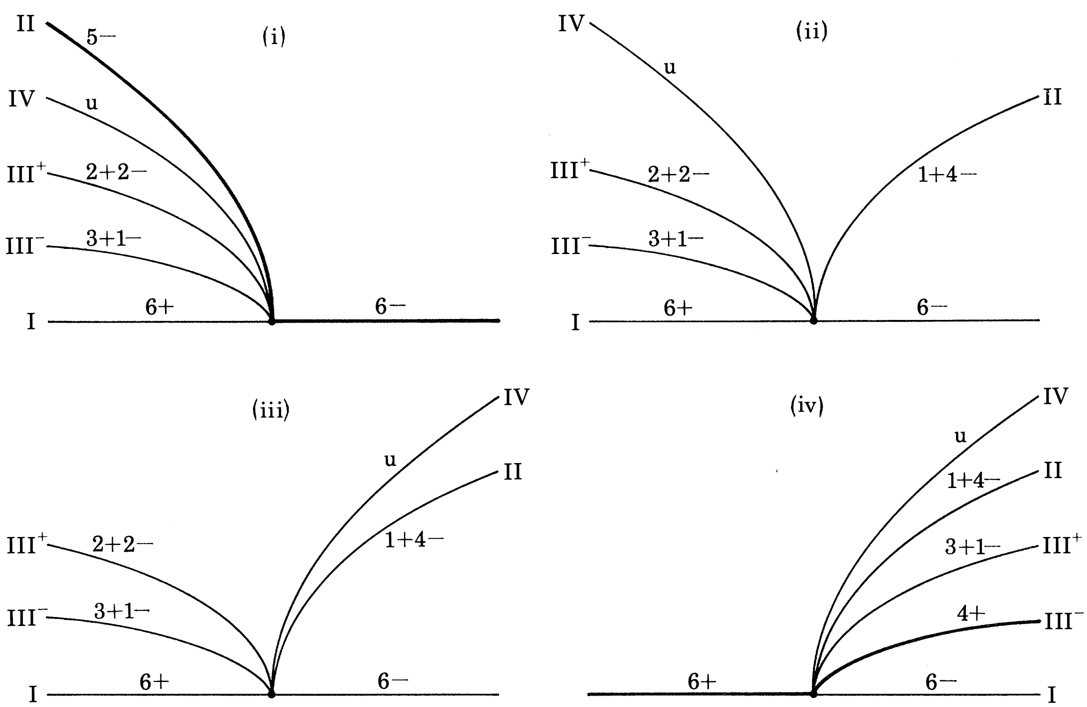

FIGURE 10. The bifurcation diagram for example C when $3b+1 > 0$, $e = 0$ and (i) $a < -1$, (ii) $-1 < a < -\frac{1}{2}$, (iii) $-\frac{1}{2} < a < -\frac{1}{3}$, (iv) $-\frac{1}{3} < a$.

Remarks

(i) The effect of the modal parameter a is clearly important; one can see geometrically the necessity of the non-degeneracy conditions on a in (6.1) as the various branches turn from subcritical to supercritical when a is varied.

(ii) The effect of the modal parameter b is subtler. There are two parts to the branch of solutions of type III, given by $x > 0$ and $x < 0$ respectively. Changing the sign of $3b+1$ interchanges the stability assignments of these branches. In general we shall draw the bifurcation diagrams for $3b+1 > 0$ as $3b+1 < 0$ leads to a similar set of figures.

(iii) To see the effect of the modal parameter c one must look at the perturbed bifurcation diagrams ($e \neq 0$).

(iv) These bifurcation problems are more interesting from the physical point of view as the possibility of stable rolls and hexagons exists. These stable solutions are indicated by a heavy black line on the figures.

(v) Observe that we have only computed four of the six eigenvalues for solutions of type IV as these are the only eigenvalues given directly by theorem 5.5. However, the two non-zero eigenvalues given in (10.7) (IV) have opposite signs; hence these solutions are unstable. We have indicated this fact on the figure by the symbol u. (The reader should note that we have only proved that the sign of one of these eigenvalues is an invariant of Γ-equivalence (see proposition 5.24). Therefore, it is possible that some $g(z, \lambda)$ with normal form (C) would have stable type IV solutions. However, for a variety of reasons, we view this as improbable.) To see that this remark is true, observe from (6.6) (IV) that

$$x_2 = (1/2b)\{1-2cx_1^2-[1-(4c+8b^2)x_1^2+4c^2x_1^4+4be]^{\frac{1}{2}}\}. \tag{10.8}$$

(We are only interested in the solution branch of the quadratic equation where $x_2 = 0$ when $x_1 = 0$ as this is a local analysis.) Now when $e = 0$, one has the estimate

$$x_2 \approx 2bx_1^2.$$

So one can see that the first eigenvalue in (10.7) (IV) is positive while the second is negative.

We continue with the analysis of example C for $e \neq 0$. The results of this analysis are given in figures 11–14. The reader will find it useful to inspect these figures before reading the detailed calculations described below. Several remarks are necessary in order to understand these figures. But first please remember that these diagrams are schematic.

Remarks

(*a*) Solution branch V is drawn as a vertical line, which is accurate only when $c = 0$. In fact this branch of solutions tilts when $c \neq 0$, and the slope of this branch depends on the sign of c (see fact 3 below).

(*b*) Bifurcation points and turning points are indicated by black dots. Other intersections on the figures are the result of projecting the bifurcation diagrams into two-dimensional space; these intersections do not occur in the actual solution set.

(*c*) Stable solutions are indicated by heavy black lines. Of course, to speak of stability, one has to choose whether the signature of the stable eigenvalue is + or −. We have made this choice in the various diagrams so as to give the most interesting physical interpretation, a steady-state theory being assumed.

(*d*) For each range of values of a there are four qualitatively different bifurcation diagrams corresponding to the choices of signs of e and $3b+1$. The differences in the cases correspond to whether or not type V solutions are present and which type III solutions ($x > 0$ or $x < 0$) have the turning point. We draw all the cases in figure 11 and only the case $3b+1 > 0$, $e > 0$ in figures 12–14.

(*e*) The bifurcation diagrams are topologically equivalent for each region of the modal parameter a with the exception of the region $-1 < a < -\frac{1}{2}$. In this region two additional degeneracies occur. First, it is possible for the λ-value, λ_3, of the intersection of branches III and IV to be either positive or negative, depending on the sign of $3a+2$ (see fact 4 below). Second, it is possible for the secondary bifurcation or branch II to occur at a λ-value either greater than or less than the λ-value of the turning point on branch III. This happens at $a = -\frac{5}{6}$ (see fact 10 below).

(*f*) The u on the unbounded part of branch IV indicates that for the normal form C the two real eigenvalues of $\mathrm{d}C$ have opposite signs. However, we have shown only that the sign of one of these eigenvalues is an invariant of Γ-equivalence. Thus it is possible, though highly unlikely, that there is a $g(z, \lambda)$ that is Γ-equivalent to C with the unbounded portion of branch IV stable.

The detailed analysis that leads to figures 11–14 involves a number of (almost unrelated small) facts, which we now give.

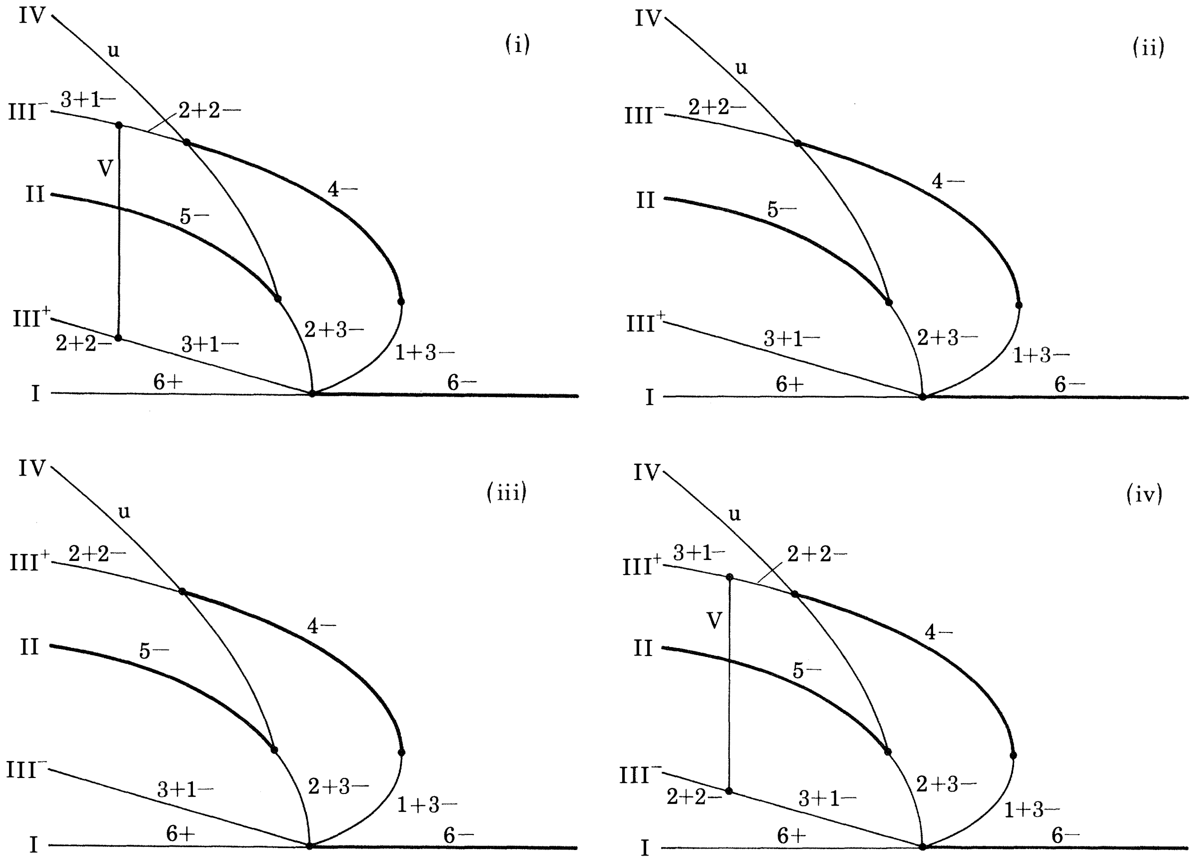

FIGURE 11. The perturbed bifurcation diagram for example C when $a < -1$ and (i) $3b+1 > 0, e > 0$; (ii) $3b+1 > 0, e < 0$; (iii) $3b+1 < 0, e > 0$; (iv) $3b+1 < 0, e < 0$.

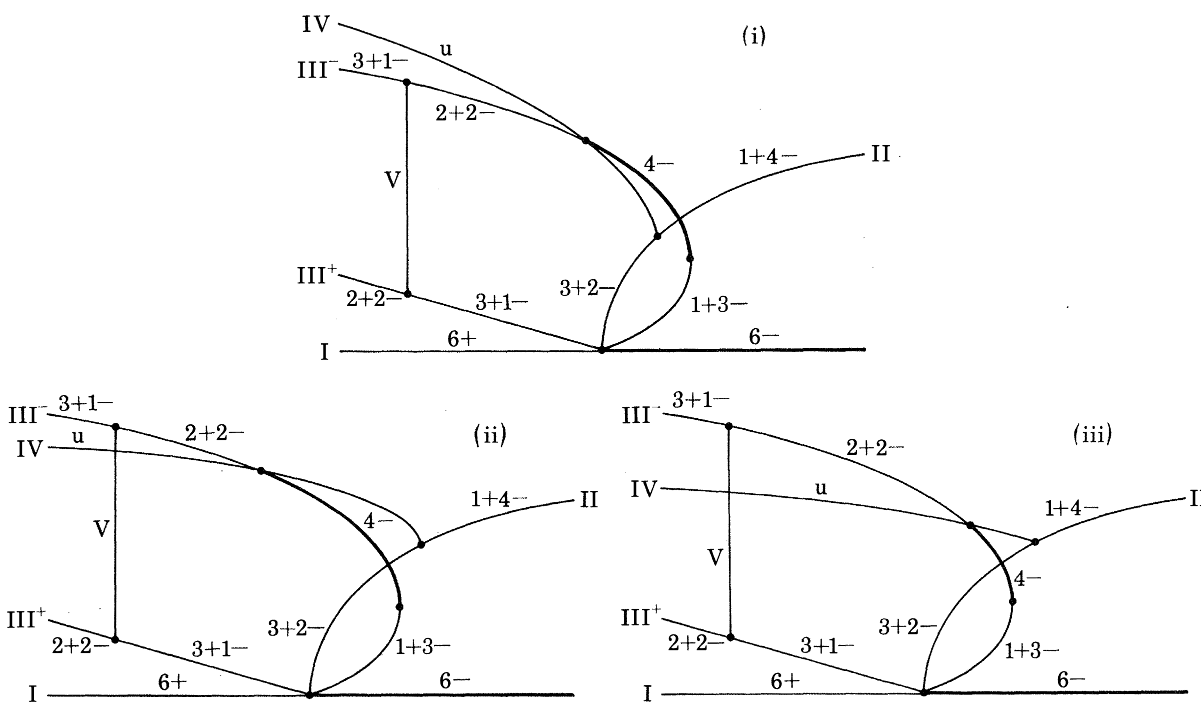

FIGURE 12. The perturbed bifurcation diagram for example C when $3b+1 > 0$, $e > 0$ and (i) $-1 < a < -\frac{5}{6}$, (ii) $-\frac{5}{6} < a < -\frac{2}{3}$, (iii) $-\frac{2}{3} < a < -\frac{1}{2}$.

Fact 1. The turning point along branch III occurs for $x \approx e/2(3a+1)$. This fact is immediate from (10.6) (III).

Fact 2. Solution branch V exists precisely when $\operatorname{sgn} e = \operatorname{sgn}(3b+1)$. This branch begins and ends at a secondary bifurcation with branch III.

The existence statement may be read directly from (10.6) (V) as $x_1^2 > 0$. Moreover, branch V is parametrized by x_2 with $|x_2| < x_1$. Let λ_b be the λ-value at which $x_2 = -x_1$ and λ_f be the λ-value at which $x_2 = x_1$. We showed in §4 that a solution where $x_1 = -x_2 = x_3 > 0$ lies on the same orbit as solutions of the form $x_1 = x_2 = x_3 < 0$. So both λ_b and λ_f are the λ-values of secondary bifurcations of branch V with branch III.

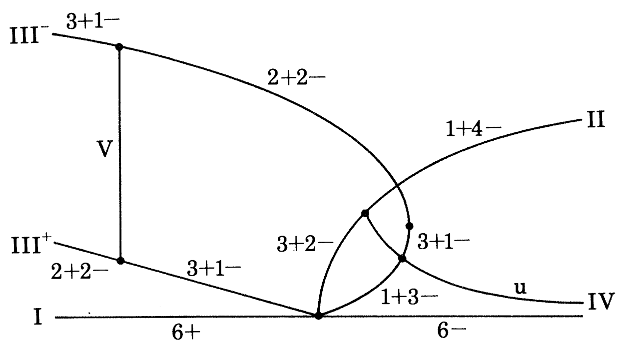

FIGURE 13. The perturbed bifurcation diagram for example C when $-\frac{1}{2} < a < -\frac{1}{3}$, $3b+1 > 0$ and $e > 0$.

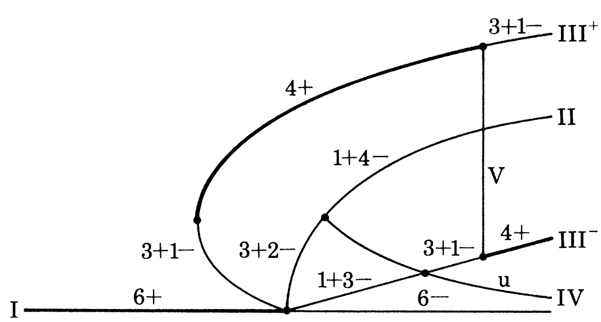

FIGURE 14. The perturbed bifurcation diagram for $-\frac{1}{3} < a$, $3b+1 > 0$ and $e > 0$.

Fact 3. Branch V is monotonic in λ and $\operatorname{sgn}(\lambda_f - \lambda_b) = -\operatorname{sgn}[(3a+1)ce]$. Moreover, one has the estimate that to first order

$$\lambda_f \approx \lambda_b \approx (3a+1)e/(3b+1). \tag{10.9}$$

The monotonicity may be obtained from (10.6) (V) by observing that x_1^2 is monotonic in x_2 (for x_2 near 0) when b, c and e are fixed and that λ is monotonic in x_1^2 (for x_1^2 near 0) when a and d are fixed.

Next, by using the defining conditions for λ_b and λ_f (namely, $x_2 = \pm x_1$) one can compute

$$\lambda_f - \lambda_b = \left(\frac{1}{3b+1+2cx_1} - \frac{1}{3b+1-2cx_1}\right)\left[(3a+1)e + 9de^2\left(\frac{1}{3b+1+2cx_1} + \frac{1}{3b+1-2cx_1}\right)\right].$$

Now the results are valid only for e small and x_1 small. Hence

$$\lambda_f - \lambda_b \approx -4c(3a+1)ex_1/(3b+1)^2$$

and $\operatorname{sgn}(\lambda_f - \lambda_b) = -\operatorname{sgn}[c(3a+1)e]$.

Finally observe that, when $x_2 = \pm x_1$, the second equation in (10.6) (V) yields the estimate

$$x_1^2 \approx e/(3b+1).$$

The first equation gives the estimate (10.9).

Fact 4. Branch IV begins with a secondary bifurcation with branch II, whose λ-value is λ_2, and has a secondary bifurcation with branch III (whose λ-value is λ_3 and whose x_1-value is x_5), and branch IV is monotonic in λ. Moreover

$$\lambda_2 \approx (a+1)\,e^2,\ \lambda_3 \approx (3a+2)\,e^2,\ x_5 \approx -e. \tag{10.10}$$

Observe that branch IV is parametrized by $x_1 > 0$; one can solve for x_2 as in equation (10.8). Moreover, $x_1 = x_3$ along such solutions; so when $x_1 = 0$ two of the z-coordinates are zero. We showed in §4 that such solutions lie on the same orbit as those of type II. When $x_1 = 0$

$$x_2 = [1-(1+4be)^{\frac{1}{2}}]/2b \approx -e,$$

yielding the estimate for λ_2 in (10.10).

We next show that there is precisely one intersection of branch IV with branch III. Type III solutions occur when $x_1^2 = x_2^2$. From the second equation in (10.6) (IV) one obtains at such a point

$$2cx_2^3 + 3bx_2^2 - x_2 - e = 0. \tag{10.11}$$

From the implicit function theorem there exists a unique solution to (10.11) satisfying the estimate $x_2 \approx -e$. From the first equation in (10.6) (IV) one obtains the estimate for λ_3 in (10.10). Note that $\operatorname{sgn} x_2 = -\operatorname{sgn} e$. When $x_2 > 0$ one has a solution $x_1 = x_2 = x_3 > 0$. When $x_2 < 0$ one has a solution that is on the same orbit as $x_1 = x_2 = x_3 < 0$. Hence the secondary bifurcation with solutions of type III has an x_1-value given by the estimate for x_5 in (10.10).

Finally, we show that branch IV is monotonic in λ by computing (using implicit differentiation)

$$\partial\lambda/\partial x_1 = 2x_1[(2a+1) + O(x_1^2) + O(x_2)].$$

In fact one sees that the slope of branch IV has the same sign as sgn $(2a+1)$.

Fact 5. The secondary bifurcations on branch V occur at $\lambda = O(e)$, see (10.9), while the secondary bifurcations associated with branch IV occur at $\lambda = O(e^2)$, see (10.10). Hence, the bifurcations on branch V occur further from the origin than the ones on branch IV.

Fact 6. The value of x at the turning point of branch III satisfies

$$-e + 2(3a+1)\,x + 3(3b+1)\,x^2 + 4(2c+9d)\,x^3 = 0. \tag{10.12}$$

The λ-value of that turning point, λ_T, is

$$\lambda_T \approx -e^2/4(3a+1). \tag{10.13}$$

One obtains (10.12) by computing $\partial\lambda/\partial x$ in (10.6) (III) and finding, therefore, that

$$x \approx e/2(3a+1).$$

Substituting this estimate for x in (10.6) (III) yields (10.13).

One now has enough information to sketch the bifurcation diagrams in figures 11–14 with the exceptions noted in remark (e). Before resolving the difficulties associated with figure 12 we discuss the stability assignments given on all the figures.

Fact 8. The bifurcation problems considered here are perturbations (given by e) of those in figure 10. Thus the stability assignments on the branches in figures 11–14, which are far from the origin, are the same as those on the corresponding branches in figure 10.

Fact 9. The eigenvalues listed in (10.7) change signs only at black dots on the figures. In (10.14) we list the points where each eigenvalue changes sign. We use the term constant to indicate that the given eigenvalue does not change sign and we ignore the zero eigenvalues. The ordering of the eigenvalues is exactly the same as given in (10.7).

$$\left.\begin{array}{ll} \text{(II)} & \text{constant, constant (2), intersection with IV (2).} \\ \text{(III)} & \text{intersection with IV (2), intersection with V, turning point.} \\ \text{(IV)} & \text{intersection with III, constant.} \end{array}\right\} \quad (10.14)$$

We have not attempted to find the eigenvalues of type V solutions. Computing the stability assignments for solutions of type IV seems to be a difficult task; moreover, we have not shown that the stability assignments for type IV solutions are invariants of Γ-equivalence.

Fact 10. Recall that λ_T is the λ-value of the turning point on branch III and λ_2 is the λ-value of the intersection of branches IV and II. Then $\lambda_T - \lambda_2$ changes sign at $a \approx -\frac{5}{6}$ and $a \approx -\frac{1}{2}$ and is negative in the interval $(-\frac{5}{6}, -\frac{1}{2})$.

Recall that $\lambda_2 \approx (a+1)\,e^2$ from (10.10) while $\lambda_T \approx -e^2/4(3a+1)$ from (10.13). Thus $\lambda_T - \lambda_2 = 0$ when

$$a+1 = -1/4(3a+1)$$

or

$$12a^2+16a+5 = (6a+5)\,(2a+1) = 0.$$

This information allows one to complete the figures.

11. Relation with the Bénard problem

As we indicated in the Introduction, there is an intimate relation between the mathematical idealization of the planar Bénard problem through the Boussinesq equations and bifurcation on the hexagonal lattice. Five pieces of information are needed to make this relation rigorous. First one has to specify boundary conditions on the upper and lower bounding planes. Second one has to show why solutions to the Boussinesq equations with the given boundary conditions are doubly periodic with respect to the hexagonal lattice. Third, even if the second step is valid, one must show why the kernel of the linearized Boussinesq equations is exactly six-dimensional. Fourth, assuming that this kernel is six-dimensional, one has to compute the Liapunov–Schmidt reduction. Finally, one should discuss the stability analysis not in terms of linearized orbital stability but in terms of stability for the partial differential equation, the Boussinesq equation. We discuss these points in order.

The specific boundary conditions imposed on the mathematical model depend on the exact experiment. For the standard planar Bénard problem there are two experimentally motivated sets of boundary conditions that are common. First, one imagines that the fluid rests on some surface and that rigid boundary conditions are appropriate below. On the upper plane, however, one has two standard choices. Either there is a free surface on top as in Bénard's original experiment or the fluid is contained between two fixed surfaces and rigid boundary conditions are also appropriate on the upper plane. As was pointed out to us by F. H. Busse these cases are quite different, depending on whether the boundary conditions on the top and the bottom are the same (the *symmetric* case) or not (the *non-symmetric* case).

The reason that these cases are different mathematically is that the group of symmetries of the problem changes. In the symmetric case the additional symmetry is given by reflecting about the midplane of the system; that is one sends X_3 to $h-X_3$ where h is the thickness of the fluid layer and X_3 is the height above the lower boundary plane. The Boussinesq equations commute with this symmetry in the symmetric case. This symmetry appears in the analysis on the hexagonal lattice. as follows. If $g(z, \lambda)$ is the function obtained by Liapunov–Schmidt then

$$g(-z, \lambda) = -g(z, \lambda). \tag{11.1}$$

The reader may recall that we showed that there is precisely one quadratic term in g denoted by $k_2(0)$ that may be non-zero. We studied in this paper the two cases (examples A and C of § 10) where $k_2(0)$ is non-zero and $k_2(0)$ is zero. In the symmetric case (11.1) implies that $k_2(0) = 0$. However, it also implies that the coefficients of all the even-order terms in g are zero. This information is sufficient to show that the non-degeneracy conditions that define normal form C fail, and the analysis presented here is *not* appropriate in the symmetric case. In a paper now in preparation Golubitsky *et al.* (1983) analyse the symmetric case showing that the results do change dramatically when this extra symmetry is added to Γ. Thus, in this paper, we have been studying implicitly the non-symmetric case where the boundary conditions on the upper and lower planes are different.

The second point that one should show in relating the Bénard problem to the hexagonal lattice is that, of necessity, solutions to the Boussinesq equations are doubly periodic with respect to the hexagonal lattice. In any technical sense, this implication must be false. However, since in many physically observed situations one finds hexagons and rolls, it is not unreasonable to look in the class of doubly periodic functions for solutions, and any solutions found in this class will indeed be solutions to the Boussinesq equations. One point worth mention is that it is not clear whether the hexagonal lattice is forced since the Boussinesq equations commute with the full Euclidean group in the plane (such a relation would be an extraordinarily powerful observation involving spontaneous symmetry breaking) or is due to the fact that experiments are never performed on the infinite plane.

Even if one assumes double periodicity there is a problem in trying to decide which lattice is relevant. For example, cellular convection in squares fits most naturally on a square lattice. We note here that one of the goals in Sattinger (1978) is to find a mechanism for selecting by stability assignments between various lattices. In this paper, we have assumed the hexagonal lattice; one could perform a similar study for bifurcation problems with respect to any planar lattice and then, perhaps, use Sattinger's ideas to select between the lattices.

The third point we wish to discuss concerns the dimensionality of the kernel of the linearized Boussinesq equations. As we described in the Introduction, the hexagonal lattice and its symmetries force a six-dimensional kernel. It is possible, however, that in some manifestation of the Bénard problem one could find a twelve-dimensional kernel occurring at the first eigenvalue for the Boussinesq equations on the hexagonal lattice. However, the generic situation is a six-dimensional kernel and the more degenerate situation might occur as some parameter in the Boussinesq equations is varied. Such a degeneracy would be found, presumably, at an isolated value of that parameter. It follows that any local bifurcation analysis performed near such a degeneracy would have to take this higher-dimensional kernel into account. We do note that Kirchgässner (1979) has begun the study of such a degeneracy and has shown that one can expect new types of solutions to appear when the kernel is twelve-dimensional. In this paper we study only the six-dimensional kernel and note that our techniques would – in theory – be applicable to the higher-dimensional cases though there is every reason to believe that the calculations involved would be very difficult. Such an application would presumably have to await further advances in the theory.

A fourth component in the connection between the fluid problem and the mathematics is the actual performance of the Liapunov–Schmidt reduction. With double periodicity, the first step would be the verification that the kernel dimension is six. This is true under reasonable hypotheses. Busse (1962) has performed part of this reduction for the case of free boundary above and rigid boundary below, assuming that the kernel is six-dimensional. In particular, he has verified that for a Boussinesq fluid the quadratic term $k_2(0)$ mentioned is zero. By the term Boussinesq fluid we mean that: (i) there are no surface tension effects; and (ii) the viscosity and thermal conductivity of the fluid are independent of the temperature. Unlike the case of symmetric boundary conditions, these assumptions do not constrain the even-order terms in $g(z, \lambda)$ to be zero. Thus, our analysis – more precisely, normal form C of § 10 – seems to apply to this case. To prove this point rigorously one would have to compute higher-order terms in the Taylor expansion of $g(z, \lambda)$ to find the exact values of the modal parameters as indicated in proposition 8.5 (iii), in particular equations (8.13). In Busse's work and in the work of Schlüter *et al.* (1965) the third-order terms have been computed. To compute explicitly the higher-order terms is extremely complicated.

The computations of the third-order terms indicate that the specific form of example C that is relevant is the one where the rolls solutions are stable supercritically. From figure 10 one can see that the only case where that situation occurs is when the modal parameter $a < -1$. In this paper, we assume that the higher-order non-degeneracy conditions are satisfied so that normal form C is appropriate.

Of course, in any real fluid there will be surface temperature effects and the viscosity will have some temperature dependence. We assume that these effects are small though non-zero. In the terms of § 10 we assume that the unfolding parameter e for example C is non-zero. This implies that the relevant bifurcation diagrams are those of figure 11, read from right to left. Certain observations can be made about this collection of diagrams. First, for small parameter ranges hexagonal solutions will be stable. Second, there will be hysteresis effects in the (implied) jumps between the trivial pure conduction solution and the hexagon solutions, and between the hexagon solutions and the roll solutions. Thus even though no stable hexagon solutions exist in the idealized problem they should exist for a small range in the Rayleigh number (λ) near criticality and should be observable if the model is correct. Third, regardless of the sign of $3b+1$ there is a sign of e for which triangle solutions exist. In any given context one must compute the signs of e and $3b+1$ to determine whether triangle solutions do exist. Whether or not triangles and wavy rolls can be stable has not been decided by our analysis. There is, however, heuristic evidence that indicates that they could be stable for one sign of the modal parameter c when $a > -\frac{1}{3}$.

The last point we make in discussing the relation between the Bénard problem and bifurcation with respect to the hexagonal lattice is the question of stability of given solutions. In § 5, we computed the linearized orbital stability for rolls and hexagon solutions. Schlüter *et al.* (1965) have shown by a perturbation calculation that whether or not a given solution is stable with respect to periodic perturbations with the correct boundary conditions may be determined by whether or not it is stable to perturbations in the six-dimensional kernel. This is exactly the information that we compute. Presumably, their results can be made rigorous by using the centre manifold theorem. This to our knowledge has not been done. However, if it can be done then one can use a result of Schaeffer & Golubitsky (1981) that states that the mapping $g(z, \lambda)$ obtained by the Liapunov–Schmidt reduction and the vector field along the centre manifold obtained from the centre manifold theorem are $(C^k)\,\Gamma$-equivalent. Next one uses proposition 5.24 to show that the stability assignments we have derived for the rolls and hexagon solutions are accurate for the original partial differential equation. No such correspondence can be made at this point for the solutions of type (IV) or (V). Moreover our analysis does not rigorously imply stability of a given solution to a more arbitrary perturbation.

In summary, there is strong evidence that normal form C of § 10 represents the most likely description of bifurcation behaviour of steady-state doubly periodic solutions to the Boussinesq equations in the non-symmetric case where the boundary conditions on the upper and lower boundaries are different. Moreover, the classification of the different types of possible bifurcation diagrams given in figures 11–14 represents a complete classification of the steady-state solutions. It is interesting to know whether each of these types of behaviour can actually be observed in experiments. As mentioned in the Introduction the most interesting non-standard example, figure 14, allows for a transition from hexagons of one type to hexagons of the other type through either a jump or a smooth transition through triangles (as described at the end of § 4).

The main ways in which our analysis is incomplete concern the stability of type IV and V solutions and the possibility of periodic solutions. We note that the 2×2 matrix listed in theorem 5.5 (IV) can have complex eigenvalues. Thus it is possible in theory for a Hopf bifurcation to occur along solution branch IV, implying the existence of time-periodic solutions to the Boussinesq equations. The reader should contrast this possibility with the results of theorem 5.5 (II) and (III) where we show that all of the eigenvalues of $\mathrm{d}g$ along solution branches of rolls and hexagons are real. Thus there is no possibility of a Hopf bifurcation along these branches. We have made little progress in rigorously determining the eigenvalues of $\mathrm{d}g$ along branches of type V solutions, though such a determination may be possible in the future with general observations about the nature of spontaneous symmetry breaking.

Our motivation for studying bifurcation on the hexagonal lattice lies in the work of David Sattinger. Our work began as an exercise to recover Sattinger's results by using a mixture of singularity theory and group theory. However, as we became more involved in the mathematical problem, the beautiful complexity of the planar Bénard problem was forced upon us. In trying to understand this relation between the mathematics and the physical problem we have sought advice and information from several persons including Fritz Busse, John Guckenheimer, Edgar Knóbloch, David Schaeffer, and Jim Swift. We are grateful to each of them for the help and perspective they have given us and hope that the exposition of the relation between the hexagonal lattice and the Bénard problem given here has profited from their advice. Finally we wish to thank Norman Dancer and Jim Swift for pointing out several errors made in the original manuscript. In particular, the observation that type V solutions have D_3 symmetry is theirs.

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