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A formula for a symmetry detective

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Abstract

Detectives are used to determine the symmetry of attractors in finite-dimensional systems. We derive an explicit formula for a symmetry detective for each representation of a finite group.

Keywords: Symmetry; Chaos; Dynamical systems

1. Introduction

Computer experimentation shows that symmetric systems may exhibit symmetric attractors (see [4,6,7,10]). For planar systems, the symmetry of an attractor is usually determined by visual inspection. For dimensions greater than two it is less clear how to calculate the symmetry group of an attractor. Barany et al. [3] devised the method of detectives for calculating the symmetries of attractors in higher dimensions; this method was expanded in [9] (see also [1,2,5,13]). The idea behind the method in [3] is that symmetries of an attractor are transferred to symmetries of a point in an associated space, by integrating an equivariant polynomial (known as an *observable*) over the given attractor. It is shown that, for an open and dense set of polynomials (known as *detectives*), the symmetry group of the attractor is generically equal to the isotropy subgroup of the associated point. Thus the symmetries of the original attractor may be calculated by finding the isotropy subgroup of a point, which is a relatively straightforward calculation.

One difficulty in applying this theory is that it is not immediate how to construct a detective for a given symmetry group. Sufficiency theorems from [3,9] allow one to decide when a given observable φ is a detective. The converse question, how to build a detective for a given system, has some partial answers. In [3,9], a detective is created for systems of coupled cells with (global) dihedral symmetry. This detective has been used by Kroon and Stewart [11] to study a model of hexapodal gaits, with some success. Tchistiakov [13] presents a detective for n coupled cell systems with full S_n symmetry, which he uses to study the dynamics of Josephson junction arrays when n is small. Tchistiakov also shows that his method of construction can be generalized to an algorithm which may be used

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to construct a detective for any finite group; however, this algorithm depends on the choice of a polynomial with certain properties, and these properties are often difficult to verify.

In this note we present a formula for a detective for finite groups. Before introducing this formula we need a few preliminary definitions. Let $\Gamma \subset \mathbf{O(n)}$ be a finite group acting on $\mathbb{R}^n$. For each isotropy subgroup $T \subset \Gamma$, define

$$\pi_T : \mathbb{R}^n \to \mathrm{Fix}(T)$$

to be orthogonal projection, and define

$$T' = N(T)/T, \tag{1.1}$$

where $N(T)$ is the normalizer of T in Γ. Note that T' acts on $\mathrm{Fix}(T)$.

Definition 1.1. The point $z \in \mathbb{R}^n$ is a *generic point* if $\pi_T(z)$ has trivial T' isotropy for all isotropy subgroups T.

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$, and let $L^2(\Gamma)$ be the vector space of all real-valued functions on Γ.

Theorem 1.2. Let $\Gamma \subset \mathbf{O(n)}$ be a finite group acting on $\mathbb{R}^n$, and let $z_0 \in \mathbb{R}^n$ be a generic point. Then the map $\varphi : \mathbb{R}^n \to L^2(\Gamma)$ defined as

$$\varphi(z)[\gamma] = (\langle z, \gamma z_0 \rangle + 1)^{|\Gamma|}$$

is an SBR detective.

The formal definition of an SBR detective is given in Definition 2.2. Roughly speaking, SBR detectives work by integrating the observable φ over an attractor A with respect to an (assumed) SBR measure.

The remainder of this paper divides into two sections. In Section 2 we give precise definitions of observables and detectives which also includes the sufficiency theorem of [9]. We then present the proof of Theorem 1.2 in Section 3.

2. Detectives

We begin with some background material on SBR (Sinai, Bowen and Ruelle) measures and attractors. Following [9], let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping.

Definition 2.1.

(a) An *SBR measure* for a mapping f with an invariant set A is an ergodic measure ρ with support equal to A and with the property that there exists an open neighborhood $U \supset A$ such that for every continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and for Lebesgue a.e. $x \in U$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} g(f^j(x)) = \int_A g \, \mathrm{d}\rho. \tag{2.1}$$

(b) An *SBR attractor* is a ω-limit point set together with an SBR measure.

Suppose that $\Gamma \subset \mathbf{O(n)}$ is a finite group, and f is Γ-equivariant with an SBR attractor A. Define the symmetry group $\Sigma(A)$ of A to be the subgroup

$$\Sigma(A) = \{\sigma \in \Gamma : \sigma A = A\}. \tag{2.2}$$

Assume that Γ also acts orthogonally on the vector space W. An *observable* is a C^∞ Γ-equivariant mapping $\varphi : \mathbb{R}^n \to W$. An *observation* K_φ is

$$K_\varphi = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \varphi(f^j(x)). \tag{2.3}$$

Since A is an SBR attractor, Definition 2.1 implies that there exists an open set $U \supset A$ such that for Lebesgue a.e. $x \in U$ the vector $K_\varphi \in W$ is independent of x. We denote the isotropy subgroup of K_φ by $\Sigma_\varphi(A)$.

Definition 2.2. The observable φ is an *SBR detective* if for each SBR attractor A there exists an open dense subset $\mathcal{N}$ in a neighborhood of the identity in (the C^k topology of) $\mathrm{Diff}_\Gamma(\mathbb{R}^n)$ such that all $\psi \in \mathcal{N}$ satisfy

$$\Sigma_\varphi(\psi(A)) = \Sigma(A).$$

In Definition 2.2, the notation $\Sigma_\varphi(\psi(A))$ stands for the isotropy subgroup of the point

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \varphi(\psi(f^j(x))) \in W.$$

We conclude this section by recalling a theorem from [9] that gives sufficient conditions for an observable φ to be a detective. Suppose T is an isotropy subgroup of Γ and define T' as in (1.1). Let $W_1', \ldots, W_s'$ be the isotypic components of the action of T' on $\mathrm{Fix}_W(T)$, and let $\rho_i : W \to W_i'$ be orthogonal projection.

Theorem 2.3 ([9]). Suppose that
(a) W contains every nontrivial irreducible representation of Γ,
(b) $\varphi : \mathbb{R}^n \to W$ is a Γ-equivariant polynomial, and
(c) for each isotropy subgroup T and for each i the subspace

$$\rho_i \circ \varphi(\mathrm{Fix}_W(T)) \subset W_i'$$

is nonzero.
Then φ is an SBR detective.

3. A general detective algorithm

In this section we present a proof of Theorem 1.2. Let $\Gamma \subset \mathbf{O(n)}$ be a finite group. The group Γ acts naturally on $L^2(\Gamma)$ as follows:

$$\sigma \cdot g[\gamma] = g[\sigma^{-1}\gamma]$$

for all $\sigma, \gamma \in \Gamma$. It is a standard fact from representation theory (see [8, p. 17]) that $L^2(\Gamma)$ contains a copy of each Γ-irreducible W_i. We use $L^2(\Gamma)$ as the representation space W in Theorem 2.3.

Define $\hat{\mathcal{P}}_m$ to be the set of polynomials $f : \mathbb{R} \to \mathbb{R}$ of degree m with all coefficients nonzero, and let $f(x) \in \hat{\mathcal{P}}_m$. Define the map $\tilde{f} : \mathbb{R}^n \to L^2(\Gamma)$ as

$$\tilde{f}(z)[\gamma] = f(\langle z, \gamma z_0 \rangle). \tag{3.1}$$

Lemma 3.1. The map

$$\tilde{f} : \mathbb{R}^n \to L^2(\Gamma)$$

defined in (3.1) is Γ-equivariant.

Proof. We need to show that, for each $\sigma \in \Gamma$, the functions $\tilde{f}(\sigma z)$ and $\sigma \tilde{f}(z)$ are equal. Note that

$$\tilde{f}(\sigma z)[\gamma] = f(\langle \sigma z, \gamma z_0 \rangle) = f(\langle z, \sigma^{-1} \gamma z_0 \rangle)$$

with the second equality following from the orthogonality of the Γ action. The action of Γ on $L^2(\Gamma)$ implies that

$$\sigma \tilde{f}(z)[\gamma] = \tilde{f}(z)[\sigma^{-1}\gamma] = f(\langle z, \sigma^{-1} \gamma z_0 \rangle)$$

and the lemma follows. □

For each $f \in \hat{\mathcal{P}}_m$, define the subspace $\mathcal{S}(\tilde{f}) \subset L^2(\Gamma)$ as

$$\mathcal{S}(\tilde{f}) = \mathrm{Span}\{\tilde{f}(z) \mid z \in \mathbb{R}^n\}.$$

Lemma 3.2. Let $f \in \hat{\mathcal{P}}_m$. Then

$$\mathcal{S}(\tilde{f}) = \mathrm{Span}\{1, \langle z, \gamma z_0 \rangle, \ldots, \langle z, \gamma z_0 \rangle^m \ : \ z \in \mathbb{R}^n\}.$$

Proof. Since $\tilde{f}$ is a polynomial of degree m in $\langle z, \gamma z_0 \rangle$, it follows that

$$\mathcal{S}(\tilde{f}) \subseteq \mathrm{Span}\{1, \langle z, \gamma z_0 \rangle, \ldots, \langle z, \gamma z_0 \rangle^m : z \in \mathbb{R}^n\}.$$

To show the reverse inclusion, let $f(x) = a_0 + a_1 x + \cdots + a_m x^m$. Since $f \in \hat{\mathcal{P}}_m$ it follows that all a_j are nonzero. Note that for every $r \in \mathbb{R}$

$$\tilde{f}(rz) = \sum a_j (\langle rz, \gamma z_0 \rangle)^j = \sum a_j r^j (\langle z, \gamma z_0 \rangle)^j.$$

Let $r_1, \ldots, r_{m+1}$ be distinct real numbers. Then

$$\begin{pmatrix} \tilde{f}(r_1 z) \\ \tilde{f}(r_2 z) \\ \vdots \\ \tilde{f}(r_{m+1} z) \end{pmatrix} = BA \begin{pmatrix} 1 \\ \langle z, \gamma z_0 \rangle \\ \vdots \\ \langle z, \gamma z_0 \rangle^m \end{pmatrix}, \tag{3.2}$$

where the $(m+1) \times (m+1)$ matrices A and B are defined as

$$B = \begin{pmatrix} 1 & r_1 & \cdots & r_1^m \\ 1 & r_2 & \cdots & r_2^m \\ \vdots & \vdots & & \vdots \\ 1 & r_{m+1} & \cdots & r_{m+1}^m \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_0 & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_m \end{pmatrix}.$$

We claim that the matrices A and B in (3.2) are invertible. The matrix A is invertible by our assumption that the a_j are nonzero. The determinant of the matrix B is the VanderMonde determinant:

$$\begin{vmatrix} 1 & r_1 & \dots & r_1^m \\ 1 & r_2 & \dots & r_2^m \\ \vdots & \vdots & & \vdots \\ 1 & r_{m+1} & \dots & r_{m+1}^m \end{vmatrix} = \prod_{i \neq j} (r_i - r_j).$$

Since the r_j are distinct, the matrix B is also invertible, and the lemma follows. □

Lemma 3.3. Let $z_0 \in \mathbb{R}^n$ be a point of trivial isotropy, and let γ be a nonidentity element of Γ. Then

$$\langle z_0, z_0 \rangle \neq \langle z_0, \gamma z_0 \rangle.$$

Proof. Assume $\langle z_0, z_0 \rangle = \langle z_0, \gamma z_0 \rangle$ and write γz_0 uniquely as

$$\gamma z_0 = a z_0 + z^{\perp},$$

where $a \in \mathbb{R}$ and $z^{\perp}$ is orthogonal to z_0. Then

$$\begin{aligned} \langle z_0, z_0 \rangle &= \langle z_0, \gamma z_0 \rangle = \langle z_0, a z_0 + z^{\perp} \rangle \\ &= \langle z_0, a z_0 \rangle + \langle z_0, z^{\perp} \rangle = a \langle z_0, z_0 \rangle \end{aligned}$$

from which it follows that $a = 1$ and $\gamma z_0 = z_0 + z^{\perp}$. By the orthogonality of the action of Γ, we have

$$\langle z_0, z_0 \rangle = \langle \gamma z_0, \gamma z_0 \rangle = \langle z_0 + z^{\perp}, z_0 + z^{\perp} \rangle = \langle z_0, z_0 \rangle + \langle z^{\perp}, z^{\perp} \rangle.$$

Thus $\langle z^{\perp}, z^{\perp} \rangle = 0$, which implies that $z^{\perp} = 0$ and $\gamma z_0 = z_0$. The assumption of trivial isotropy on z_0 implies that $\gamma = e$, which proves the lemma. □

Lemma 3.4. Let $f \in \hat{\mathcal{P}}_N$ where $N \geq |\Gamma|$. Assume $z_0 \in \mathbb{R}^n$ has trivial isotropy. Then

$$\mathcal{S}(\tilde{f}) = L^2(\Gamma).$$

Proof. Let

$$C = \prod_{\sigma \in \Gamma - e} \frac{1}{(\langle z_0, z_0 \rangle - \langle z_0, \sigma z_0 \rangle)}.$$

It follows from Lemma 3.3 that C is well-defined. Define

$$h(x) = C \prod_{\sigma \in \Gamma - e} (x - \langle z_0, \sigma z_0 \rangle).$$

Note that h is a polynomial of degree $|\Gamma| - 1$ and that

$$h(\langle z_0, \gamma z_0 \rangle) = \begin{cases} 1, & \gamma = e, \\ 0, & \gamma \neq e. \end{cases}$$

The polynomial h generates a map $\tilde{h} : \mathbb{R}^n \to L^2(\Gamma)$ as in (3.1), and

$$\begin{aligned} \mathcal{S}(\tilde{h}) &= \operatorname{Span}\{\tilde{h}(z) \mid z \in \mathbb{R}^n\} \\ &\subseteq \operatorname{Span}\{1, \langle z, \gamma z_0 \rangle, \dots, \langle z, \gamma z_0 \rangle^{|\Gamma|-1} \mid z \in \mathbb{R}^n\} \\ &\subseteq \operatorname{Span}\{1, \langle z, \gamma z_0 \rangle, \dots, \langle z, \gamma z_0 \rangle^{N} \mid z \in \mathbb{R}^n\} \end{aligned}$$

since $N \geq |\Gamma|$ by assumption. It follows from Lemma 3.2 that

$$\mathcal{S}(\tilde{h}) \subset \mathcal{S}(\tilde{f}) \subset L^2(\Gamma).$$

We claim that $\mathcal{S}(\tilde{h}) = L^2(\Gamma)$. To verify this claim, let $\{f_g \mid g \in \Gamma\}$ be the canonical basis of $L^2(\Gamma)$ where

$$f_g[\gamma] = \begin{cases} 1, & \gamma = g, \\ 0, & \gamma \neq g. \end{cases}$$

Then

$$\begin{aligned} \tilde{h}(gz_0)[\gamma] &= h(\langle gz_0, \gamma z_0\rangle) = h(\langle z_0, g^{-1}\gamma z_0\rangle) \\ &= \begin{cases} 1, & g^{-1}\gamma = e, \\ 0, & g^{-1}\gamma \neq e \end{cases} \\ &= f_g[\gamma], \end{aligned}$$

which proves the claim. Thus $L^2(\Gamma) \subset \mathcal{S}(\tilde{f}) \subset L^2(\Gamma)$. □

Lemma 3.5. Let $f \in \hat{\mathcal{P}}_m$. Assume $T \subset \Gamma$ is an isotropy subgroup and define $T' = N(T)/T$. Then the map $\tilde{f}$ restricts to a T'-equivariant map

$$\tilde{f}_T : \mathrm{Fix}_{\mathbb{R}^n}(T) \to \mathrm{Fix}_{L^2(\Gamma)}(T).$$

Proof. Let $\tau \in T$ and $v \in \mathrm{Fix}_{\mathbb{R}^n}(T)$. We need to show that $\tilde{f}_T(v) \in \mathrm{Fix}_{L^2(\Gamma)}(T)$. Since

$$\tilde{f}(v) = \tilde{f}(\tau v) = \tau \tilde{f}(v),$$

the lemma follows. □

Our last step is to show that $\mathrm{Fix}_{L^2(\Gamma)}(T)$ contains a copy of each T' irreducible representation. The following lemma is proved in [9, Lemma 2.3].

Lemma 3.6.

$$L^2(T') \subset \mathrm{Fix}_{L^2(\Gamma)}(T).$$

Proof. By the definition of the action of Γ on $L^2(\Gamma)$ we have

$$\mathrm{Fix}_{L^2(\Gamma)}(T) = \{g : \Gamma \to \mathbb{R} \text{ such that } g \text{ is constant on } T \text{ cosets}\}.$$

Each such g induces a map $\hat{g} : N_\Gamma(T)/T \to \mathbb{R}$ in $L^2(T')$. We get all such maps $N_\Gamma(T)/T \to \mathbb{R}$ in this way, which proves the lemma. □

Proof of Theorem 1.2. Let $f(x) = (x+1)^{|\Gamma|}$ and note that $f \in \hat{\mathcal{P}}_{|\Gamma|}$. Lemma 3.6 implies the orthogonal projection $\pi : \mathrm{Fix}_{L^2(\Gamma)}(T) \to L^2(T')$ can be defined. Lemma 3.5 implies that

$$\tilde{\psi} = \pi \circ \tilde{f}_T : \mathrm{Fix}_{\mathbb{R}^n}(T) \to L^2(T')$$

is a T'-equivariant of the form (3.1). Indeed

$$\tilde{\psi}(z) = f(\langle z, \gamma z_0\rangle).$$

It follows that

$$\mathcal{S}(\tilde{\psi}) = \mathrm{Span}\{\tilde{f}(z) \mid z \in \mathrm{Fix}_{\mathbb{R}^n}(T)\}.$$

Lemma 3.4 and the assumption that z_0 is a generic point together imply that $\mathcal{S}(\tilde{\psi}) = L^2(T')$ for each isotropy subgroup T. Thus the linear span

$$\mathrm{Span}\{\tilde{f}(z) \mid z \in \mathrm{Fix}_{\mathbb{R}^n}(T)\}$$

contains a copy of each irreducible T' representation, and the conclusion of the theorem follows from Theorem 2.3. □

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