A Symmetry Classification of Columns

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1 Introduction

The Yale art historian George Hersey showed us the columns in Figure 1 and asked us whether the ideas of symmetry breaking could be used to help classify architectural columns. Provoked by this question and the intriguing columns in that figure, we attempted to answer Hersey in the following way. We view a column as a deformed cylinder and column symmetries as the subgroup of the symmetries of the cylinder that preserve the column.

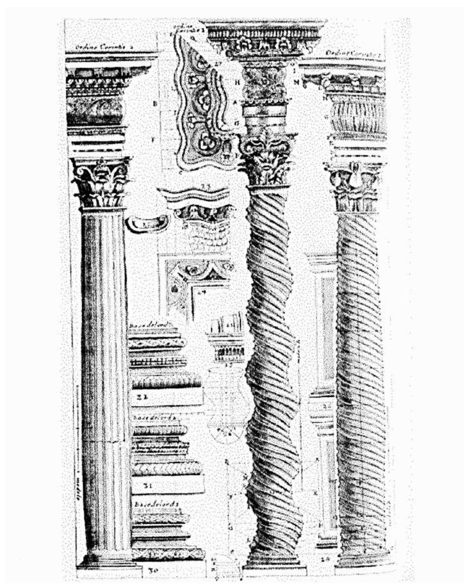

Figure 1: Plate xv from G. Guarini [2]. (Left) Fluted column (see Figure 7). (Right) Spiral column with k large (see Figure 9). (Center) Spiral column with $k = 1$. See Table 2 for a definition of k.

More precisely, we think of a column as a function on a cylinder (either finite or infinite) where the function tells us how far to deform the cylinder in the direction normal to the cylinder. The *symmetries of a column* are the symmetries that preserve the level contours of the function, that is, the isotropy subgroup of the defining function. In this paper we present the mathematical classification of the 29 different types of column symmetry. We note that there is a related classification of the *rod groups* that corresponds to the columns with discrete symmetry. See [1]. The classification theorem is stated and proved in Section 2. Level contours (drawn on a flattened cylinder using MATLAB) of representatives of the twenty-eight nontrivial column symmetry types are presented in Section 3, as are three-dimensional images of columns drawn using geomview.

2 Symmetries of Columns

We define a *column* by a real-valued function f on the cylinder $\mathcal{C} = \mathbf{S}^1 \times \mathbf{R}$. Let $(\varphi, z) \in \mathcal{C}$. The function $f(\varphi, z)$ measures the height of the column in the direction normal to the cylinder at the point (φ, z).

The group of symmetries of the cylinder is

$$\Gamma = \mathbf{D}_2(\tau, \kappa) \dot{+} (\mathbf{SO(2)} \oplus \mathbf{R})$$

where Γ acts on $(\varphi, z) \in \mathcal{C}$ by

$$\begin{array}{rcll} (\theta, t)(\varphi, z) & = & (\varphi + \theta, z + t) & \qquad (\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R} \\ \tau(\varphi, z) & = & (-\varphi, z) & \\ \kappa(\varphi, z) & = & (\varphi, -z). & \end{array}$$

Multiplication in Γ follows from the definition of the action. Suppose that $(A_j, (\theta_j, t_j))$ is in Γ for $j = 1, 2$, where $A_j \in \mathbf{D}_2$, $\theta_j \in \mathbf{SO(2)}$ and $t_j \in \mathbf{R}$. Then multiplication is given by

$$(A_2, (\theta_2, t_2)) \cdot (A_1, (\theta_1, t_1)) = (A_2 A_1 \, , \, A_2(\theta_1, t_1) + (\theta_2, t_2)). \tag{1}$$

We wish to classify columns by their symmetries. A *symmetry* of the column $f : \mathcal{C} \to \mathbf{R}$ is $\gamma \in \Gamma$ such that

$$f(\gamma(\varphi, z)) = f(\varphi, z) \quad \forall (\varphi, z) \in \mathcal{C}.$$

The symmetry group $\Sigma_f \subset \Gamma$ is the collection of all symmetries of f. We classify all subgroups Σ which are symmetry subgroups for some column f.

Our classification proceeds as follows. To each subgroup $\Sigma \subset \Gamma$, we can associate the normal subgroup

$$\Sigma_0 = \Sigma \cap (\mathbf{SO(2)} \oplus \mathbf{R}). \tag{2}$$

(So Σ_0 consists of the pure 'translations' in Σ.) Thus it suffices to

(i) classify the closed subgroups Σ_0 of $\mathbf{SO(2)} \oplus \mathbf{R}$,

(ii) for each subgroup Σ_0 in (i), compute the subgroups $\Sigma \subset \Gamma$ that satisfy (2).

The calculation in (ii) is simplified by observing that Σ is contained in the normalizer of Σ_0.

As usual, we identify conjugate subgroups of Γ. In addition, we identify subgroups that are related by axial scalings. More precisely, we define the scaling transformation $s_\alpha : \Gamma \to \Gamma$ by

$$s_\alpha(A, \theta, t) = (A, \theta, \alpha t), \qquad (A, \theta, t) \in \Gamma.$$

Provided $\alpha \neq 0$, this is an isomorphism. We say that two subgroups Σ, Σ' are related by a scaling if $s_\alpha \Sigma = \Sigma'$ for some nonzero α.

Classification of Subgroups of $\mathbf{SO(2)} \oplus \mathbf{R}$.

In this section, we classify the closed subgroups of $\mathbf{SO(2)} \oplus \mathbf{R}$ up to scaling and conjugacy in Γ. Also, we compute the normalizers of these subgroups in Γ.

Lemma 2.1 *Suppose that C is a compact subgroup of $\mathbf{SO(2)} \oplus \mathbf{R}$. Then $C \subset \mathbf{SO(2)} \oplus \mathbf{1}$.*

Proof: If $(\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R}$ and $t \neq 0$, then (θ, t) generates a noncompact subgroup of $\mathbf{SO(2)} \oplus \mathbf{R}$ (isomorphic to $\mathbf{Z}$). It follows that $(\theta, t) \notin C$. ∎

Proposition 2.2 *Suppose that G is a closed connected subgroup of $\mathbf{SO(2)} \oplus \mathbf{R}$. Then, up to conjugacy and scaling, G is one of the subgroups*

$$\mathbf{SO(2)} \oplus \mathbf{R}, \quad \mathbf{SO(2)} \oplus \mathbf{1}, \quad \mathbf{1} \oplus \mathbf{R}, \quad \mathbf{L}, \quad \mathbf{1},$$

where

$$\mathbf{L} = \{(t, t) \in \mathbf{SO(2)} \oplus \mathbf{R} : t \in \mathbf{R}\}.$$

Proof: If $\dim G = 2$, then connectivity implies that $G = \mathbf{SO(2)} \oplus \mathbf{R}$. If $\dim G = 1$, then connectivity implies that G is group isomorphic to either $\mathbf{SO(2)}$ or $\mathbf{R}$. In the first case, it follows from Lemma 2.1 that $G = \mathbf{SO(2)} \oplus \mathbf{1}$. In the second case, there is a smooth isomorphism $h : \mathbf{R} \to G \subset \mathbf{SO(2)} \oplus \mathbf{R}$. This isomorphism is given by $h(t) = (\theta_0 t, a_0 t)$ for some $(\theta_0, a_0) \in \mathbf{SO(2)} \oplus \mathbf{R}$ (defined as $h(1)$). By assumption $a_0 \neq 0$. If $\theta_0 = 0$, then $G = \mathbf{1} \oplus \mathbf{R}$. If $\theta_0 \neq 0$, then by axial scaling we can arrange that $a_0 = \theta_0$ and $G = \mathbf{L}$. ∎

From now on, we use the abbreviations $\mathbf{R} = \mathbf{1} \oplus \mathbf{R}$ and $\mathbf{SO(2)} = \mathbf{SO(2)} \oplus \mathbf{1}$. The proper closed subgroups of $\mathbf{SO(2)}$ are given by $\mathbf{Z}_k$, $k \geq 1$: the subgroup of rotations of the cylinder through angles which are multiples of $2\pi/k$. In addition, we set $\mathbf{Z} \subset \mathbf{R}$ to be the subgroup of unit axial translations of the cylinder generated by the element $(0, 1) \in \mathbf{SO(2)} \oplus \mathbf{R}$. Finally, for any $\omega \in \mathbf{R}$, we define

$$\mathbf{N}_\omega = \{(\omega n, n) \in \mathbf{SO(2)} \oplus \mathbf{R} : n \in \mathbf{Z}\}.$$

Of course, $\mathbf{N}_0 = \mathbf{Z}$.

Theorem 2.3 *Up to axial scaling and conjugacy, the closed subgroups $\Sigma_0 \subset \mathbf{SO(2)} \oplus \mathbf{R}$ are listed in Table 1.*

Proof: Since Σ_0 is abelian, we can write $\Sigma_0 \cong C \oplus \mathbf{Z}^p \oplus \mathbf{R}^q$ where C is compact and $p, q \geq 0$. Clearly, $p + q \leq 1$. By Lemma 2.1, $C = \mathbf{SO(2)}$ or $C = \mathbf{Z}_k$.

Assume that $C = \mathbf{SO(2)}$. Since $\mathbf{SO(2)} \oplus \mathbf{R}$ is connected, the only subgroup satisfying $\dim \Sigma_0 = 2$ is $\Sigma_0 = \mathbf{SO(2)} \oplus \mathbf{R}$. Suppose next that $\dim \Sigma_0 = 1$. We claim that $\Sigma_0 = \mathbf{SO(2)}$ or $\Sigma_0 = \mathbf{SO(2)} \oplus \mathbf{Z}$. Choose the smallest positive $t \in \mathbf{R}$ such that there is $\theta \in \mathbf{SO(2)}$ with $(\theta, t) \in \Sigma_0$. Since $(\theta, 0) \subset \Sigma_0$, it follows that $\Sigma_0 = \mathbf{SO(2)} \oplus t\mathbf{Z}$, where $t\mathbf{Z}$ is the subgroup of $\mathbf{SO(2)} \oplus \mathbf{R}$ generated by $(0, t)$. By making an axial scaling, we can set $t = 1$ so that $\Sigma_0 = \mathbf{SO(2)} \oplus \mathbf{Z}$.

Now assume that $C = \mathbf{Z}_k$. If $\dim \Sigma_0 = 1$, then it follows from Proposition 2.2 that $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{R}$ or $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{L}$. If $\dim \Sigma_0 = 0$, then either $\Sigma_0 = \mathbf{Z}_k$ or $\Sigma_0 \cong \mathbf{Z}_k \oplus \mathbf{Z}$. In the

$\dim\Sigma_0$	Σ_0	H
2	$\mathbf{SO(2)}\oplus\mathbf{R}$	$\mathbf{D}_2$
1	$\mathbf{SO(2)}$	$\mathbf{D}_2$
	$\mathbf{SO(2)}\oplus\mathbf{Z}$	$\mathbf{D}_2$
	$\mathbf{Z}_k\oplus\mathbf{R}$	$\mathbf{D}_2$
	$\mathbf{Z}_k\oplus\mathbf{L}$	$\mathbf{Z}_2(\tau\kappa)$
0	$\mathbf{Z}_k$	$\mathbf{D}_2$
	$\mathbf{Z}_k\oplus\mathbf{Z}$	$\mathbf{D}_2$
	$\mathbf{Z}_k\oplus\mathbf{N}_\omega \qquad 0<\omega<\pi/k$	$\mathbf{Z}_2(\tau\kappa)$
	$\mathbf{Z}_k\oplus\mathbf{N}_{\pi/k}$	$\mathbf{D}_2$

Table 1: Classification of closed subgroups $\Sigma_0\subset\Gamma$ up to scaling and conjugacy. The normalizers are given by $N(\Sigma_0)=H\oplus(\mathbf{SO(2)}\oplus\mathbf{R})$

latter case, we can choose a generator $(a,b)\in\mathbf{Z}\subset\mathbf{SO(2)}\oplus\mathbf{R}$ with smallest $b>0$. Making an axial scaling, we can suppose that the generator is of the form $(\omega,1)$ for some $\omega\in\mathbf{R}$. In other words, $\Sigma_0=\mathbf{Z}_k\oplus\mathbf{N}_\omega$. Note that $\mathbf{Z}_k\oplus\mathbf{N}_{\omega+2\pi/k}=\mathbf{Z}_k\oplus\mathbf{N}_\omega$, so we can suppose that $|\omega|\le\pi/k$. Using formula (1) we compute that

$$\tau\cdot(\omega t,t)\cdot\tau^{-1}=(-\omega t,t),$$

where $\tau\cdot(\omega t,t)$ is an abbreviation for $(\tau,(0,0))\cdot(1,(\omega t,t))$. Hence up to conjugacy, we may suppose that $\omega\ge 0$. The case $\omega=0$ is the distinguished case $\mathbf{N}_0=\mathbf{Z}$. ∎

Proposition 2.4 *The normalizers of the subgroups* $\Sigma_0\subset\mathbf{SO(2)}\oplus\mathbf{R}$ *have the form*

$$N(\Sigma_0)=H\dot{+}(\mathbf{SO(2)}\oplus\mathbf{R}),$$

where the subgroup $H\subset\mathbf{D}_2$ *is as given in Table 1.*

Proof: Since $\mathbf{SO(2)}\oplus\mathbf{R}$ is abelian, it is clear that $\mathbf{SO(2)}\oplus\mathbf{R}\subset N(\Sigma_0)$. Hence $N(\Sigma_0)=H\dot{+}(\mathbf{SO(2)}\oplus\mathbf{R})$ for some subgroup $H\subset\mathbf{D}_2$. We compute that $A\cdot(\theta,t)\cdot A^{-1}$ is the element $A(\theta,t)\in\mathbf{SO(2)}\oplus\mathbf{R}$. Hence, H consists of those elements $A\in\mathbf{D}_2$ that preserve Σ_0. The element $\tau\kappa$ acts as $-I$ on $\mathbf{SO(2)}\oplus\mathbf{R}$ and so is always contained in H. It follows that $H=\mathbf{Z}_2(\tau\kappa)$ or $H=\mathbf{D}_2$. It now suffices to determine whether or not τ preserves Σ_0, that is, whether or not Σ_0 is preserved by the transformation $(\theta,t)\mapsto(-\theta,t)$. ∎

Untwisted Symmetry Groups

Suppose that $\Sigma\subset\Gamma$ is a symmetry group. Then $\Sigma_0=\Sigma\cap(\mathbf{SO(2)}\oplus\mathbf{R})$ is one of the subgroups listed in Table 1. We say that Σ is an *untwisted* subgroup of Γ if Σ is conjugate to a subgroup of the form $K\dot{+}\Sigma_0$ where K is contained in the subgroup H given in Table 1. The untwisted symmetry groups are listed in Table 2.

It is not the case that every subgroup $K\subset H$ produces a symmetry group. For example, when $\Sigma_0=\mathbf{SO(2)}\oplus\mathbf{R}$, the only symmetry group Σ corresponding to Σ_0 is $\Sigma=\Gamma$. (This is

Σ_0	Σ		
$\mathbf{SO(2)} \oplus \mathbf{R}$	Γ		
$\mathbf{SO(2)}$	$\mathbf{Z}_2(\tau)\dot{+}\mathbf{SO(2)}$	$\mathbf{D}_2\dot{+}\mathbf{SO(2)}$	
$\mathbf{SO(2)} \oplus \mathbf{Z}$	$\mathbf{Z}_2(\tau)\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$	$\mathbf{D}_2\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$	
$\mathbf{Z}_k \oplus \mathbf{R}$	$\mathbf{Z}_2(\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{R})$	$\mathbf{D}_2\dot{+}(\mathbf{Z}_k \oplus \mathbf{R})$	
$\mathbf{Z}_k \oplus \mathbf{L}$	$\mathbf{Z}_k \oplus \mathbf{L}$	$\mathbf{Z}_2(\tau\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{L})$	
$\mathbf{Z}_k$	$\mathbf{Z}_k$	$\mathbf{Z}_2(\tau)\dot{+}\mathbf{Z}_k$	$\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_k$
	$\mathbf{Z}_2(\tau\kappa)\dot{+}\mathbf{Z}_k$	$\mathbf{D}_2\dot{+}\mathbf{Z}_k$	
$\mathbf{Z}_k \oplus \mathbf{N}_\omega \quad 0 \le \omega \le \pi/k$	$\mathbf{Z}_k \oplus \mathbf{N}_\omega$	$\mathbf{Z}_2(\tau\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_\omega)$	
$\mathbf{Z}_k \oplus \mathbf{Z}$	$\mathbf{Z}_2(\tau)\dot{+}(\mathbf{Z}_k \oplus \mathbf{Z})$	$\mathbf{Z}_2(\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{Z})$	$\mathbf{D}_2\dot{+}(\mathbf{Z}_k \oplus \mathbf{Z})$
$\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$	$\mathbf{Z}_2(\tau)\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k})$	$\mathbf{Z}_2(\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k})$	$\mathbf{D}_2\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k})$

Table 2: The 22 untwisted symmetry groups $\Sigma \subset \Gamma$

independent of the restriction to untwisted symmetry groups.) To verify this point, observe that $\mathbf{SO(2)} \oplus \mathbf{R}$ acts transitively on the cylinder $\mathcal{C}$. Hence if Σ is the symmetry group of a function $f : \mathcal{C} \to \mathbf{R}$, then f is the constant function. It follows that f is invariant under Γ, and that the symmetry subgroup $\Sigma = \Gamma$.

When Σ_0 contains $\mathbf{SO(2)}$, the function f is constant on each horizontal cross-section of $\mathcal{C}$ and hence automatically has the symmetry τ. In these cases, the only possibilities are $K = \mathbf{Z}_2(\tau)$ and $K = \mathbf{D}_2$. Similarly, when Σ contains $\mathbf{R}$ then automatically $\kappa \in \Sigma$ and the only possibilities are $K = \mathbf{Z}_2(\kappa)$ and $K = \mathbf{D}_2$.

In all other cases, there are no restrictions on K other than the condition $K \subset H$.

Twisted Symmetry Groups

We continue to suppose that Σ is a symmetry group with $\Sigma_0 = \Sigma \cap (\mathbf{SO(2)} \oplus \mathbf{R})$. We have $\Sigma \subset H\dot{+}(\mathbf{SO(2)} \oplus \mathbf{R})$ where H is given in Table 1. The canonical projection $\pi : \Gamma \to D_2$ induces a projection $\pi : \Sigma \to H$.

We say that a symmetry group $\Sigma \subset \Gamma$ is *twisted* if it is not conjugate to an untwisted symmetry group. Equivalently, there exists an $A \in \pi(\Sigma)$ such that $A \notin \Sigma$.

The next lemma states that, without loss of generality, we can always suppose that the element $A = \tau\kappa$ is not responsible for twisting.

Lemma 2.5 *Suppose that Σ is a symmetry group and that $\tau\kappa \in \pi(\Sigma)$. Then there is a subgroup of Γ that is conjugate to Σ and contains $\tau\kappa$. The conjugacy leaves Σ_0 unchanged.*

Proof: Recall that $\tau\kappa$ acts as $-I$ on $\mathbf{SO(2)} \oplus \mathbf{R}$. By assumption $(\tau\kappa, \theta, t) \in \Sigma$ for some $(\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R}$. We conjugate by the element $(-\theta/2, -t/2) \in \mathbf{SO(2)} \oplus \mathbf{R}$. Compute that

$$(1\,,\,(-\theta/2, -t/2)) \cdot (\tau\kappa\,,\,(\theta, t)) \cdot (1\,,\,(\theta/2, t/2)) = (\tau\kappa\,,\,(0, 0)),$$

as required. ∎

Σ_0	$\pi(\Sigma)$	generators of Σ/Σ_0
$\mathbf{Z}_k$	$\mathbf{Z}_2(\kappa)$	$\tilde{\kappa}$
	$\mathbf{D}_2$	$\tau, \quad \tilde{\kappa}$
$\mathbf{Z}_k \oplus \mathbf{Z}$	$\mathbf{Z}_2(\tau)$	$\tilde{\tau}$
	$\mathbf{Z}_2(\kappa)$	$\tilde{\kappa}$
	$\mathbf{D}_2$	$\tau, \quad \tilde{\kappa}$
	$\mathbf{D}_2$	$\tilde{\tau}, \quad \kappa$
	$\mathbf{D}_2$	$\tilde{\tau}, \quad \tilde{\kappa}$

Table 3: The 7 twisted symmetry groups $\Sigma \subset \Gamma$. Σ is generated by Σ_0 together with the generators of Σ/Σ_0. Notation: $\tilde{\kappa} = (\kappa, (\pi/k, 0))$, $\tilde{\tau} = (\tau, (0, 1/2))$

Proposition 2.6 *Let Σ be a twisted symmetry group. Then either $\Sigma_0 = \mathbf{Z}_k$, $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$ or $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. In addition, $\pi(\Sigma)$ is one of the three subgroups $\mathbf{Z}_2(\tau)$, $\mathbf{Z}_2(\kappa)$ and $\mathbf{D}_2$.*

Remark: The possibility $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$ will be eliminated in the proof of Theorem 2.7.

Proof: It follows from Lemma 2.5 that we can eliminate the subgroups Σ_0 for which $H = \mathbf{Z}_2(\tau\kappa)$, that is we can eliminate $\mathbf{Z}_k \oplus \mathbf{L}$ and $\mathbf{Z}_k \oplus \mathbf{N}_\omega$.

Next, suppose that Σ_0 contains $\mathbf{SO(2)}$. As observed in the previous subsection, Σ contains τ. If Σ is larger than $\mathbf{Z}_2(\tau)\dot{+}\mathbf{SO(2)}$, then $\pi(\Sigma) = \mathbf{D}_2$. It follows from Lemma 2.5 that $\tau\kappa \in \Sigma$ and hence $\Sigma = \mathbf{D}_2\dot{+}\Sigma_0$. In either case, Σ is untwisted. The possibility that Σ_0 contains $\mathbf{R}$ can be eliminated similarly. This completes the proof that Σ_0 is one of the groups $\mathbf{Z}_k$, $\mathbf{Z}_k \oplus \mathbf{Z}$ or $\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$.

Recall that $\pi(\Sigma)$ is a subgroup of $\mathbf{D}_2$. If $\pi(\Sigma) = \mathbf{1}$, then $\Sigma = \Sigma_0$. If $\pi(\Sigma) = \mathbf{Z}_2(\tau\kappa)$, then Σ is conjugate to $\mathbf{Z}_2(\tau\kappa)\dot{+}\Sigma_0$ by Lemma 2.5. Hence, for Σ to be twisted, $\pi(\Sigma)$ must be one of the three remaining subgroups of $\mathbf{D}_2$. ∎

Theorem 2.7 *Up to conjugacy and scaling, there are seven twisted symmetry groups in Γ. These are as listed in Table 3.*

Proof: By Proposition 2.6, we can assume that $\Sigma_0 = \mathbf{Z}_k$, $\mathbf{Z}_k \oplus \mathbf{Z}$ or $\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$ and that $K = \pi(\Sigma)$ is one of the subgroups $\mathbf{Z}_2(\tau)$, $\mathbf{Z}_2(\kappa)$ or $\mathbf{D}_2$. We consider the three possibilities for K in turn.

Suppose that $K = \mathbf{Z}_2(\tau)$. Then $\sigma = (\tau, (\theta, t)) \in \Sigma$ for some $(\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R}$. Conjugating by $(-\theta/2, 0) \in \mathbf{SO(2)} \oplus \mathbf{R}$, we can set $\theta = 0$. Note that

$$\sigma^2 = (1, (0, 2t)) \in \Sigma_0.$$

When $\Sigma_0 = \mathbf{Z}_k$, it follows that $t = 0$ in which case $\sigma = \tau$, and there is no twisting. When $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$, there is the additional possibility that $2t \in \mathbf{Z}$ but $t \notin \mathbf{Z}$. Since $\mathbf{Z} \subset \Sigma$, this reduces to the case $t = 1/2$. The argument is more complicated when $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. Squaring yields the condition $(0, 2t) \in \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. Working modulo $\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$, we can choose σ so that $t = 1$. But still working modulo $\mathbf{N}_{\pi/k}$, we can replace σ by $\sigma = (\tau, (\pi/k, 0))$. Conjugating once again, we have $\sigma = \tau$ and there is no twisting.

The case $K = \mathbf{Z}_2(\kappa)$ is similar. Conjugation reduces to $\sigma = (\kappa, (\theta, 0))$ and squaring yields the condition $2\theta \in \mathbf{Z}_k$. Twisting occurs when $\theta = \pi/k$ but only for $\Sigma_0 = \mathbf{Z}_k$ and $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$.

Finally, suppose that $K = \mathbf{D}_2$. We concentrate attention on the two generators

$$\sigma_1 = (\tau, (\theta_1, t_1)) \qquad \sigma_2 = (\kappa, (\theta_2, t_2))$$

of Σ modulo Σ_0. Since the reflections are orthogonal, we can simultaneously conjugate so that $\theta_1 = t_2 = 0$. Squaring the generators, we obtain that $\theta_2 \in \mathbf{Z}_{2k}$ and either $t_1 = 0$, $2t_1 \in \mathbf{Z}$ or $t_1 \in \mathbf{Z}$ depending on whether $\Sigma_0 = \mathbf{Z}_k$, $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$ or $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. The various combinations of generators yield one untwisted subgroup and one twisted subgroup for $\Sigma_0 = \mathbf{Z}_k$, and one untwisted subgroup and three twisted subgroups for $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$. Once again, there is no twisting when $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. The arguments are similar to the previous cases of K; we replace σ_j by untwisted group elements. ∎

3 Classification of Columns

The results of the previous section show that there are twenty-nine symmetry classes of columns. The symmetry class of a column can be determined by answering a sequence of questions. The most important question is:

> Are the symmetries of the column continuous, discrete and infinite, or finite?

The column has continuous symmetries when the column can be slid along itself. These symmetries can occur either using either axial translations or rotations about the axis, or by a combination of the two. With two exceptions infinite discrete symmetry groups occur when the column is axially periodic but has no continuous symmetries. Both of the first two types of symmetry groups are infinite. If the symmetry group of a column is not infinite, then it is finite.

3.1 Columns with Continuous Symmetry

If the column has both axial-translation and rotation symmetry, than the column is a cylinder with symmetry group Γ. Continuous symmetries come in three types: rotations about the column axis (columns of revolution), translations along the column axis (fluted columns), or corkscrew symmetries which are a mixture of the two (spiral columns).

Columns of Revolution — Four Types

There are four types of column with rotational $\mathbf{SO(2)}$ symmetry. Two types are periodic in the axial direction and two are not. The nonperiodic columns may have a reflection symmetry in the horizontal plane ($\mathbf{D}_2\dot{+}\mathbf{SO(2)}$ — for example a column which is bowed out at the center) or not ($\mathbf{Z}_2(\tau)\dot{+}\mathbf{SO(2)}$ — a column which widens at the base). See Figures 2 and 3.

The periodic columns of revolution may have an up-down symmetry ($\mathbf{D}_2\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$) or not ($\mathbf{Z}_2(\tau)\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$). See Figures 4 and 5.

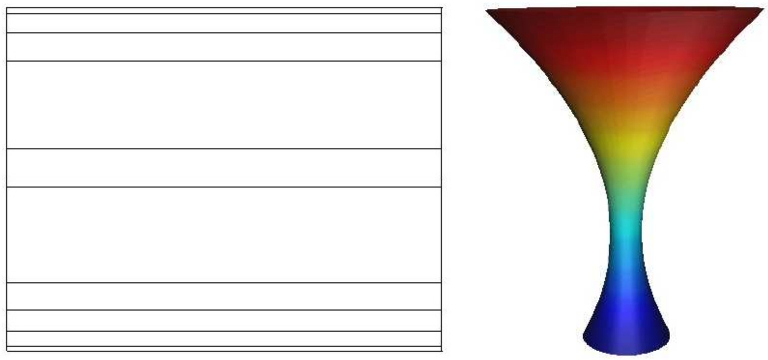

Figure 2: Nonperiodic column of revolution with no up-down reflection.

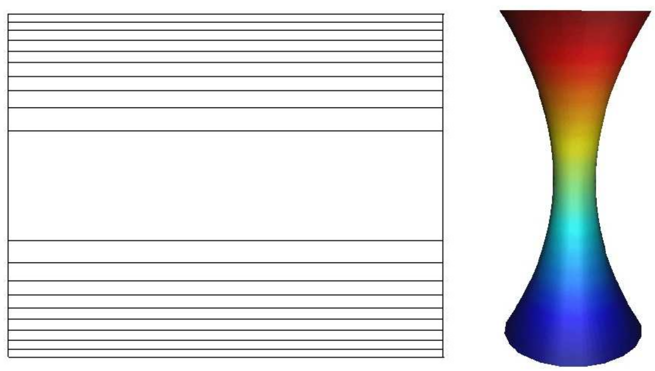

Figure 3: Nonperiodic column of revolution with up-down reflection.

Fluted Columns — Two Types

All remaining symmetry groups have at least $\mathbf{Z}_k$ symmetry for some k, that is, rotation symmetry through an angle $2\pi/k$. In our description of this classification we now set $k = 1$ with the understanding that there is a version of each of the remaining columns for each natural number k. Indeed, the pictures we show all have $k = 2$.

There are two types of columns with axial translation symmetry: those which have a plane of reflection across a plane containing the axis of the cylinder ($\mathbf{D}_2\dot{+}\mathbf{R}$) and those that do not ($\mathbf{Z}_2\dot{+}\mathbf{R}$). See Figures 6 and 7.

Spiral Columns — Two Types

There are two types of spirals — both of which have twisted translation symmetry. There are the spirals that are symmetric when the column is rotated by 180° in a plane containing the axis of the cylinder ($\mathbf{Z}_2(\tau\kappa)\dot{+}\mathbf{L}$) and those that do not have this symmetry ($\mathbf{L}$). See Figures 8 and 9.

Figure 4: Periodic column of revolution with no up-down reflection.

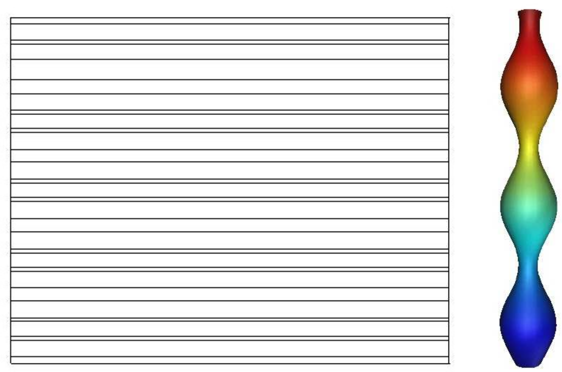

Figure 5: Periodic column of revolution with up-down reflection.

3.2 Columns with Discrete Symmetry

There are two types of symmetry groups that are infinite and discrete — those with corkscrew symmetries and those without.

Periodic Columns with No Corkscrew Symmetry — Eight Types

Recall that τ is a reflection through a plane containing the axis of the cylinder and κ is the reflection through the midplane – the up-down symmetry. Each of these symmetries has a glide reflection version

$$\tilde{\tau} = (\tau, (0, 1/2)) \qquad \tilde{\kappa} = (\kappa, (\pi, 0)).$$

There are ten subsets $G \subset \{\tau, \tilde{\tau}, \kappa, \tilde{\kappa}\}$ that form symmetry groups when coupled with $\mathbf{Z}$. These subsets are:

$$\{\kappa\} \quad \{\tau\} \quad \{\tilde{\kappa}\} \quad \{\tilde{\tau}\} \quad \{\tau, \kappa\} \quad \{\tilde{\tau}, \kappa\} \quad \{\tau, \tilde{\kappa}\} \quad \{\tilde{\tau}, \tilde{\kappa}\} \quad \emptyset \quad \{\tau\kappa\}.$$

The symmetry groups of the corresponding periodic columns are: $< G, \mathbf{Z} >$ — the group generated by G and $\mathbf{Z}$. Examples of columns having one pure reflection symmetry are found in Figures 10 and 11. Examples of columns having precisely one glide reflection are given in Figures 12 and 13. Columns having two reflections or glide reflections are shown

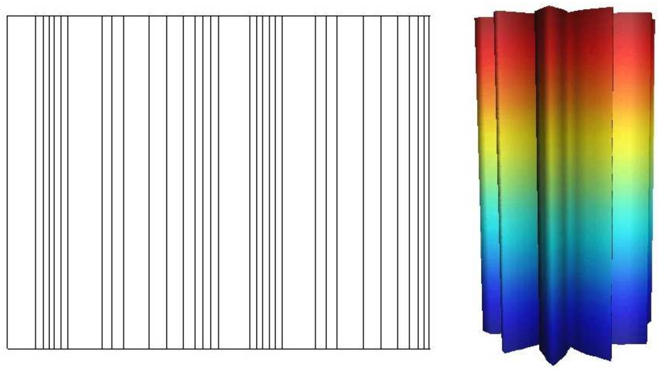

Figure 6: Fluted column with no left-right reflection.

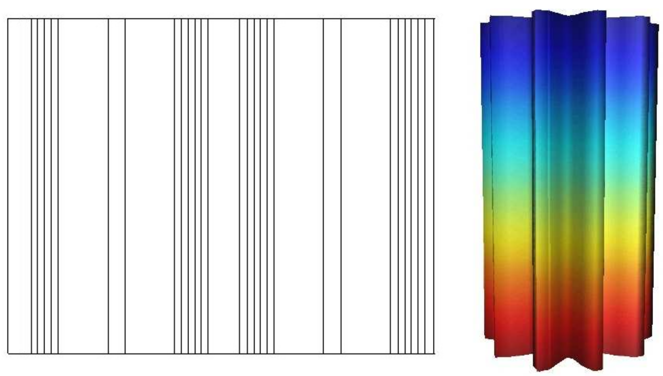

Figure 7: Fluted column with left-right reflection.

in Figures 14, 15, 16 and 17. The last two subsets correspond to symmetry groups that lie in infinite families and these infinite families have corkscrew symmetries (see Figures 21 and 22).

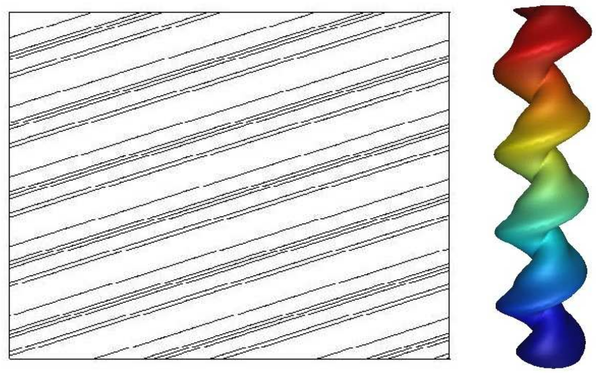

Figure 8: Spiral column with no up-down rotation.

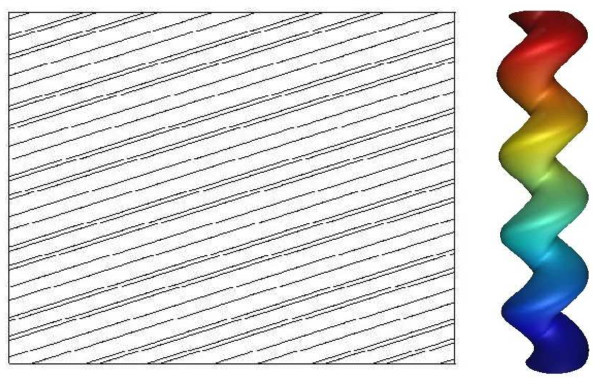

Figure 9: Spiral column with up-down rotation.

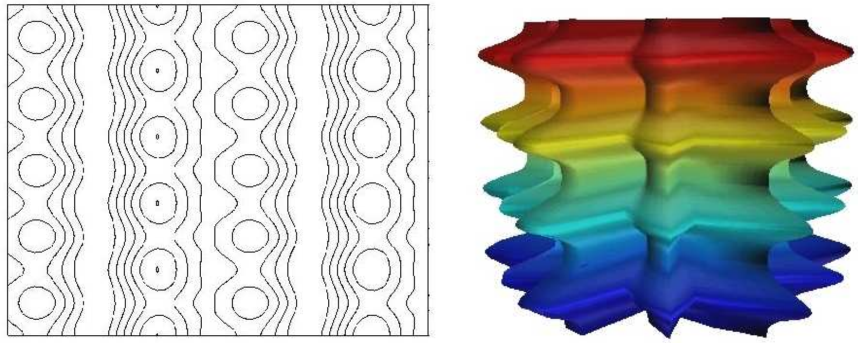

Figure 10: Periodic column with up-down reflection.

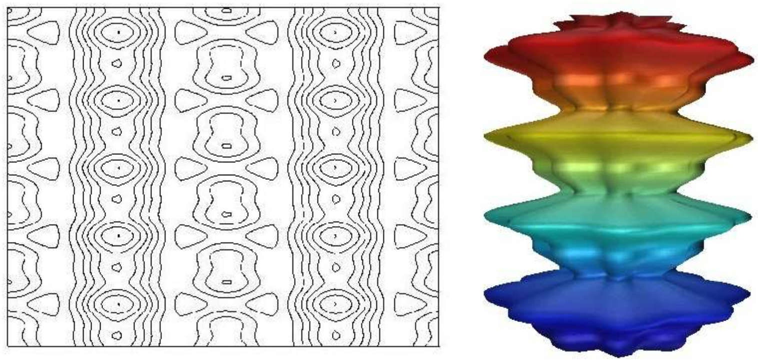

Figure 11: Periodic column with left-right reflection.

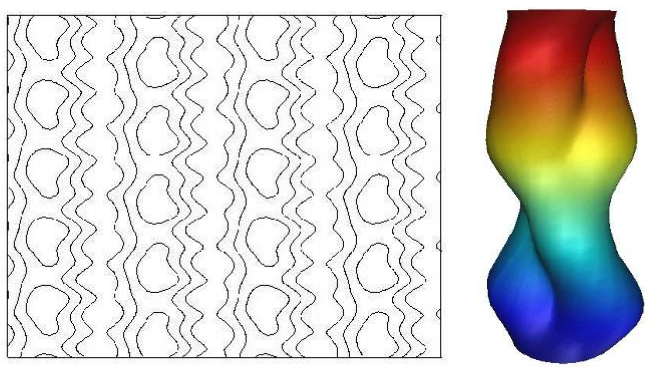

Figure 12: Periodic column with up-down glide reflection.

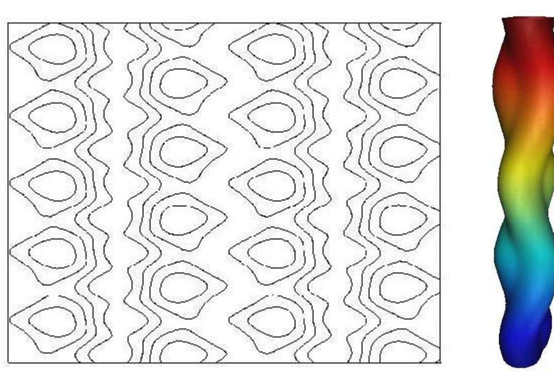

Figure 13: Periodic column with left-right glide reflection.

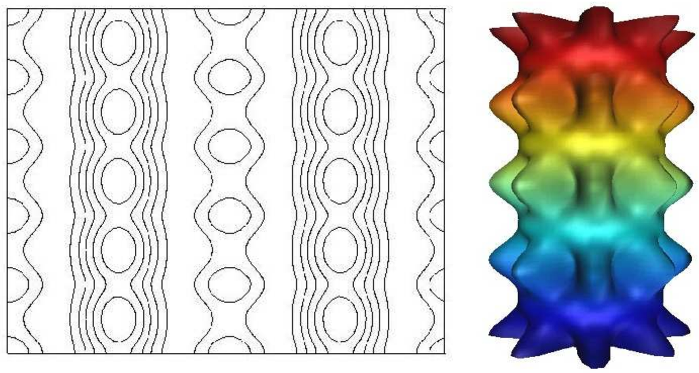

Figure 14: Periodic column with up-down and left-right reflections.

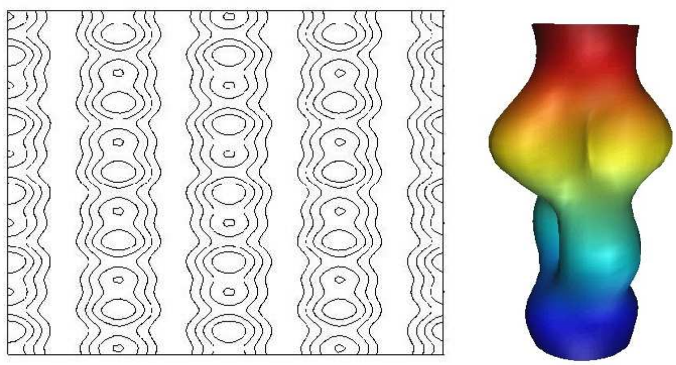

Figure 15: Periodic column with up-down glide reflection and left-right reflection.

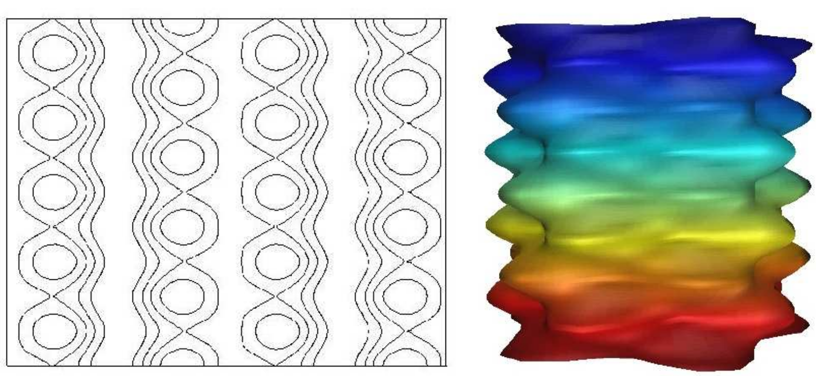

Figure 16: Periodic column with up-down reflection and left-right glide reflection.

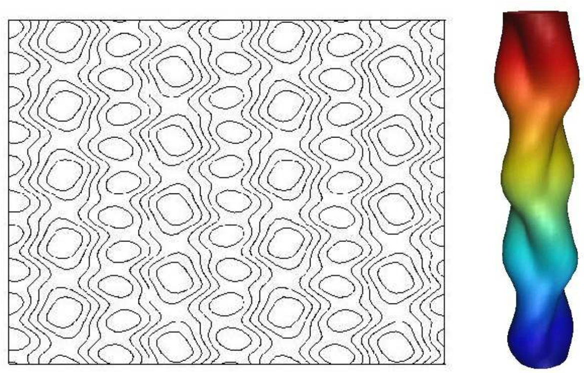

Figure 17: Periodic column with up-down and left-right glide reflections.

Discrete Corkscrew Columns — Five Types

There are three column types having $\mathbf{N}_\pi$ symmetry. These columns remain the same when translated in the axial direction a unit distance and simultaneously rotated through the angle 180° (π/k, in general). Among these columns are those that are invariant under reflection through the centerplane of the column ($\mathbf{Z}_2(\kappa)$), those that are invariant under reflection through a plane containing the cylinder axis ($\mathbf{Z}_2(\tau)$) and those that are invariant under both reflections. See Figures 18, 19 and 20.

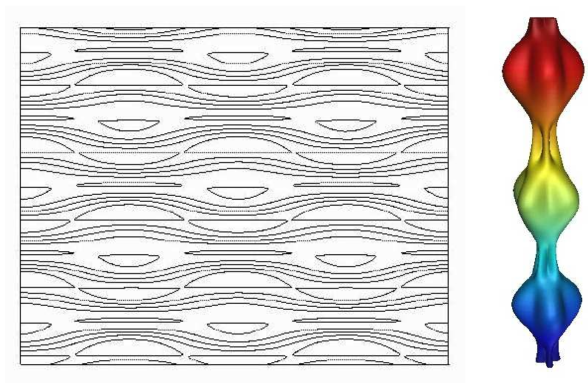

Figure 18: Corkscrew column with $\pi/2$ rotation and left-right reflection.

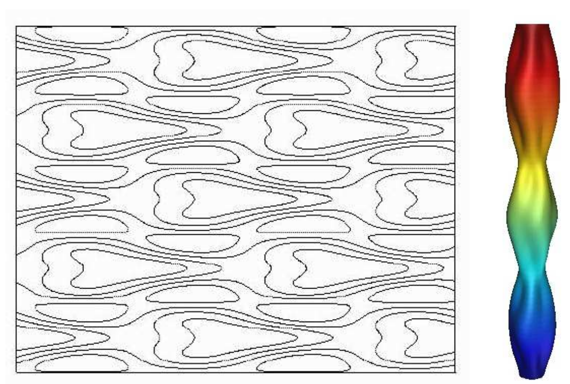

Figure 19: Corkscrew column with $\pi/2$ rotation and up-down reflection.

There are two continuous families depending on ω with discrete corkscrew motions (those with $\mathbf{N}_\omega$ symmetry). See Figures 21 and 22.

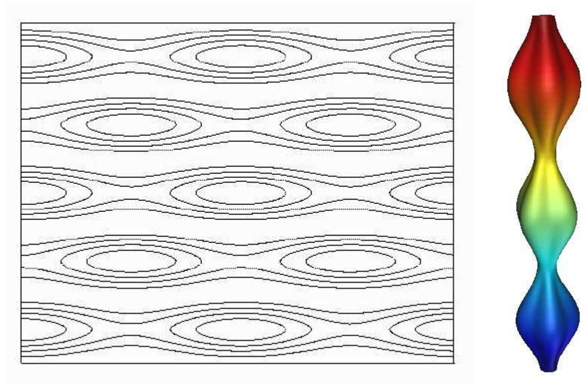

Figure 20: Corkscrew column with $\pi/2$ rotation and left-right and up-down reflections.

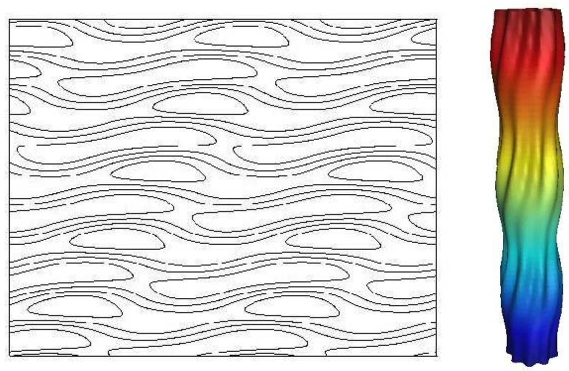

Figure 21: Corkscrew column with ω rotation where $0^\circ \leq \omega \leq \frac{180^\circ}{k}$ and no additional symmetry.

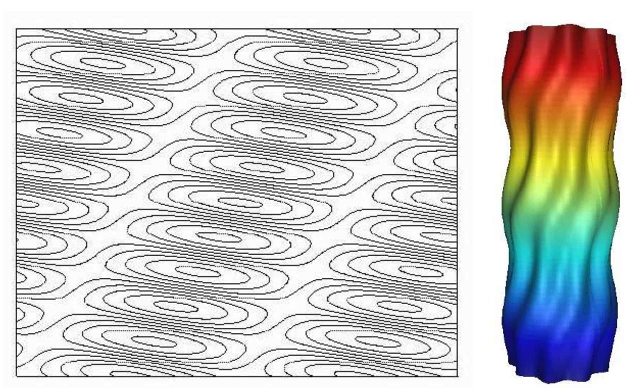

Figure 22: Corkscrew column with ω rotation where $0^\circ \leq \omega \leq \frac{180^\circ}{k}$ and up-down rotation.

3.3 Columns with Finite Symmetry — Seven Types

This types of column have neither a pure translation symmetry nor any symmetry that includes a translation symmetry. There are seven possible symmetry groups:

$$\mathbf{1} \quad <\kappa> \quad <\tilde{\kappa}> \quad <\tau\kappa> \quad <\tau> \quad <\tau,\kappa> \quad <\tau,\tilde{\kappa}>.$$

An example of a column with no symmetry is given in Figure 23. Columns with just a single reflection or glide reflection are shown in Figure 24, 25, 26 and 27, while columns with exactly two reflection or glide reflection symmetries are shown in Figure 28 and 29.

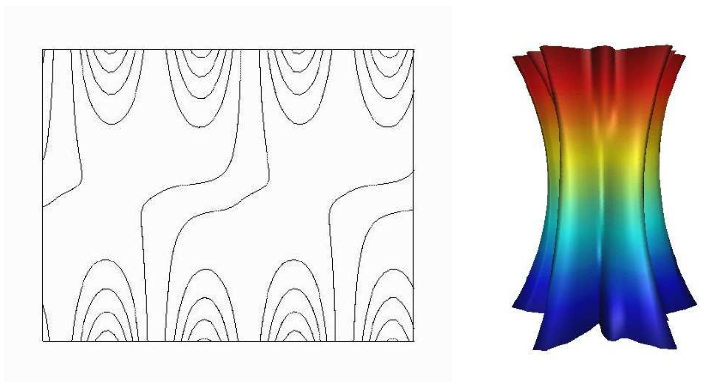

Figure 23: Column with no symmetries.

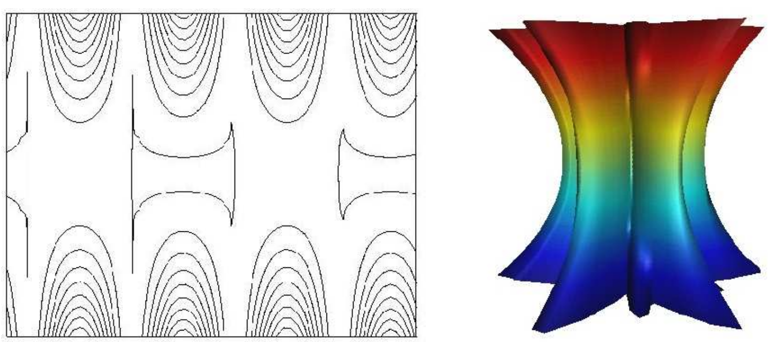

Figure 24: Column with up-down reflection.

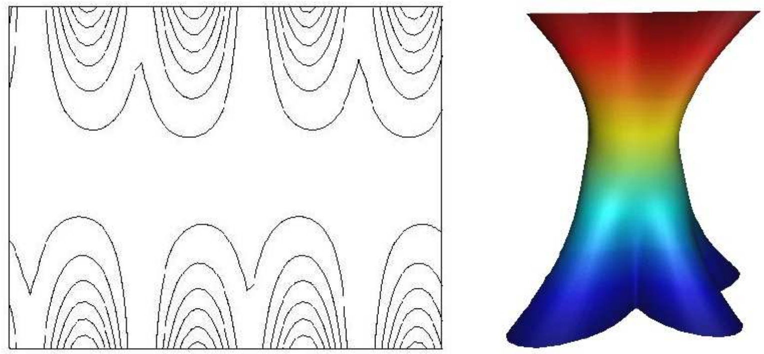

Figure 25: Column with up-down glide reflection.

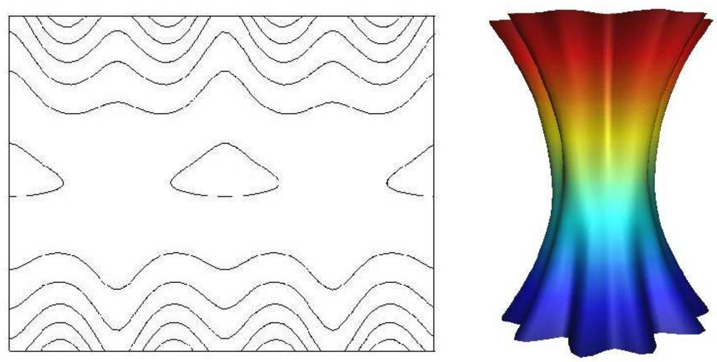

Figure 26: Column with left-right reflection.

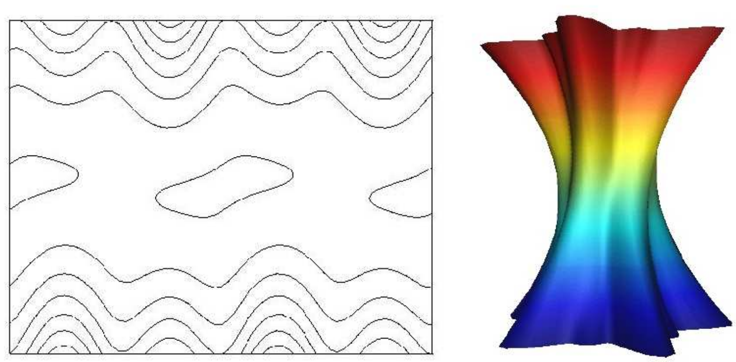

Figure 27: Column with up-down rotation.

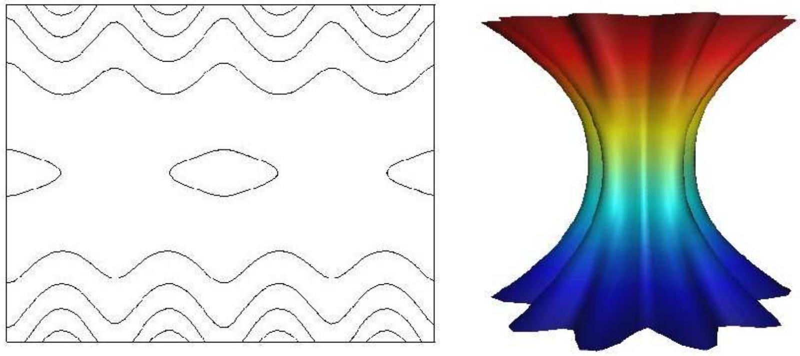

Figure 28: Column with up-down and left-right reflections.

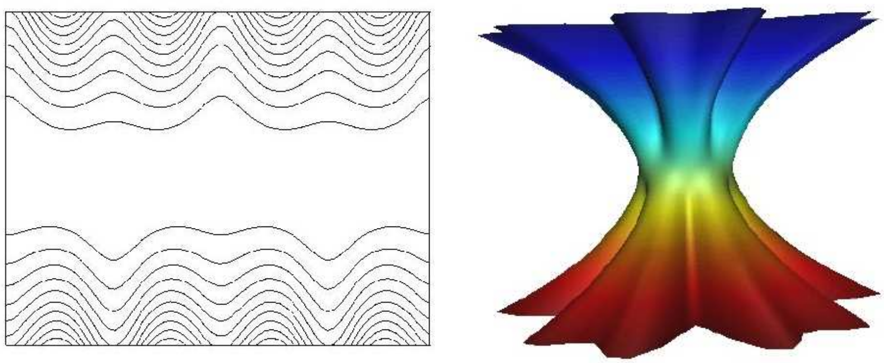

Figure 29: Column with up-down glide reflection and left-right reflection.

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[1] J.A. Baglivo and J.E. Graver. *Incidence and Symmetry in Design and Architecture*, Cambridge University Press, Cambridge, 1983, pp. 259–265.

[2] G. Guarini. *Architettura Civile* [1686], Milan: Polifilo, 1968.