A REMARK ON PERIODICALLY PERTURBED BIFURCATION

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Consider the autonomous differential equation

$$x' = f(x,\mu) \tag{1.1}$$

where $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth. Assume that for each μ there is a steady state solution to (1.1) $x(\mu)$ which we assume to be identically zero. In addition we assume that this steady state is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$. More precisely, it will be assumed that 0 is a simple eigenvalue of $f_x(0,0)$ and that $\frac{d\lambda}{d\mu}(0) > 0$ where $\lambda = \lambda(\mu)$, $\lambda(0) = 0$, is the smooth extension of the zero eigenvalue of $f_x(0,0)$ for $\mu \neq 0$. All other eigenvalues of $f_x(0,0)$ are assumed to lie in the negative half-plane. These assumptions give a well-known sufficient condition for the bifurcation of a steady state from the zero

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steady state at $\mu = 0$.

The problem considered here is to periodically perturb (1.1) and to consider the existence of 2π-periodic solutions. More precisely let $F: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^n$

$$F(x,\mu,t,\alpha) = F(x,\mu,t+2\pi,\alpha) \qquad (1.2)$$
$$F(x,\mu,t,0) = f(x,\mu)$$

We consider perturbations depending on the k-parameters α. Our goal is to find all 2π-periodic solutions to

$$x' = F(x,\mu,t,\alpha) \qquad (1.3)$$

having small (supremum) norm for (μ,α) near 0. It should be noted that if $f_x(0,0)$ were to have no eigenvalues of the form ni, $n = 0, \pm 1, \pm 2, \ldots$ then a straightforward implicit function theorem argument would imply the existence of a unique small norm 2π-periodic solution $x = x(\mu,\alpha)$, $x(0,\alpha) \equiv 0$ for small μ and α.

This problem has also been considered by Rosenblat and Cohen [2] for some specific "model" equations. Their approach is formal and involves matching an outer solution for $\mu \neq 0$ obtained as a perturbation expansion (generated by the implicit function theorem) with an inner solution in the limit $\mu \to 0$. In this paper we extend and give rigorous proofs of results of Rosenblat and Cohen by using the singularity theory techniques of [1]. Besides putting these results in what we feel is a more natural setting we extend and simplify them. We also show why a perturbation theory based on one parameter as used in [2] leads to difficulty.

Our approach will be to reduce the problem of finding 2π-

periodic solutions of (1.3) to finding zeros of a scalar "bifurcation" equation

$$A(z,\mu,\alpha) = 0 \tag{1.4}$$

by the classical Liapunov-Schmidt Technique. In (1.4) z is the average of a 2π-periodic solution measured along the direction $\phi \neq 0$ where $f_x(0,0)\phi = 0$. We will exploit the very simple observation that

$$a(z,\mu) \equiv A(z,\mu,0) = 0 \tag{1.5}$$

is precisely the bifurcation equation arising from the steady state bifurcation problem (1.1). In (1.5), z represents the component of the bifurcating steady states in the ϕ direction. In the language of singularity theory, $A(z,\mu,\alpha)$ is a k-parameter unfolding of $a(z,\mu)$. We consider two situations. First, very briefly we consider the case that f has "effective" quadratic terms so that the bifurcation of steady states of (1.1) is transcritical near $(x,\mu) = (0,0)$ (see Figure 1.1). Secondly, we consider in more detail the case that $f(\cdot,\mu)$ has no quadratic terms and that cubic terms of f determine the direction of the bifurcating steady states. In this last case the bifurcating steady states exist for either $\mu > 0$ or $\mu < 0$ but not both, i.e., we have the pitchfork diagram (see Figure 1.2). This situation would arise naturally if there were a reflectional symmetry in the differential equation (1.1) which caused f to be odd in x. While our discussion will be limited to these two cases, our techniques apply in more complicated situations (the essential point is that $a(z,\mu)$ has finite codimension, see [1]).

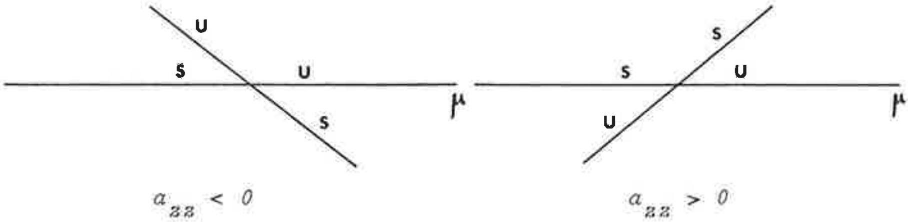

Figure 1.1. Solutions of (1.5) when f *has effective quadratic terms.*

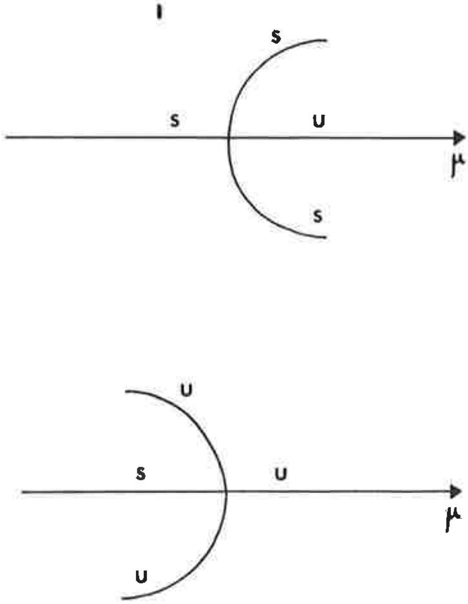

Figure 1.2. Solutions of (1.5) when cubic terms determine direction of bifurcation. The pitchfork.

For $\alpha \neq 0$, the equation $A(\cdot,\cdot,\alpha) = 0$ can be viewed as a perturbation of (1.5). Hence the bifurcation diagram corresponding to (1.4) will be perturbations of those in Figures 1.1 and 1.2. Our first objective will be to classify qualitatively all possible perturbations of the bifurcation diagrams in Figures 1.1 and 1.2 which can be realized by (1.4).

The question of which particular perturbed diagram corresponds to a particular perturbation F depends on various assumptions on the form of F. We return to this question briefly in a later section in order to compare our results with those in [2]. In pursuit of our first objective we will be led to determine conditions on F so that (1.4) will be a universal unfolding of (1.5) in the two cases which we consider. Elementary results of singularity theory will be used in this determination. The possible (stable) perturbed bifurcation diagrams (1.4) that we find are shown below. The stability assignments are obtained by straightforward consideration of the dynamics on the one dimensional center manifold and continuity arguments. Note that solution branches correspond to 2π-periodic solutions in these figures.

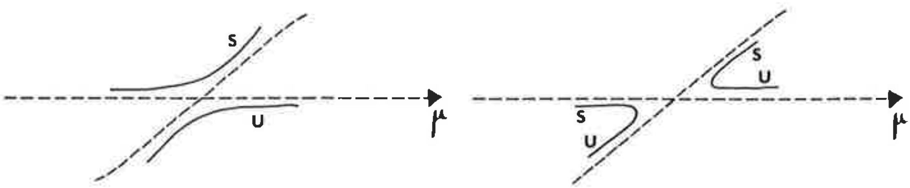

Figure 1.3. Perturbations (1.4) of Figure 1.1, $a_{zz} < 0$.

The possibility of hysteresis phenomena as in Figure 1.4(b) and (d) is overlooked in [2].

In the next section we recall some elementary definitions and results from singularity theory. For a more detailed treatment the reader is referred to [1]. In section three the bifurcation equations (1.4) and (1.5) are derived and singularity theory is applied to obtain the perturbed diagrams in Figures 1.3 and 1.4. Moreover conditions on $F(x,\mu,t,\alpha)$ are presented which insure that the various diagrams result.

Finally, we show how to obtain rigorously the results of Rosenblat and Cohen [2].

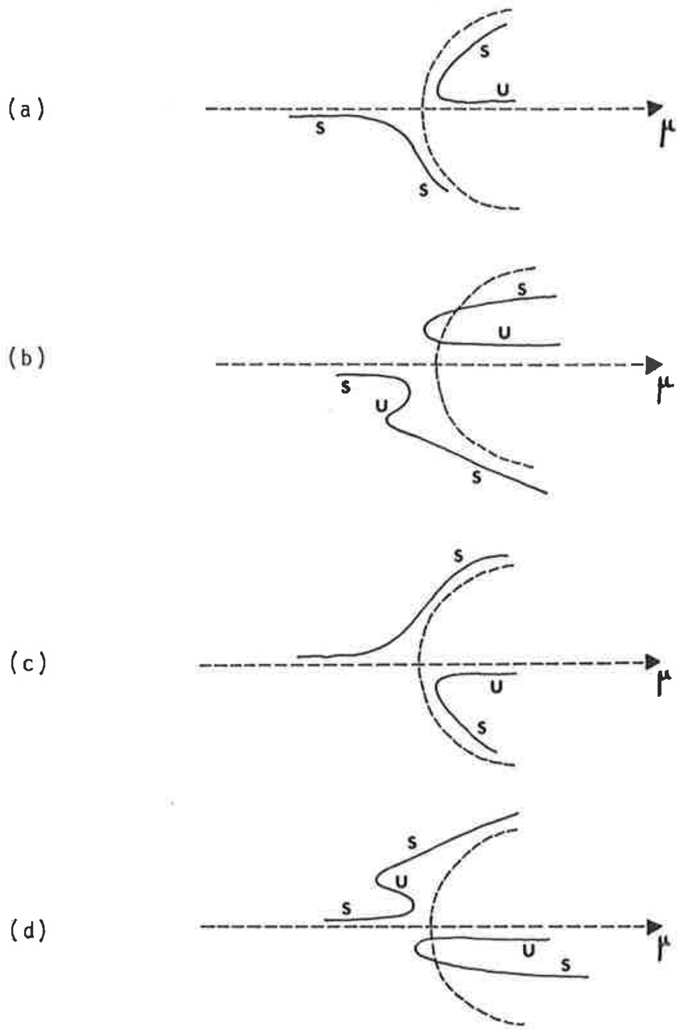

Figure 1.4. Perturbations (1.4) of Figure 1.2, $a_{zzz} < 0$

II. SOME ELEMENTARY RESULTS OF SINGULARITY THEORY

The singularity theory approach, as described in [1], to the bifurcation problem

$$G(z,u) = 0 \tag{2.1}$$

where $G:\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $G(0,0) = 0$ is defined in some neighborhood of the origin in $\mathbb{R} \times \mathbb{R}$, is to change variables in order to put (2.1) into a normal (standard) form. The normal form can be taken to be a polynomial map whose zeros are readily found. The changes of variable that will be allowed are now described. Two such maps, G and H, are said to be contact equivalent, written $G \sim H$, if there exists a smooth map $\tau:\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\tau(0,0) \neq 0$ defined on a neighborhood of the origin and a diffeomorphism of a neighborhood of the origin in $\mathbb{R} \times \mathbb{R}$ of the form $(z,\mu) \to (\rho(z,\mu), \Lambda(\mu))$, with $\rho(0,0) = 0$ and $\Lambda(0) = 0$ such that

$$H(z,\mu) = \tau(z,\mu)G(\rho(z,\mu),\Lambda(\mu))$$

We assume $\rho(\cdot,\mu)$ and Λ are orientation preserving ($\rho_z(0,0) > 0$, $\Lambda_\mu(0) > 0$). It is easily seen that if H and G are contact equivalent then their bifurcation diagrams are equivalent in the sense that for each μ, the number of solutions z of $H(z,\mu) = 0$ is the same as the number of solutions z of $G(z,\Lambda(\mu)) = 0$ is some neighborhood of the origin. Furthermore the bifurcation diagrams are diffeomorphic. More motivation for the form of the allowed changes of coordinates is given in [1].

The following proposition can be found in [1].

PROPOSITION 2.1.

(a) If $a(z,\mu)$ *satisfies* $a(0,0) = a_z(0,0) = a_\mu(0,0) = 0$, $a_{z\mu}(0,0) \neq 0$, $a_{zz}(0,0) \neq 0$ *then* $a(z,\mu) \sim z^2 + \mu z$.

(b) If $a(0,0) = a_z(0,0) = a_\mu(0,0) = a_{zz}(0,0) = 0$,

$a_{z\mu}(0,0) \neq 0$, $a_{zzz}(0,0) \neq 0$ *then* $a(z,\mu) \sim z^3 \pm \mu z$ *where* $+$ *if* $a_{zzz}a_{z\mu} > 0$ *and* $-$ *if* $a_{zzz}a_{z\mu} < 0$.

In fact, the hypotheses in parts (a) and (b) are also necessary. Proposition 2.1 contains the answers to the questions: When do we have a transcritical bifurcation as in Figure 1.1? When do we have the pitchfork bifurcation as in Figure 1.2?

A principal aim of singularity theory is to describe the perturbations of a particular bifurcation problem. The singularity theory approach to a perturbation is the notion of an unfolding. $A: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is said to be k-parameter unfolding of $a(z,\mu)$ if $A(z,\mu,0) \equiv a(z,\mu)$. Singularity theory provides a method for describing the bifurcation diagrams of all perturbations (unfoldings), modulo contact equivalence, through the idea of a universal unfolding. More precisely, we call $A(z,\mu,\alpha)$ a universal unfolding of $a(z,\mu)$ provided any other unfolding $B(z,\mu,\beta)$, $\beta \in \mathbb{R}^\ell$ for β near 0, $B(z,\mu,0) \equiv a(z,\mu)$, has the property that for each $\beta \in \mathbb{R}^\ell$, $B(\cdot,\cdot,\beta) \sim A(\cdot,\cdot,\psi(\beta))$ where $\psi: \mathbb{R}^\ell \to \mathbb{R}^k$ is a smooth map. We allow the parameters of contact equivalence (τ,Λ,ρ) to depend on β (see [1] for a precise definition). In other words, if A is a universal unfolding of a and B is any other unfolding (perturbation) of a then each bifurcation diagram $B(\cdot,\cdot,\beta) = 0$ is equivalent to a bifurcation diagram $A(\cdot,\cdot,\psi(\beta)) = 0$. Thus understanding all bifurcation diagram $A(\cdot,\cdot,\alpha)$ for α near zero is tantamount to understanding all perturbations of the bifurcation diagram corresponding to a.

If either of the hypotheses in proposition 2.1 (a) or (b) hold then the following result of singularity theory [1] implies that a has a universal folding.

PROPOSITION 2.2

(a) A universal folding with the minimum number of parameters for $z^2 + \mu z$ *is given by* $z^2 + \mu z + \alpha_1$. *If (a) of proposition 2.1 holds,* $A:\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a universal unfolding of* $a(z,\mu)$ *if and only if* $A_\alpha(0,0,0) \neq 0$.

(b) A universal unfolding with the minimum number of parameters for $z^3 \pm \mu z$ *is given by* $z^3 \pm \mu z + \alpha_2 z^2 + \alpha_1$. *When (b) of proposition 2.1 holds,* $A:\mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ *is a universal unfolding of* $a(z,\mu)$ *if and only if*

$$\det \begin{pmatrix} 0 & 0 & a_{z\mu} & a_{zzz} \\ 0 & a_{z\mu} & a_{zz} & a_{zz\mu} \\ A_{\alpha_1} & A_{z\alpha_1} & A_{\alpha_1\mu} & A_{\alpha_1 zz} \\ A_{\alpha_2} & A_{z\alpha_2} & A_{\alpha_2\mu} & A_{\alpha_2 zz} \end{pmatrix} \tag{2.2}$$

where all derivatives are evaluated at $(z,\mu,\alpha) = (0,0,0)$.

For the normal form given in (b) above one can ask which bifurcation diagrams occur for which values of (α_1,α_2). This information is summarized in figure 1.5. The important observation here is that the separating curves shown in Figure 1.5 are tangent at $(\alpha_1,\alpha_2) = (0,0)$. The universal unfolding theorem also implies that there is a diffeomorphic copy of Figure 1.5 occuring in the parameter space of any universal unfolding. The order two contact of the two curves $\alpha_1 = 0$ and $\alpha_1 = \alpha_2^3/27$ in Figure 1.5 implies that very careful scaling is required in order to observe the hysteresis diagrams in Figure 1.4 (b) and (d) with a one parameter pertur-

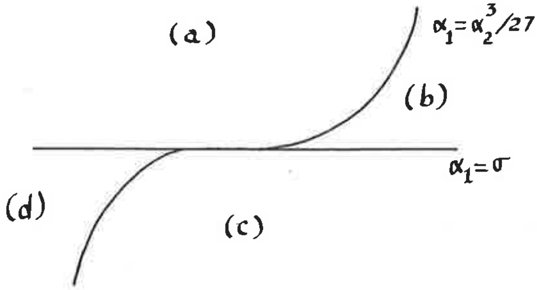

FIGURE 1.5. Catalog of bifurcation diagrams for $z^3 - \mu z + \alpha_2 z^2 + \alpha_1$. *The letters in the figure refer to the bifurcation diagrams listed in Figure 1.4; i.e., if* (α_1,α_2) *is in region (a) then the associated bifurcation diagram is equivalent to Figure 1.4 (a). See [1] for details.*

bation (unfolding) of the pitchfork diagram. Indeed, if $\bar{A}(z,\mu,\varepsilon)$ is a one parameter unfolding of $a(z,\mu)$ then by proposition 2.2, $\bar{A}(\cdot,\cdot,\varepsilon) \sim A(\cdot,\cdot,\alpha_1(\varepsilon),\alpha_2(\varepsilon))$, where $A(z,\mu,\alpha_1,\alpha_2)$ is the universal unfolding of proposition 2.2 and $(\alpha_1(0),\alpha_2(0)) = (0,0)$. Thus the observed bifurcation diagram for $\bar{A}(\cdot,\cdot,\varepsilon)$ depends crucially on the location of the curve $\varepsilon \to (\alpha_1(\varepsilon),\alpha_2(\varepsilon))$ passing through the origin at $\varepsilon = 0$. Since "most" curves are transverse to $\alpha_1 = 0$ at the origin, it is very easy using a one parameter perturbation to miss (as Rosenblat and Cohen do) the possibility of a hysteresis loop in the bifurcation diagram. As a final remark we note that if A satisfies the conditions of proposition 2.2 (b) then the curve $\alpha_1 = 0$ in the standard normal form corresponds to the curve in the (α_1,α_2) space defined by eliminating z and μ from

$$A(z,\mu,\alpha) = A_z(z,\mu,\alpha) = A_\mu(z,\mu,\alpha) = 0 \qquad (2.3)$$

while the curve $\alpha_1 = \alpha_2^3/27$ corresponds to the curve defined by eliminating z and μ from

$$A(z,\mu,\alpha) = A_z(z,\mu,\alpha) = A_{zz}(z,\mu,\alpha) = 0 \tag{2.4}$$

This can be checked for the given universal unfolding.

III. THE BIFURCATION EQUATIONS

We begin by considering the bifurcation of steady states of (1.1). Let ϕ be an eigenvector corresponding to the eigenvalue 0 of $f_x(0,0)$ and ψ be an eigenvector for $f_x(0,0)^+$ satisfying $\psi\cdot\psi = 1$, $\psi\cdot\phi > 0$. Then the equation

$$f(x,\mu) = 0$$

is equivalent to the pair of equations

$$\begin{aligned} &\text{(a)} \quad f(z\phi + w,\mu) - (f(z\phi + w,\mu)\cdot\psi)\psi = 0 \\ &\text{(b)} \quad f(z\phi + w,\mu)\cdot\psi = 0 \end{aligned} \tag{3.1}$$

where $x = z\phi + w$, $w\varepsilon<\phi>^{\perp}$. The first of the above equations can be solved, by the implicit function theorem, for $w = w(z,\mu)\ \varepsilon <\phi>^{\perp}$ satisfying $w(0,\mu) \equiv 0$ and $w_z(0,0) = 0$. The latter equality is easily checked by implicit differentiation, the former follows from $f(0,\mu) \equiv 0$. The bifurcation equation is given by

$$a(z,\mu) = f(z\phi + w(z,\mu),\mu)\cdot\psi = 0. \tag{3.2}$$

It is clear that $a(0,\mu) \equiv 0$. We consider two cases

(H1) $f_{xx}(0,0)(\phi,\phi)\cdot\psi \neq 0$,

(H2) $f_{xx}(0,\mu)\equiv 0$, $f_{xxx}(0,0)(\phi,\phi,\phi)\cdot\psi \neq 0$.

In either case, $a_z(0,0) = f_x(0,0)\phi\cdot\psi = 0$ and $a_{z\mu}(0,0) = \frac{d\lambda}{d\mu}(0)\phi\cdot\psi > 0$. Straightforward differentiation of (3.2) yields the following consequences of (H1) and (H2) on (3.2)

(H1) $a_z(0,0) = a_\mu(0,0) = a(0,0) = 0, \quad a_{z\mu}(0,0) > 0,$
$a_{zz}(0,0) = f_{xx}(0,0)(\phi,\phi)\cdot\psi \neq 0.$

(H2) $0 = a(0,0) = a_z(0,0) = a_\mu(0,0) = a_{zz}(0,0) = a_{\mu\mu}(0,0)$
$= a_{zz\mu}(0,0) = a_{\mu\mu\mu}(0,0)$ and
$a_{z\mu}(0,0) > 0$ and $a_{zzz}(0,0) = f_{xxx}(0,0)(\phi,\phi,\phi)\cdot\psi \neq 0$

It follows immediately from proposition (2.1) that if (H1) holds $a(z,\mu) \sim z^2 + \mu z$ and if (H2) holds then $a(z,\mu) \sim z^3 \pm \mu z$ ($\pm$ depending on the sign of $f_{xxx}(0,0)(\phi,\phi,\phi)\cdot\psi$). The bifurcation diagram corresponding to (3.2) is given by Figure 1.1 in case (H1) holds and by Figure 1.2 in case (H2) holds.

Now consider the problem of 2π-periodic solutions of (1.3). Let $N: C^1_{2\pi} \times \mathbb{R} \times \mathbb{R}^k \to C^0_{2\pi}$ defined on the C^1-2π-periodic functions into the continuous 2π-periodic functions be given by

$$N(x,\mu,\alpha) = x' - F(x,t,\mu,\alpha)$$

We are interested in solutions of

$$N(x,\mu,\alpha) = 0.$$

Let $L = N_x(0,0,0) = \frac{d}{dt} - f_x(0,0)$ be the Frechet derivative and $Ker\ L = \{z\phi : z \in \mathbb{R}\}$. We may write $C^1_{2\pi} = Ker\ L \oplus W$ where $W = \{x \in C^1_{2\pi} : \int_0^{2\pi} x(t)\cdot\phi\, dt = 0\}$. Similarly $C^0_2 = \{r\psi : r \in \mathbb{R}\} \oplus Y$ where $Y = Range\ L = \{y \in C^0_{2\pi} : \int_0^{2\pi} y\cdot\psi\, dt = 0\}$. The equation $N = 0$ is equivalent to the pair of equations

$$\begin{aligned} &\text{(a)} \quad w' - F(z\phi + w,\mu,t,\alpha) + \left(\frac{1}{2\pi}\int_0^{2\pi} F(z\phi + w,\mu,t,\alpha)\cdot\psi\, dt\right)\psi = 0 \\ &\text{(b)} \quad \frac{1}{2\pi}\int_0^{2\pi} F(z\phi + w,\mu,t,\alpha)\cdot\psi\, dt = 0 \end{aligned} \tag{3.3}$$

where $x = z\phi + w$, $w \in W$. As before (3.3) (a) can be solved for $w = w(z,\mu,\alpha) \in W$. An important point to observe is that $w(z,\mu,0) = w(z,\mu)$ where the w on the right is the w solved for in (3.1) (a). The bifurcation equation is given by

$$A(x,\mu,\alpha) \equiv \frac{1}{2\pi}\int_0^{2\pi} F(z\phi + w(z,\mu,\alpha),\mu,t,\alpha)\cdot\psi\, dt = 0 \tag{3.4}$$

If (z,μ,α) satisfies (3.4) then $x = z\phi + w(z,\mu,\alpha)$ satisfies (1.3). It is immediate that

$$A(z,\mu,0) = a(z,\mu),$$

i.e., $A(z,\mu,\alpha)$ is a k-parameter unfolding of $a(z,\mu)$. We now seek conditions on F so that A is a universal unfolding of a, i.e., we ask what are all possible perturbations of the bifurcation diagram associated with (3.2). We may as well assume that A depends on no more parameters than the minimum required for a universal unfolding. Thus if (H1) holds we take $\alpha \in \mathbb{R}$ and if (H2) holds we assume $\alpha \in \mathbb{R}^2$.

If (H1) holds, proposition 2.2 gives that A is a universal unfolding of a if and only if

$$A_\alpha(0,0,0) = \frac{1}{2\pi}\int_0^{2\pi} F_\alpha(0,0,t,0)\cdot\psi\, dt \neq 0$$

If, for example

$$F(x,\mu,t,\alpha) = f(x,\mu) + \alpha g(x,\mu,t)$$

the requirement is simply that $g(0,0,t)\cdot\psi$ has nonzero mean.

In this case one gets bifurcation diagrams as in Figure 1.3. If $g(0,0,t)\cdot\psi$ has mean zero one either gets bifurcation diagrams as in Figure 1.3 or the transverse crossings as in Figure 1.1. This last situation could arise if, for instance, $g(0,\mu,t) \equiv 0$. We do not pursue further the degenerate case that $A_\alpha = 0$ here as it is straightforward (see [2]). Rather we will consider later the more interesting degenerate case when $A(z;\mu,\alpha)$ fails to be a universal unfolding of the pitchfork.

If (H2) holds, proposition 2.2 implies that A is a universal unfolding of a if and only if (2.2) holds. Since $a_{zz} = a_{zz\mu} = 0$ and $a_{z\mu} \neq 0$ (2.2) is equivalent to

$$\det \begin{pmatrix} 0 & a_{z\mu} & a_{zzz} \\ A_{\alpha_1} & A_{\alpha_1\mu} & A_{\alpha_1 zz} \\ A_{\alpha_2} & A_{\alpha_2\mu} & A_{\alpha_2 zz} \end{pmatrix}$$

Clearly, this inequality requires either $A_{\alpha_1} \neq 0$ or $A_{\alpha_2} \neq 0$ or both. But no new phenomena will be obtained by requiring both to be nonzero. Hence we assume $A_{\alpha_1} \neq 0$, $A_{\alpha_2} = 0$. The inequality above is then reduced to

$$\begin{vmatrix} a_{z\mu} & a_{zzz} \\ A_{\alpha_2\mu} & A_{\alpha_2 zz} \end{vmatrix} \neq 0 \qquad (3.5)$$

In view of (3.5), A will be a universal unfolding of a if, for example,

$$F(x,\mu,t,\alpha_1,\alpha_2) = f(x,\mu) + \alpha_2 g(x,t) + \alpha_1 h(x,t,\mu) \qquad (3.6)$$

where

$$A_{\alpha_2} = \frac{1}{2\pi}\int_0^{2\pi} g(0,t)\cdot\psi\, dt = 0, \quad A_{\alpha_2\mu} = 0$$

$$A_{\alpha_1} = \frac{1}{2\pi}\int_0^{2\pi} h(0,t,0)\cdot\psi\, dt \neq 0 \tag{3.7}$$

$$A_{\alpha_2 zz} = \frac{1}{2\pi}\int_0^{2\pi} g_{xx}(0,t)(\phi,\phi)\cdot\psi\, dt \neq 0$$

The bifurcation diagrams corresponding to (3.4) when (3.6), (3.7) hold are given in Figure 1.4. Which particular diagram one sees depends on where (α_1,α_2) lies in relation to the curves (2.3) and (2.4) in (α_1,α_2) space.

It should be noted that all the perturbed bifurcation diagrams in Figure 1.3 and 1.4 can be the result of a purely autonomous perturbation of (1.1), in which case the solution branches correspond to steady state solutions of (1.3).

We end this note by observing that the results of Rosenblat and Cohen [2] can be recovered rather easily from the above considerations. They study one parameter periodic perturbations of (1.1) so that their results can be expected to be imbedded in our universal unfolding. We shall see that this is the case. In particular, Rosenblat and Cohen consider the model equation

$$x' = \mu x - x^3 + \varepsilon\{g(t) + xh(t)\} \tag{3.8}$$

In this case clearly $a(z,\mu) = \mu z - z^3$. To simplify notation we use $\langle f\rangle$ $\frac{1}{2\pi}\int_0^{2\pi} f(t)dt$; if the 2π-periodic function f satisfies $\langle f\rangle = 0$ we denote by $\tilde{f}$ the unique 2π-periodic, mean-zero antiderivative of f. The equation (3.3) (a) becomes

$$w' = \mu w - 3z(w^2 - <w^2>) - 3z^2 w - (w^3 - <w^3>)$$
$$+ \varepsilon[(g(t) - <g>) + z(h(t) - <h>) + (wh - <wh>)]$$

It follows that $w \equiv 0$ if $\varepsilon = 0$, i.e., $w(z,\mu,\varepsilon) = \varepsilon\bar{w}(z,\mu,\varepsilon)$ the resulting equation for $\bar{w}$ is then given by

$$\begin{aligned}\bar{w}' &= \mu\bar{w} - 3z\varepsilon(\bar{w}^2 - <\bar{w}^2>) - 3z^2\bar{w} - \varepsilon^2(\bar{w}^3 - <\bar{w}^3>)\\ &+ (g(t) - <g>) + z(h(t) - <h>) + \varepsilon(\bar{w}h - <\bar{w}h>)\end{aligned} \tag{3.9}$$

At this point the essential observation one makes from (3.9) is that

$$\bar{w}(0,0,0) = (\widetilde{g - <g>}). \tag{3.10}$$

The bifurcation equation (3.4) is given by

$$\begin{aligned}A(z,\mu,\varepsilon) &= \mu z - z^3 - 3\varepsilon^2 z<\bar{w}^2> - \varepsilon^3<\bar{w}^3> + \varepsilon<g>\\ &+ \varepsilon z<h> + \varepsilon^2<\bar{w}h>\ ,\end{aligned} \tag{3.11}$$

i.e., A is a one parameter unfolding of $\mu z - z^3$. By proposition 2.2 (b), $A(z,\mu,\varepsilon) \sim \mu z - z^3 + \alpha_2(\varepsilon)z^3 + \alpha_1(\varepsilon)$. More precisely

$$A(z,\mu,\varepsilon) = \tau(z,\mu,\varepsilon)(\Lambda(\mu,\varepsilon)Z(z,\mu,\varepsilon) - Z^3 + \alpha_2(\varepsilon)Z^2+\alpha_1) \tag{3.12}$$

where $\tau(z,\mu,0) = 1$, $\Lambda(\mu,0) = \mu$, $Z(z,\mu,0) = z$, $\alpha_1(0) = \alpha_2(0) = 0$. Which bifurcation diagrams are observed will depend crucially on the placement of the curve

$$\varepsilon \to (\alpha_2(\varepsilon),\alpha_1(\varepsilon)) \tag{3.13}$$

which passes through the origin for $\varepsilon = 0$ in Figure 1.5. The table below summarizes the results of our calculations.

Table 1.6

Perturbation $\varepsilon\{g(t)+xh(t)\}$	Curve $\varepsilon\to(\alpha_2(\varepsilon),\alpha_1(\varepsilon))$	Bifurcation diagram
$<g> \neq 0$	$\alpha_1'(0) \neq 0$	(a) or (b)
$<g> = 0, \quad <\bar{g}h> \neq 0$	$\alpha_1'(0) = 0$ $\alpha_1''(0) \neq 0$	(a) or (c)
$<g> = 0, \quad h \equiv 0$ $<g^3> \neq 0$	$\alpha_2'(0) = 0$ $\alpha_1'(0) = \alpha_1''(0) = 0$ $\alpha_1'''(0) = 0$	(a) or (c)
$<g> = 0, \quad h \equiv 0$ $<\tilde{g}^3> = 0$ $<\tilde{g}^2(\widetilde{\tilde{g}^3})> \neq 0$	$\alpha_2'(0) = 0$ $\alpha_1^{(k)}(0) = 0, \ k = 0,1,2,3,4$ $\alpha_1^{(5)}(0) \neq 0$	(a) or (c)

In Table 1.6, the first column, labeled 'perturbation', consists of various sets of assumptions on the functions f and g. Column two, labeled 'curve', contains the implications of these assumptions on the derivatives of the $\alpha_i(\varepsilon)$. In all cases, the derivative asserted to be nonzero is proportion to the corresponding non-zero mean in the perturbation column (i.e., $\alpha_1'(0) = <g>$). In column three, the bifurcation diagram corresponding to the particular set of hypotheses in the perturbation column is specified. In all cases considered, diagrams (a) or (c) in Figure 1.4 are obtained. This follows immediately from the information in in Figure 1.5. In fact, if $\alpha_1 = O(\varepsilon^k)$ and $\alpha_2 = O(\varepsilon^\ell)$ near $\varepsilon = 0$ then $k \geq 3\ell$ is a necessary condition in order that the bifurcation diagrams (b) or (d), the hysteresis loops, occur.

The results summarized in the first three rows of table 1.6 confirm the formal results of Rosenblat and Cohen. In the last row, if $\langle g \rangle = \langle \tilde{g}^3 \rangle = 0$, $h \equiv 0$, Rosenblat and Cohen contend that the resulting bifurcation diagram is the displaced pitchford (see Figure 4 [2]). Clearly this is false if $\langle \tilde{g}^2 (\widetilde{\tilde{g}^3}) \rangle \neq 0$. In fact, the displaced pitchfork can occur if and only if $\alpha_1(\varepsilon) \equiv \alpha_2(\varepsilon) \equiv 0$!

Table 1.6 motivates the following interesting open problem: can conditions be found on the perturbation $\varepsilon\{g(t) + xh(t)\}$ which result in a bifurcation diagram having a hysteresis loop?

We indicate briefly in the paragraphs below how the results in table 1.6 are determined.

The first row of table 1.6 obtains since when $z = \mu = 0$ in (3.12), the left hand side begins with $\varepsilon\langle g \rangle$, while the right hand side is $\alpha_1(\varepsilon) + O(\varepsilon^2)$.

If $\langle g \rangle = 0$, one shows $Z_\varepsilon(0,\mu,0) = 0$. (To see this, differentiate (3.12) with respect to ε and set $z = \varepsilon = 0$.) Then, setting $z = \mu = 0$ in (3.12) one observes (recall (3.10)) that the left hand side begins with $\varepsilon^2\langle \tilde{g}h \rangle$ while the right hand side is $\alpha_1(\varepsilon) + O(\varepsilon^3)$.

If $\langle g \rangle = 0$ and $h \equiv 0$ one shows $\Lambda_\varepsilon(0,0) = 0$. Setting $z = \mu = 0$ in (3.12), one observes that the left hand side begins with $-\varepsilon^3\langle \tilde{g}^3 \rangle$ and the right hand side is $\alpha_1(\varepsilon)+O(\varepsilon^4)$. To see that $\alpha_2 = O(\varepsilon^2)$, differentiate (3.12) twice with respect to z, set $z = \mu = 0$, and evaluate module ε^2.

To complete the calculation, first show that $Z_{\varepsilon\varepsilon}(0,\mu,0) = 0$ and $\Lambda_{\varepsilon\varepsilon}(0,0) = 0$. The first equality is obtained by differentiating (3.12) twice with respect to ε and setting $z = \varepsilon = 0$. The second equality is obtained by evaluating $(3.12)_{z\varepsilon\varepsilon}$ at $z = \mu = \varepsilon = 0$. To compute α_1, set $\mu = z = 0$ in (3.12) and observe that the RHS is $\alpha_1 + O(\varepsilon^6)$ while the LHS is $-\varepsilon^2 \langle\bar{w}^3\rangle$. Then set $\mu = z = 0$ in (3.9) to obtain

$$\bar{w}(0,0,\varepsilon) = \tilde{g} - \widetilde{\varepsilon^2 \tilde{g}^3} + O(\varepsilon^4).$$

The result follows.

REFERENCES

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2 S. Rosenblat and D. Cohen, 1980, "Periodically Perturbed Bifurcation -- 1". Simple Bifurcation, *Studies in Applied Math.*, 63, p. 1-23.