Physica 13D (1984) 1–33
North-Holland, Amsterdam

PATTERN SELECTION WITH O(3) SYMMETRY†

E. IHRIG
Department of Mathematics, Arizona State University, Tempe, Arizona 85287, USA

and

M. GOLUBITSKY
Department of Mathematics, University of Houston-University Park, Houston, Texas 77004, USA

Received 29 July 1983
Revised manuscript received 27 February 1984

For each irreducible representation of SO(3) and O(3) we determine, up to conjugacy, all isotropy subgroups and identify, in particular, the maximal isotropy subgroups. Each isotropy subgroup corresponds to a possible planform for the spherical Bénard problem. Using an equivariant branching lemma of Cicogna [1] we prove, for each of these representations, the existence of solutions corresponding to a number of different planforms, thus extending substantially the work of Busse [2, 3] and Sattinger [4]. We also give a useful criterion for showing when solutions obtained by the equivariant branching lemma must be unstable.

1. Introduction

The study of the steady-states in the buckling of a spherical shell of finite thickness and in thermal conduction and convection of a fluid confined between two concentric spherical shells have one important point in common: the partial differential equations which model these phenomena are invariant under the action of the orthogonal group O(3). This invariance implies that the mathematical study of the two problems will have much in common. It is the purpose of this paper to indicate, using group theoretic results alone, some of the common structure induced on these problems by the existence of the symmetry group O(3).

See Busse [2] and Knightly and Sather [5] and references therein for precise mathematical description of the fluids problem and the buckling problem, respectively. Technically, the way in which the symmetry group enters each of these problems is through what is classically called the Liapunov–Schmidt reduction. The idea behind this reduction is to view the system of PDE's as an operator

$$\Phi:\ B \times \mathrm{R} \to B',$$

where B and B' are (appropriately chosen) Banach spaces and the scalar variable λ is the bifurcation parameter. In the buckling problem we assume a uniform load applied to the outer shell of strength λ; in the fluids problem we assume constant temperature source on the inner shell whose difference from the temperature of the outer shell is λ. The steady state solutions are given mathematically by solving the equation

$$\Phi(b, \lambda) = 0.$$

The invariance of Φ with respect to O(3) takes the

†Research supported in part by N.S.F. Grant MCS-8101580.

form

$$\Phi(\gamma \cdot b, \lambda) = \gamma \cdot \Phi(b, \lambda) \quad \gamma \in \mathrm{O}(3). \tag{1.1}$$

We assume that the only point in B fixed by O(3) is $b = 0$. This invariance then implies that $b = 0$ is always a solution, called the *trivial solution*; i.e.

$$\Phi(0, \lambda) \equiv 0.$$

The trivial solution in the buckling problem is the undeformed shell; the trivial solution in the fluids problem is the pure conduction solution where the fluid remains stationary but heat is conducted radially, by thermal diffusion, from the inner sphere to the outer sphere.

The Liapunov–Schmidt method works as follows. Let $L(\lambda)$ be the linearization of the (nonlinear) operator Φ with λ held fixed about the trivial solution $b = 0$. Typically, L is a Fredholm operator of index zero and this is indeed true for the two problems described above. The values λ where $\dim \ker L(\lambda) > 0$ are called *eigenvalues* for Φ. Our interest is in studying the solution structure of $\Phi = 0$ locally near $b = 0$ and $\lambda = \lambda_0$, where λ_0 is the first eigenvalue of Φ. Using the implicit function theorem, one shows that the solutions to $\Phi = 0$ in $B \times \mathrm{R}$ are parametized by the zeros of a smooth mapping

$$g\colon V \times \mathrm{R} \to V,$$

where $V = \ker L(\lambda_0)$. Moreover, V is invariant under the action of O(3) and the commutativity of Φ with O(3), eq. (1.1), implies that g commutes with the representation of O(3) on V; i.e.

$$g(\gamma \cdot x, \lambda) = \gamma \cdot g(x, \lambda), \quad \gamma \in \mathrm{O}(3). \tag{1.2}$$

(See Sattinger [4].)

In order to understand (1.2) one must identify which representation of O(3) actually occurs in V. Typically, such actions are irreducible. This implies that $\ker L(\lambda_0)$ is isomorphic to the space of spherical harmonics of some order l which we denote by V_l. Recall that $\dim V_l = 2l + 1$ and that there are two irreducible actions of O(3) on V_l. This point is explained in section 3. The actual value of l which occurs depends on a geometric parameter called the aspect ratio. The aspect ratio p is the ratio of the radius of the inner sphere to the radius of the outer sphere. In fact, in each of these physical problems, $\lim_{p \to 1} l = \infty$; thus, each of the irreducible representations of O(3) on spherical harmonics appears for some choice of p. See Chossat [6] for fluids and Knightly and Sather [5] for buckling. For the fluids problem, one is typically interested in relatively thick shells – the inner mantle of the Earth, for example – while for the buckling problem one is typically interested in thin shells where the aspect ratio is near unity and l is quite large.

There are two questions regarding the physical problems which we discuss in this paper. First, for each λ how many distinct solutions to $g(x, \lambda) = 0$ exist. Note that the commutativity condition (1.2) implies that g vanishes on orbits of the action of O(3). We consider two solutions to be distinct if they lie on different orbits. Second, can one make any inferences about the form of the corresponding solutions to $\Phi = 0$? The answers to both of these questions involve the understanding of the isotropy subgroups of the action of O(3) on V_l.

Recall that if Γ is a group acting linearly on a vector space V then the *isotropy subgroup* (or little group) of a point x in V is

$$\Sigma_x = \{\gamma \in \Gamma \mid \gamma x = x\} \tag{1.3}$$

consisting of those group elements which fix x. The isotropy subgroup may be interpreted as the symmetries of the point x. For example, axisymmetric steady state solutions to the physical problems correspond to solutions x in V_l whose isotropy subgroup Σ_x contains the circle group SO(2).

Isotropy subgroups also appear in the discussion of when two solutions are distinct. In particular, two points in the same orbit have conjugate isotropy subgroups; that is,

$$\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}.$$

Thus, two solutions are distinct if their isotropy subgroups are not conjugate. We use this fact to enumerate solutions.

There is one last point in our general discussion. Isotropy subgroups are not equally likely to occur as solutions to $g = 0$. To understand this statement we need to discuss the lattice of isotropy subgroups. We say that one conjugacy class of isotropy subgroups represented by Σ_x is contained in another, represented by Σ_y, if some conjugate of Σ_x is contained in Σ_y. We denote the containment of conjugacy classes by $\Sigma_x < \Sigma_y$. In this way, we can make the set of conjugacy classes for a fixed representation of Γ into a lattice. There is much evidence for the following statement – but no proof. For a generic set of g's satisfying (1.2) the only solutions to $g = 0$ near the trivial solution have isotropy subgroups which are maximal subgroups. An isotropy subgroup Σ is maximal if Σ is proper and the only isotropy subgroup containing Σ is Γ. See Golubitsky [7] and lemma 3.1.

There is a physically motivated plausibility argument supporting this conjecture. Loosely speaking, it takes energy to break symmetries and thus, it seems reasonable that it is harder for a system to break more symmetries than less. This conjecture is proved in the case $l = 2$ in Golubitsky and Schaeffer [8].

Conversely, one must ask whether there is a method for guaranteeing the existence of solutions to $g = 0$ which have a given isotropy subgroup. The answer is yes for a special class of maximal isotropy subgroups with one-dimensional fixed point subspaces. See Cicogna [1]. A proof of this equivariant branching lemma is also given (with slightly different hypothesis) in Sattinger [5] and Golubitsky [7]. Related ideas are considered in Michel [9] where a restricted version of Cicogna's result is obtained. The lemma is proved by an elementary application of the implicit function theorem once the appropriate setting has been described. We will give a proof in section 4 as the details are needed in our discussion of linearized stability. Now we state that lemma.

Let Γ be a Lie group acting *absolutely irreducibly* on the space V; that is, the only linear mappings on V which commute with the given representation of Γ are scalar multiples of the identity. Note that (1.2) implies that the Jacobian $(d_x g)_{0,\lambda} = c(\lambda)I$ where $c(\lambda)$ is a scalar. For a bifurcation of solutions to occur at $\lambda = 0$ one must have $c(\lambda_0) = 0$.

Let $\Sigma \subset \Gamma$ be an isotropy subgroup and let

$$V^{\Sigma} = \{ y \in V_m | \sigma y = y \text{ for each } \sigma \in \Sigma \} \tag{1.4}$$

be the fixed point set of Σ. Assume

(a) $c'(\lambda_0) \neq 0$; i.e., the trivial solution changes stability in a non-degenerate fashion; (1.5)

(b) $\dim V^{\Sigma} = 1$.

The equivariant branching lemma states that there exists a unique branch of solutions to $g(x, \lambda) = 0$ given by $x = \Lambda(\lambda) \in V^{\Sigma}$, $\Lambda'(0) = 0$. That is, there is a unique solution branch having isotropy subgroup Σ. For example, axisymmetric solutions have one-dimensional fixed point sets. See Sattinger [4].

We note that condition (1.5a) is satisfied by the mathematical models of both the fluids and buckling problems described above. In addition (1.5b) implies that Σ is a maximal isotropy subgroup of Γ.

In this paper we consider several questions concerning the isotropy subgroups of SO(3) and O(3) in each irreducible representation:

I) What is the lattice of closed subgroups of SO(3) and O(3), up to conjugacy? The complete description of the conjugacy classes of subgroups of O(3) which are known to chemists (Cotton [10]) as crystallographic groups and physicists as point groups is given in section 2. The subgroups of SO(3) – D_n, Z_n, (dihedral and cyclic subgroups, respectively) and the exceptional groups O, T and I (octahedral, tetrahedral and icosahedral, respectively) – are well known (cf. Wolf [11]). In O(3), there are several classes of non conjugate but isomorphic subgroups – which we denote by various superscripts. It is these subgroups, in particular, that we describe in section 2.

II) What are the dimensions of the fixed point sets for each closed subgroup of O(3) and each irreducible representation? In particular, which of these subgroups have one-dimensional fixed point sets? For these subgroups, the equivariant branching lemma proves under generic hypotheses the existence of a unique solution branch. In section 3 we give a complete enumeration of the dimensions of the fixed point sets. Michel [12] computes the dimensions of the fixed point subspaces for subgroups of SO(3). The formulas for the subgroups of O(3), when $-I$ acts trivially, are identical to the ones Michel obtains. We complete the computations for these dimensions for subgroups of O(3) when $-I$ acts as minus the identity. See section 3 for a careful discussion of these representations of O(3).

Sattinger [4], by a slightly different method, has found the dimensions for the fixed point subspaces of the octahedral subgroup of SO(3). In particular, he has calculated for which l the fixed point subspace of O is one dimensional.

Busse [2] and Busse and Riahi [3] found, implicitly, the isotropy subgroups with one-dimensional fixed point subspaces when $l = 2$, 3, 4 or 6.

III) What are the maximal isotropy subgroups of O(3)? This question is answered as a corollary to our computation of the dimensions of fixed point sets in section 3.

IV) What is the full lattice of isotropy subgroups of O(3)? This is a much more difficult calculation. Our results are given in theorems 6.6 and 6.8. Sample results for small l are given in tables I and II. For large l, the results are simpler. See corollaries 6.7 and 6.9.

Michel [12], in appendix A, outlines a method for calculating the isotropy subgroups of the irreducible representations of SO(3). We use a similar approach with modifications for the actual calculation of the dimensions of the fixed point subspaces of SO(3). Our results differ from Michel's in two points:

a) Unfortunately, Michel's criterion for determining when a subgroup is actually an isotropy subgroup (lemma 2, p. 639) is incorrect as stated.

Table I
Lattice of isotropy subgroups of SO(3) acting on V_l for $l = 2, 4, 6$.

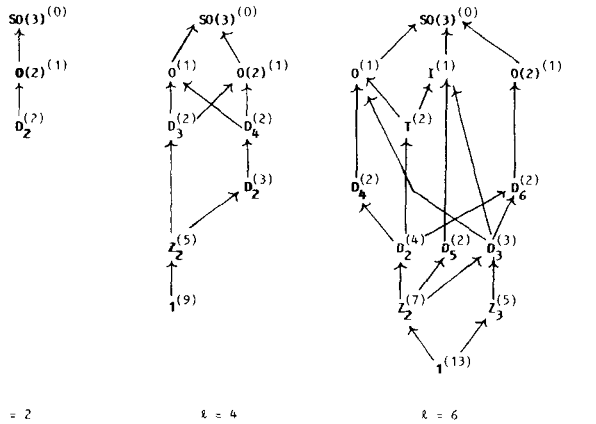
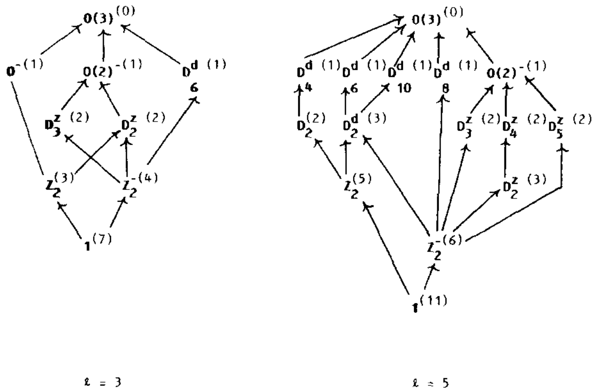


Table II
Lattice of isotropy subgroups of O(3) acting on V_l for $l = 3, 5$.



(For example, it is incorrect for the identity subgroup in SO(3).) In section 5 we give a correct version of this lemma. See lemma 5.3. Its proof is involved. It seems likely that the condition we give is both necessary and sufficient though we have not been able to prove this.

b) We complete the calculation of the dimensions of the fixed point subspaces for the irreducible representations of O(3).

V) Are the solutions obtained in the equivariant branching lemma linearly (orbitally) stable? Our results, given in section 4, are partial and negative.

Loosely speaking, if there exists a mapping g_2: $V \to V$ which commutes with Γ and is homogeneous of degree 2, then the solutions given by the equivariant branching lemma are usually unstable. See theorem 4.2 for a precise statement of this result. In particular, when l is even generically the solutions whose existence is guaranteed by the equivariant branching lemma are unstable. In such a case, to find physically meaningful solutions one must study degenerate bifurcation problems. See the discussion in section 4.

We now describe our results for the irreducible representations of O(3) on V_l, the spherical harmonics of order l, in more detail. The exact form of these representations are given in section 3. We begin by presenting the list of maximal isotropy subgroups. These results follow directly form propositions 3.6–3.9.

Theorem 1.1. Let O(3) act on the spherical harmonics of order l in the standard way (see section 3). The maximal isotropy subgroups are

$l = 2$: $\mathrm{O}(2) \oplus \mathrm{Z}_2^{\mathrm{c}}$;

$l = 4, 8, 14$: $\mathrm{O}(2) \oplus \mathrm{Z}_2^{\mathrm{c}}$ and $\mathrm{O} \oplus \mathrm{Z}_2^{\mathrm{c}}$;

all other even l: $\mathrm{O}(2) \oplus \mathrm{Z}_2^{\mathrm{c}}$, $\mathrm{O} \oplus \mathrm{Z}_2^{\mathrm{c}}$ and $\mathrm{I} \oplus \mathrm{Z}_2^{\mathrm{c}}$;

$l = 1$: $\mathrm{O}(2)^-$;

$l = 3, 7, 11$: $\mathrm{O}(2)^-$, O^- and D_{2n}^{2} $\left(\frac{l}{2} < n \le l\right)$;

$l = 5$: $\mathrm{O}(2)^-$, $\mathrm{D}_6^{\mathrm{d}}$ and $\mathrm{D}_8^{\mathrm{d}}$;

$l = 9, 13, 17, 19, 23, 29$:

$\mathrm{O}(2)^-$, O^-, O and $\mathrm{D}_{2n}^{\mathrm{d}}$ $\left(\frac{l}{2} < n \le l\right)$;

all other odd l:

$\mathrm{O}(2)^-$, O^-, O, I and $\mathrm{D}_{2n}^{\mathrm{d}}\left(\frac{l}{2} < n \le l\right)$.

See section 2 for the precise definition of the subgroups listed here. Note that $\mathrm{Z}_2^{\mathrm{c}}$ is the center of O(3) and the superscripts c and d indicate nonconjugate but isomorphic subgroups of O(3).

Theorem 1.2. A complete listing of the isotropy subgroups of O(3) acting on V_l whose fixed point sets are one-dimensional is given as follows:

(a) $\mathrm{O}(2) \oplus \mathrm{Z}_2^{\mathrm{c}}$: all even l,

(b) $\mathrm{O} \oplus \mathrm{Z}_2^{\mathrm{c}}$: $l = 4, 6, 8, 10, 14$,

(c) $\mathrm{I} \oplus \mathrm{Z}_2^{\mathrm{c}}$: $l = 6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 32, 34, 38, 44$,

(d) $\mathrm{O}(2)^-$: all odd l,

(e) O: $9, 13, 15, 17, 19, 23$,

(f) O^-: $3, 7, 9, 11, 13, 17$,

(g) I: $15, 21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59$,

(h) $\mathrm{D}_{2n}^{\mathrm{d}}$: $l/2 < n \le l$, all odd $l \ge 3$.

Remarks.

a) It is interesting to note that the isomorphic but non-conjugate octahedral subgroups O and O^- are both maximal isotropy subgroups with one-dimensional fixed point sets when $l = 9$. Thus, there exist, by the equivariant branching lemma, two distinct branches of solutions with octahedral symmetry for (almost) all g's commuting with O(3) when $l = 9$.

b) As a corollary of the above theorems one finds that *all* of the maximal isotropy subgroups of O(3) acting on V_l have fixed point subspaces which are one-dimensional when $l \le 11$. Thus, the equivariant branching lemma guarantees the existence of solution branches for each of these maximal isotropy subgroups. If the conjecture that generically only maximal isotropy subgroups appear as solutions is true then for $l \le 11$ one has generically a complete description of the solution branches. This calculation extends the work of Busse and Riahi [3] about as far as seems reasonable.

c) Sattinger [4] found only the instances where $\mathrm{O} \oplus \mathrm{Z}_2^{\mathrm{c}}$ or O were isotropy subgroups with one-dimensional fixed point subspaces. Our results on O^- in theorem 3.5f augments his list when considering octahedral symmetry.

In tables I and II we list the lattice of isotropy subgroups of O(3) acting on the spherical harmonics of order l when $l = 2$, 3, 4, 5 and 6. These tables are compiled from the information listed in theorems 6.6 and 6.8. For even l each isotropy subgroup has the form $K \oplus \mathrm{Z}_2^c$ where K is a subgroup of SO(3) and $\mathrm{Z}_2^c = \{\pm I\}$ is the center of O(3). In table I we list only the K's. The dimensions of the fixed point subspaces are listed as superscripts in parentheses on the individual groups. This information is obtained using theorems 3.2 and 3.5.

2. The lattice of closed subgroups of O(3): the point groups

2.1. *The subgroups of* SO(3)

The set of conjugacy classes of subgroups of O(3) is a known object; our purposes here are to familiarize the reader with this classification and to set notation. Each subgroup of SO(3) may be viewed as the symmetry group of a regular planar or solid figure in R^3 with the planar figures having both a directed and undirected version. In figs. 1 and 2 we picture certain of these regular figures. The notation for the symmetry groups associated with each of these figures is also given. Note that the octahedral group O is the symmetry group of both the cube and the regular octahedron while I is the symmetry group of both the regular icosahedron and the regular dodecahedron. We call the symmetry groups of the platonic solids – T (the tetrahedral group), O, and I – the *exceptional subgroups* of SO(3).

For definiteness we let

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

be the rotation of the xy-plane counterclockwise through the angle θ. Then

$$\mathrm{SO}(2) = \{R_\theta | 0 \le \theta < 2\pi\} \tag{2.2}$$

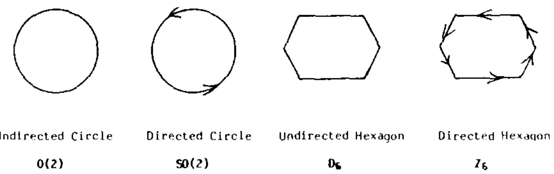


Fig. 1. The regular planar figures.

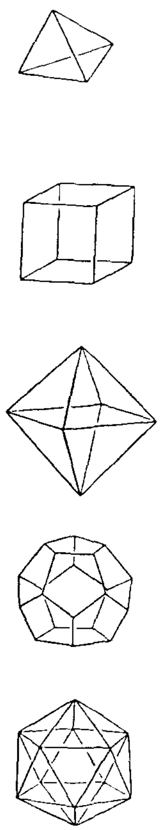


Fig. 2. The platonic solids.

is one realization of the rotation group. Clearly, SO(2) leaves the unit circle and its orientation invariant. There is, however, an element of SO(3) which maps the circle onto itself but changes the orientation; namely, rotation through the angle π in the yz-plane. Let

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.3}$$

The group generated by SO(2) and C in SO(3) is the symmetry group of the undirected circle and is isomorphic to O(2). We emphasize that this realization of O(2) lies inside the connected group SO(3).

It is now easy to see that

$$Z_n = \{ R_{2\pi k/n} | k \in Z \} \tag{2.4}$$

is the symmetry group of the directed regular n-gon. Augmenting Z_n by the rotation C yields the subgroup D_n of SO(3), the symmetry group of the undirected n-gon. We will not give specific realizations for the exceptional subgroups.

Note. D_{2n} and O(2) have two reflectional symmetries R_π and C which are conjugate in O(3) but not in O(2).

The following theorem is proved by combining results in Bredon [13, p. 153] and Wolf [11, p. 85]:

Theorem 2.1. Every proper closed subgroup of SO(3) is conjugate to one of the following subgroups:

$$\mathrm{O(2), SO(2), I, O, T, D}_n \text{ or } \mathrm{Z}_n.$$

For future reference we list here the normalizer of each subgroup Δ, denoted by $N(\Delta)$ and the order, when finite, $|\Delta|$.

Lemma 2.2.

(a) $N(Z_n) = N(SO(2)) = N(O(2)) = O(2)$,
$N(T) = N(O) = N(D_2) = O$,
$N(I) = I, \quad N(D_n) = D_{2n}, \quad n \neq 2$;

(b) $|Z_n| = n, \quad |D_n| = 2n, \quad |T| = 12, \quad |O| = 24$,
$|I| = 60$.

Having enumerated the closed subgroups of SO(3), up to conjugacy, we now describe the lattice of subgroups. The lattice structure is defined as follows. Let H and K be (conjugacy classes of) subgroups in SO(3). We say that H is *contained* in K, denoted $H < K$, if some conjugate of H is actually contained in K.

The containment relations among the D_n and Z_n subgroups are easily obtained; namely,

$$\begin{aligned} &\text{(a)} \quad Z_n \subset D_n \subset O(2), \quad \text{for all } n; \\ &\text{(b)} \quad Z_n \subset Z_m \quad \text{iff } n \text{ divides } m; \\ &\text{(c)} \quad Z_n \subset SO(2) \subset O(2), \quad \text{for all } n; \\ &\text{(d)} \quad Z_2 \subset D_m, \quad \text{for all } m \\ &\text{and} \quad Z_n \subset D_m \quad \text{when } n \text{ divides } m. \end{aligned} \tag{2.5}$$

Note. In (d) the copy of Z_2 we refer to is generated by C.

These containment relations are indicated by arrows in fig. 3. Of course, (2.5b) is impossible to sketch.

The real difficulty in determining the containment relations involves the exceptional subgroups. These relations are indicated in fig. 4. Observe that O(2), I and O are maximal subgroups of SO(3) and that T is contained only in I and O.

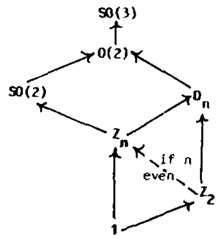


Fig. 3. Lattice of the non-exceptional subgroups of SO(3).

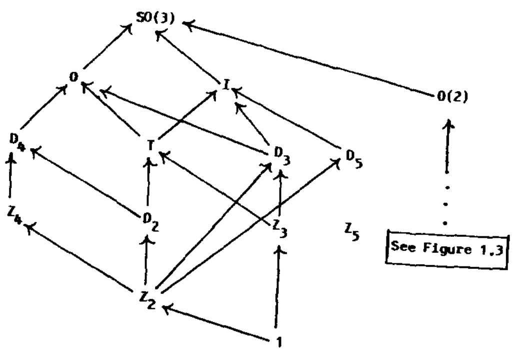


Fig. 4. Lattice of subgroups of SO(3) involving exceptional subgroups.

There is a simple decomposition of the exceptional groups which enables one to determine the lattice structure of fig. 4. More importantly, this decomposition is also the key to determining which irreducible representations of SO(3) and O(3) have the exceptional groups as isotropy subgroups.

Definition 2.3. Let $K_1, \ldots, K_s$ be subgroups of H. Then H is the *disjoint union* of the K_i's if

(a) $H = \bigcup_{i=1}^{s} K_i$,

(b) $K_i \cap K_j = \{I\}$, for all $i \neq j$.

We use the notation $H = \dot{\cup}_{i=1}^{s} K_i$ to indicate disjoint unions.

It is unusual for a group to be a disjoint union; nevertheless, both D_n and the exceptional subgroups have disjoint union decompositions into cyclic subgroups. These decompositions are rooted in the regular polyhedra.

Lemma 2.4.

(a) $D_n = Z_n \dot{\cup}^n Z_2$,

(b) $T = \dot{\cup}^4 Z_3 \dot{\cup}^3 Z_2$,

(c) $I = \dot{\cup}^6 Z_5 \dot{\cup}^{10} Z_3 \dot{\cup}^{15} Z_2$,

(d) $O = \dot{\cup}^3 Z_4 \dot{\cup}^4 Z_3 \dot{\cup}^6 Z_2$,

where the rotation $\dot{\cup}^k Z_l$ indicates the disjoint union of k subgroups all conjugate to Z_l.

Proof. We give here only the basic ideas involved in the proof. The decomposition for D_n is easy to verify. We now discuss the exceptional subgroups.

Each rotation in SO(3) has an axis of symmetry. That axis must intersect the invariant polyhedron in either a face, an edge or a vertex. Moreover, to be a symmetry of that polyhedron it must intersect the center of the face, the center of an edge or a vertex. The idea behind this lemma is to classify the elements in the exceptional groups by their axes of symmetry.

For example, the octahedral group O is the symmetry group of the cube. The rotations in SO(3) which have an axis of symmetry intersecting the center of a face are generated by rotation by 90° about the axis. This axis generates the group Z_4. Similarly, axes intersecting the center of an edge generate the group Z_2 and axes intersecting a vertex generates the group Z_3. There are 3 axes intersecting faces, 6 axes intersecting edges and 4 axes intersecting vertices. This leads to the decomposition

$$O = \dot{\cup}^3 Z_4 \dot{\cup}^6 Z_2 \dot{\cup}^4 Z_3.$$

Similarly, symmetries of the dodecahedron, pictured in fig. 2, leads to the decomposition of the icosahedral group I. The decomposition of the tetrahedral group is obtained by a slightly different enumeration. Here, the axes of symmetries fall into two classes, those connecting a vertex and the center of the opposing face and those connecting the centers of opposing edges. □

We now prove that the containments indicated in fig. 4 are correct.

Lemma 2.5. The conjugacy classes of proper subgroups of the exceptional subgroups are

(a) T: Z_3, D_2 and Z_2;

(b) O: D_4, Z_4, D_3 and the subgroups of T;

(c) I: D_5, Z_5, D_3 and the subgroups of T.

Proof. We sketch a geometric proof of these facts. The elements of a cyclic subgroup of SO(3) must share a common axis. These cyclic subgroups have been identified in lemma 2.4. If Z_n appears as a subgroup of T, O or I, then the corresponding D_n appears only if the axis of symmetry of Z_n connects like cells; e.g., vertex to vertex or face to face. The reasoning is simply that one can invert the axis of rotation by a rotation in a plane containing the axis and then rotate about the axis to align the cells. This information is contained in the proof of lemma 2.4.

By simply comparing the orders of the exceptional subgroups, we see that the only containments which are possible are $\mathrm{T} < \mathrm{O}$ and $\mathrm{T} < \mathrm{I}$. One can show that both of these containments occur by embedding regular tetrahedrons in the cube and the icosahedron so that the vertices of the tetrahedron lie on vertices of the containing figures. □

2.2. *The subgroups of O(3)*

To describe the structure of the closed subgroups of O(3) we first note that O(3) splits as

$$\mathrm{O}(3) = \mathrm{SO}(3) \oplus \mathrm{Z}_2^c, \tag{2.6}$$

where

$$\mathrm{Z}_2^c = \{\pm I\}$$

is the center of O(3).

We may now divide the subgroups of O(3) into two classes, those that contain $-I$ and those that do not.

Consider the first case and let H be a subgroup of O(3) containing $-I$. It follows from (2.6) that $H = K \oplus \mathrm{Z}_2^c$ for some subgroup K of SO(3). In fact, if we let π: $\mathrm{O}(3) \to \mathrm{SO}(3)$ be the group homomorphism whose kernel is Z_2^c; i.e., π is the projection onto SO(3) stemming from (2.6), then K is just $\pi(H)$.

The subgroups of O(3) which do not contain $-I$ fall into two classes: those that are in SO(3) and those that are not. The first class has been enumerated in section 2.1; the second class is the source of all of the essential difficulties in describing the subgroups of O(3).

The discussion above is summarized as follows. The closed subgroups of O(3) divide into three classes:

(a) Closed subgroups K of SO(3);

(b) $K \oplus \mathrm{Z}_2^c$ with K a closed subgroup of SO(3);

(c) Subgroups H of O(3) not containing $-I$ and not contained in SO(3). (2.7)

We call the subgroups of class (2.7c) the *class III* subgroups of O(3). Closed subgroups of types (2.7a) and (2.7b) are clearly classified, up to conjugacy, by theorem 2.1. We shall show that the class III subgroups H are determined by pairs of subgroups (K, L) in SO(3) where $K = \pi(H)$ and $L = H \cap \mathrm{SO}(3)$.

Lemma 2.6. Let H be a class III subgroup of O(3). Then

a) H and $K = \pi(H)$ are isomorphic subgroups; and

b) $L = H \cap \mathrm{SO}(3)$ is a subgroup of index 2 in H.

Proof. Since π is a group homomorphism, $K = \pi(H)$ is a subgroup of SO(3). Since $-I$ is not in H, it follows that $\ker \pi \cap H = \{I\}$ and that H and K are isomorphic.

To prove part (b), observe that if g and h are in $\mathrm{O}(3) \sim \mathrm{SO}(3)$ then gh is in SO(3) since $\mathrm{O}(3)/\mathrm{SO}(3) \cong \mathrm{Z}_2$. Now suppose g and h are in $H \sim L$, then $h^{-1}g \in L$ and $g \in hl$. Thus, there are precisely two cosets in H/L. □

We now show that $K = \pi(H)$ and $L = H \cap \mathrm{SO}(3)$ *uniquely* determine H.

Lemma 2.7. Let $L \subset K \subset \mathrm{SO}(3)$ be subgroups with the index of L in K being 2. Then, there is a unique subgroup H of O(3) satisfying $\pi(H) = K$ and $H \cap \mathrm{SO}(3) = L$.

Note. H has to be a class III subgroup. For if $-I \in H$ then $\pi(H) = H \cap \mathrm{SO}(3)$ which is impossible since $K \neq L$.

Proof. First we show that H exists. Since L has index 2 in K the cosets of K/L are L and gL where g is in $K \sim L$. Using the direct sum decomposition $\mathrm{O}(3) = \mathrm{SO}(3) \oplus \mathrm{Z}_2^c$ we let $h = (g, -I)$ and define $H = L \cup hL$. Since $h^2 = g^2 \in L$, it follows that H is a subgroup of O(3). Moreover, $\pi(H)=K$ and $H\cap \mathrm{SO}(3)=L$.

Next, we prove that H is unique. Let $\tilde{H}$ be another subgroup of O(3) satisfying $\pi(\tilde{H})=K$ and $\tilde{H}\cap \mathrm{SO}(3)=L$. Note that π induces an isomorphism of H with K and $\tilde{H}$ with K. Hence L is a subgroup of $\tilde{H}$ of index 2. It follows that $\tilde{H}=L\cup \tilde{h}L$ where $\tilde{h}\in \tilde{H}-L$. Let $\tilde{g}=\pi(\tilde{h})$ and note that $\tilde{g}\in K\sim L$. It follows that $g=\tilde{g}l$ where g is the group element used in the construction of H and l is in L. Hence, $\tilde{H}=L\cup hL=H$, as desired. □

It follows from lemma 2.7 that we can enumerate, up to conjugacy, all class III subgroups of O(3) by enumerating all pairs of closed subgroups (K,L) in SO(3), up to conjugacy, where L has index 2 in K. One can do this using theorem 2.1, lemma 2.5 and the lattice of closed subgroups of SO(3) pictured in figs. 3 and 4. These pairs are

$$\mathrm{O(2)}\supset \mathrm{SO(2)},\quad \mathrm{O}\supset \mathrm{T},\quad \mathrm{D}_n\supset \mathrm{Z}_n,\quad \mathrm{D}_{2n}\supset \mathrm{D}_n,$$
$$\mathrm{Z}_{2n}\supset \mathrm{Z}_n \quad\text{and}\quad \mathrm{Z}_2^{\mathrm{s}}\supset 1. \tag{2.8}$$

There is one subtlety which we have noted by $\mathrm{Z}_2^{\mathrm{s}}$, (s denoting subtlety!). Inside of O(2) , there are two subgroups Z_2 which are conjugate in O(3) but which are not conjugate in O(2). To see this, observe that a fixed choice of O(2) is equivalent to choosing a plane in which the rotations of O(2) act. The first choice of Z_2 is generated by a rotation through the angle π in this plane. This choice has been assigned the pair $(\mathrm{Z}_2,\mathrm{Z}_1)=(\mathrm{Z}_2,1)$. The other choice of Z_2 is made by considering rotation through the angle π in a plane perpendicular to the one fixed by O(2). This we have denoted by the pair $(\mathrm{Z}_2^{\mathrm{s}},1)$. Note that $\mathrm{Z}_2^{\mathrm{s}}$ is a fixed cyclic subgroup in O(2) which is not in SO(2); this is the only special case. Our concerns about conjugacy of pairs will make more sense to the reader after seeing lemma 2.11. This point effects the determination of the lattice of closed subgroups of O(3). See lemma 2.12.

For a class III subgroup H we use the rotation $\pi(H)^-$, with one exception. The group D_{2n} has two isomorphic subgroups of index 2, Z_{2n} and D_n. We let $\mathrm{D}_n^{\mathrm{z}}$ denote the class III subgroup corresponding to the pair $\mathrm{D}_n\supset \mathrm{Z}_n$ and $\mathrm{D}_{2n}^{\mathrm{d}}$ denote the class III subgroup corresponding to the pair $\mathrm{D}_{2n}\supset \mathrm{D}_n$.

We now list the conjugacy classes of subgroups of O(3).

Theorem 2.8. Every closed subgroup is conjugate to precisely one of the following:

(a) $\mathrm{SO(3)},\mathrm{O(2)},\mathrm{SO(2)},\mathrm{I},\mathrm{O},\mathrm{T},\mathrm{D}_n,\mathrm{Z}_n$, or $1,(n>1)$;

(b) $K\oplus \mathrm{Z}_2^{\mathrm{c}}$ where K is a subgroup listed in (a) and $\mathrm{Z}_2^{\mathrm{c}}=\{\pm I\}$.

(c) $\mathrm{O(2)}^-,\mathrm{O}^-,\mathrm{D}_n^{\mathrm{z}}(n\geq 2),\mathrm{D}_{2n}^{\mathrm{d}}(n\geq 2)$ or $\mathrm{Z}_{2n}^-(n>1)$.

Notes.

1) $\mathrm{D}_2^{\mathrm{d}}$ is conjugate to $\mathrm{D}_2^{\mathrm{z}}$.

2) It is important to emphasize that the subgroups O and O^-, etc., are isomorphic as groups but are not conjugate as subgroups of O(3).

3) There are three non-conjugate 2 element subgroups of O(3): $\mathrm{Z}_2^{\mathrm{c}}$, Z_2 and Z_2^-. These subgroups are generated by

$$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},$$
$$\text{and}\quad \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \text{respectively.}$$

There are three non-conjugate subgroups of O(3) isomorphic to D_{2n}, D_{2n}, $\mathrm{D}_{2n}^{\mathrm{z}}$ and $\mathrm{D}_{2n}^{\mathrm{d}}$. It is amusing to see how each of these is a different realization of the symmetries of the $2n$-gon in $\mathbf{R}^3$. In the first two cases, one generates the cyclic subgroup Z_{2n} by rotation through the angle π/n. In D_{2n} one generates the planar reflection by a rotation of π in a perpendicular plane while in $\mathrm{D}_{2n}^{\mathrm{z}}$ one generates the planar reflection by a reflection in O(3) across a perpendicular plane. In $\mathrm{D}_{2n}^{\mathrm{d}}$ the cyclic subgroup is generated by the rotation through the angle π/n in the plane of the n-gon followed by a reflection across that plane. This copy of Z_{2n} is not contained in SO(3). The reflection in D_{2n}^d is the rotation described in D_{2n}. We denote the normalizer of a subgroup K of O(3) by $N_O(K)$. The proof of the following lemma is left to the reader.

Lemma 2.9. Let K be a subgroup of O(3). Then $N_O(K) = N(\pi(K)) \oplus Z_2^c$.

To begin our discussion of the lattice of closed subgroups of O(3) we first give the disjoint union decompositions of the finite class III subgroups into cyclic subgroups.

Lemma 2.10.

(a) $O^- = \dot{\cup}^3 Z_4^- \dot{\cup}^4 Z_3 \dot{\cup}^6 Z_2^-$;

(b) $D_n^z = Z_n \dot{\cup}^n Z_2^-$;

(c) $D_{2n}^d = Z_{2n}^- \dot{\cup}^n Z_2 \dot{\cup}^n Z_2^-$.

Proof. Each of these cases involves a combinatorial argument based on the disjoint unions of O and D_n given in lemma 2.4 and the fact that exactly half of the elements of each of these groups must not be in SO(3).

In (b) the fact that $D_n = Z_n \dot{\cup}^n Z_2$ implies that the only cyclic subgroups of D_n^z are isomorphic to Z_n and Z_2. Since $D_n^z \cap SO(3) = Z_n$ it follows that the remaining n elements of D_n^z are not in SO(3). Since $-I \in D_n^z$ the only possibility for the Z_2 factors is that they are all Z_2^-. Similarly, in (c) $D_{2n}^d \cap SO(3) = D_n$. Since one factor of D_{2n}^d is isomorphic to Z_{2n} and Z_{2n} is not in D_n it follows that that factor is Z_{2n}^-. Since exactly half of the elements of Z_{2n}^- are in SO(3) it follows that exactly half of the elements in $D_{2n}^d \sim Z_{2n}^-$ are in SO(3) and that $D_{2n}^d = Z_{2n}^- \dot{\cup}^n Z_2 \dot{\cup}^n Z_2^-$.

The argument for O^- is similar. Recall that $O = \dot{\cup}^3 Z_4 \dot{\cup}^4 Z_3 \dot{\cup}^6 Z_2$ and note that Z_3 must lie in SO(3). Hence, O^- has $\dot{\cup}^4 Z_3$ as part of its disjoint union decomposition. Moreover, both Z_4 and Z_4^- contain Z_2 and hence have one non-identity element in SO(3). This implies that $\dot{\cup}^3 Z_4^- \dot{\cup}^4 Z_3$ has 12 elements in SO(3). Since $|O| = 24$ it follows that $O^- = \dot{\cup}^3 Z_4^- \dot{\cup}^4 Z_3 \dot{\cup}^6 Z_2^-$. □

In discussing the lattice of closed subgroups of O(3) we make two simplifications. First, we ignore all subgroups of the form $K \oplus Z_2^c$ where K is a subgroup of SO(3). In the representation theory results described in the next section, these groups enter in only a trivial way. Second, we describe only that part of the lattice that includes the class III subgroups. The remainder of the lattice has been given in the lattice for SO(3).

In fig. 5, we describe that part of the lattice contained in the maximal subgroup O^- and in fig. 6 we indicate relations among the other class III subgroups.

In order to determine the inclusion relations, up to conjugacy, of the class III subgroups we need the following lemma:

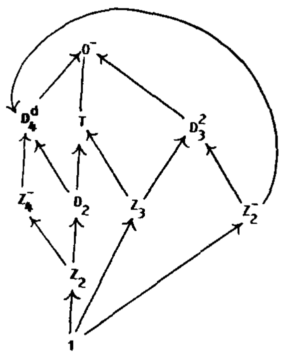


Fig. 5. Lattice of subgroups of O^-.

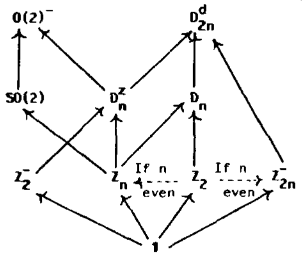


Fig. 6. Part of the lattice of class III subgroups.

Lemma 2.11. Let H_1 and H_2 be class III subgroups with $K_i = \pi(H_i)$ and $L_i = H_i \cap \mathrm{SO}(3)$. Then $H_2 \subset H_1$ if and only if

(a) $K_2 \subset K_1$ and $L_2 \subset L_1$;

and

(b) $K_2 \not\subset L_1$.

Proof. The necessity of (a) is obvious. The point of interest in the lemma is the need for condition (b). Now assume that $H_2 \subset H_1$ and that $K_2 \subset L_1$. We claim that these inclusions imply that $-I \in H_1$ which contradicts the assumption that H_1 is class III.

Let h be in $H_2 \sim L_2$. Then $\pi(h) \in K_2 \subset L_1 \subset H_1$. Since $H_2 \subset H_1$ it follows that $h \in H_1$ and that

$$\pi(h)^{-1}h \in H_1.$$

We now compute using the direct sum decomposition $\mathrm{O}(3) = \mathrm{SO}(3) \oplus \mathrm{Z}_2^c$. Since $h \in H_2 \sim L_2$ it follows that $h = (g, -I)$ and that $\pi(h)^{-1} = (g^{-1}, I)$. Thus

$$\pi(h)^{-1}h = (g^{-1}, I)(g, -I) = -I \in H_1$$

yielding the desired contradiction.

Conversely, suppose that (a) and (b) are satisfied. Using condition (b) we may choose $h \in K_2 \sim L_1$. Since $L_2 \subset L_1$ it follows that $h \in K_2 \sim L_2$ and that $H_2 = L_2 \cup hL_2$. Since $K_2 \subset K_1$ it follows that $h \in K_1 \sim L_1$ and that $H_1 = L_1 \cup hL_1$. Since $L_2 \subset L_1$ we have proved $H_2 \subset H_1$ as desired. □

We can now list the subgroups of the class III subgroups.

Lemma 2.12. The conjugacy classes of the subgroups of the class III groups are

(a) O^-: $\mathrm{D}_4^d, \mathrm{Z}_4^-, \mathrm{D}_3^z, \mathrm{Z}_2^-$ and subgroups of T;

(b) $\mathrm{O}(2)^-$: $\mathrm{D}_n^z, \mathrm{Z}_2^-$ and subgroups of $\mathrm{SO}(2)$;

(c) Z_{2n}^-: Z_{2m}^- where m divides n and $2m$ does not divide n and subgroups of Z_n;

(d) D_{2n}^d: $\mathrm{D}_{2m}^d, \mathrm{Z}_{2m}^-$ where m divides n and $2m$ does not divide n, Z_2^- and subgroups of D_n;

(e) D_n^z: D_m^z where m divides n, Z_2^- and subgroups of Z_n.

Proof. There are two general comments in determining the subgroups of a class III subgroup H. First $H \cap \mathrm{SO}(3)$ and all of its subgroups are contained in H. Second, the class III subgroups can be obtained by use of lemma 2.11 and the listing of pairs of subgroups (K, L) of SO(3) of index 2 in (2.8). We give the arguments for (a) and (b).

a) Since $\mathrm{T} = \mathrm{O}^- \cap \mathrm{SO}(3)$, all of the subgroups of T are contained in O^-. Let H be a class III subgroup with $K = \pi(H)$ and $L = H \cap \mathrm{SO}(3)$. Lemma 2.11 states that $H \subset \mathrm{O}^-$ if and only if

$$K \subset \mathrm{O}, \quad L \subset \mathrm{T} \quad \text{and} \quad K \not\subset \mathrm{T}.$$

Inspection of the list of subgroups of O and T in lemma 2.5a and 2.5b and the possible pairs (K, L) in (2.8) shows that eligible pairs are $(\mathrm{D}_4, \mathrm{D}_2)$ $(\mathrm{Z}_4, \mathrm{Z}_2)$ and $(\mathrm{D}_3, \mathrm{Z}_3)$. In addition, there is a reflectional group Z_2^s which is not contained in T. Thus, one sees that D_4^d, Z_4^-, D_3^z and Z_2^- are subgroups of O^-.

b) Since $\mathrm{O}(2)^- \cap \mathrm{SO}(3) = \mathrm{SO}(2)$ we see that all subgroups of SO(2) are contained in $\mathrm{O}(2)^-$. Lemma 2.11 states that if a subgroup H of class III is contained in $\mathrm{O}(2)^-$, with $K = \pi(H)$ and $L = H \cap \mathrm{SO}(3)$, then

$$L \subset \mathrm{SO}(2), \quad K \subset \mathrm{O}(2) \quad \text{and} \quad K \not\subset \mathrm{SO}(2).$$

From (2.8) we see that the only possibilities are $(\mathrm{D}_n, \mathrm{Z}_n)$ and the spurious Z_2^s sitting in $\mathrm{O}(2) \sim \mathrm{Z}_2$. Thus, D_n^z and Z_2^- are in $\mathrm{O}(2)^-$. □

3. Fixed point spaces and maximal isotropy subgroups

Our method for finding the maximal isotropy subgroups for the irreducible representations of

SO(3) and O(3) is quite simple. First, we enumerate the dimensions of the fixed points spaces for each closed subgroup. To fix notation let Γ be a Lie group acting linearly on the vector space V and let Δ be a subgroup of Γ. Let

$$V^{\Delta} = \{ y \in V | \Delta y = y \} \tag{3.1a}$$

be the fixed point subspace of Δ and let

$$d(\Delta) = \dim V^{\Delta} \tag{3.1b}$$

be the dimension of that subspace.

There is a simply proved characterization of maximal isotropy subgroups given in terms of the dimensions of fixed point sets.

Lemma 3.1. Let Δ be a subgroup of Γ. Then Δ is a maximal isotropy subgroup if and only if

$$\begin{aligned} &\text{(a)} \quad d(\Delta) > 0, \\ &\text{(b)} \quad d(\Sigma) = 0 \text{ for every subgroup } \Sigma \supsetneqq \Delta. \end{aligned} \tag{3.2}$$

Proof. Suppose Δ is a maximal isotropy subgroup. Then $d(\Delta) > 0$ since, by definition of isotropy subgroup, Δ must fix some non-zero vector in V. Moreover, if $d(\Sigma) > 0$ then Σ fixes some non-zero vector in v in V. Thus, the isotropy subgroup of v contains Σ and hence Δ; so Δ is not maximal as an isotropy subgroup.

Conversely, if (3.2) holds then there is a non-zero v in V which is fixed by Δ. The isotropy subgroup Γ_v of v contains Δ and $d(\Gamma_v) \neq 0$. By (3.2b) Γ_v must equal Δ so Δ is an isotropy subgroup. The same argument shows that Δ is also maximal as an isotropy subgroup. □

Our strategy for finding maximal isotropy subgroups for SO(3) and O(3) is as follows. We compute the dimensions of the fixed point spaces for each of the closed subgroups, using in equal parts the decomposition of finite subgroups of O(3) into a disjoint union of cyclic subgroups, the trace formula and the weight space decomposition of the irreducible representations of SO(3). Then we use the description of the lattices of closed subgroups of SO(3) and O(3) presented in section 2 to find the largest subgroups which have positive dimensional fixed point sets and apply lemma 3.1.

We begin by describing the irreducible representations of SO(3) and O(3) along with some well-known facts about these representations. There is, up to conjugacy, a unique irreducible representation for SO(3) in each odd dimension. A standard presentation for each of these representations is given by the action of SO(3) on V_l, the space of spherical harmonics of order l. The dimension of V_l is $2l+1$. Recall that the spherical harmonics of order l may themselves be realized as the restriction to the sphere of those polynomials p: $\mathbf{R}^3 \to \mathbf{R}$ which are homogeneous of degree l. The action of γ in O(3) on $p(x)$ is given by

$$(\gamma \cdot p)(x) \equiv p(\gamma^{-1} x). \tag{3.3}$$

There are two distinct irreducible representations of O(3) in each odd dimension. In each of these SO(3) acts irreducibly and the distinction between the representations of O(3) depends only on whether $-I$ acts as the identity on V or as minus the identity. In the first case $-I$ lies in every isotropy subgroup of O(3) and thus the isotropy subgroups have the form $K \oplus \mathbf{Z}_2^c$ where K is an isotropy subgroup of SO(3) acting on V_l. The lattice of isotropy subgroups for these representations of O(3) may be determined directly from the representations of SO(3). In the second case $-I$ fixes no element of V_l, save the origin, and occurs in no isotropy subgroup of O(3), save O(3) itself. For these representations the isotropy subgroups are either subgroups of SO(3) or class III subgroups of O(3). It is for this reason that we have ignored the subgroups of the form $K \oplus \mathbf{Z}_2^c$ in our description of the lattice of closed subgroups of O(3) given in fig. 5.

We are particularly interested in the action of O(3) on V_l defined by (3.3) as it is these representations that often occur in applications. For these representations $(-I) \cdot p(x) = p(-x) = (-1)^l p(x)$ since p is homogeneous of degree l. Thus, the parity of l determines which of the two irreducible representations of O(3) described above occurs in the representation of O(3) on spherical harmonics.

We begin the determination of the maximal isotropy subgroups by finding the dimensions of the fixed point sets for the irreducible representations of SO(3) and then discuss the corresponding result for the non-trivial irreducible representation of O(3).

Theorem 3.2. Let SO(3) act irreducibly on V_l, the space of spherical harmonics on order l. The dimensions of the fixed point sets of the closed subgroups of SO(3) are

(a) $d(\mathrm{Z}_n) = 2\left[\frac{l}{n}\right] + 1, \quad (n \geq 1);$

(b) $d(\mathrm{D}_n) = \begin{cases} \left[\frac{l}{n}\right], & l \text{ odd,} \\ & (n \geq 2) \\ \left[\frac{l}{n}\right] + 1, & l \text{ even;} \end{cases}$

(c) $d(\mathrm{SO}(2)) = 1;$

(d) $d(\mathrm{O}(2)) = \begin{cases} 0, & l \text{ odd,} \\ 1, & l \text{ even;} \end{cases}$

(e) $D(\mathrm{T}) = 2\left[\frac{l}{3}\right] + \left[\frac{l}{2}\right] - l + 1;$

(f) $d(\mathrm{O}) = \left[\frac{l}{4}\right] + \left[\frac{l}{3}\right] + \left[\frac{l}{2}\right] - l + 1;$

(g) $d(\mathrm{I}) = \left[\frac{l}{5}\right] + \left[\frac{l}{3}\right] + \left[\frac{l}{2}\right] - l + 1;$

where $[x]$ is the greatest integer less than or equal to x.

The proof of theorem 3.2 divides into two steps. First, one uses the weight space decomposition of the representation to compute the dimensions of the fixed point spaces of SO(2) and Z_n. Then, one uses the Weyl trace formula and the decomposition of the finite subgroups into a disjoint union of cyclic subgroups to determine the dimensions of the fixed point spaces for the remaining groups.

Recall that the Cartan subgroup of SO(3) is SO(2) and the root space decomposition of V_l is

$$V_l = W_0 \oplus W_1 \oplus \cdots \oplus W_l, \tag{3.4}$$

where $\dim W_0 = 1$ and $\dim W_k = 2$ for $k > 0$. Moreover, if $\theta \in \mathrm{SO}(3)$ and $w \in W_k$ then θ acts on w by rotation through the angle $k\theta$.

Remark. In the standard basis for the (complex) spherical harmonics of order l. $\{Y_{-l}, \ldots, Y_0, \ldots, Y_l\}$, W_k is the real subspace in the span of Y_{-k} and Y_k.

From this decomposition one sees immediately that $V^{\mathrm{SO}(2)} = W_0$ and $d(\mathrm{SO}(2)) = 1$ as desired. Recall that Z_n is generated by $R_{2\pi/n}$; i.e., rotation through the angle $2\pi n$. Therefore, the group Z_n fixes a non-zero vector in W_k (and hence all of W_k) only if n divides k. There are $[l/n]$ integers k between 1 and l for which n divides k. Each of these adds two dimensions to the fixed point set. Adding 1 for the dimension of W_0 yields the formula

$$d(\mathrm{Z}_n) = 2\left[\frac{l}{n}\right] + 1. \tag{3.5}$$

We claim that the fixed point sets for each of the remaining subgroups of SO(3) may be computed from (3.5) using the decomposition of finite subgroups into disjoint unions and the trace formula.

Theorem 3.3. (Trace formula). Let ρ be a representation of a finite group H on the finite dimensional vector space V. Then

$$\dim V^H = \frac{1}{|H|} \sum_{h \in H} \mathrm{Tr}\,\rho(h).$$

Proof. Define the linear transformation on V

$$A = \frac{1}{|H|} \sum_{h \in H} \rho(h)$$

and observe that $A^2 = A$. Here one uses the fact

$$\sum_{h \in H} \rho(h) = \sum_{h \in H} \rho(h'h) \tag{3.6}$$

for each $h' \in H$. It follows that $V = \ker A \oplus \operatorname{Im} A$ and $A|_{\operatorname{Im} A} =$ identity. Moreover, $\operatorname{Tr} A = \dim \operatorname{Im} A$.

We claim that $V^H = \operatorname{Im} A$ from which the theorem follows. It is easy to show that $V^H \subset \operatorname{Im} A$. For the converse, let v be in $\operatorname{Im} A$. So $v = Aw$ for some w. Now compute using (3.6) that

$$\begin{aligned}\rho(h)v &= \rho(h)Aw = \frac{\rho(h)}{|H|} \sum_{h' \in H} \rho(h')v \\ &= \frac{1}{|H|} \sum_{h' \in H} \rho(hh')w = \frac{1}{|H|} \sum_{h' \in H} \rho(h')w \\ &= Aw = v.\end{aligned}$$

Hence $v \in V^H$ and the claim is verified. □

Corollary 3.4. Let $H = H_1 \dot{\cup} H_2 \dot{\cup} \cdots \dot{\cup} H_k$. Then

$$\dim V^H = \frac{1}{|H|} \sum_{i=1}^{k} |H_i| \dim V^{H_i} - (k-1) \dim V.$$

Proof. From theorem 3.3 we have

$$\dim V^H = \frac{1}{|H|} \sum_{h \in H} \operatorname{Tr} \rho(h). \tag{3.7}$$

The RHS of (3.7) can be divided into the sums of elements in H_i; however, by doing so, one counts $\operatorname{Tr}\rho(e)$, where e is the identity in H, k times, whereas it should only be counted once. Thus,

$$\dim V^H = \frac{1}{|H|} \left[\sum_{i=1}^{k} \sum_{h \in H_i} \operatorname{Tr}\rho(h) - (k-1) \operatorname{Tr}\rho(e) \right].$$

Now $\rho(e) = I_V$, so $\operatorname{Tr}\rho(e) = \dim V$. We apply the trace formula to the first term on the RHS of (3.2) to obtain

$$\dim V^H = \frac{1}{|H|} \left[\sum_{i=1}^{k} |H_i| \dim V^{H_i} - (k-1) \dim V \right]$$

as desired.

Proof of theorem 3.2. First, one uses corollary 3.4 and the disjoint union decompositions of lemma 2.4 to derive the formulas for the dimensions of the fixed point subspaces of D_n, T, I and O using the calculation for Z_n in (3.5). Second, one makes the simple observation that

$$\begin{aligned}&\text{(a)} \quad V_l^{\mathrm{SO}(2)} = \bigcap_{n=2}^{\infty} V_l^{\mathrm{Z}_n}, \\ &\text{(b)} \quad V_l^{\mathrm{O}(2)} = \bigcap_{n=2}^{\infty} V_l^{\mathrm{D}_n}.\end{aligned} \tag{3.8}$$

To prove the validity of (3.8a), observe that since $\mathrm{Z}_n \subset \mathrm{SO}(2)$ for each n we have $V_l^{\mathrm{SO}(2)} \subset V_l^{\mathrm{Z}_n}$ for each n. Conversely, if a vector v is fixed by Z_n for each n (that is, v is fixed by rotation through every rational angle) then, by continuity, v must be fixed by SO(2). The argument proving (3.3b) is similar. It follows that

$$d(\mathrm{O}(2)) = \lim_{n \to \infty} d(\mathrm{D}_n) = \begin{cases} 0, & l \text{ odd}, \\ 1, & l \text{ even}. \end{cases}$$

(This last computation could have been done directly. The expense would have been the introduction of more specific details about the action of SO(3) on spherical harmonics.)

We now discuss the dimensions of the fixed point sets for the irreducible representations of O(3) for which $-I$ acts as minus the identity. As discussed in the beginning of this section we need only consider the subgroups of SO(3) and the class III subgroups. The results for the subgroups of SO(3) are identical with those of theorem 3.2. We now present the results for the class III subgroups.

Theorem 3.5. Let O(3) act irreducibly on V_l with $-I$ acting as minus the identity. Then the dimensions of the fixed point spaces for the class III subgroups are

(a) $d(\mathrm{Z}_{2n}^-) = 2\left[\frac{l+n}{2n}\right]$;

(b) $d(\mathrm{D}_n^z) = \begin{cases} \left[\frac{l}{n}\right], & l \text{ even}, \\ \left[\frac{l}{n}\right] + 1, & l \text{ odd}; \end{cases}$

(c) $d(\mathrm{D}_{2n}^d) = \left[\frac{l+n}{2n}\right]$;

(d) $d(\mathrm{O}^-) = \left[\frac{l+2}{4}\right] + \left[\frac{l}{3}\right] + \left[\frac{l+1}{2}\right] - l$;

(e) $d(\mathrm{O(2)}^-) = \begin{cases} 0, & l \text{ even}, \\ 1, & l \text{ odd}. \end{cases}$

Proof. To compute $d(\mathrm{Z}_{2n}^-)$ observe that Z_{2n}^- is generated by a rotation through an angle π/n, $R_{\pi/n}$ followed by $-I$. The only way a vector v can be fixed by this element, and hence by the cyclic group Z_{2n}^-, is for $R_{\pi/n}v = -v$. Since $R_{\pi/n}$ is in SO(2) we may use the root space decomposition of V_l (3.4) to determine the possible v's. Since $R_{\pi/n}$ acts on W_k by $R_{k\pi/n}$ we see that v in W_k is fixed if and only if k/n is an odd integer. Thus, $d(\mathrm{Z}_{2n}^-)$ is equal to twice the number of integers $k \leq l$ for which k/n is an odd integer. (Recall that $\dim W_k = 2$ for $k > 0$.) One can now show that $d(\mathrm{Z}_{2n}^-)$ is $2[(l+n)/2n]$.

Next, one uses corollary 3.4 and the decomposition of the finite class III subgroups into a disjoint union of cyclic subgroups, lemma 2.10, to compute $d(\mathrm{D}_n^z)$, $d(\mathrm{D}_{2n}^d)$ and $d(\mathrm{O}^-)$. Finally, one shows that

$$V^{\mathrm{O(2)}^-} = \bigcup_{n=2}^{\infty} V^{\mathrm{D}_n^z},$$

so that

$$d(\mathrm{O(2)}^-) = \lim_{n\to\infty} d(\mathrm{D}_n^z).$$

The crucial point here is the observation that elements of $\mathrm{D}_n^z \sim \mathrm{Z}_n$ actually lie in $\mathrm{O(2)}^-$ and this fact follows from the construction of D_n^z and $\mathrm{O(2)}^-$ using lemma 2.7. □

We begin our discussion of the maximal isotropy subgroups of SO(3) with the observation that if a maximal subgroup has a positive dimensional fixed point set then it is a maximal isotropy subgroup. There are three maximal subgroups of SO(3): O(2), O, and I. One can see directly that there is a basic difference in the description of the maximal isotropy subgroups when l is even and when l is odd. From theorem 3.2d we see that O(2) is a maximal isotropy subgroup when l is even and is not an isotropy subgroup when n is odd.

Proposition 3.6. Let SO(3) act irreducibly on V_l with l even and positive. With four exceptions, the maximal isotropy subgroups are

O(2), O and I.

The exceptional cases, along with the maximal isotropy subgroups, are

$l = 2$: O(2);

$l = 4, 8, 14$: O(2) and O.

Proof. The idea of the proof is to show that, in general, the fixed point subspaces of the maximal closed subgroups are all positive dimensional. As remarked above, $d(\mathrm{O(2)}) = 1$ for all even l so that O(2) is a maximal isotropy subgroup for all even l. From theorem 3.2f and g, we see that the functions $d(\mathrm{O})$ and $d(\mathrm{I})$ have a kind of periodicity in l. In particular $d(\mathrm{O})\ (l+12) = d(\mathrm{O})\ (l) + 1$ and $d(\mathrm{I})\ (l+30) = d(\mathrm{I}) + 1$. It follows that O is a maximal isotropy subgroup whenever $l > 12$ and I is a maximal isotropy subgroup whenever $l > 30$. To prove precisely our results we list $d(\mathrm{O})$ $(1 \leq l \leq 12)$ and $d(\mathrm{I})$ $(1 \leq l \leq 30)$ in table III. Note the only even l for which O is not a maximal isotropy subgroup is $l = 2$ and the only even l's for which I is not a maximal isotropy subgroup are $l = 2, 4, 8$ and 14.

To complete the proof note that the only subgroups of SO(3) which are not contained in O(2) are the exceptional subgroups. Thus, if O(2) is an isotropy subgroup the only possibilities for maximal isotropy subgroups are the exceptional sub-

Table III

l	d(O)	l	d(I)	l	d(I)	l	d(T)
1	0	1	0	16	1	1	0
2	0	2	0	17	0	2	0
3	0	3	0	18	1	3	1
4	1	4	0	19	0	4	1
5	0	5	0	20	1	5	0
6	1	6	1	21	1	6	2
7	0	7	0	22	1		
8	1	8	0	23	0		
9	1	9	0	24	1		
10	1	10	1	25	1		
11	0	11	0	26	1		
12	2	12	1	27	1		
		13	0	28	1		
		14	0	29	0		
		15	1	30	2		

groups. When O is also a maximal isotropy subgroup it follows that since $T \subset O$ only I can be a maximal isotropy subgroup. It only remains to note that in the case $l = 2$, T is not an isotropy subgroup as $d(T) = 0$. □

We now consider the case when l is odd. The distinction here is that SO(2), rather than O(2), is always a maximal isotropy subgroup.

Proposition 3.7. Let SO(3) act irreducibly on V_l with l odd. In general, the maximal isotropy subgroups are

$$\mathrm{SO}(2), \mathrm{O}, \mathrm{I} \text{ and } \mathrm{D}_n \quad \left(\frac{l}{2} < n \le l\right).$$

The exceptional cases are

$l = 9, 13, 17, 19, 23, 29$:

$$\mathrm{SO}(2), \mathrm{O} \text{ and } \mathrm{D}_n \quad \left(\frac{l}{2} < n \le l\right);$$

$$l = 3, 7, 11: \quad \mathrm{SO}(2), \mathrm{T} \text{ and } \mathrm{D}_n \quad \left(\frac{l}{2} < n \le l\right);$$

$$l = 5: \quad \mathrm{SO}(2), \mathrm{D}_3, \mathrm{D}_4;$$

$$l = 1: \quad \mathrm{SO}(2).$$

Proof. First, we consider the stable picture. From table III we see that T is a isotropy subgroup for all odd l except

$l = 1, 3, 5, 7, 9, 11, 13, 17, 19, 23$ and 29

and that O is a isotropy subgroup for all odd l except

$l = 1, 3, 5, 7$ and 11.

It is possible for T to be a maximal isotropy subgroup only when both O and I are not isotropy subgroups (lemma 3.1). From table III it follows that T is a maximal isotropy subgroup only when $l = 3$, 7 and 11.

Since SO(2) is a maximal isotropy subgroup it follows that only the D_n's are possible maximal isotropy subgroups. From theorem 3.2b we see that when l is odd

$$d(\mathrm{D}_n) = \left[\frac{l}{n}\right].$$

Thus, D_n can be an isotropy subgroup only when $n \le l$. Moreover, when $n \le l/2$ D_n is contained in D_{2n} whose fixed point space has positive dimension. Hence, D_n is not a maximal isotropy subgroup (lemma 3.1).

The remaining point is that some of the D_n's are also contained in the exceptional subgroups which could prevent certain D_n from being a maximal isotropy subgroup. This does not happen but one must check for the possibility. The containment information we need may be found in fig. 4. It is

$$\mathrm{I} \supset \mathrm{D}_5, \mathrm{D}_3, \mathrm{D}_2; \quad \mathrm{O} \supset \mathrm{D}_4, \mathrm{D}_2; \quad \text{and} \quad \mathrm{T} \supset \mathrm{D}_2.$$

Combining the information above yields the proof of proposition 3.7. □

Proposition 3.8. We enumerate which of the maximal isotropy subgroups of SO(3) have fixed point sets which are precisely 1-dimensional:

O(2): all l even;

SO(2): all l odd;

D_n: l odd, $l/2 < n \le l$;

I: $l = 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53$, and 59;

O: $l = 4, 6, 8, 9, 10, 13, 14, 15, 17, 19$ and 23;

T: $l = 3, 7, 11$.

The proof involves combining the results of propositions 3.6 and 3.7 and table III.

Next we consider the non-trivial irreducible representations of O(3): that is, those representations where $-I$ acts as minus the identity of V_l. As was discussed in the beginning of this section, the isotropy subgroups for the non-trivial representations of O(3) are either subgroups of SO(3) or class III subgroups. (Thus, we do not consider subgroups of the form $K \oplus \mathrm{Z}_2^c$.) In this class of subgroups of O(3) there are three maximal closed subgroups: SO(3), $\mathrm{O}(2)^-$ and O^-. However, SO(3) acts irreducibly on V_l and cannot be an isotropy subgroup. Thus, the obvious candidates for maximal isotropy subgroups are: O(2), $\mathrm{O}(2)^-$, O, O^- and I. Immediately, we again see a difference depending on the parity of l. For l even O(2) is a maximal isotropy subgroup and $\mathrm{O}(2)^-$ is not while the reverse is true for l odd.

Proposition 3.9. Let O(3) act irreducibly on V_l with l odd and let $-I$ act as minus the identity on V_l. Then, in general, the maximal isotropy subgroups are

$$\mathrm{O}(2)^-, \mathrm{O}^-, \mathrm{O}, \mathrm{I} \quad \text{and} \quad \mathrm{D}_{2n}^{\mathrm{d}} \quad \left(\frac{l}{3} < n \le l\right).$$

The exceptional cases are

$l = 9, 13, 17, 19, 23, 29$:
$\mathrm{O}(2)^-$, O^-, O and $\mathrm{D}_{2n}^{\mathrm{d}}$ $\left(\frac{l}{3} < n \le l\right)$;

$l = 3, 7, 11$: $\mathrm{O}(2)^-$, O^- and $\mathrm{D}_{2n}^{\mathrm{d}}$ $\left(\frac{l}{3} < n \le l\right)$;

$l = 5$: $\mathrm{O}(2)^-$, $\mathrm{D}_4^{\mathrm{d}}$, $\mathrm{D}_6^{\mathrm{d}}$, $\mathrm{D}_8^{\mathrm{d}}$ and $\mathrm{D}_{10}^{\mathrm{d}}$;

$l = 1$: $\mathrm{O}(2)^-$.

Proof. Since we assume l is odd, $\mathrm{O}(2)^-$ is always a maximal isotropy subgroup. Since Z_{2n}^- and D_n^z are contained in $\mathrm{O}(2)^-$ they cannot be maximal isotropy subgroups. The only possibility for maximal isotropy subgroups of class III are O^- and $\mathrm{D}_{2n}^{\mathrm{d}}$ $(n \ge 2)$.

In table IV we enumerate $d(\mathrm{O}^-)$ as a function of l noting the periodicity

$$d(\mathrm{O}^-)(l+12) = d(\mathrm{O}^-)(l) + 1.$$

By inspection of this table it follows that O^- is a maximal isotropy subgroup for all odd l except $l = 1$ and $l = 5$.

From theorem 3.5c we see that $\mathrm{D}_{2n}^{\mathrm{d}}$ has a non-trivial fixed point subspace when $n \le l$. In addition, $\mathrm{D}_{2n}^{\mathrm{d}} \subset \mathrm{D}_{6n}^{\mathrm{d}}$ implies that $\mathrm{D}_{2n}^{\mathrm{d}}$ can be a maximal isotropy subgroup only if $n > l/3$. We claim that each such subgroup is a maximal isotropy subgroup. We use lemma 3.1. Observe that the only subgroups containing $\mathrm{D}_{2n}^{\mathrm{d}}$ are those $\mathrm{D}_{2m}^{\mathrm{d}}$ where n divides m except for $\mathrm{D}_4^{\mathrm{d}}$ which is contained in O^-. Now O^- is not an isotropy subgroup when $l = 5$ and when $l \ge 5$ the condition $n > l/3$ rules out $\mathrm{D}_4^{\mathrm{d}}$. So the claim is valid.

We now consider the subgroups of SO(3). Using lemma 3.1, we may rule out $\mathrm{Z}_n \subset \mathrm{O}(2)^-$, $\mathrm{SO}(2) \subset \mathrm{O}(2)^-$ and $\mathrm{D}_n \subset \mathrm{D}_{2n}^{\mathrm{d}}$. The only possible additional maximal isotropy subgroups are the exceptional groups. Note that T is contained in O, O^- and I. Thus, T can be a maximal isotropy subgroup only when none of O, O^- and I are isotropy subgroups. This happens only when $l = 1$ or $l = 5$ and in neither case can T be an isotropy subgroup. See table III.

Finally, note that O and I are maximal isotropy subgroups of O(3) precisely when they are maxi-

Table IV

l	$d(\mathrm{O}^-)$
1	0
2	0
3	1
4	0
5	0
6	1
7	1
8	0
9	1
10	1
11	1
12	1

mal isotropy subgroups of SO(3) and this information is contained in proposition 3.7.

In a similar vein, we state without proof

Proposition 3.10. Let O(3) act irreducibly on V_l with l even and let $-I$ act as minus the identity on V_l. Then, in general, the maximal isotropy subgroups are

$$\mathrm{O}(2), \mathrm{O}^-, \mathrm{O}, \mathrm{I} \text{ and } \mathrm{D}_{2n}^{\mathrm{d}} \left(\frac{l}{3} < n \leq l\right).$$

The exceptional cases are

$$l = 14: \quad \mathrm{O}(2), \mathrm{O}^-, \mathrm{O} \text{ and } \mathrm{D}_{2n}^{\mathrm{d}} \quad (5 \leq n \leq 14);$$

$$l = 4, 8: \quad \mathrm{O}(2), \mathrm{O} \text{ and } \mathrm{D}_{2n}^{\mathrm{d}} \quad \left(\frac{l}{3} < n \leq l\right);$$

$$l = 2: \quad \mathrm{O}(2), \mathrm{D}_2^{\mathrm{d}} \text{ and } \mathrm{D}_4^{\mathrm{d}}.$$

Finally, we enumerate those maximal isotropy subgroups of O(3) which have one dimensional fixed point sets.

Proposition 3.11. Let O(3) act irreducibly on V_l with $-I$ acting as minus the identity. The following is a complete list of those maximal isotropy subgroups of O(3) which have one-dimensional fixed point sets:

$\mathrm{O}(2)$: all even l;

$\mathrm{O}(2)^-$: all odd l;

O^-: $l = 3, 6, 7, 9, 10, 11, 12, 13, 14, 16,$ 17 and 20;

$\mathrm{D}_{2n}^{\mathrm{d}}$: $\frac{l}{3} < n \leq l$, all $l \geq 2$;

O: see proposition 3.8;

I: see proposition 3.8.

The proof involves combining the results of propositions 3.9 and 3.10 and table IV.

4. The equivariant branching lemma and asymptotic instability

In section 3 we enumerated those maximal isotropy subgroups whose fixed point sets are 1-dimensional. As indicated in the introduction, solution branches corresponding to such isotropy subgroups exist under very mild assumptions. The proof of this fact is quite simple once one determines the correct setting. In this section we show that for many group actions (generically) the solutions found by Cicogna's [1] equivariant branching lemma must be linearly unstable. In the context of the present paper, it will follow that generically the resolutions we have found for O(3) acting on the spherical harmonics of order l must be unstable if l is even.

There are four hypotheses needed for the equivariant branching lemma.

(H1) Let Γ be a compact Lie group acting absolutely irreducibly on the vector space V.

By absolute irreducibility we mean that the only linear maps on V commuting with Γ are multiples of the identity. (Over C Schur's lemma states that absolute irreducibility is equivalent to irreducibility. Over R a representation is absolutely irreducible precisely when its complexification is irreducible.)

(H2) Let g: $V \times \mathrm{R} \to V$ be a smooth mapping commuting with Γ; that is $g(\gamma v, \lambda) = \gamma g(v, \lambda)$ for all $\gamma \in \Gamma$.

The absolute irreducibility of the action of Γ on V along with (H2) imply

$$\begin{aligned} &\text{(a)} \quad g(0, \lambda) \equiv 0, \\ &\text{(b)} \quad (dg)_{0,\lambda} \equiv c(\lambda) I. \end{aligned} \tag{4.1}$$

Thus, g has a trivial solution $v = 0$ (4.1a) and along the trivial solution the Jacobian matrix in the V-direction dg is a multiple of the identity. (To prove (4.1b) apply the chain rule to the commutivity relation (H2) to obtain for each γ in Γ

$$(dg)_{\gamma v, \lambda} \gamma = \gamma (dg)_{v, \lambda} \tag{4.2}$$

and evaluate at $v = 0$.) It follows that

$$g(v,\lambda) = c(\lambda)v + g_2(v,\lambda) + \cdots + g_k(v,\lambda) + \cdots \tag{4.3}$$

when g_k is homogenous of degree k in v.

(H3) Assume that $c(0) = 0$ and that $c'(0) < 0$ where $c(\lambda)$ is defined in (4.1b).

The assumption that $c(0) = 0$ is just the statement that there is a bifurcation along the trivial solution at $\lambda = 0$. We consider the system of ODE's

$$\frac{\mathrm{d}v}{\mathrm{d}t} + g(v,\lambda) = 0.$$

For such a system a steady state solution $g(v_0, \lambda_0)$ is *linearly (asymptotically) stable* if the eigenvalues of $(dg)_{v_0,\lambda_0}$ all have real parts which are positive. The solution is *unstable* if there is one eigenvalue of $(dg)_{v_0,\lambda_0}$ with real part negative.

The assumption that $c'(0) < 0$ means that the trivial solution is linearly stable for $\lambda < 0$ and unstable for $\lambda > 0$ with a non-degenerate change in stability occurring at $\lambda = 0$.

(H4) Let Σ be an isotropy subgroup of Γ with $\dim V^{\Sigma} = 1$.

Assumption (H4) is the crucial hypothesis. In particular, one should note that (H4) implies that Σ is a maximal isotropy subgroup of Γ. Note that if Σ is an isotropy subgroup of Γ and V^{Σ} is its fixed point set then (H2) implies

$$g\colon V^{\Sigma} \times \mathrm{R} \to V^{\Sigma}. \tag{4.4}$$

The proof is quite simple. If $\gamma v = v$ then

$$\gamma g(v,\lambda) = g(\gamma v,\lambda) = g(v,\lambda);$$

hence $g(v, \lambda)$ is fixed by γ. Note also that $N(\Sigma)$ is the largest subgroup of Γ whose elements leave V^{Σ} invariant. If we let

$$h = g|V^{\Sigma}, \tag{4.5}$$

then h: $V^{\Sigma} \times \mathrm{R} \to V^{\Sigma}$ commutes with $N(\Sigma)$.

The following theorem has appeared in Cicogna [1], Sattinger [4] and Golubitsky [7]:

Theorem 4.1. Assume hypotheses (H1)–(H4). Then there is a unique branch of solutions to $g = 0$ with isotropy subgroup Σ.

Proof of theorem 4.1. Let v_0 be a non-zero vector in V^{Σ}. Let tv_0 coordinatize the one-dimensional space V^{Σ} and let

$$h(t,\lambda)v_0 = g(tv_0,\lambda). \tag{4.7}$$

It follows from (4.1a) that $h(0, \lambda) \equiv 0$, hence

$$h(t,\lambda) = tk(t,\lambda) \tag{4.8}$$

by Taylor's theorem. From (4.1b) and (H3) it follows that

$$k(0,0) = 0 \quad \text{and} \quad k_{\lambda}(0,0) = c'(0) < 0. \tag{4.9}$$

Using (4.9) and the implicit function theorem, there exists a unique function $\Lambda(t)$ satisfying

$$k(t,\Lambda(t)) \equiv 0, \quad \Lambda(0) = 0. \tag{4.10}$$

It follows that there is a unique nontrivial solution branch to $g = 0$ in $V^{\Sigma} \times \mathrm{R}$ given by $t \to (tv_0, \Lambda(t))$. For $t \neq 0$ these solutions all have isotropy subgroup Σ. $\square$

Theorem 4.2. Assume hypotheses (H1)–(H4). Then the branch of solutions with isotropy subgroup Σ obtained in theorem 4.1 is unstable if either

(A) some term in the Taylor expansion of $h(v,0) = g|V^{\Sigma} \times \{0\}$ is non-zero and the branch is subcritical; or

(B) $(dg_2)_{v_0,0}$ has an eigenvalue with nonzero real part where $v_0 \in V^{\Sigma}$ is nonzero.

Remarks 4.3.

(a) The instability of subcritical bifurcation is well known (cf. Crandall and Rabinowitz [14]). Since g: $V^{\Sigma} \times \mathrm{R} \to V^{\Sigma}$ by (4.4) and $\dim V^{\Sigma} = 1$ by (H4) it follows that v_0 is an eigenvector for $(dg)_{v_0,\lambda}$. The claim of theorem 4.2A is that the corresponding eigenvalue is negative when the branch is subcritical.

(b) The proof of instability in theorem 4.2B requires information about eigenvalues of dg which are not associated with V^{Σ}.

(c) The assumption of theorem 4.2B implies, in particular, that g_2 is nonzero. In some sense the simplest way in which g_2 may be nonzero is for $(d^2g_2)_{0,0}(v_0, v_0)$ to be nonzero. Since the eigenvalue of $(dg_2)_{v_0,0}$ corresponding to v_0 is just $(d^2g_2)_{0,0}(v_0, v_0)$ we see that in this case the hypothesis in (B) is satisfied. We claim that the assumption that $(d^2g_2)_{0,0}(v_0, v_0) \neq 0$ is equivalent to the solution branch being transcritical, i.e., $\Lambda'(0) \neq 0$. Thus, we are led to the conclusion, surprising to those who only consider bifurcation problems in one state variable, that transcritical branches are unstable.

To prove the claim, note that it follows by implicit differentiation of (4.10) that

$$\Lambda'(0) = -k_t(0,0)/k_\lambda(0,0),$$

where $k_\lambda(0,0) < 0$ by (4.9). Moreover,

$$k_t(0,0) = h_{tt}(0,0) = (d^2g)_{0,0}(v_0, v_0).$$

Thus $\Lambda'(0)$ is nonzero if and only if $(d^2g)_{0,0}(v_0, v_0)$ is nonzero, as desired.

(d) Recall from the discussion after H4 that $h = g|V^{\Sigma} \times \mathrm{R}$ commutes with the action of $N(\Sigma)$ on V^{Σ}. Since Σ acts as the identity h actually commutes with $D(\Sigma) = N(\Sigma)/\Sigma$. Since we assume V^{Σ} is one-dimensional there are only two possibilities for $D(\Sigma)$; it is either trivial or equal to Z_2. In the latter case h is forced to be odd in v and $(d^2g)_{0,0}(v_0, v_0)$ is forced to be zero. In such cases, it is still possible for the assumption of theorem 4.2B to be valid. For example, see the study of the planar Bénard problem given in Buzano and Golubitsky [15]. In that context there are two maximal isotropy subgroups; one corresponding to rolls and the other to hexagons. The hexagons give a transcritical branch and are unstable but the rolls have $D(\Sigma) = \mathrm{Z}_2$ and the instability of that branch may be deduced from the more general hypothesis.

(e) On the other hand, the simplest way for the hypothesis in theorem 4.2B to fail is for there to exist a group element in Γ which acts as minus the identity on V. Then (H2) implies that g is an odd function in v and that $g_2 \equiv 0$. Such a group element appears in the representations of O(3) on V_l when l is odd, and the instability results (B) do not apply to these representations. In Golubitsky, Swift and Knobloch [16] another instance of the existence of a group element acting as minus the identity occurs. There, several (orbitally) stable planforms may be stable; however, to deduce this fact one must consider third and fifth order terms in g.

(f) The theorem we would like to prove would state that if $g_2 \equiv 0$ then all of the nontrivial solutions to $g = 0$ are unstable. Here we are, of course, assuming (H1)–(H3). We have not proved such a theorem for two reasons. First, in the proof that transcritical solutions are unstable we will need to know that subcritical solutions are unstable whose proof relies on the assumption (H4). Second, it may be possible for $(dg_2)_{v_0,0}$ to have all of its eigenvalues on the imaginary axis. It is true that for many group actions, dg_2, must have real eigenvalues. Hence, hypothesis (B) is satisfied if g_2 is nonzero.

(g) There are two distinct criteria which imply that dg_2 has real eigenvalues. We consider first that g_2 is the gradient of an invariant homogeneous cubic function. Then $(dg_2)_{v_0,\lambda}$ is a symmetric matrix and has real eigenvalues. Sattinger [4] observed that for the representations of O(3) on the spherical harmonics of order l where l is even there is one quadratic mapping $g_2(v)$ satisfying the commutativity property (H2) which can be nonzero and that mapping is the gradient of an invariant function. This observation implies that

$$g(v,\lambda) = c(\lambda)v + c_2(\lambda)g_2(v) + \cdots. \tag{4.11}$$

Thus, assuming that $c_2(0) \neq 0$ implies that all of the solutions corresponding to isotropy subgroups whose fixed point sets are one-dimensional are unstable.

(h) One should keep firmly in mind that the proof of instability holds generically for V_l, l even, but not *always*. For example, in the spherical

Bénard problem when $l = 2$, Chossat [6] has shown that under certain circumstances (self-adjointness of the linearized problem) $c_2(0) = 0$ in (4.11) and that there exists an (orbitally) stable (axisymmetric) solution whose existence was implied by theorem 4.2.

We have shown that if one wishes to find physically relevant (stable) solutions for l even by local (perturbation) arguments one *must consider special cases*. Then, one may use more sophisticated perturbation theorems such as found in singularity theory to universally unfold the (degenerate) special case and to determine qualitative behavior near the degeneracy. See Golubitsky and Schaeffer [8] for a discussion of the unfolding of Chossat's example when $l = 2$. A similar example occurs in the planar Bénard problem. See Buzano and Golubitsky [15].

(i) A second way in which hypothesis (B) may be satisfied is for (Γ, Σ) to form a V Gelfand pair. That is, let

$$V = V_1 \oplus \cdots \oplus V_k$$

be the decomposition of V into the direct sum of irreducible representations of Σ. Then (Γ, Σ) is a *V Gelfand pair* if all of the representations V_j's are distinct and Σ acts absolutely irreducibly on each V_j. It follows from (4.2) that if $\gamma \in \Sigma$ and $v \in V^{\Sigma}$ then

$$(dg)_{v,\lambda}\gamma = \gamma(dg)_{v,\lambda}.$$

Thus,

$$(dg)_{v,\lambda}: V_j \to V_j$$

and $(dg)_{v,\lambda}|V_j$ is a multiple of the identity, this means $(dg)_{v,\lambda}$ has real eigenvalues and is diagonalizable. Moreover, the same argument shows that any two equivariant maps A and B have real eigenvalues, are diagonalizable and commute. This means in particular the eigenvalues of AB are products of eigenvalues of A with eigenvalues of B.

Examples of this approach are given in Buzano and Golubitsky [15] and Golubitsky, Swift and Knobloch [16].

Proof of theorem 4.2A. Note that the eigenvalue of $(dg)_{tv_0, \Lambda(t)}$ corresponding to eigenvector v_0 is $h_t(t, \Lambda(t))$. Since $k(t, \lambda)$ vanishes along the solution branch, we have

$$h_t(t, \Lambda(t)) = -t\Lambda'(t)k_\lambda(t, \Lambda(t))$$

and that

$$\operatorname{sgn}(h_t(t, \Lambda(t))) = \operatorname{sgn}(t\Lambda'(t)), \tag{4.12}$$

since $k_\lambda(0,0) < 0$.

The assumption that some derivative of $h(t,0)$ is nonzero implies that on one side of the origin, but near 0, $\Lambda'(t)$ has a definite sign. It is now easy to check that the solution branch $t \to (tv, \Lambda(t))$, $t > 0$ or $t < 0$, is subcritical precisely when

$$\operatorname{sgn}[t\Lambda'(t)] = -1. \tag{4.13}$$

coupled with (4.12) we have proved that the eigenvalue in the direction of V^{Σ} is negative and the solution branch consists of unstable solutions.

Before proving theorem 4.2B we need the following observations:

Lemma 4.4. a) Let f: $V \to \mathrm{R}$ be invariant and homogeneous of degree one. Then $f \equiv 0$;

b) If $(dg_2)_{v,\lambda}$ has an eigenvalue with positive real part then it has an eigenvalue with negative real part.

Proof. a) Let $G(v) = (\nabla f)_v$. Note that G commutes with Γ and is homogenous of degree 0. The irreducibility of the action of Γ implies that $G(0) = 0$. Hence, by homogeneity, $G \equiv 0$. It follows that f is constant. However, $f(0) = 0$ by linearity and $f \equiv 0$ as desired.

b) Define

$$f(v, \lambda) = \operatorname{Tr}(dg_2)_{v,\lambda}.$$

It follows from (4.2) that f is invariant and homogenous of degree 1 in v. By (a) $f(v, \lambda) \equiv 0$. Since the trace is the sum of the real parts of the eigenvalues it follows that if one eigenvalue has a positive real part then one has a negative real part.

Proof of theorem 4.2B. We first consider the case of transcritical bifurcation, $\Lambda'(0) \neq 0$. See remark 4.3c. Since $\Lambda'(0) \neq 0$ the branch of solutions in $V^{\Sigma} \times \mathrm{R}$ has both a subcritical and a supercritical part. From (A) we know that the subcritical part is unstable; we concentrate on the supercritical part where from (4.13) we have

$$t\Lambda'(t) > 0. \tag{4.14}$$

Define $T(v, \lambda) = \mathrm{Tr}(dg)_{v,\lambda}$ and let

$$m(t) = T(tv_0, \Lambda(t)).$$

Using the Taylor expansion of g in (4.3) and lemma 4.4a we see that

$$m(t) = nc(\Lambda(t)) + \mathcal{O}(t^2).$$

where $n = \dim V$. It follows that

$$m(t) = nc'(0)\Lambda'(0)t + \mathcal{O}(t^2).$$

It follows from (4.14) and $\Lambda'(0) \neq 0$ that

$$\mathrm{sgn}(\Lambda'(0)t) = \mathrm{sgn}(t\Lambda'(t)) = +1.$$

on the supercritical branch. Since $c'(0) < 0$ (H3) we see that $m(t) < 0$ for all t near 0 on the supercritical side. This can happen only if there is some eigenvalue of $(dg)_{tv_0,\Lambda(t)}$ which has a negative real part. So the supercritical solution branch consists of unstable solutions for t near 0.

We may now assume that $(d^2g)_{0,0}(v_0, v_0) = 0$; that is, $\Lambda'(0) = 0$. Consider the Jacobian matrix of g expanded along the solution branch using (4.3),

$$(dg)_{tv_0,\Lambda(t)} = c(\Lambda(t))I + t(dg_2)_{v_0,\Lambda(t)} + \mathcal{O}(t^2). \tag{4.15}$$

Here we have used the fact that $(dg_k)_{v,t}$ is homogenous of degree $k-1$. Since $c(0) = \Lambda(0) = \Lambda'(0) = 0$ it follows that the first term on the RHS of (4.15) is also $\mathcal{O}(t^2)$; hence

$$(dg)_{tv_0,\Lambda(t)} = t\left[(dg_2)_{v_0,0} + tJ(t)\right],$$

where $J(t)$ is an $n \times n$ matrix depending smoothly on t. By assumption $(dg_2)_{v_0,0}$ has an eigenvalue with nonzero real part and by lemma 4.4b that it has eigenvalues with real parts negative and positive. By continuity the same can be said for

$$(dg_2)_{v_0,0} + tJ(t);$$

the eigenvalues of a matrix vary continuously with parameters. Thus $(dg)_{tv_0,\Lambda(t)}$ has at least one eigenvalue with a negative real part when $t \neq 0$ and the corresponding solutions are unstable.

We finish this section with a brief discussion of when the signs of the eigenvalues of $(dg)_{v,\lambda}$ are preserved under equivalence. The basic result in this direction (4.4) states that this is the case when the isotropy subgroup Γ_v of the solution forms a V_l Gelfand pair with Γ. Remark 4.5 gives the V_l Gelfand pairs for SO(3) and O(3). Unfortunately, very few solutions other than the axisymmetric ones have isotropy subgroups with this property. However, there is some indication that the assumption of being a Gelfand pair may be necessary for stability to be preserved under equivalence for all bifurcation problems. If we insist that stability be an equivalence invariant only generically then more situations may become permitted.

Proposition 4.4. Suppose (Γ, Γ_v) is a V Gelfand pair. Then the eigenvalues of $(dg)_{v,\lambda}$ are real and their signs are preserved under equivalence.

Proof. We will suppress λ here since it plays no essential role. For $\gamma \in \Gamma_v$ we have

$$\gamma\, dg_v \gamma^{-1} = \gamma\, dg_{\gamma v} \gamma^{-1} = dg_v.$$

Thus by remark 4.3h dg_v has real eigenvalues. We must only show their signs are preserved under equivalence. Suppose g is equivalent to g'. Then there are an X: $V \to V$ and a T: $V \times V \to V$ with

$$g'(x) = T(x) g(X(x)),$$

where $T(x)y \equiv T(x, y)$. Moreover, X is an equivariant diffeomorphism with $\det(dX_0) > 0$, and $T(x)$ is an invertable linear transformation for all x such that

$$\gamma T(\gamma x)\gamma^{-1} = T(x),$$
$$\gamma \in \Gamma, x \in V \text{ and } \det(T(0)) > 0.$$

We now calculate dg'_v using the product and chain rule:

$$dg'_v = dT_v g(X(v)) + T(v)\, dg_{x(v)}\, dX_v,$$

where $g'(v) = 0$. Since $g(X(v)) = T(v)^{-1} g'(v) = 0$, we have $dg'_v = T(v)\, dg_{x(v)}\, dX_v$. Each of these four matrices are Δ equivariant so Remark 3.3(h) shows that the eigenvalue of dg'_v are a product of eigenvalues of $T(v)$, dX_v and $dg_{x(v)}$. Since both $T(0)$ and dX_0 are multiples of I (Γ acts absolutely irreducibly on V), the determinant conditions above imply T and dX have all positive real eigenvalues at 0 and thus in a neighborhood of 0. Therefore, the signs of the eigenvalue of dg and dg' are the same near the 0.

Remark 4.5. a) The Σ for which (Σ; SO(3)) are V_l Gelfand pairs are the following:

SO(2) for all odd l;
O(2) for all even l;
I for $l = 6$;
O for $l = 4$.

b) The Σ for which (Σ, O(3)) are V_l Gelfand pairs (for the non-trivial representations of O(3)) are the following:

O(2) for all even l;
$\mathrm{O}^-(2)$ for all odd l;
I for $l = 6$;
O for $l = 4$;
O^- for $l = 3$;
$\mathrm{D}^{\mathrm{d}}_{2l}$ for all l.

The proof of this remark is straightforward. It involves only reducing the representation of Δ on V_l. For $\Delta \subset \mathrm{O}(2)$ or $\mathrm{O}^-(2)$. This reduction can be done using the weight space decomposition. For the exceptional groups, an upper bound on the dimension of V_l is given by the sum of the dimensions of all the distinct irreducible representations of Γ. These bounds are 16, 10, 10 and 6 for I, O, O^-, and T respectively. This, together with the constraint $\dim(V_l^{\Gamma}) = 1$ leaves only the possibilities of $l = 6$, 4 and 3 for I, O and O^- respectively. T is ruled out altogether. A simple character check shows that these cases actually give V_l Gelfand pairs.

5. The relationship between isotropy subgroups and fixed point sets

At first glance there would seem to be a very simple method for determining when a closed subgroup Σ of Γ is an isotropy subgroup. The idea is to consider in the lattice of closed subgroups each subgroup $\Delta \supsetneqq \Sigma$. If $V^{\Delta} \subsetneqq V^{\Sigma}$ for each such Δ then one is tempted to conclude that Σ is, in fact, an isotropy subgroup. Certainly, the condition is necessary; for if $V^{\Delta} = V^{\Sigma}$ and Σ fixes a point v in V then so does Δ; thus, Σ cannot be an isotropy subgroup. However, it may happen that each point in V^{Σ} is also contained in some V^{Δ} with Δ depending on v, so that Σ is *not* the isotropy subgroup for any v in V^{Σ}. Let $I(\Gamma, \Sigma)$ be the set of all isotropy subgroups Δ of Γ such that $\Delta \subsetneqq \Sigma$. The appropriate statement is

Lemma 5.1. Σ is an isotropy subgroup of Γ if and only if

$$\bigcup_{\Delta \in I(\Gamma, \Sigma)} V^{\Delta} \subsetneqq V^{\Sigma}. \tag{5.1}$$

Proof. The necessity of (5.1) was shown above. Assume that (5.1) is valid and let v be in $V^{\Sigma} \sim \cup_{\Delta \in I(\Gamma,\Sigma)} V^{\Delta}$. We claim that Σ is the isotropy subgroup of v, Σ_v. Clearly, $\Sigma \subset \Sigma_v$. Moreover, if $\Sigma_v \neq \Sigma$ then Σ_v is one of the isotropy subgroups in the union considered in (5.1). Since $v \in V^{\Sigma_v}$ we have a contradiction and $\Sigma = \Sigma_v$. □

Lemma 5.1 suggests an inductive procedure for determining the lattice of isotropy subgroups. First, one finds the maximal isotropy subgroups, which may be determined using only the computation of dimensions of fixed point sets. See lemma 5.2. In theory, one may then use lemma 5.1 to find the submaximal isotropy subgroups, etc. However, lemma 5.1 is inadequate in two distinct ways. First, although the dimension of V^{Δ} is often easily determined, it is rarely the case that V^{Δ} itself is known explicitly. Second, although the lattice of conjugacy classes of all isotropy subgroups of a Lie group Γ is often known, it is rarely the case that all isotropy subgroups of Γ can be given explicitly. Since lemma 5.1 uses this information it is not particularly useful as it stands.

Our first task is to understand better the construction of $\cup_{\Delta \in I(\Gamma,\Sigma)} V^{\Delta}$. Suppose $\Delta \supset \Sigma$, let

$$N(\Sigma, \Delta) = \{\gamma \in \Gamma | \gamma \Delta \gamma^{-1} \supset \Sigma\}. \tag{5.2}$$

Lemma 5.2. a) $N(\Sigma, \Delta)$ is closed under multiplication from the right by elements of $N(\Delta)$,

b) $$\bigcup_{\Delta \in I(\Gamma, \Sigma)} V^{\Delta}$$

$$= \bigcup_{\Delta \in CI(\Gamma,\Sigma), \tilde{\gamma} \in N(\Sigma,\Delta)/N(\Delta)} \bigcup \gamma(V^{\Delta}), \tag{5.3}$$

where $CI(\Gamma, \Sigma)$ is the set of conjugacy classes of isotropy subgroups in $I(\Gamma, \Sigma)$ and $\tilde{\gamma}$ is the projection of $\gamma \in \Gamma$ into $\Gamma/N(\Delta)$; and

c) $N(\Sigma, \Delta)$ is closed under multiplication from the left by elements of $N(\Sigma)$.

Assuming for the moment the validity of lemma 5.2 and that $N(\Sigma, \Delta)$ is a sufficiently nice subset of Γ (see lemma 5.4) one is led to

Proposition 5.3. Assume that $CI(\Gamma, \Sigma)$ is at most countable. If for every conjugacy class of Δ in $CI(\Gamma, \Sigma)$ one has

$$\dim V^{\Delta} + \dim N(\Sigma, \Delta) - \dim N(\Delta) < \dim V^{\Sigma}, \tag{5.4}$$

then Σ is an isotropy subgroup of Γ.

Remarks. Crudely, the proof of proposition 5.3 is obtained from lemmas 5.1 and 5.2 as follows. The dimension of the set

$$S_{\Delta} = \bigcup_{\tilde{\gamma} \in N(\Sigma,\Delta)/N(\Delta)} \gamma(V^{\Delta})$$

can be no larger than $\dim V^{\Delta} + \dim N(\Sigma, \Delta)/N(\Delta)$. Moreover, if $N(\Sigma, \Delta)$ is sufficiently nice, then $\dim N(\Sigma, \Delta)/N(\Delta)$ will equal $\dim N(\Sigma, \Delta) - \dim N(\Delta)$. Thus, if eq. (5.4) is valid we see that $\dim S_{\Delta} < \dim V^{\Sigma}$ and that S_{Δ} has measure zero in V^{Σ}. If (5.4) is valid for all $\Delta > \Sigma$ and these are only a countable number of such Δ's then the RHS of (5.3) has measure zero in V^{Σ}. Thus, (5.1) is valid and, using lemma 5.1, Σ is an isotropy subgroup of Γ.

It appears likely that the converse of proposition 5.3 will be true under fairly general circumstances but we have been unable to find a proof of this. However, a converse is not needed for O(3) since the fixed point sets of the subgroups that do not satisfy (5.4) may be easily calculated directly.

Proof of lemma 5.2. To prove (a) we must show that if $\sigma \in N(\Sigma, \Delta)$ and $\delta \in N(\Delta)$ then $\sigma\delta \in N(\Sigma, \Delta)$. Observe that

$$(\sigma\delta)\Delta(\sigma\delta)^{-1} = \sigma(\delta\Delta\delta^{-1})\sigma^{-1} = \sigma\Delta\sigma^{-1}$$

since σ is in the normalizer of Δ. It follows from the definition of $N(\Sigma, \Delta)$, eq. (5.2), that $\sigma\Delta\sigma^{-1} \supset \Sigma$ and that $\sigma\delta \in N(\Sigma, \Delta)$. Note that (a) implies that $N(\Sigma, \Delta)$ is invariant under $N(\Delta)$ and that $N(\Sigma, \Delta)/N(\Delta)$ is a well-defined subset of the coset space $\Gamma/N(\Delta)$.

To prove that (b) is valid, we first show that the RHS of (5.3) makes sense. In particular, suppose that γ_1 and γ_2 are in $N(\Sigma, \Delta)$ and that $\tilde{\gamma}_1 = \tilde{\gamma}_2$ in $N(\Sigma, \Delta)/N(\Delta)$. We claim that $\gamma_1(V^\Delta) = \gamma_2(V^\Delta)$. This claim follows from

$$N(\Delta) \subset \{\gamma \in \Gamma | \gamma(V^\Delta) = V^\Delta\}. \tag{5.5}$$

the assumptions on γ_1 and γ_2 imply that there is a τ in $N(\Delta)$ such that $\gamma_1 = \gamma_2\tau$. Assuming (5.5) we see that

$$\gamma_1(V^\Delta) = \gamma_2\tau(V^\Delta) = \gamma_2(V^\Delta),$$

as claimed. To verify (5.5), note that if τ is in the normalizer of Δ and if δ is in Δ then $\delta\tau = \tau\delta'$ for some δ' in Δ. It follows that if v is in V^Δ that

$$\tau v = \tau\delta' v = \delta\tau v.$$

So τv is fixed by each δ in Δ and τv is in V^Δ. (Note that if Δ is an isotropy subgroup then

$$N(\Delta) = \{\gamma \in \Gamma | \gamma(V^\Delta) = V^\Delta\}. \tag{5.6}$$

For suppose $\Delta = \Sigma_v$, the isotropy subgroup for $v \in V^\Delta$. Since $\gamma v \in V^\Delta$ it follows that the isotropy subgroup $\Sigma_{\gamma v}$ certainly contains Δ. However,

$$\Sigma_{\gamma v} = \gamma\Sigma_v\gamma^{-1} = \gamma\Delta\gamma^{-1}.$$

Thus, the only possibility is that $\Sigma_{\gamma v}$ is Δ. This means γ is in the normalizer of Δ, as stated.)

Finally, observe that if Δ and Δ' are conjugate isotropy subgroups of Γ, both containing Σ, then there is a γ in Γ such that $\Delta' = \gamma\Delta\gamma^{-1}$. Moreover, since $\Delta' \supset \Sigma$ we see that γ is in $N(\Sigma, \Delta)$, by definition. Thus,

$$\begin{aligned}\bigcup_{\Delta \in I(\Gamma,\Sigma)} V^\Delta &= \bigcup_{\Delta \in CI(\Gamma,\Sigma)} \bigcup_{\substack{\Delta' \in I(\Gamma,\Sigma)\\ \Delta' \text{ conjugate to } \Delta}} V^{\Delta'} \\ &= \bigcup_{\Delta \in CI(\Gamma,\Sigma)} \bigcup_{\gamma \in N(\Sigma,\Delta)} \gamma(V^\Delta) \\ &= \bigcup_{\Delta \in CI(\Gamma,\Sigma)} \bigcup_{\tilde{\gamma} \in N(\Sigma,\Delta)/N(\Delta)} \gamma(V^\Delta),\end{aligned}$$

as desired.

We finish by showing (c). Let σ be in $N(\Sigma)$ and let γ be in $N(\Sigma, \Delta)$. Then $(\sigma\gamma)\Delta(\sigma\gamma)^{-1} = \sigma(\gamma\Delta\gamma^{-1})\sigma^{-1} \supset \sigma\Sigma\sigma^{-1} = \Sigma$. Hence, $\sigma\gamma \in N(\Sigma, \Delta)$ as claimed. □

As indicated in remark (a) above, the proof of proposition 5.3 requires knowing that $N(\Sigma, \Delta)$ is a sufficiently nice subset of Γ so that its dimension behaves well. This requirement is satisfied by the following result:

Lemma 5.4. Let Δ be an isotropy subgroup of Γ containing Σ, a closed subgroup. Then

a) $N(\Sigma, \Delta) = \{\gamma \in \Gamma | \gamma(V^\Delta) \subset V^\Sigma\}$;

b) $N(\Sigma, \Delta)$ is a real analytic subvariety of Γ; and

c) $N(\Sigma, \Delta)$ is compact in Γ.

Proof. First, note

$$\Delta \supset \Sigma \text{ if and only if } V^\Delta \subset V^\Sigma. \tag{5.7}$$

The necessity of (5.7) is obvious. To prove the sufficiency, suppose that $V^\Delta \subset V^\Sigma$ and let σ be in Σ. Then $\sigma v = v$ for all v in V^Δ. In particular σ fixes V^Δ. Now choose x in V^Δ such that Δ is the isotropy subgroup for x. Then, since $\sigma x = x$, it follows that σ is in Δ. Thus, $\Sigma \subset \Delta$.

Using (5.7) we see that

$$\gamma\Delta\gamma^{-1} \supset \Sigma \quad \text{if and only if } V^{\gamma\Delta\gamma^{-1}} \subset V^\Sigma.$$

But $V^{\gamma\Delta\gamma^{-1}} = \gamma(V^\Delta)$ and (a) is proved.

To prove (b) and (c), let π: $V \to V/V^{\Sigma}$ be the cononical projection. Consider the mapping

$$\Phi: \Gamma \to \mathrm{Hom}(V^{\Delta}, V/V^{\Sigma}) \quad \Phi(\gamma) = \pi \circ \gamma. \quad (5.8)$$

The representation of Γ on V is real analytic because Γ is a Lie group and Γ acts by linear transformations which are real analytic (see Montgomery and Zippen [17, p. 212]). It then follows that Φ is real analytic. Moreover, observe that $\Phi(\gamma) = 0$ if and only if $\gamma(V^{\Delta}) \subset V^{\Sigma}$; that is, γ is in $N(\Sigma, \Delta)$. Thus, $N(\Sigma, \Delta) = \Phi^{-1}(0)$ is a real analytic subvariety of Γ (consider that the coordinate functions of Φ define $N(\Sigma, \Delta)$) and $N(\Sigma, \Delta)$ is closed. Since Γ is compact $N(\Sigma, \Delta)$ is also compact.

The fact that $N(\Sigma, \Delta)$ is a compact real analytic variety has several important consequences. The basic reference is Lojasiewicz [18]. The most important point is: $N(\Sigma, \Delta)$ is a stratified set with a finite number of strata, each of which is a smooth manifold. Thus, the dimension of $N(\Sigma, \Delta)$ is the maximum of the dimensions of these strata.

Proof of proposition 5.3. The idea behind the proof is to show that for each $\Delta \in I(\Gamma, \Sigma)$

$$S_{\Delta} = \bigcup_{\tilde{\gamma} \in N(\Sigma, \Delta)/N(\Delta)} \gamma(V^{\Delta})$$

is a set of measure zero in V^{Σ}. If this claim is valid, then the RHS of (5.3) has measure zero since we assume that $CI(\Gamma, \Sigma)$ is at most countable. It follows from lemma 5.2 that $\cup_{\Delta \in I(\Gamma, \Sigma)} V^{\Delta} \neq V^{\Sigma}$ and from lemma 5.1 that Σ is an isotropy subgroup of Γ.

There are three additional facts which we need to complete the proof:

(a) $N(\Sigma, \Delta)/N(\Delta)$ is an analytic variety;

(b) $\dim N(\Sigma, \Delta)/N(\Delta) = \dim N(\Sigma, \Delta) - \dim N(\Delta)$; (5.9)

(c) Locally, on each stratum S of $N(\Sigma, \Delta)/N(\Delta)$, there is a smooth section s mapping S into $N(\Sigma, \Delta)$.

Assuming the validity of (5.9), there is a smooth mapping defined locally on a stratum S of $N(\Sigma, \Delta)/N(\Delta)$,

$$\psi: S \times V^{\Delta} \to V^{\Sigma}, \quad (\tilde{\gamma}, v) \mapsto s(\tilde{\gamma}) \cdot v;$$

where s: $S \to N(\Sigma, \Delta)$ is the smooth local section whose existence is guaranteed by (5.9b). Sard's theorem states that the image of a smooth mapping has measure zero if the dimension of the domain is less than the dimension of the range (cf. Golubitsky and Guillemin [19, p. 31]). Now

$$\begin{aligned} \dim S \times V^{\Delta} &\leq \dim N(\Sigma, \Delta)/N(\Delta) + \dim V^{\Delta} \\ &= \dim N(\Sigma, \Delta) + \dim V^{\Delta} \\ &\quad - \dim N(\Delta) \end{aligned}$$

using (5.9c). Thus, assumption (5.4) implies that $\dim S \times V^{\Delta} < \dim V^{\Sigma}$ and the image of ψ has measure zero. We can cover each stratum S by an at most countable collection of open sets, each having a local section; moreover, there exists a finite number of strata by (5.9a). Thus, taking the union of the images of ψ for each open set and each stratum still yields a set of measure zero. However, that union is just S_{Δ}.

To prove (5.9), note that locally Γ is (analytically) isomorphic to $[\Gamma/N(\Delta)] \times N(\Delta)$. Using this isomorphism and fact (lemma 5.3a) that $N(\Sigma, \Delta)$ is invariant under multiplication on the right by $N(\Delta)$, we see that locally

$$N(\Sigma, \Delta) \simeq N(\Sigma, \Delta)/N(\Delta) \times N(\Delta).$$

Recall that $N(\Sigma, \Delta) = \Phi^{-1}(0)$ for the Φ defined in (5.8). Write Φ locally in the cross-product $[\Gamma/N(\Delta)] \times N(\Delta)$ and let $\psi = \Phi | \Gamma/N(\Delta) \times \{e\}$ where e is the identity in Γ. Note that ψ is a real analytic mapping on $\Gamma/N(\Delta)$ and that $\psi^{-1}(0) = N(\Sigma, \Delta)/N(\Delta)$. Thus, $N(\Sigma, \Delta)/N(\Delta)$ is a real analytic variety and, by Lojasiewicz, is a stratified set. Moreover, one can choose the strata to be the images of the strata of $N(\Sigma, \Delta)$ under the projection π. It is easy to see that (5.9) follows from this local cross-product presentation of $N(\Sigma, \Delta)$. □

We end this section with a lemma which simplifies the calculation of $N(\Sigma, \Delta)$ where Δ is finite. Since most of the subgroups of O(3) are finite this result will prove quite useful in the next section. We define

$$C(\Sigma) = \{ \gamma \in \Gamma | \gamma\sigma = \sigma\gamma \text{ for all } \sigma \in \Sigma \}. \qquad (5.10)$$

Lemma 5.5. Let Σ and Δ be finite subgroups of Γ. Assume Δ is an isotropy subgroup. Then

$$\dim N(\Sigma, \Delta) = \dim C(\Sigma).$$

Proof. Let $s = |\Sigma|$ and let $\Sigma = \{ \sigma_1, \dots, \sigma_s \}$. Define

$$F\colon\ N(\Sigma, \Delta) \to \Delta^s = \underbrace{\Delta \times \cdots \times \Delta}_{s\text{-times}},$$

$$F(\gamma) = (\gamma^{-1}\sigma_1\gamma, \dots, \gamma^{-1}\sigma_s\gamma).$$

Recall that for each $\gamma \in N(\Sigma, \Delta), \gamma\delta\gamma^{-1} \supset \Sigma$. Hence, $\gamma^{-1}\Sigma\gamma \subset \Delta$ and the image of F lies in Δ^s. Moreover, $F(\gamma_1) = F(\gamma_2)$ precisely when $\gamma_2^{-1}\gamma_1$ is in $C(\Sigma)$. Thus, F induces a 1:1 mapping

$$\tilde{F}\colon\ C(\Sigma) \setminus N(\Sigma, \Delta) \to \Delta^s.$$

Since Δ^s is finite so is $C(\Sigma) \setminus N(\Sigma, \Delta)$ and $\dim N(\Sigma, \Delta) = \dim C(\Sigma)$. □

6. The isotropy subgroups of SO(3) and O(3)

We finish this paper by given the full lattice of isotropy subgroups for all of the irreducible representations of SO(3) and O(3). The major work remaining is the calculation of $\dim N(\Sigma, \Delta)$; then proposition 5.3 may be used. These results are given in theorems 6.3 and 6.5. Then all the terms in the inequality (5.4) will have been calculated and it becomes a straightforward, although tedious, task to determine when this inequality is satisfied. The final result is given in theorems 6.6 and 6.8. We begin with a basic lemma.

Lemma 6.1. (a) Each nontrivial element σ in SO(3) lies in a unique torus; that is, a unique conjugate of SO(2);

(b) $\dim C(\mathrm{Z}_n) = 1$;

(c) let Σ be a noncyclic finite subgroup of SO(3). Then $C(\Sigma)$ is finite.

Proof. (a) If σ is in a torus T, then σ commutes with all of T since SO(2) is an abelian group. If σ is contained in two tori T_1 and T_2 then σ commutes with the group generated by T_1 and T_2. Since $\dim(T_1 \cdot T_2) = 2$; it follows that the group spanned by T_1 and T_2 is SO(3). However, no nontrivial element in SO(3) commutes with SO(3).

(b) Since SO(2) is abelian it follows that $C(\mathrm{Z}_n)$ SO(2). Conversely, if $\gamma \mathrm{Z}_n \gamma^{-1} = \mathrm{Z}_n$ then $\gamma \mathrm{SO}(2) \gamma^{-1} = \mathrm{SO}(2)$ since Z_n lies in a unique torus. It follows that $\gamma \in N(\mathrm{SO}(2)) = \mathrm{O}(2)$. So $C(\mathrm{Z}_n) \subset \mathrm{O}(2)$ and $\dim C(\mathrm{Z}_n) = 1$.

(c) Let $\Sigma \neq \mathrm{Z}_n$ be a finite subgroup of SO(3). Since $\Sigma \neq \mathrm{Z}_n$, Σ intersects two distinct tori T_1 and T_2 in nontrivial elements σ_1 and σ_2. After conjugation, if necessary, we may assume that $T_1 = \mathrm{SO}(2)$. Now let γ be in $C(\Sigma)$. By definition, $\gamma\sigma_i\gamma^{-1} = \sigma_i$ and hence $\gamma T_i \gamma^{-1} = T_i$. It follows that γ is in $N(T_1) \cap N(T_2) = \mathrm{O}(2) \cap N(T_2)$ which is a subgroup of O(2). This subgroup is not O(2) or SO(2) since $T_1 \neq T_2$. Thus, $N(T_1) \cap N(T_2)$ is finite and $C(\Sigma) \subset N(T_1) \cap N(T_2)$ is also finite. □

When trying to use proposition 5.3, in particular, when implementing condition (5.4) one needs to compute

$$d(\Sigma, \Delta) \equiv \dim N(\Sigma, \Delta) - \dim N(\Delta). \qquad (6.1)$$

When Σ and Δ are subgroups of SO(3) there is the possibility of confusion over whether $d(\Sigma, \Delta)$ is being computed relative to SO(3) or O(3). However,

$$d(\Sigma, \Delta) = d(\pi(\Sigma), \pi(\Delta)), \qquad (6.2)$$

where the LHS of (6.2) is computed with subgroups of O(3) and the RHS of (6.2) is computed as subgroups of SO(3). To prove (6.2) observe from lemma 2.9 that $\dim N(\Delta) = \dim n(\pi(\Delta))$. Moreover, one can use the direct sum $O(3) = SO(3) \oplus Z_2^c$ to show that

$$N(\Sigma, \Delta) = N(\pi(\Sigma), \pi(\Delta)) \oplus Z_2^c.$$

Thus, $\dim N(\Sigma, \Delta) = \dim N(\pi(\Sigma), \pi(\Delta))$ and (6.2) is proved.

Proposition 6.2. Let Σ and Δ be subgroups of SO(3) with $\Sigma \subsetneqq \Delta \subsetneqq SO(3)$. Then $d(\Sigma, \Delta) = 0$ except for the following:

(A1) $d(\{1\}; \Delta) = 3 - \dim N(\Delta)$,

(A2) $d(Z_2, \Delta) = 1$, if $\Delta = O(2), D_m, T, O$ or I;

(A3) $d(Z_3, \Delta) = 1$, if $\Delta = D_m, T, O$ or I;

(A4) $d(Z_4, \Delta) = 1$, if $\Delta = D_m$ or O;

(A5) $d(Z_5, \Delta) = 1$, if $\Delta = D_m$ or I;

(A6) $d(Z_n, D_m) = 1$.

Proof. We begin by considering the cases when both Σ and Δ are finite. If $\Sigma \neq Z_n$ then lemmas 5.5 and 5.6 imply that $\dim N(\Sigma, \Delta) = 0$. This fact implies that $d(\Sigma, \Delta) = 0$. If $\Sigma = Z_n$ $(n \geq 2)$ then lemmas 5.5 and 6.1 imply that $\dim N(\Sigma, \Delta) = 1$. Since $\dim N(Z_m) = 1$ and $\dim N(\Delta) = 0$ if Δ is not cyclic we obtain $d(Z_n, Z_m) = 0$ and $d(Z_n, \Delta) = 1$ when Δ is not cyclic. The various subgroups Δ which are possible are listed in (A2)–(A6). Note there is the remaining case of $d(Z_2, O(2))$ listed under (A2).

Next, note that $N(\{1\}, \Delta) = SO(3)$ and hence $d(\{1\}, \Delta) = 3\text{-}\dim N(\Delta)$ which yields the last exception (A1).

The infinite subgroups of SO(3) are all conjugate to either SO(2) or O(2). We claim that $\dim N(SO(2), O(2)) = 1$ from which it follows that $d(SO(2), O(2)) = 0$. Suppose that $\gamma \in N(SO(2), O(2))$. Then $\gamma O(2)\gamma^{-1} \supset SO(2)$ There is only one torus in $\gamma O(2)\gamma^{-1}$ we must have $\gamma SO(2)\gamma^{-1} = SO(2)$. Thus, $\gamma \in O(2)$ and $N(SO(2), O(2)) = O(2)$.

We now consider the case when Σ is finite and Δ is infinite. If $\Delta = SO(2)$ then $\Sigma = Z_n$. Using a proof similar to the last one shows that $N(Z_n, SO(2)) = O(2)$ and hence $d(Z_n, SO(2)) = 0$. We now assume that $\Delta = O(2)$. There is one exceptional case $\Sigma = Z_2$ which we consider first. We claim that $\dim N(Z_2, O(2)) = 2$ and hence $d(Z_2, O(2)) = 1$ as claimed in (A2). Note that if $\gamma \in N(Z_2, O(2))$ then $\gamma^{-1} Z_2 \gamma \subset O(2)$. Let σ be the nontrivial element in Z_2. Since σ is contained in exactly one torus SO(2) we see that each $\gamma \in N(Z_2, O(2))$ corresponds to a unique torus $\gamma^{-1} SO(2) \gamma$ which has a non-trivial intersection with O(2). Since the space of tori which intersect O(2) nontrivially is two-dimensional we see that $\dim N(Z_2, O(2)) \leq 2$. One can show that any $\gamma \in SO(3)$ whose axis of rotation lies in the plane of O(2) is in $N(Z_2, O(2))$. The invariance of $N(Z_2, O(2))$ under multiplication by O(2) on the right, lemma 5.4a, shows that $\dim N(Z_2, O(2)) \geq 2$.

We now show that $\dim N(\Sigma, O(3)) = 1$ if Σ is a finite subgroup of SO(3) equal to Z_2. The salient feature of such Σ is the existence of σ_1 and σ_2 in Σ satisfying $\sigma_1, \sigma_2 \neq 1$. Since Σ is finite, there are a finite number of tori which intersect Σ and since all tori are conjugate there exist $\gamma_1, \ldots, \gamma_s$ in Γ such that

$$\Sigma \subset \gamma_1 SO(2) \gamma_1^{-1} \cup \cdots \cup \gamma_s SO(2) \gamma_s^{-1}.$$

We claim that

$$O(2) \subset N(\Sigma, O(2)) \subset \gamma_1 O(2) \cup \cdots \cup \gamma_s O(2). \tag{6.3}$$

Note that the first inclusion follows trivially from lemma 5.2a as $N(\Sigma, O(2)) \supset N(O(2)) = O(2)$. Assuming the validity of (5.3) we see immediately that $\dim N(\Sigma, O(2)) = 1$ as desired.

We now prove that the second inclusion in (6.3) is valid. Let γ be in $N(\Sigma, O(2))$. Then, by definition, $\gamma O(2) \gamma^{-1} \supset \Sigma$. We claim that $\gamma SO(2) \gamma^{-1} \cap \Sigma \neq \{1\}$. Since $\gamma O(2) \gamma^{-1} \supset \Sigma$ it follows that

$$\sigma_i = \gamma \tau_i \gamma^{-1}, \quad i = 1, 2,$$

for some τ_i in O(2). If either τ_i is in SO(2) the claim is satisfied. If both τ_i are in $\mathrm{O}(2) \sim \mathrm{SO}(2)$ then

$$\sigma_1\sigma_2 = \gamma\tau_1\tau_2\gamma^{-1} \in \gamma \mathrm{SO}(2)\gamma^{-1} \cap \Sigma,$$

since $\tau_1\tau_2 \in \mathrm{SO}(2)$. Since we have assumed $\sigma_1\sigma_2 \neq 1$ the claim is valid.

Note that $\gamma\mathrm{SO}(2)\gamma^{-1}$ is a torus which intersects Σ in a nontrivial element. As such, it must equal one of the tori $\gamma_i\mathrm{SO}(2)\gamma_i^{-1}$. This equality implies that $\gamma_i^{-1}\gamma \in N(\mathrm{SO}(2)) = \mathrm{O}(2)$. Thus, $\gamma \in \gamma_i\mathrm{O}(2)$ proving (6.3). □

We now apply propositions 5.3 and 6.2 to $\Gamma = \mathrm{SO}(3)$ and $\Gamma = \mathrm{O}(3)$. Recall the notation $d(\Sigma) = \dim V^{\Sigma}$.

Theorem 6.3. Let Σ be a closed subgroup of SO(3) and assume that SO(3) acts irreducibly on V_l.

(I) If Σ is an isotropy subgroup of SO(3) then

(B1) $d(\Sigma) > 0$; and

(B2) for each proper isotropy subgroup $\Delta \supsetneqq \Sigma$ $d(\Delta) < d(\Sigma)$.

(II) If $\Sigma \neq \mathrm{Z}_n$ then Σ is an isotropy subgroup if conditions (B) are satisfied.

(III) $\{1\}$ is an isotropy subgroup of SO(3) if for every proper isotropy subgroup Δ,

$$d(\Delta) - \dim N(\Delta) < 2l - 2.$$

(IV) Z_n is an isotropy subgroup of SO(3) if (B) and

$$(C)_\Delta \; d(\Delta) + 1 < d(\mathrm{Z}_n)$$

are valid for

$n \geq 6$: $\Delta = \mathrm{D}_{nk}$,

$n = 5$: $\Delta = \mathrm{D}_{5k}$ and I,

$n = 4$: $\Delta = \mathrm{D}_{4k}$ and O,

$n = 3$: $\Delta = \mathrm{D}_{3k}$, O, I and T,

$n = 2$: $\Delta = \mathrm{O}(2)$, D_{3k}, O, I and T.

Proof. The necessity of conditions (B) follow directly from the definition of isotropy subgroups. To prove the sufficiency we must use proposition 5.3 and the formula (5.4) which has the form

$$d(\Delta) + d(\Sigma, \Delta) < d(\Sigma).$$

The computations of proposition 6.2 are now sufficient to prove the remainder of theorem 6.3. We just make two observations. When $\Delta = \Gamma$ then (5.4) is just $d(\Sigma) > 0$. So, we may assume Δ is a proper isotropy subgroup. At the other extreme $\Sigma = \{1\}$. It is obvious that $d(\{1\}) = 2l + 1$, which when coupled with proposition 5.7 (A1) yields part (III). □

Remark. The situation with O(3) is similar. Note that when $-I$ in O(3) acts as the identity on V_l then considerations of isotropy subgroups of O(3) reduce to questions about isotropy subgroups of SO(3). So we now restrict our attention to the irreducible representations of O(3) in which $-I$ acts as minus the identity on V_l.

Proposition 6.4. Let $\Sigma \subsetneqq \Delta$ be subgroups of O(3) with Δ a class III subgroup. Then $d(\Sigma, \Delta) = 0$ except for the following:

(D1) $d(\{1\}, \Delta) = 3 - \dim N(\Delta)$, for all Δ;

(D2) $d(\mathrm{Z}_n, \Delta) = 1 \quad (n \geq 2)$

if $\Delta = \mathrm{O}(2)^-$, $\mathrm{D}_{2m}^{\mathrm{d}}$, $\mathrm{D}_m^{\mathrm{z}}$ or O^-.

$d(\mathrm{Z}_{2n}^-, \Delta) = 1 \quad (n \geq 1)$

Note. We have not listed which subgroups Δ actually contain Z_n or Z_{2n}^- in this proposition.

Proof. Recall (6.2) which states that $d(\Sigma, \Delta) = d(\pi(\Sigma), \pi(\Delta))$. From proposition 6.2 we see that $d(\Sigma, \Delta) = 0$ unless $\pi(\Sigma) = \mathrm{Z}_n$. Hence, we have three possibilities: $\Sigma = \{1\}$, $\Sigma = \mathrm{Z}_n$ or $\Sigma = \mathrm{Z}_{2n}^-$. The first case is simple. Next, note that the class III subgroups Δ must have $\pi(\Delta)$ equal to one of the groups listed in (A2)–(A6). The possibilities are O(2) (for Z_2 and Z_2^-), $\mathrm{D}_{2m}^{\mathrm{d}}$, $\mathrm{D}_m^{\mathrm{z}}$ and O^-. □

Theorem 6.5. Let Σ be a closed subgroup of O(3) and assume that O(3) acts irreducibly on V_l with $-I$ acting as minus the identity on V_l.

(I) If Σ is an isotropy subgroup of O(3) then

(B1) $d(\Sigma) > 0$; and

(B2) for each proper isotropy subgroup $\Delta \supsetneqq \Sigma$ $d(\Delta) < d(\Sigma)$.

(II) If $\Sigma \neq \{1\}$, Z_n, Z_{2n}^- then Σ is an isotropy subgroup if conditions (B) are satisfied.

(III) $\{1\}$ is an isotropy subgroup of O(3) if for every proper isotropy subgroup Δ

$$d(\Delta) - \dim N(\Delta) < 2l - 2.$$

(IV) Z_m is an isotropy subgroup of O(3) if Z_n is an isotropy subgroup of SO(3) and, in addition, conditions (B) and

$$(C)_\Delta \; d(\Delta) + 1 < d(Z_n)$$

are valid for

$n \geq 4$: $\Delta = D_{2nm}^{d}$ or D_{nm}^{z};

$n = 3$: $\Delta = D_{6m}^{d}, D_{3m}^{z}$ or O^-;

$n = 2$: $\Delta = O(2)^-, D_{4m}^{d}, D_{2m}^{z}$ or D^-.

(V) Z_{2n}^- is an isotropy subgroup of O(3) if conditions (B) and $(C)_\Delta$ are valid for

$n = 1$: $\Delta = O(2)^-, D_{2m}^{d}, D_{m}^{z}$ or O^-;

$n = 2$: $\Delta = D_{4(2m+1)}^{d}$ $(m \geq 1)$ or O^-;

$n \geq 3$: $\Delta = D_{2n(2m+1)}^{d}$ $(m \geq 1)$.

Proof. One combines the results of proposition 6.4 and the enumeration of the subgroups of class III groups given in lemma 2.12 and mixes with the statement and proof of theorem 6.3. □

At this point we have all the information we need to determine when a subgroup is an isotropy subgroup. It only remains to compare the dimensions of the fixed points sets calculated in section 4. This process, while rather lengthy, is straightforward. We have two comments to make which may prove helpful. The first is that

$$d(\Sigma) = [2l/|\Sigma|] \text{ or } [2l/|\Sigma|] + 1$$

for every finite Σ. This observation (proven by direct comparison with the formulas for the dimensions of the fixed point sets already obtained) serves to help organize the computations and to prove the result that the exceptional groups are all isotropy subgroups for large l. The second comment concerns the fact that we have not demonstrated the converse of proposition 5.3.

Theorems 6.3 and 6.5 state that this only presents a problem for $\Sigma = Z_n$. In this case the fixed point set can be easily computed explicitly using the weight space decomposition of V_l. In the cases in which (5.4) of proposition 5.3 is not satisfied, it can be easily seen that each vector in the fixed point set of Σ has a larger isotropy subgroup. Thus, (5.4) does give a necessary and sufficient condition for Σ being an isotropy subgroup in O(3).

As the calculations described above are straightforward, we will leave them to the reader. We give the results in theorems 6.6 and 6.8.

Theorem 6.6. The following are the isotropy subgroups of SO(3) acting on V_l with $l > 0$:

(a) 1, for $l \geq 3$;

(b) Z_n $(n \geq 2)$, for $\begin{cases} n \leq l, & \text{when } l \text{ is odd,} \\ n \leq \dfrac{l}{2}, & \text{when } l \text{ is even;} \end{cases}$

(c) D_n $(n \geq 2)$, for $n \leq l$;

(d) T, for $l = 6, 7$ or $l \geq 9$;

(e) O, for $l \neq 1, 2, 5, 7, 11$;

(f) I, for $l = 6, 10, 12, 15, 16, 18$ or $l \geq 20$ with $l \neq 23, 29$;

(g) SO(2), for l odd,

(h) O(2), for l even.

The exceptional groups appear as isotropy subgroups in a somewhat irregular fashion. However, for large l the picture stabilizes.

Corollary 6.7. Let $l \geq 30$. The following are the SO(3) isotropy subgroups:

(a) When l is odd:
  1, T, O, I, SO(2), Z_n $(n \leq l)$, D_n $(n \leq l)$;

(b) when l is even:
  1, T, O, I, O(2), Z_n $(n \leq l/2)$, D_n $(n \leq l)$.

We finish with the isotropy subgroups of O(3).

Theorem 6.8. The following are the isotropy subgroups of O(3) acting on V_l with $l > 0$ (the nontrivial representations):

(a) 1, for $l \geq 3$;

(b) Z_n, for $2 \leq n \leq l/2$;

(c) Z_{2n}^-, for $n \leq l/3$;

(d) D_n, for $\begin{cases} 1 < n \leq l/2, & \text{if } l \text{ is odd,} \\ 1 < n \leq l, & \text{if } l \text{ is even;} \end{cases}$

(e) D_n^z, for $\begin{cases} 1 < n \leq l, & \text{if } l \text{ is odd,} \\ 1 < n \leq l/2, & \text{if } l \text{ is even;} \end{cases}$

(f) D_{2n}^d, for $1 < n < l$ except D_4^d when $l = 3$;

(g) T, for $l \neq 1,2,5,7,8,11$;

(h) O, for $l \neq 1,2,3,5,7,11$;

(i) O^-, for $l \neq 1,2,4,5,8$;

(j) I, for $l = 6, 10, 12, 15, 16, 18$ or $l \geq 20$ with $l \neq 23, 29$;

(k) O(2), for l even;

(l) $O(2)^-$, for l odd.

Corollary 6.9. Let $l > 30$. The following are the O(3) isotropy subgroups.

(a) When l is odd:
1. 1, T, O, O^-, I, $O(2)^-$;
2. Z_n, for $n \leq l/2$;
3. Z_{2n}^-, for $n \leq l/3$;
4. D_n, for $n \leq l/2$;
5. D_n^z, for $n \leq l$;
6. D_{2n}^d, for $n \leq l$;

(b) When l is even:
1. 1, T, O, O^-, I, O(2);
2. Z, for $n \leq l/2$;
3. Z_{2n}^-, for $n \leq l/3$;
4. D_n, for $n \leq l$;
5. D_n^z, for $n < l/2$;
6. D_{2n}^d, for $1 < n \leq l$.

References

[1] G. Cicogna, Symmetry breakdown from bifurcations. Lettere al Nuovo Cimento 31 (1981) 600–602.
[2] F.H. Busse, Pattern of convection in spherical shells, J. Fluid Mech. 72 (1975) 65–85.
[3] F.H. Busse and N. Riahi, Pattern of convection in spherical shells, II, J. Fluid Mech. 123 (1982) 283–391.
[4] D.H. Sattinger, Branching in the presence of symmetry. CBMS-NSF regional conference series in Applied Mathematics 40 (SIAM, New York, 1983).
[5] G.H. Knightly and D. Sather, Buckled states of a spherical shell under uniform external pressure, Arch. Rat. Mech. Anal. 72 (1980) 315–380.
[6] P. Chossat, Bifurcation and stability of convective flows in a rotating or not rotating spherical shell, SIAM J. Appl. Math. 37 (1975) 624–647.
[7] M. Golubitsky, The Bénard problem, symmetry and the lattice of isotropy subgroups, in Bifurcation Theory, Mechanics and Physics. C.P. Borter et al., eds. (Reidel, Dordrecht, 1983) 225–256.
[8] M. Golubitsky, and D. Schaeffer, Bifurcation with O(3) symmetry including applications to the Bénard problem. Communs. Pure. Appl. Math. 35 (1982) 81–111.
[9] L. Michel, Nonlinear Group Action. Smooth action of compact Lie groups on manifolds, in: Statistical Mechanics and Field Theory, R.N. Sen and C. Weil, eds. (Israel Univ. Press, Jerusalem, 1972) pp. 133–150.
[10] F.A. Cotton, Chemical Applications of Group Theory (Interscience, New York, 1971).
[11] T.A. Wolf, Spaces of Constant Curvature (McGraw-Hill, New York, 1967).
[12] L. Michel, Symmetry defects and broken symmetry, configurations. Hidden symmetry, Rev. Mod. Phys. 52 (1980) 617–651.
[13] G.E. Bredon, Introduction to Compact Transformation Groups (Academic Press, New York, 1972).
[14] M. Crandall and P. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, Arch. Rational Mech. Anal. 52 (1973) 161–180.
[15] E. Buzano and M. Golubitsky, Bifurcation on the hexagonal lattice and the planar Bénard problem, Phil. Trans. R. Soc. Lond. A 308 (1983) 617–667.

[16] M. Golubitsky, J. Swift and E. Knobloch, Symmetries and pattern selection in Rayleigh–Bénard convection, Physica 10D (1984) 249.
[17] D. Montgomery and L. Zippin, Topological Transformation Groups (Interscience, New York, 1955).
[18] S. Lojasiewicz, Sur les ensembles semi-analytiques, Actes, Congrés Intern. Math. Nice 2 (1970) 237–241.
[19] M. Golubitsky and V. Guillemin, Stable Mappings and Their Singularities (Springer, New York, 1973).
[20] D.H. Sattinger, Group Theoretic Methods in Bifurcation Theory. Lec. Notes Math. #762 (Springer, New York, 1979).