Proceedings of the Royal Society of Edinburgh, **113A**, 315–345, *1989*

Heteroclinic cycles involving periodic solutions in mode interactions with O(2) symmetry

I. Melbourne

Department of Mathematics, University of Houston, Houston, TX 77204–3746, U.S.A

P. Chossat

I.M.S.P., Université de Nice, Parc Valrose, F-06034 Nice Cedex, France

and

M. Golubitsky

Department of Mathematics, University of Houston, Houston, TX 77204-3746, U.S.A.

(MS received 11 November 1988. Revised MS received 10 April 1989)

Dedicated to Professor Jack K. Hale on the occasion of his 60th birthday

Synopsis

In this paper we show that in O(2) symmetric systems, structurally stable, asymptotically stable, heteroclinic cycles can be found which connect periodic solutions with steady states and periodic solutions with periodic solutions. These cycles are found in the third-order truncated normal forms of specific codimension two steady-state/Hopf and Hopf/Hopf mode interactions.

We find these cycles using group-theoretic techniques; in particular, we look for certain patterns in the lattice of isotropy subgroups. Once the pattern has been identified, the heteroclinic cycle can be constructed by decomposing the vector field on fixed-point subspaces into phase/amplitude equations (it is here that we use the assumption of normal form). The final proof of existence (and stability) relies on explicit calculations showing that certain eigenvalue restrictions can be satisfied.

1. Introduction

Let $S_1, \ldots, S_k$ be flow invariant sets for some fixed system of ODEs. A *heteroclinic cycle* of the S_js is a collection of trajectories $x_j(t)$ ($j = 1, \ldots, k$) such that each x_j is asymptotic to S_{j+1} as $t \to +\infty$ and S_j as $t \to -\infty$. (Here we use the convention $S_{k+1} = S_1$.) Typically one does not expect heteroclinic cycles to exist for general systems.

For symmetric systems however, Field [7] has shown that heteroclinic cycles between saddle points can be structurally stable. More recently, Guckenheimer and Holmes [15] have shown that a primary branch of asymptotically stable heteroclinic cycles can appear in steady-state bifurcation in the presence of $\mathbb{T} \oplus \mathbb{Z}_2$ symmetry, where $\mathbb{T}$ is the twelve-element group of orientation-preserving symmetries of the tetrahedron. (In fact, this bifurcation was noted earlier by Busse and Clever [3] and Busse and Heikes [4] in the specific context of rotating Rayleigh–Bernard convection. The bifurcation was also noted by May and Leonard [21] in the population dynamics of three competing species. See also

Swift [**26**].) Finally, Field and Richardson [**10**] have shown the existence of primary bifurcations to branches of heteroclinic cycles for a number of different finite symmetry groups. It is worth noting that structurally stable connections in symmetric systems sometimes have implications for systems of ODEs without symmetry. Melbourne [**22**] shows that a variant of Guckenheimer and Holmes' example implies the existence of heteroclinic type behaviour of a cycle between three distinct periodic orbits in systems with three parameters.

Symmetric systems have the property that if S is a flow invariant set, then so is the set γS for any γ in the group of symmetries. We say that S and γS are *conjugate*. When attempting to enumerate the types of dynamic behaviour possible in symmetric systems, it seems appropriate to identify conjugate attractors, since these conjugate attractors all have the same (dynamic) properties. With this in mind we make the following definition.

Definition 1.1. Let $S_1, \ldots, S_{k+1}$ be flow invariant sets for some fixed system of ODEs with symmetry group Γ where S_{k+1} is conjugate to S_1. A *heteroclinic cycle* of the S_js is a collection of trajectories $x_j(t)$ $(j = 1, \ldots, k)$ where x_j connects S_j to S_{j+1}.

Remark 1.2. If the group Γ is finite, then the heteroclinic cycle will eventually return to the given S_1, not just a conjugate S_1.

In another direction, Jones and Proctor [**17, 23**] (in the context of the Bénard problem) and Armbruster, Guckenheimer and Holmes [**2**] have shown the existence of structurally stable, asymptotically stable, secondary branches of heteroclinic cycles connecting saddle points in 2:1 resonant steady-state/steady-state mode interactions with O(2) symmetry. In the latter reference, these solutions were then used to describe certain phenomena in the Kuramoto–Sivashinsky equation (with periodic boundary conditions) observed numerically by Kevrekidis, Nicolaenko and Scovel [**18**]. Chossat and Armbruster have found heteroclinic cycles in the corresponding mode interaction with O(3) symmetry [**1**].

In this paper, we continue the study of heteroclinic cycles in mode interactions with O(2) symmetry. In Section 3, we show that steady-state/Hopf mode interactions can produce heteroclinic cycles connecting the primary branches of 2-tori foliated by periodic standing waves and circles of steady states. The steady-state and periodic solutions of such systems are described in [**13**]. (See also [**14**].) In Section 4, we show that Hopf/Hopf mode interactions with O(2)-symmetry can produce heteroclinic cycles connecting two branches of rotating waves and cycles connecting two branches of 2-tori of standing waves. Cycles connecting these branches of rotating waves and standing waves are also possible. The basic structure of these Hopf/Hopf mode interactions is discussed in [**6**]. Further examples of connections between limit cycles may be found in work of Armbruster [**1**].

The basic idea in our analysis is that certain structures in the lattice of isotropy subgroups lead one to suspect the existence of heteroclinic cycles. We discuss these structures in Section 2, along with a general condition which is sufficient to prove asymptotic stability for heteroclinic cycles. The actual computations verifying the existence of heteroclinic cycles in the two mode interactions we consider is based on the phase-amplitude decomposition of the centre manifold, normal form, vector field. The computations for steady-state/Hopf and Hopf/Hopf mode interactions will be presented in Sections 3 and 4, respectively.

2. Heteroclinic connections and the lattice of isotropy subgroups

Let

$$\frac{dx}{dt}=f(x), \quad x \in \mathbb{R}^n \tag{2.1}$$

be an Γ-equivariant system of ODEs, where $\Gamma \subset \mathrm{O}(n)$ is a compact Lie group. Thus

$$f(\gamma x) = \gamma f(x) \tag{2.2}$$

for all $\gamma \in \Gamma$. The feature of symmetric systems that allows the existence of structurally stable heteroclinic cycles is the existence of flow invariant subspaces. These so-called *fixed-point subspaces* are found as follows. Let $\Sigma \subset \Gamma$ be a subgroup; then

$$\mathrm{Fix}\,(\Sigma) = \{y \in \mathbb{R}^n\colon \sigma y = y \;\forall \sigma \in \Sigma\} \tag{2.3}$$

is flow invariant for (2.1). More precisely, (2.2) implies

$$f\colon \mathrm{Fix}\,(\Sigma) \to \mathrm{Fix}\,(\Sigma). \tag{2.4}$$

Typically, we apply (2.4) to those Σ which are *isotropy* subgroups, that is, subgroups of the form $\Sigma_y = \{\sigma \in \Gamma\colon \sigma y = y\}$ for some fixed $y \in \mathbb{R}^n$.

To understand how (2.4) can force the existence of a heteroclinic cycle, imagine the following situation. There exist two nonzero saddle points A and B for (2.1) and two subgroups Σ and T such that

$$A,\, B \in \mathrm{Fix}\,(\Sigma) \cap \mathrm{Fix}\,(T), \tag{2.5a}$$

$$\dim \mathrm{Fix}\,(\Sigma) = 2 = \dim \mathrm{Fix}\,(T), \tag{2.5b}$$

$$\mathrm{Fix}\,(\Sigma) \neq \mathrm{Fix}\,(T). \tag{2.5c}$$

See Figure 2.1. Imagine that in the plane $\mathrm{Fix}\,(\Sigma)$: A is a saddle, B is a sink and

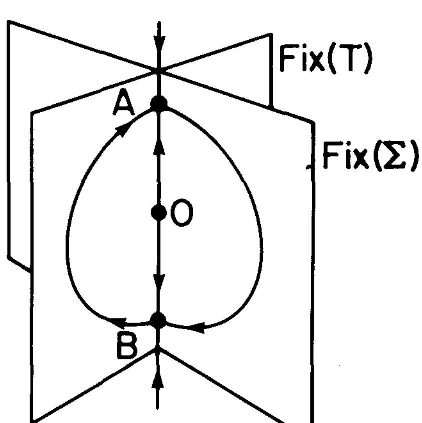

Figure 2.1. Heteroclinic cycle between two equilibria.

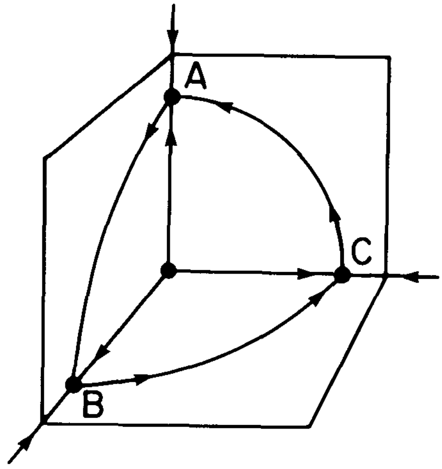

Figure 2.2. Heteroclinic cycle between three equilibria.

the unstable manifold of the saddle A connects to the sink B. Simultaneously, it is possible that in the plane $\mathrm{Fix}\,(T)$: B is a saddle, A is a sink and the unstable manifold of B connects B to A. In this way one can construct a heteroclinic cycle connecting A to B and back to A. This cycle is structurally stable since equivariant perturbations of f will still have $\mathrm{Fix}\,(\Sigma)$ and $\mathrm{Fix}\,(T)$ as invariant planes and saddle sink connections are structurally stable in $\mathbb{R}^2$. This situation is precisely the one found in the 2:1 resonant steady-state/steady-state O(2) mode interaction example.

Indeed, one can imagine longer strings of saddles connected through more complicated configurations of fixed-point subspaces. The Guckenheimer–Holmes example is based on a connection sequence like that pictured in Figure 2.2.

The proofs of the existence of these heteroclinic cycles in bifurcation problems proceed along the following lines. Truncate the general f satisfying (2.2) at lowest nontrivial order (usually third order). Find the cycle in the truncated system and use structural stability arguments to show that the cycle persists for general f. This argument seems straightforward enough. There is, however, a simple obstruction to finding heteroclinic cycles in the truncated system – the truncated system may be forced by symmetry to be a gradient system, and heteroclinic cycles cannot occur in gradient systems. In fact, the only symmetry group which admits this construction of a primary branch of heteroclinic cycles in a generic one-parameter bifurcation problem in three dimensions is the group $\mathbb{T} \oplus \mathbb{Z}_2$ considered by Guckenheimer and Holmes. In two-parameter systems (where mode interactions are generic possibilities) and in higher dimensional systems, more examples can be constructed.

We now abstract certain properties of the cycles connecting saddle points pictured in Figures 2.1 and 2.2:

(i) each saddle sits on a flow invariant line and each such line is the fixed-point subspace for the isotropy subgroup of that saddle,

(ii) the isotropy subgroups of the invariant lines are all maximal isotropy subgroups,

(iii) the invariant plane containing the invariant line is the fixed-point subspace of a submaximal isotropy subgroup,

where we define inclusion with respect to the *lattice* of isotropy subgroups. This lattice consists of conjugacy classes of isotropy subgroups ordered by inclusion as follows:

$$\Sigma_1 < \Sigma_2 \quad \Leftrightarrow \quad \Sigma_1 \text{ is contained in some conjugate of } \Sigma_2. \tag{2.6}$$

Group theoretically we can identify a certain class of heteroclinic cycles.

DEFINITION 2.1. A *homoclinic cycle* is a heteroclinic cycle connecting saddle points whose isotropy subgroups are all conjugate.

The examples given in [**2**] and [**15**] are examples of homoclinic cycles.

Next we discuss when a connection between two saddles on invariant lines is possible.

DEFINITION 2.2. Two flow invariant lines l_1 and l_2 are *adjacent* inside the invariant plane P if there exists a wedge region W as in Figure 2.3(a) such that no invariant line, besides l_1 and l_2, intersects W.

Remark 2.3. In this definition, we allow the possibility that $l_1 = l_2 = l$ (see Figure 2.3(b)). In this case W is a half plane.

We address the following question. Suppose we are given two adjacent, flow-invariant lines $l_j = \mathrm{Fix}\,(\Sigma_j)$ contained in a flow-invariant plane $P = \mathrm{Fix}\,(T)$ where Σ_1, Σ_2 and T are isotropy subgroups. When do there exist saddle points in l_1 and l_2 and a trajectory in P connecting these saddles? Moreover, we pose this question in the sense of bifurcation theory. Thus we assume that (2.1) depends explicitly on a parameter λ; that is,

$$\frac{dx}{dt} = f(x, \lambda), \tag{2.7}$$

where $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is Γ-equivariant. We also assume that (2.7) has a trivial solution ($f(0, \lambda) \equiv 0$) and undergoes a steady-state bifurcation at $\lambda = 0$; that is,

$$\left.\begin{aligned} &\mathrm{Fix}\,(\Gamma) = \{0\}, && (2.8a)\\ &(df)_{0,0} = 0. && (2.8b)\end{aligned}\right\}$$

We begin our discussion by making assumptions which imply the existence of the desired equilibria. Since $\dim \mathrm{Fix}\,(\Sigma_j) = 1$ the equivariant branching lemma [**14**] implies that there exists a unique branch of equilibria bifurcating from the

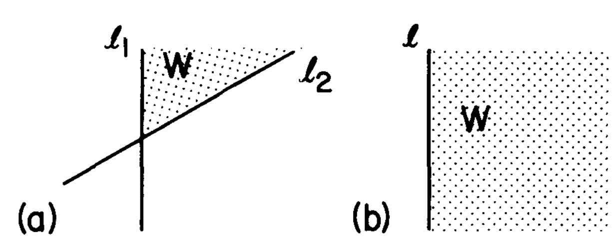

Figure 2.3. Wedge region of Definition 2.2.

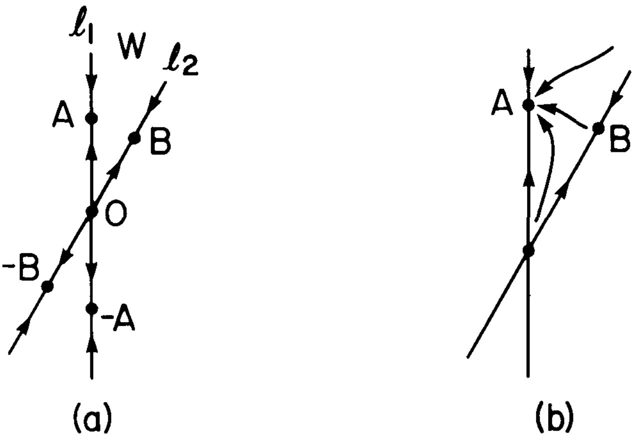

Figure 2.4. Dynamics in a wedge region.

origin in l_j as long as

$$\frac{\partial}{\partial\lambda}(df)_{0,\lambda}(v_j)|_{\lambda=0} > 0, \tag{2.9}$$

where $v_j \in l_j$ is nonzero. We assume $>$ rather than $\neq$ in (2.9) to ensure that the trivial solution is subcritically asymptotically stable in l_j. Next we assume that

$$N(\Sigma_j)/\Sigma_j \cong \mathbb{Z}_2, \tag{2.10}$$

where $N(\Sigma)$ is the normaliser of Σ in Γ. Assumption (2.10) implies that the bifurcations in l_j are pitchfork bifurcations. Finally, we assume

$$(d^3f)_{0,0}(v_j, v_j, v_j) < 0, \tag{2.11}$$

so that the pitchfork bifurcations are supercritical. Thus, in P, the dynamics for $\lambda > 0$ are as pictured in Figure 2.4(a).

To ensure that there is a trajectory connecting B to A, we need to assume

there are no equilibria inside W (that is, no equilibria with submaximal isotropy Δ in W) (2.12a)

and

orbits of $f(\cdot, \lambda)$ remain bounded inside W. (2.12b)

Then the Poincaré–Bendixson theorem implies dynamics like that pictured in Figure 2.4(b). (Of course, the connecting trajectory could go from A to B.)

In fact, when the action of Γ is absolutely irreducible and there do not exist quadratic Γ-equivariant mappings, conditions (2.12) can be replaced by conditions on coefficients in the Taylor expansion of $f(x, 0)$. Field [**8**] shows that conditions on d^3f consistent with (2.11) imply the existence of an attracting flow-invariant $(n-1)$-sphere in $\mathbb{R}^n$; thus, trajectories starting near 0 in W remain bounded in W. A more complicated argument of Field and Richardson [**9**] shows that the (generic) existence of submaximal solutions in P depends on equations involving the nonidentity generators of the module of Γ-equivariant mappings. This implies, in particular, that generically the existence of equilibria in W is determined by inequalities in a finite number of Taylor coefficients which are independent of those needed to ensure the existence of bounded trajectories.

At this stage, we assume

$$\text{conditions on } d^3f \text{ ensuring the existence of a flow-invariant } (n-1)\text{-sphere.} \tag{2.13}$$

PROPOSITION 2.4. *Let l_1 and l_2 be adjacent flow-invariant lines in a flow-invariant plane P and let $W \subset P$ be the wedge between l_1 and l_2. Assume that Γ and the system of ODEs* (2.7) *satisfy* (2.8)–(2.11) *and* (2.13). *Finally assume that*

$$\Sigma_1 \text{ and } \Sigma_2 \text{ are conjugate isotropy subgroups.} \tag{2.14}$$

Then (for an open set of coefficients in a truncated Taylor expansion of f) either there exist equilibria of (2.7) *inside W (with submaximal isotropy Δ) or there exist homoclinic cycles in* (2.7) *connecting saddles in l_1 and l_2.*

Proof. In our discussion above, generically, either the conditions of Field and Richardson **[9]** imply the existence of equilibria with submaximal isotropy in W or (2.12) is satisfied. In the latter case, a trajectory connecting A with B exists.

To complete our proof, we need only show that the equilibria A and B lie on the same group orbit. By (2.14), there exists $\gamma \in \Gamma$ such that $\gamma B = A$ or $\gamma B = -A$. Assumption (2.10), however, implies that $\delta \in N(\Sigma_1) \sim \Sigma_1$ satisfies $\delta A = -A$. Thus, in the second case $\delta\gamma B = A$. So A and B are on the same group orbit and the homoclinic cycle between saddles exists. □

Remark 2.5. In the example given in **[15]**, both homoclinic cycles and submaximal branches may exist, depending on the coefficients of f at third order. This is consistent with Proposition 2.4.

Proposition 2.4 can be extended to heteroclinic cycles in conceptually a straightforward manner. Suppose that $l_1, \ldots, l_{k+1}$ are flow invariant lines with isotropy $\Sigma_1, \ldots, \Sigma_{k+1}$ and suppose that Σ_{k+1} is conjugate to Σ_1. Suppose that l_j and l_{j+1} are adjacent lines in a flow invariant plane $P_j = \mathrm{Fix}\,(T_j)$ and that there do not exist equilibria in P_j with submaximal isotropy T_j. Finally suppose (2.8)–(2.10) and (2.13) are valid. Then there exist trajectories connecting l_1 and l_2, l_2 and $l_3, \ldots, l_k$ and l_{k+1}. Of course one must show that these connections can be made in the correct directions l_1 to l_2, etc., to prove the existence of the heteroclinic cycle.

In subsequent sections, we show how such heteroclinic cycles can be constructed explicitly in the normal forms of steady-state/Hopf and Hopf/Hopf mode interactions with O(2) symmetry. Some of the details will differ from our description of heteroclinic cycles given here, since we construct connections between steady-state and periodic solutions. Nevertheless, the essence of the argument remains the same: should there exist a part of the lattice of isotropy subgroups, as in Figure 2.5, then the possibility for a heteroclinic cycle exists. The proof of the existence of such a cycle requires explicit control over the equilibria in $\mathrm{Fix}\,(\Sigma_j)$ and $\mathrm{Fix}\,(T_j)$ and the boundedness of orbits.

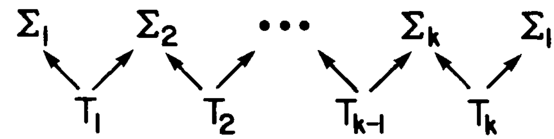

Figure 2.5. Structure within the isotropy lattice suggesting possibility of a structurally stable heteroclinic cycle.

One technical difficulty that we find when considering mode interactions is that the action of Γ will be reducible; hence Field's results on boundedness of trajectories will not apply. Fortunately, we do not need the full generality of the conclusion to Field's theorem to assert the existence of bounded orbits. In Proposition 2.6 below, we derive sufficient conditions for showing that the flow in certain two-dimensional fixed point subspaces (of the type occurring in the examples of later sections) remains bounded. As in [**8**], these conditions depend only on terms to degree three in the Taylor expansion of f.

PROPOSITION 2.6. *Consider the system of ODE*

$$\left.\begin{aligned} \frac{dx}{dt} &= (a_1\lambda + b_1x^2 + c_1y^2)x, \\ \frac{dy}{dt} &= (a_2\lambda + c_2x^2 + b_2y^2)y, \end{aligned}\right\} \tag{2.15}$$

where $a_1,\ a_2>0$ *and* $b_1,\ b_2<0$. *Then all trajectories starting within a circle of radius* $O(\sqrt{\lambda})$ *stay bounded near the origin if*:

$$C \equiv \frac{c_1a_2}{b_2a_1} + \frac{c_2a_1}{b_1a_2} > -2 \tag{2.16}$$

Remarks 2.7. Equation (2.15) has other noteworthy properties:

(a) If the equilibria on the axes are a pair of saddles and a pair of sinks, then there are no equilibria off the axes.

(b) If $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, there are invariant lines $x = \pm y$, each with two equilibria corresponding to maximal isotropy subgroups. In this case, generically, there are no submaximal equilibria in this plane. Moreover, if all four maximal branches of equilibria bifurcate supercritically, then condition (2.16) is automatically satisfied.

Proof. Rescale (2.15) so that $b_1 = -a_1$ and $b_2 = -a_2$ and consider the Liapunov function $N = (x^2/a_1 + y^2/a_2)/2$. To see that N is a Liapunov function, calculate $dN/dt = \lambda(x^2+y^2) - (x^2+y^2)^2 + (2-C)x^2y^2$. If $2-C<0$, then $dN/dt<0$ when $\lambda < x^2+y^2$, as desired. If $2-C>0$, then we use the identity $x^2y^2 \leqq (x^2+y^2)^2/4$ to show that $dN/dt \leqq (x^2+y^2)\{\lambda - (2+C)(x^2+y^2)/4\}$. Thus $dN/dt<0$ when $\lambda < (2+C)(x^2+y^2)/4$, which proves the proposition upon assuming (2.16). □

We remark that it is possible for heteroclinic cycles such as those pictured in Figures 2.1 and 2.2 to be asymptotically stable. This point will be addressed below in Theorem 2.10, but only after we have discussed more general heteroclinic cycles.

2.1. Heteroclinic cycles involving periodic solutions

In this section, we concentrate on homoclinic cycles and heteroclinic cycles connecting equilibria lying in invariant planes. Now reconsider Figure 2.5, but suppose that the fixed-point subspaces of the depicted isotropy subgroups are of arbitrary dimension. It is still the case that if there is an attractor S_i in each $\mathrm{Fix}(\Sigma_i)$ which is attracting in $\mathrm{Fix}(T_{i-1})$, but repelling in the transverse directions to $\mathrm{Fix}(\Sigma_i)$ in $\mathrm{Fix}(T_i)$, then there is the possibility of a heteroclinic cycle connecting the sets S_i.

The general possibility of the existence of such a cycle is analogous to the cases already considered in this section. However, the problem of actually finding sufficient conditions for the cycle to exist is more difficult. Our method for establishing heteroclinic connections relied on the two-dimensional Poincaré–Bendixson theory and it is this theory that we can extend, in certain instances, to higher dimensions.

As we have seen in dimension two, symmetry permits the existence of structurally stable heteroclinic cycles by forcing certain subspaces to be invariant under the flow of an equivariant vector field. In higher-dimensional subspaces, symmetry can also simplify the flow on a flow-invariant subspace; it is this observation that we use in the examples of Sections 3 and 4. (We note that it may be the case that an equivariant version of the Poincaré–Bendixson theory **[20]** could be applied to give general results; in our examples, however, a more straightforward approach applies.) In these examples, we find that there is, in each three- or four-dimensional $\mathrm{Fix}(T_i)$ which we consider, a two-dimensional subspace which intersects every group orbit in $\mathrm{Fix}(T_i)$. Moreover, the vector field restricted to $\mathrm{Fix}(T_i)$ decouples into phase-amplitude equations, and the dynamics in $\mathrm{Fix}(T_i)$ are determined by the dynamics of the amplitude equations on the two-dimensional cross-section. The flow-invariant sets S_i in our examples are either periodic solutions or (group orbits of) equilibria. These both reduce to isolated equilibria of the (two-dimensional) amplitude equations and the methods of this section are then applicable.

The theorem on asymptotic stability rests on information about the flow in a neighbourhood of each flow-invariant set S_i, whereas the phase-amplitude reductions used in this paper only hold in a proper subspace of this neighbourhood. However, as we now describe, we have recourse to a more general decomposition of the vector field, analogous to, although less explicit than, the decoupling into phase-amplitude equations. This decomposition holds in a full neighbourhood of S_i.

Krupa **[19]** shows that if S is a group orbit, then in a neighbourhood of S, the vector field f can be decomposed as $f = f_N + f_T$, where both f_T and f_N are equivariant, f_T is tangential to group orbits, and f_N is transverse to group orbits. Moreover, the dynamics of f may be understood as the dynamics of f_N coupled with drift along group orbits. In particular, if the group orbit S is invariant under the flow of f, then it corresponds to an equilibrium of f_N and the (orbital) asymptotic stability of S is given by the asymptotic stability of this equilibrium.

Definition 2.8. The flow-invariant set S is a *relative equilibrium* if S is a group orbit under the action of Γ. The *isotropy subgroup* Σ of S consists of all group elements in Γ which fix S pointwise. We call S *hyperbolic* if $f_N \mid \{\text{normal fibres}\}$ has hyperbolic zeros at points in S.

Remarks 2.9. (a) If S has isotropy Σ, then by definition $S \subset \mathrm{Fix}(\Sigma)$.

(b) Our examples of relative equilibria in Sections 3 and 4 involve periodic solutions obtained through (generalised) Hopf bifurcation. In Birkhoff normal form, these solutions lie on group orbits given by the phase shift S^1 introduced in the normal form, and it is this normal form that we study.

THEOREM 2.10. *Suppose that there exists a structurally stable heteroclinic cycle connecting hyperbolic relative equilibria S_j and that the connecting trajectories lie in fixed-point subspaces P_j. For $x_j \in S_j$, suppose further that*:

(a) $(df_N)_{x_j}$ *has precisely three eigenvalues corresponding to directions in* $P_{j-1} + P_j$, *and*

(b) *these eigenvalues have real parts* $a_j < b_j < 0$ *and* $c_j > 0$.

Suppose that the eigenspace of $(df_N)_{x_j}$ corresponding to c_j (possibly multiple due to symmetry) is contained in the stable manifold of S_{j+1}. Let μ_j denote the maximum of the real parts of the remaining eigenvalues of $(df_N)_{x_j}$ which are not forced by the group action to equal c_j. If the conditions

(c) $\mu_j < 0$ *for each* j, *and*

(d) $\prod_{j=1}^{k} \min(-b_j, c_j - \mu_j) > \prod_{j=1}^{k} c_j$,

hold, then the heteroclinic cycle is generically asymptotically stable.

A proof of this theorem including a discussion of genericity is outlined in the Appendix (Section 5). We note that this theorem includes the intuitively plausible case when, at each node S_j, the contracting eigenvalues in P_{j-1} (a_j and b_j) are stronger than the expanding eigenvalue in P_j (c_j), and that the remaining eigenvalues are contracting.

The hypotheses of this theorem are sufficient but not necessary. We have striven for a generality that includes the variety of examples considered in this paper and in the references cited in this paper. However, for many of the examples in these references, it is known that asymptotic stability does not depend on the value of the contracting eigenvalue in $P_{j-1} \cap P_j$. It may also be possible that condition (c) can be weakened.

3. O(2) steady-state/Hopf mode interactions

In two-parameter families of vector fields, one may expect to find points where a steady state loses stability by having eigenvalues of the linearised equation simultaneously at 0 and $\pm\omega i$. In systems with symmetry, this so-called *codimension two* point is further complicated by the fact that these eigenvalues may each be multiple. In this section, we show the existence of heteroclinic cycles in the unfolding of certain codimension two singularities in the presence of O(2) symmetry. Unlike our discussion in the previous section, these cycles will connect equilibria with periodic solutions. Through the use of phase-amplitude equations, our analysis, however, will follow the discussion in Section 1. We assume that the reader is familiar with the discussion of such singularities given in [**14**, XX, Section 2] and begin by briefly reviewing the necessary background material.

We assume that the system of ODEs

$$\frac{dx}{dt} = F(x, \lambda, \mu), \quad x \in \mathbb{R}^N \tag{3.1}$$

is O(2)-equivariant and that $x = 0$ is a 'trivial' equilibrium. We also assume that the Jacobian $(d_xF)_{0,0,0}$ has eigenvalues 0, $\pm\omega i$ each of which is double (due to the O(2)-symmetry).

Under these assumptions on (3.1), we may perform a centre manifold reduction, arriving at a system of ODEs

$$\frac{dz}{dt} = g(z, \lambda, \mu), \quad z \in \mathbb{C}^3, \tag{3.2}$$

where $g(0, \lambda, \mu) \equiv 0$ and $(d_zg)_{0,0,0}$ has double eigenvalues at 0 and $\pm\omega i$. One can always choose coordinates $z = (z_0, z_1, z_2)$ such that the O(2)-action has the form

$$\phi \,.\, (z_0, z_1, z_2) = (e^{ki\phi}z_0, e^{li\phi}z_1, e^{-li\phi}z_2) \quad \phi \in \mathrm{SO}(2), \tag{3.3a}$$

$$\kappa \,.\, (z_0, z_1, z_2) = (\bar{z}_0, z_2, z_1), \tag{3.3b}$$

where k and l are positive, coprime integers. Note that the z_0-coordinate corresponds to the eigenvectors associated with the zero eigenvalue and leads to steady-state solutions, while (z_1, z_2) corresponds to the $\pm\omega i$ eigenvalues and leads to periodic solutions. Although our investigation could, in principle, be carried out for all k and l, this would be a somewhat tedious exercise. Thus we study here only the case $k = l = 1$, which occurs for instance in the Taylor–Couette system **[11, 13]**.

Next we assume that (3.2) is transformed into Poincaré–Birkhoff normal form, arriving at the system

$$\frac{dz}{dt} = f(z, \lambda, \mu) + \ldots, \tag{3.4}$$

where f is the normal form of g. As shown in **[12]**, 'normal form' in this instance may be formulated as: f is also SO(2)-equivariant where the (phase shift) SO(2) acts by

$$\theta \,.\, (z_0, z_1, z_2) = (z_0, e^{i\theta}z_1, e^{i\theta}z_2). \tag{3.5}$$

(We remark that the symmetry group of the Taylor–Couette system is actually $\mathrm{O}(2) \times \mathrm{SO}(2)$, and thus the centre manifold vector field g is automatically in normal form as in (3.4).)

Our demonstration of the existence of heteroclinic cycles will be for the normal form system

$$\frac{dz}{dt} = f(z, \lambda, \mu), \quad z \in \mathbb{C}^3, \tag{3.6}$$

where f is $\mathrm{O}(2) \times \mathrm{SO}(2)$-equivariant. The equivariance is generated by

$$(\phi, \theta) \,.\, (z_0, z_1, z_2) = (e^{i\phi}z_0, e^{i(\theta+\phi)}z_1, e^{i(\theta-\phi)}z_2), \tag{3.7a}$$

$$\kappa \,.\, (z_0, z_1, z_2) = (\bar{z}_0, z_2, z_1). \tag{3.7b}$$

Now the general O(2) × SO(2)-equivariant mapping has the form

$$f(z,\lambda,\mu)=(c^1+i\delta c^2)\begin{bmatrix} z_0\\ 0\\ 0\end{bmatrix}+(c^3+i\delta c^4)\begin{bmatrix}\bar z_0 z_1\bar z_2\\ 0\\ 0\end{bmatrix}$$
$$+P^1\begin{bmatrix}0\\ z_1\\ z_2\end{bmatrix}+P^2\delta\begin{bmatrix}0\\ z_1\\ -z_2\end{bmatrix}+P^3\begin{bmatrix}0\\ z_0^2 z_2\\ \bar z_0^2 z_1\end{bmatrix}+P^4\delta\begin{bmatrix}0\\ z_0^2 z_2\\ -\bar z_0^2 z_1\end{bmatrix} \tag{3.8}$$

where $\delta=|z_2|^2-|z_1|^2$ and $c^j\in\mathbb{R}$, $P^j\in\mathbb{C}$ are functions of

$$\rho=|z_0|^2,\quad N=|z_1|^2+|z_2|^2,\quad \Delta=\delta^2,\quad \Phi=\mathrm{Re}\,(A),\quad \Psi=\delta\,\mathrm{Im}\,(A) \tag{3.9}$$

and the parameters λ, μ. Here $A=z_0^2\bar z_1 z_2$. Note that the eigenvalue structure of F leads to:

$$c^1(0)=0, \tag{3.10a}$$

$$P^1(0)=\omega i. \tag{3.10b}$$

Finding heteroclinic cycles in (3.6), even when using the explicit description of f in (3.8), is not easy. However, our task is made easier by help from the lattice of isotropy subgroups. This lattice is given in Figure 3.1. Observe that this lattice suggests a possible heteroclinic cycle connecting (steady states with) isotropy $\mathbb{Z}_2(\kappa)\times SO(2)$ to (periodic solutions with) isotropy $\mathbb{Z}_2(\kappa)\oplus\mathbb{Z}_2^c$ with a connecting trajectory going one way through isotropy $\mathbb{Z}_2(\kappa)$, and the other way through isotropy $\mathbb{Z}_2(\kappa\,.\,(\pi,\pi))$. The various subgroups are defined as follows:

$$\mathbb{Z}_2(\kappa)=\{1,\kappa\}, \tag{3.11a}$$

$$\mathbb{Z}_2^c=\{1,(\pi,\pi)\}, \tag{3.11b}$$

$$\mathbb{Z}_2(\kappa\,.\,(\pi,\pi))=\{1,\kappa\,.\,(\pi,\pi)\}, \tag{3.11c}$$

$$\tilde{S}\tilde{O}(2)=\{(\phi,-\phi)\colon \phi\in SO(2)\}. \tag{3.11d}$$

In Table 3.1, we list the four relevant isotropy subgroups, their fixed-point subspaces and the restriction of equation (3.6) to each of these subspaces. The restricted equations can be further simplified by observing that on the fixed-point

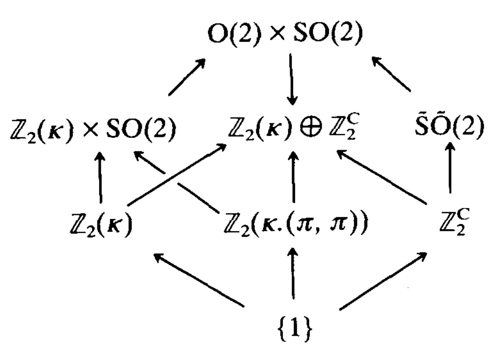

Figure 3.1. Lattice of isotropy subgroups of O(2) × SO(2) acting on $\mathbb{C}^3$.

TABLE 3.1
Restricted equations on Fix (Σ)

Isotropy subgroup	Fixed-point subspace	Restricted equations
$\mathbb{Z}_2(\kappa)\times \mathrm{SO}(2)$	$(x, 0, 0)$	$\dfrac{dx}{dt}=c^1x$
$\mathbb{Z}_2(\kappa)\oplus\mathbb{Z}_2^c$	$(0, z_1, z_1)$	$\dfrac{dz_1}{dt}=P^1z_1$
$\mathbb{Z}_2(\kappa)$	(x, z_1, z_1)	$\dfrac{dx}{dt}=(c^1+c^3\lvert z_1\rvert^2)x$
		$\dfrac{dz_1}{dt}=(P^1+P^3x^2)z_1$
$\mathbb{Z}_2(\kappa\,.\,(\pi, \pi))$	(iy, z_1, z_1)	$\dfrac{dy}{dt}=(c^1-c^3\lvert z_1\rvert^2)y$
		$\dfrac{dz_1}{dt}=(P^1-P^3y^2)z_1$

subspaces the complex parts of these equations decouple into amplitude/phase equations. Specifically, if we write $z_1=re^{i\Phi}$ then the equations in the amplitude variables are those given in Table 3.2. There

$$p^1=\mathrm{Re}\,P^1 \quad\text{and}\quad p^3=\mathrm{Re}\,P^3. \tag{3.12}$$

Note that zeros (x, r) of the amplitude equations correspond to steady states of (3.6) if $r=0$, and periodic solutions of (3.6) if $r\neq 0$.

The equations in Table 3.2 are now in the form discussed in Section 2. The 'effective' dimensions of the fixed-point subspaces of $\mathbb{Z}_2(\kappa)\times\mathrm{SO}(2)$ and $\mathbb{Z}_2(\kappa)\oplus\mathbb{Z}_2^c$ are one and the 'effective' dimensions of the fixed-point subspaces of $\mathbb{Z}_2(\kappa)$ and $\mathbb{Z}_2(\kappa\,.\,(\pi, \pi))$ are two. Observe that the flow-invariant lines $(x, 0)$ and $(0, r)$ are adjacent in $\mathrm{Fix}\,(\mathbb{Z}_2(\kappa))$ while $(y, 0)$ and $(0, r)$ are adjacent in $\mathrm{Fix}\,(\mathbb{Z}_2(\kappa\,.\,(\pi, \pi)))$. Also note that $(x, 0, 0)$ and $(iy, 0, 0)$ are conjugate fixed-point subspaces in $\mathbb{C}^3$.

TABLE 3.2
Amplitude equations on Fix (Σ)

Isotropy subgroup	Amplitude equations
$\mathbb{Z}_2(\kappa)\times\mathrm{SO}(2)$	$\dfrac{dx}{dt}=c^1(x^2, 0, 0, 0, 0, \lambda, \mu)x$
$\mathbb{Z}_2(\kappa)\oplus\mathbb{Z}_2^c$	$\dfrac{dr}{dt}=p^1(0, 2r^2, 0, 0, 0, \lambda, \mu)r$
$\mathbb{Z}_2(\kappa)$	$\dfrac{dx}{dt}=(c^1+c^3r^2)x$
	$\dfrac{dr}{dt}=(p^1+p^3x^2)r$
$\mathbb{Z}_2(\kappa\,.\,(\pi, \pi))$	$\dfrac{dy}{dt}=(c^1-c^3r^2)y$
	$\dfrac{dr}{dt}=(p^1-p^3y^2)r$

To demonstrate the existence of heteroclinic cycles, we need to make certain hypotheses on the function c^1, c^3, p^1 and p^3. Our strategy is as follows. We set the second parameter μ to zero; then as λ is varied through zero, a double bifurcation occurs. We shall find conditions which guarantee the existence of structurally stable heteroclinic cycles when $\lambda > 0$ (analogous conditions hold when $\lambda < 0$). Structural stability will then guarantee the existence of heteroclinic cycles in an open region of the (λ, μ) plane abutting on the codimension two point $(0, 0)$. We make no effort here to determine how large this region of existence is as (λ, μ) approaches the origin. The results in **[22]** suggest, however, that this region will be a large wedge rather than a thin cusp near the origin.

To state our theorem, we define:

$$\delta_1 = (c^1_\lambda(p^1_\rho + p^3) - p^1_\lambda c^1_\rho)/c^1_\lambda, \tag{3.13a}$$

$$\delta_2 = (c^1_\lambda(p^1_\rho - p^3) - p^1_\lambda c^1_\rho)/c^1_\lambda, \tag{3.13b}$$

$$\delta_3 = (p^1_\lambda(2c^1_N + c^3) - 2c^1_\lambda p^1_N)/p^1_\lambda, \tag{3.13c}$$

$$\delta_4 = (p^1_\lambda(2c^1_N - c^3) - 2c^1_\lambda p^1_N)/p^1_\lambda, \tag{3.13d}$$

$$\delta_5 = \frac{c^1_\lambda(p^1_\rho + p^3)}{p^1_\lambda c^1_\rho} + \frac{p^1_\lambda(2c^1_N + c^3)}{2c^1_\lambda p^1_N}, \tag{3.13e}$$

$$\delta_6 = \frac{c^1_\lambda(p^1_\rho - p^3)}{p^1_\lambda c^1_\rho} + \frac{p^1_\lambda(2c^1_N - c^3)}{2c^1_\lambda p^1_N}. \tag{3.13f}$$

THEOREM 3.1. *Fix* $\mu = 0$. *There exists a structurally stable branch of heteroclinic cycles in* (3.6) *for small* $\lambda > 0$ *connecting an equilibrium in* $\mathrm{Fix}(\mathbb{Z}_2(\kappa) \times \mathrm{SO}(2))$ *with a periodic solution in* $\mathrm{Fix}(\mathbb{Z}_2(\kappa) \oplus \mathbb{Z}_2^c)$ *through* $\mathrm{Fix}(\mathbb{Z}_2(\kappa))$ *and* $\mathrm{Fix}(\mathbb{Z}_2(\kappa\,.\,(\pi, \pi)))$ *if*:

$$c^1_\lambda(0) > 0, \quad c^1_\rho(0) < 0, \tag{3.14a}$$

$$p^1_\lambda(0) > 0, \quad p^1_N(0) < 0, \tag{3.14b}$$

$$\mathrm{sgn}(\delta_1) = \mathrm{sgn}(\delta_4) = -\mathrm{sgn}(\delta_2) = -\mathrm{sgn}(\delta_3), \tag{3.14c}$$

$$\delta_5 > -2, \tag{3.14d}$$

$$\delta_6 > -2. \tag{3.14e}$$

Note 3.2. In order to have a heteroclinic cycle, conditions (3.14a–c) must be satisfied.

Proof. The normaliser condition (2.10) is satisfied for the maximal isotropy subgroup, as indicated by the fact that the primary bifurcations are of pitchfork type. To ensure that these primary bifurcations are supercritical, and that the dynamics in the two planes (when $\lambda > 0$) are like those in Figure 2.4(a), we assume (3.14a,b). These assumptions are just (2.9) and (2.11) interpreted for this example.

We need to establish three points about the amplitude equations in $\mathrm{Fix}(\mathbb{Z}_2(\kappa))$ and $\mathrm{Fix}(\mathbb{Z}_2(\kappa\,.\,(\pi, \pi)))$:

(a) the equilibrium in $\mathrm{Fix}(\mathbb{Z}_2(\kappa) \times \mathrm{SO}(2))$ is a saddle (respectively sink) in $\mathrm{Fix}(\mathbb{Z}_2(\kappa))$ and a sink (respectively saddle) in $\mathrm{Fix}(\mathbb{Z}_2(\kappa\,.\,(\pi, \pi)))$ while the

TABLE 3.3
Other eigenvalues for amplitude equations in fixed-point planes

Equilibrium in	Eigenvalue in Fix $(\mathbb{Z}_2(\kappa))$	Fix $(\mathbb{Z}_2(\kappa\,.\,(\pi,\,\pi)))$
Fix $(\mathbb{Z}_2(\kappa)\times \mathrm{SO}(2))$	$\delta_1 x^2$	$\delta_2 x^2$
Fix $(\mathbb{Z}_2(\kappa)\oplus \mathbb{Z}_2^c)$	$\delta_3 r^2$	$\delta_4 r^2$

equilibrium in Fix $(\mathbb{Z}_2(\kappa)\oplus \mathbb{Z}_2^c)$ is a sink (respectively saddle) in Fix $(\mathbb{Z}_2(\kappa))$ and a saddle (respectively sink) in Fix $(\mathbb{Z}_2(\kappa\,.\,(\pi,\,\pi)))$;

(b) there are no other equilibria in Fix $(\mathbb{Z}_2(\kappa))$ and Fix $(\mathbb{Z}_2(\kappa\,.\,(\pi,\,\pi)))$;

(c) solutions starting near the origin in the fixed-point planes remain bounded.

Firstly, we consider whether the equilibria in Fix $(\mathbb{Z}_2(\kappa)\times \mathrm{SO}(2))$ and the periodic solution in Fix $(\mathbb{Z}_2(\kappa)\oplus \mathbb{Z}_2^c)$ (represented as equilibria in the amplitude coordinate r) are saddles or sinks in the two fixed-point planes Fix $(\mathbb{Z}_2(\kappa))$ and Fix $(\mathbb{Z}_2(\kappa\,.\,(\pi,\,\pi)))$. Assumptions (3.14a,b) imply that the eigenvalues of df at these equilibria corresponding to eigenvectors in the (corresponding) fixed point subspaces of maximal isotropy are negative. Symmetry forces the eigenvectors corresponding to the other eigenvalues of df to be perpendicular to the maximal isotropy subspaces. Thus, we can compute the leading term of the expansion of this eigenvalue and the sign of this term determines whether the equilibrium is a saddle or a sink. These leading terms are recorded in Table 3.3. From Table 3.3 we see that point (a) holds when (3.14c) is assumed.

Point (b) follows from Remark 2.7 when point (a) is satisfied. To establish point (c), we must show that trajectories of (3.6) with initial point near the origin in Fix $(\mathbb{Z}_2(\kappa))$ and Fix $(\mathbb{Z}_2(\kappa\,.\,(\pi,\,\pi)))$ remain bounded. Applying Proposition 2.6 to the amplitude equations (3.15) gives a sufficient condition for proving boundedness in Fix $(\mathbb{Z}_2(\kappa))$.

$$[c_p^1(0)x^2+(c^3(0)+2c_N^1(0))r^2+c_\lambda^1(0)\lambda]x=0, \tag{3.15a}$$

$$[(p_p^1(0)+p^3(0))x^2+2p_N^1(0)r^2+p_\lambda^1(0)\lambda]r=0. \tag{3.15b}$$

This condition yields (3.14d). A similar calculation using the amplitude equations in Fix $(\mathbb{Z}_2(\kappa\,.\,(\pi,\,\pi)))$ yields (3.14e).

These arguments complete the proof of existence of a branch of structurally stable heteroclinic cycles when $\lambda>0$ in the normal form equations (3.6) truncated at third order. The structural stability of these cycles allows us to conclude that the existence of higher-order terms in the normal form equations does not alter the conclusion, thus proving the theorem. □

Next we determine conditions when the heteroclinic cycles found in Theorem 3.1 are asymptotically stable. Assumptions (a)–(d) of Theorem 2.10 give sufficient conditions for heteroclinic cycles to be stable. We apply these assumptions here to determine conditions when the heteroclinic cycles just constructed are asymptotically stable.

Asymptotic stability will follow from assuming (3.16) and either (3.17a) or (3.17b), where

$$p^2>0, \tag{3.16}$$

$$\delta_1\delta_4<\min\{-2c_p^1,\,-\delta_2\}\min\{-4p_N^1,\,-\delta_3,\,\delta_4+4p^2\} \tag{3.17a}$$

$$\delta_2\delta_3<\min\{-2c_p^1,\,-\delta_1\}\min\{-4p_N^1,\,-\delta_4,\,\delta_3+4p^2\}. \tag{3.17b}$$

TABLE 3.4
Eigenvalues at equilibria

$A_1 \in \mathrm{Fix}\,(\mathbb{Z}_2(\kappa)\times \mathrm{SO}(2))$	$A_2 \in \mathrm{Fix}\,(\mathbb{Z}_2(\kappa)\oplus \mathbb{Z}_2^c)$
$2c_p^1 x^2+\ldots$	$4p_N^1 r^2+\ldots$
0	0 (twice)
$\delta_1 x^2+\ldots$ (twice)	$-4p^2r^2+\ldots$
$\delta_2 x^2+\ldots$ (twice)	$\delta_3 r^2+\ldots$
	$\delta_4 r^2+\ldots$

THEOREM 3.3. *Assume* (3.14), (3.16) *and either* (3.17a) *or* (3.17b) *are valid. Then the heteroclinic cycle whose existence is asserted in Theorem* 3.1 *is generically asymptotically stable.*

Remark 3.4. The hypotheses of Theorem 3.3 are simultaneously valid for a nonempty open subset of the coefficients of the linear and cubic terms in $f(z, \lambda)$.

Proof. The eigenvalues of the Jacobian of f at 'equilibria' A_1 in $l_1 = \mathrm{Fix}\,(\mathbb{Z}_2(\kappa)\times \mathrm{SO}(2))$ and A_2 in $l_2 = \mathrm{Fix}\,(\mathbb{Z}_2(\kappa)\oplus \mathbb{Z}_2^c)$ may be computed from [**14**, XX, Table 2.9]. The results are presented in Table 3.4. Let $P_1 = \mathrm{Fix}\,(\mathbb{Z}_2(\kappa))$ and $P_2 = \mathrm{Fix}\,(\mathbb{Z}_2(\kappa\,.\,(\pi,\pi)))$. Except for $-4p^2(0)$, all of the nonzero eigenvalues of df are determined by $df \mid (P_1+P_2)$ and multiplicity (forced by isotropy).

It follows from Table 3.4, (3.14b,c) and (3.16) that Theorem 2.10 (a,b,c) are valid.

The trajectories connecting A_1 to A_2 can be made in either forward ($\delta_1>0$) or backward ($\delta_1<0$) time in $\mathrm{Fix}\,(\mathbb{Z}_2(\kappa))$. (Assumptions (3.14) imply that this connection is made in the opposite direction in $\mathrm{Fix}\,(\mathbb{Z}_2(\kappa\,.\,(\pi,\pi)))$. When $\delta_1>0$, the contracting eigenvalues at A_1 are δ_2 and $2c_p^1$ and the expanding eigenvalue is δ_1. Similarly at A_2, the contracting eigenvalues are δ_3 and $4p_N^1$ and the expanding eigenvalue is δ_4. The fact that Theorem 2.10(d) is valid follows from (3.17a). Similarly, if $\delta_1<0$, then the roles of δ_1 and δ_2 and the roles of δ_3 and δ_4 are reversed, yielding conditions (3.17b). □

4. O(2) Hopf/Hopf mode interaction

In this section, we discuss the existence of heteroclinic cycles connecting periodic solutions in two-parameter families of vector fields, whose linearisation about a trivial steady state has eigenvalues simultaneously at $\pm i\omega_1$ and $\pm i\omega_2$. As in the previous section, we study this interaction using 'amplitude equations' associated with Birkhoff normal form on a centre manifold.

More precisely, we again consider (3.1), the O(2)-equivariant system of ODEs in $\mathbb{R}^N$, for which $x=0$ is a 'trivial' equilibrium. We now assume, however, that $(d_xF)_{0,0,0}$ has eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, where ω_1/ω_2 is irrational. Due to the O(2) symmetry, we expect these eigenvalues to be either simple or double, and, to avoid trivial situations, we assume that $\pm i\omega_2$ are each double eigenvalues. Thus, we discuss two cases: $\pm i\omega_1$ are simple eigenvalues, and $\pm i\omega_1$ are double eigenvalues. We show that, in each case, there are configurations in the isotropy subgroup lattice which indicate the possible existence of heteroclinic cycles. In the first case, however, there are restrictions on the eigenvalues, forced by symmetry, which prohibit the existence of cycles. In the second case, we isolate three configurations in the lattice which suggest heteroclinic cycles, and demonstrate that each of these connections actually exists and may be asymptotically stable. Our results rely heavily on the discussion of the codimension two mode interactions given in **[6]**. We freely reference results in that paper.

A centre manifold reduction transforms (3.1) to an equation

$$\frac{dz}{dt} = g(z, \lambda, \mu), \quad z \in \mathbb{C}^n, \tag{4.1}$$

where $n = 3$ in the first case, and $n = 4$ in the second. We refer to these mode interations as the six- and eight-dimensional interactions, respectively. Next we transform (4.1) to Birkhoff normal form, obtaining

$$\frac{dz}{dt} = f(z, \lambda, \mu) + \dots, \tag{4.2}$$

where f commutes with $O(2) \times T^2$, the 2-torus of symmetries of normal form coming from the two rationally independent frequencies ω_1 and ω_2. Finally, as in Section 3, we study only the dynamics of the normal form equation

$$\frac{dz}{dt} = f(z, \lambda, \mu), \quad z = (z_0, z_1, z_2) \in \mathbb{C}^3 \tag{4.3}$$

by ignoring the higher-order terms in (4.2).

This section is divided into two main subsections: subsection 4.1 examines possible heteroclinic cycles in the six-dimensional Hopf bifurcation, while subsection 4.2 deals with the eight-dimensional case.

4.1. The six-dimensional interaction

In the six-dimensional case, the $O(2) \times T^2$ symmetry allows us to rewrite (4.3) in phase amplitude equations. Set $z_j = r_j \exp(i\theta_j)$. Then the amplitude equations have the form

$$\frac{dr_0}{dt} = p_0 r_0, \tag{4.4a}$$

$$\frac{dr_1}{dt} = (p_1 + \delta p_2) r_1, \tag{4.4b}$$

$$\frac{dr_2}{dt} = (p_1 - \delta p_2) r_2, \tag{4.4c}$$

where the p_j are functions of $\rho = r_0^2$, $N = r_1^2 + r_2^2$, $\Delta = \delta^2$ and where $\delta = r_2^2 - r_1^2$. The amplitude equations (4.4) retain the symmetries $\mathbb{Z}_2 \times D_4$ from $O(2) \times T^2$ where $\mathbb{Z}_2 = \{1, F_0\}$, D_4 is generated by the reflections F, F_1, F_2 and

$$F_0(r_0, r_1, r_2) = (-r_0, r_1, r_2), \tag{4.5a}$$

$$F_1(r_0, r_1, r_2) = (r_0, -r_1, -r_2), \tag{4.5b}$$

$$F_2(r_0, r_1, r_2) = (r_0, r_1, -r_2), \tag{4.5c}$$

$$F(r_0, r_1, r_2) = (r_0, r_2, r_1). \tag{4.5d}$$

$$
\begin{tikzcd}
 & & \mathbb{Z}_2 \times \mathrm{D}_4 & & \\
\mathbb{Z}_2 \times \{\mathrm{F}\} \arrow[urr] & & \mathrm{D}_4 \arrow[u] & & \mathbb{Z}_2 \times \{\mathrm{F}_2\} \arrow[ull] \\
\{\mathrm{F}\} \arrow[u] \arrow[urr] & & \mathbb{Z}_2 \arrow[ull] \arrow[urr] & & \{\mathrm{F}_2\} \arrow[ull] \arrow[u] \\
 & & \{1\} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

Figure 4.1. Lattice of isotropy subgroups in the six-dimensional case.

Zeros of (4.4) correspond to periodic solutions and invariant tori of (4.3). A heteroclinic cycle connecting equilibria in (4.4) corresponds to a heteroclinic cycle connecting periodic solutions (or tori) in (4.3).

The lattice of isotropy subgroups of $\mathbb{Z}_2 \times D_4$ is given in Figure 4.1. Note that, in principle, a heteroclinic connection from $\mathrm{Fix}\,(\mathbb{Z}_2 \times \{F\})$ to $\mathrm{Fix}\,(\mathbb{Z}_2 \times \{F_2\})$ (through $\mathrm{Fix}\,(\mathbb{Z}_2)$) to $\mathrm{Fix}\,(D_4)$ (through $\mathrm{Fix}\,(\{F_2\})$) to $\mathrm{Fix}\,(\mathbb{Z}_2 \times \{F\})$ (through $\mathrm{Fix}\,(\{F\})$) is possible. To have such a cycle, the fixed point in $\mathrm{Fix}\,(D_4)$ must be a saddle in one of the planes $\mathrm{Fix}\,(\{F\})$ and $\mathrm{Fix}\,(\{F_2\})$ and a sink in the other. Symmetry, however, forces the equilibrium in $\mathrm{Fix}\,(D_4)$ to be of the same type in both planes.

Thus we have an example where the lattice suggests a possible heteroclinic cycle, but fine structure of the symmetries precludes its existence.

4.2. The eight-dimensional interaction

We now assume that both $\pm i\omega_1$ and $\pm i\omega_2$ are double eigenvalues. We can choose coordinates $z = (z_1, z_2, z_3, z_4)$ on $\mathbb{C}^4$ so that the O(2) action is generated by:

$$\phi \,.\, z = (e^{li\phi} z_1,\, e^{-li\phi} z_2,\, e^{mi\phi} z_3,\, e^{-mi\phi} z_4), \tag{4.6a}$$

$$\kappa \,.\, z = (z_2, z_1, z_4, z_3). \tag{4.6b}$$

Factoring out by the kernel of the representation, we may assume that l and m are coprime and that $l \leqq m$. In addition, the normal form equation

$$\frac{dz}{dt} = f(z, \lambda, \mu), \quad z \in \mathbb{C}^4 \tag{4.7}$$

commutes with the action of T^2 defined by $(\psi_1, \psi_2) \,.\, z = (e^{i\psi_1} z_1, e^{i\psi_1} z_2, e^{i\psi_2} z_3, e^{i\psi_2} z_4)$.

Due to symmetry, the vector field f in (4.7) has the form $f = (f_1, f_2, f_3, f_4)$ where

$$f_1(z) = P^1 z_1 + R^1 \bar{z}_1^{m-1} z_2^m (z_3 \bar{z}_4)^l, \tag{4.8a}$$

$$f_2(z) = f_1(\kappa \,.\, z), \tag{4.8b}$$

$$f_3(z) = P^3 z_3 + R^3 (z_1 \bar{z}_2)^m \bar{z}_3^{l-1} z_4^l, \tag{4.8c}$$

$$f_4(z) = f_3(\kappa \,.\, z). \tag{4.8d}$$

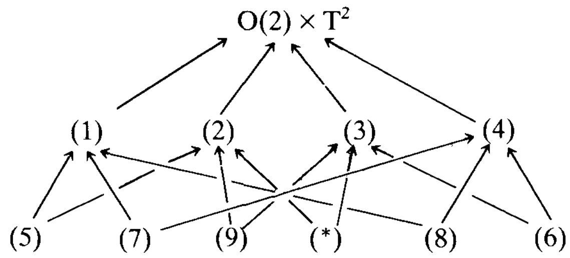

Figure 4.2. Upper part of lattice of isotropy subgroups of O(2) × T^2 acting on $\mathbb{C}^4$. Isotropy subgroups (1) and (4) correspond to rotating waves and (2) and (3) to standing waves (of wave numbers l and m respectively). Isotropy (*) is (10) if m odd and (11) if m even.

The functions P^1, P^3, R^1, R^3 are complex-valued functions of $\rho_k = |z_k|^2$ ($k = 1, \ldots, 4$), Re α, Im α, where $\alpha = (z_1\bar{z}_2)^m(\bar{z}_3 z_4)^l$. Moreover, $P^1(0) = i\omega_1$ and $P^3(0) = i\omega_2$.

The top part of the lattice of isotropy subgroups of O(2) × T^2 is given in Figure 4.2. The actual subgroups are listed in Table 4.1 using the following notation:

$$\mathbb{Z}(\phi, \psi_1, \psi_2) = \text{group generated by } (\phi, \psi_1, \psi_2) \in \mathrm{SO}(2) \times T^2, \tag{4.9a}$$

$$\mathbb{Z}_\kappa(\phi, \psi_1, \psi_2) = \text{group generated by } \kappa\,.\,(\phi, \psi_1, \psi_2), \tag{4.9b}$$

$$S(k, l, m) = \{(k\theta, l\theta, m\theta) \in \mathrm{SO}(2) \times T^2 \colon \theta \in S^1\}. \tag{4.9c}$$

From the lattice in Figure 4.2, we can isolate three different types of possible heteroclinic cycles. These are shown in Figure 4.3.

The observation that makes analysis of heteroclinic cycles possible in this case is: the restriction of the vector field f in (4.8) to any four-dimensional fixed-point subspace of a (submaximal) isotropy subgroup yields a vector field that decouples into phase/amplitude equations. Thus the effective dimension of these fixed-point subspaces is two, and the observations of Section 2 apply.

We show below in subsections 4.2.1–4.2.3 that cycles inspired by Figure 4.3 can both exist and be asymptotically stable. To do this we must compute the

TABLE 4.1
Isotropy subgroups in eight-dimensional case

	Isotropy subgroup	Fixed-point subspace	Dimension
(0)	O(2) × T^2	$\{0\}$	0
(1)	$S(0, 0, 1) \times S(1, -l, 0)$	$z_2 = z_3 = z_4 = 0$	2
(2)	$S(0, 0, 1) \times \mathbb{Z}_\kappa \times \mathbb{Z}(\pi/l, \pi, 0)$	$z_1 = z_2,\ z_3 = z_4 = 0$	2
(3)	$S(0, 1, 0) \times \mathbb{Z}_\kappa \times \mathbb{Z}(\pi/m, 0, \pi)$	$z_1 = z_2 = 0,\ z_3 = z_4$	2
(4)	$S(0, 1, 0) \times S(1, 0, m)$	$z_1 = z_2 = z_3 = 0$	2
(5)	$S(0, 0, 1) \times \mathbb{Z}(\pi/l, \pi, 0)$	$z_3 = z_4 = 0$	4
(6)	$S(0, 1, 0) \times \mathbb{Z}(\pi/m, 0, \pi)$	$z_1 = z_2 = 0$	4
(7)	$S(1, l, m)$	$z_1 = z_3 = 0$	4
(8)	$S(1, l, -m)$	$z_1 = z_4 = 0$	4
(9)	$\mathbb{Z}_\kappa \times \mathbb{Z}(\pi, l\pi, m\pi)$	$z_1 = z_2,\ z_3 = z_4$	4
(10)	$\mathbb{Z}_\kappa(0, \pi, 0) \times \mathbb{Z}(\pi, l\pi, m\pi)$ (m odd)	$z_1 = -z_2,\ z_3 = z_4$	4
(11)	$\mathbb{Z}_\kappa(0, 0, \pi) \times \mathbb{Z}(\pi, l\pi, m\pi)$ (l odd, m even)	$z_1 = z_2,\ z_3 = -z_4$	4

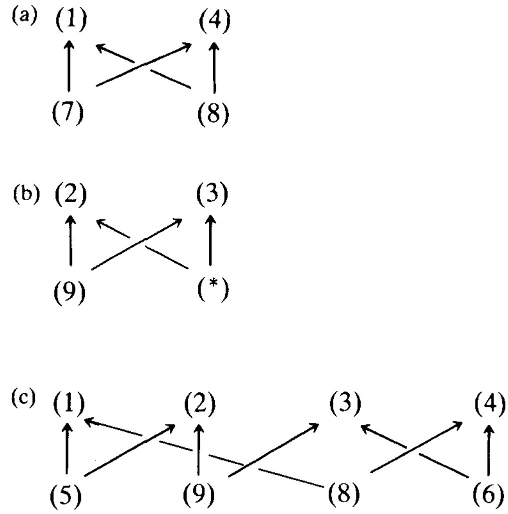

Figure 4.3. Isotropy connections indicative of possible heteroclinic cycles: (a) cycle of rotating waves, (b) cycle of standing waves, and (c) cycle of rotating and standing waves.

eigenvalues of the 8×8 matrix df, where f is defined in (4.8), at solutions with isotropy (1)–(4). This calculation is made easier by using some elementary representation theory and the complex notation of (4.8). Here we follow **[6]** (see also **[14]**).

The isotypic decomposition of $\mathbb{C}^4$ by each of the isotropy subgroups Σ in (1)–(4) has the form $\mathrm{Fix}(\Sigma) \oplus V_2 \oplus V_3 \oplus V_4$ where each of the summands is two-dimensional and Σ acts irreducibly on the V_j. These decompositions are presented in Table 4.2 when $l = m = 1$. Otherwise, for the standing waves, the isotypic components are actually V_1, V_2, $V_3 \oplus V_4$. (This has the result that the eigenvalues of df are the same in V_3 as V_4.) One can check, moreover, that on each of these two-dimensional isotypic components, either $(\mathrm{O}(2) \times T^2)/\Sigma$ forces one eigenvalue of df to be zero, or the effective action of Σ is by the rotation group SO(2), which forces the eigenvalues of df to be complex conjugates. In either case stability is determined by $\mathrm{tr}\,(df \mid V_j)$.

In complex coordinates df, where $f = (f_1, f_2, f_3, f_4)$, has the form

$$(df)(W) = \left[\sum_{i=1}^{4}\left[\frac{\partial f_i}{\partial z_1} w_i + \frac{\partial f_i}{\partial \bar{z}_1} \bar{w}_i\right], \ldots, \sum_{i=1}^{4}\left[\frac{\partial f_i}{\partial z_4} \omega_i + \frac{\partial f_i}{\partial \bar{z}_4} \bar{w}_i\right]\right] \tag{4.10}$$

TABLE 4.2
Isotypic decompositions if maximal isotropy subgroups (here $w \in \mathbb{C}$)

	$\mathrm{Fix}(\Sigma) = V_1$	V_2	V_3	V_4
(1)	$(w, 0, 0, 0)^*$	$(0, w, 0, 0)$	$(0, 0, w, 0)$	$(0, 0, 0, w)$
(2)	$(w, w, 0, 0)^*$	$(w, -w, 0, 0)^*$	$(0, 0, w, w)$	$(0, 0, w, -w)$
(3)	$(0, 0, w, w)^*$	$(0, 0, w, -w)^*$	$(w, w, 0, 0)$	$(w, -w, 0, 0)$
(4)	$(0, 0, 0, w)^*$	$(0, 0, w, 0)$	$(w, 0, 0, 0)$	$(0, w, 0, 0)$

* *Existence of one null vector for df.*

TABLE 4.3
Eigenvalues of $df \mid V_j$ at solutions with maximal isotropy. In fact, $\operatorname{tr}(df \mid V_j)/2$ is given by the real part of the entries in this table

Isotropy subgroup	V_1	V_2	V_3	V_4
(1)	$\frac{\partial f_1}{\partial z_1}$	$\frac{\partial f_2}{\partial z_2}$	$\frac{\partial f_3}{\partial z_3}$	$\frac{\partial f_4}{\partial z_4}$
(2)	$\frac{\partial}{\partial z_1}(f_1+f_2)$	$\frac{\partial}{\partial z_1}(f_1-f_2)$	$\frac{\partial}{\partial z_3}(f_3+f_4)$	$\frac{\partial}{\partial z_3}(f_3-f_4)$
(3)	$\frac{\partial}{\partial z_3}(f_3+f_4)$	$\frac{\partial}{\partial z_3}(f_3-f_4)$	$\frac{\partial}{\partial z_1}(f_1+f_2)$	$\frac{\partial}{\partial z_1}(f_1-f_2)$
(4)	$\frac{\partial f_4}{\partial z_4}$	$\frac{\partial f_3}{\partial z_3}$	$\frac{\partial f_1}{\partial z_1}$	$\frac{\partial f_2}{\partial z_2}$

where $W=(w_1, w_2, w_3, w_4) \in \mathbb{C}^4$. In this form, it is easy to compute $df \mid V_j$ for each of the isotypic components. Since df is a real linear map on the two-dimensional subspace V_j, $df \mid V_j$ has the form $w \mapsto \alpha w + \beta \bar{w}$, where $\alpha, \beta \in \mathbb{C}$. The trace of this linear mapping is just $2\operatorname{Re}(\alpha)$. The calculation of $\operatorname{tr}(df \mid V_j)$ from (4.10) is now possible; the results are recorded in Table 4.3.

Next we compute the eigenvalues of (df) using the form of f in (4.8). These results are recorded in Table 4.4, where $p^j = \operatorname{Re} P^j$ and $r^j = \operatorname{Re}(R^j)$.

Finally, we expand the entries in Table 4.4 to lowest order in a^2. To do this, we need to know the equations defining the solutions (1)–(4) to lowest order. They are:

$$\left.\begin{aligned}
&(1)\quad p^1=0, \quad \lambda=-\frac{p^1_{\rho_1}}{p^1_\lambda}a^2+\dots,\\
&(2)\quad p^1=0, \quad \lambda=-\frac{p^1_{\rho_1}+p^1_{\rho_2}}{p^1_\lambda}a^2+\dots,\\
&(3)\quad p^3=0, \quad \lambda=-\frac{p^3_{\rho_3}+p^3_{\rho_4}}{p^3_\lambda}a^2+\dots,\\
&(4)\quad p^4=0, \quad \lambda=-\frac{p^3_{\rho_3}}{p^3_\lambda}a^2+\dots.
\end{aligned}\right\} \tag{4.11}$$

We calculate the real part of the eigenvalue of $(df) \mid V_i$ at a solution with isotropy (j) to be $\varepsilon_{ij}a^2+\dots$, where the 4×4 matrix (ε_{ij}) is given in Table 4.5.

TABLE 4.4
Computation of eigenvalues of df at maximal isotropy in terms of p^j, r^j

Isotropy at $z=$	V_1	V_2	V_3	V_4
(1) $(a, 0, 0, 0)$	$2(p^1_{z_1}a+p^1)$	p^2	p^3	p^4
(2) $(a, a, 0, 0)$	$2(p^1+a(p^1+p^2)_{z_1})$	$2(p^1+a(p^1-p^2)_{z_1})$	$p^3+a^{2m}r^4\delta_{l1}$	$p^3-a^{2m}r^4\delta_{l1}$
(3) $(0, 0, a, a)$	$2(p^3+a(p^3+p^4)_{z_3})$	$2(p^3+a(p^3-p^4)_{z_3})$	$p^1+a^{2l}r^2\delta_{m1}$	$p^1-a^{2l}r^2\delta_{m1}$
(4) $(0, 0, 0, a)$	$2(p^4_{z_4}a+p^4)$	p^3	p^1	p^2

Note: $p^2(z)=p^1(\kappa\,.\,z)$ and $p^4(z)=p^3(\kappa\,.\,z)$ (similarly for r^2, r^4). δ_{j1} is the Kronecker delta function.

TABLE 4.5
Coefficients giving eigenvalues of $(df) \mid V_i$ for solutions with isotropy (j) to lowest order

$$\begin{aligned}
&\varepsilon_{11} = 2p^1_{\rho_1}\\
&\varepsilon_{12} = 2(p^1_{\rho_1} + p^1_{\rho_2})\\
&\varepsilon_{13} = 2(p^3_{\rho_3} + p^3_{\rho_4})\\
&\varepsilon_{14} = 2p^3_{\rho_3}\\
&\varepsilon_{21} = p^1_{\rho 2} - p^1_{\rho_1}\\
&\varepsilon_{22} = 2(p^1_{\rho_1} - p^1_{\rho_2})\\
&\varepsilon_{23} = 2(p^3_{\rho_3} - p^3_{\rho_4})\\
&\varepsilon_{24} = p^3_{\rho_4} - p^3_{\rho_3}\\
&\varepsilon_{31} = (p^1_\lambda p^3_{\rho_1} - p^3_\lambda p^1_{\rho_1})/p^1_\lambda\\
&\varepsilon_{32} = [p^1_\lambda(p^3_{\rho_1} + p^3_{\rho_2} + r^3\delta_{l1}\delta_{m1}) - p^3_\lambda(p^1_{\rho_1} + p^1_{\rho_2})]/p^1_\lambda\\
&\varepsilon_{33} = [p^3_\lambda(p^1_{\rho_3} + p^1_{\rho_4} + r^1\delta_{l1}\delta_{m1}) - p^1_\lambda(p^3_{\rho_3} + p^3_{\rho_4})]/p^3_\lambda\\
&\varepsilon_{34} = (p^3_\lambda p^1_{\rho_4} - p^1_\lambda p^3_{\rho_3})/p^3_\lambda\\
&\varepsilon_{41} = (p^1_\lambda p^3_{\rho_2} - p^3_\lambda p^1_{\rho_1})/p^1_\lambda\\
&\varepsilon_{42} = [p^1_\lambda(p^3_{\rho_1} + p^3_{\rho_2} - r^3\delta_{l1}\delta_{m1}) - p^3_\lambda(p^1_{\rho_1} + p^1_{\rho_2})]/p^1_\lambda\\
&\varepsilon_{43} = [p^3_\lambda(p^1_{\rho_3} + p^1_{\rho_4} - r^1\delta_{l1}\delta_{m1}) - p^1_\lambda(p^3_{\rho_3} + p^3_{\rho_4})]/p^3_\lambda\\
&\varepsilon_{44} = (p^3_\lambda p^1_{\rho_3} - p^1_\lambda p^3_{\rho_3})/p^3_\lambda
\end{aligned}$$

4.2.1. *Heteroclinic cycles between rotating waves*

Define:

$$\varepsilon_5 = \frac{p^3_\lambda p^1_{\rho_3}}{p^1_\lambda p^3_{\rho_3}} + \frac{p^1_\lambda p^3_{\rho_1}}{p^3_\lambda p^1_{\rho_1}} \tag{4.12a}$$

$$\varepsilon_6 = \frac{p^3_\lambda p^1_{\rho_4}}{p^1_\lambda p^3_{\rho_3}} + \frac{p^1_\lambda p^3_{\rho_2}}{p^3_\lambda p^1_{\rho_1}} \tag{4.12b}$$

THEOREM 4.1. *Fix $\mu = 0$. There exists a structurally stable branch of heteroclinic cycles in* (4.3) *for $\lambda > 0$, connecting rotating waves* (1) *to rotating waves* (4) *through* Fix (7) *and* Fix (8) *if*:

$$p^1_\lambda(0) > 0, \quad p^3_\lambda(0) > 0, \tag{4.13a}$$

$$\varepsilon_{11} < 0, \quad \varepsilon_{14} < 0, \tag{4.13b}$$

$$\operatorname{sgn}(\varepsilon_{31}) = \operatorname{sgn}(\varepsilon_{34}) = -\operatorname{sgn}(\varepsilon_{41}) = -\operatorname{sgn}(\varepsilon_{44}), \tag{4.13c}$$

$$\varepsilon_5 > -2, \quad \varepsilon_6 > -2. \tag{4.13d}$$

Note 4.2. Conditions (4.13a,b,c) must be assumed in order for the heteroclinic cycle to exist.

Proof. The existence of a heteroclinic cycle connecting (1) and (4) is proved in a fashion similar to the proof of Theorem 3.1. The basic observation is that when f in (4.8) is restricted to Fix (7) or Fix (8), it decouples into phase-amplitude equations and periodic solutions (1) and (4) correspond to equilibria of the two-dimensional system of amplitude equations. These amplitude equations may be written in terms of the ρ_j and are:

$$\left.\begin{aligned}\frac{d\rho_2}{dt}&=p^1(\rho_2,0,\rho_4,0,0,0,\lambda,\mu)\rho_2\\ \frac{d\rho_4}{dt}&=p^3(\rho_2,0,\rho_4,0,0,0,\lambda,\mu)\rho_4\end{aligned}\right\}\quad \text{in Fix}\,(7), \tag{4.14a}$$

$$\left.\begin{aligned}\frac{d\rho_2}{dt}&=p^1(\rho_2,0,0,\rho_3,0,0,\lambda,\mu)\rho_2\\ \frac{d\rho_3}{dt}&=p^3(0,\rho_2,\rho_3,0,0,0,\lambda,\mu)\rho_3\end{aligned}\right\}\quad \text{in Fix}\,(8). \tag{4.14b}$$

As in Theorem 3.1, we set $\mu=0$, and note that (4.13a,b) imply that solutions (1) and (4) occur supercritically in λ and are asymptotically stable in Fix (1) and Fix (4), respectively. We need to establish three points:

(a) solution (4) is a saddle in one of the planes Fix (7) and Fix (8) and a sink in the other,
(b) there are no other equilibria of the amplitude equations in Fix (7) and Fix (8), and
(c) solutions starting near the origin in Fix (7) and Fix (8) stay bounded so that connecting trajectories actually exist.

Using the data in Table 4.6, one may check that point (a) is established if (4.13c) is assumed. Point (b), the nonexistence of equilibria in (4.14) with submaximal isotropy, follows directly from Remarks 2.7. Finally, point (c) is established using Proposition 2.6, as in Theorem 3.1, by assuming (4.13d). □

Similarly, one can derive conditions sufficient to imply asymptotic stability.

THEOREM 4.3. *The branch of heteroclinic cycles between rotating waves in* (4.3) *found in Theorem* 4.1 *generically consists of asymptotically stable cycles if* (4.15) *and either* (4.16a) *or* (4.16b) *is valid where*:

$$\varepsilon_{21}<0,\ \varepsilon_{24}<0, \tag{4.15}$$

$$\min\{-\varepsilon_{11},-\varepsilon_{41},\varepsilon_{31}-\varepsilon_{21}\}\,.\,\min\{-\varepsilon_{14},-\varepsilon_{44},\varepsilon_{34}-\varepsilon_{24}\}>\varepsilon_{31}\varepsilon_{34}, \tag{4.16a}$$

$$\min\{-\varepsilon_{11},-\varepsilon_{31},\varepsilon_{41}-\varepsilon_{21}\}\,.\,\min\{-\varepsilon_{14},-\varepsilon_{34},\varepsilon_{44}-\varepsilon_{24}\}>\varepsilon_{41}\varepsilon_{44}. \tag{4.16b}$$

Proof. The proof proceeds like that of Theorem 3.3.

4.2.2. *Heteroclinic cycles between standing waves*

Define:

$$\varepsilon_7=\frac{p^1_\lambda(p^3_{\rho_1}+p^3_{\rho_2}+r^3)}{p^3_\lambda(p^1_{\rho_1}+p^1_{\rho_2})}+\frac{p^3_\lambda(p^1_{\rho_3}+p^1_{\rho_4}+r^1)}{p^1_\lambda(p^3_{\rho_3}+p^3_{\rho_4})}, \tag{4.17a}$$

$$\varepsilon_8=\frac{p^1_\lambda(p^3_{\rho_1}+p^3_{\rho_2}-r^3)}{p^3_\lambda(p^1_{\rho_1}+p^1_{\rho_2})}+\frac{p^3_\lambda(p^1_{\rho_3}+p^1_{\rho_4}-r^1)}{p^1_\lambda(p^3_{\rho_3}+p^3_{\rho_4})}. \tag{4.17b}$$

TABLE 4.6
Other eigenvalues for amplitude equations in fixed-point planes

	Fix (7)	Fix (8)
(1)	ε_{31}	ε_{41}
(4)	ε_{44}	ε_{34}

TABLE 4.7
Other eigenvalues for amplitude equations in fixed-point planes

	Fix (9)	Fix (10)
(2)	ε_{32}	ε_{42}
(3)	ε_{33}	ε_{43}

THEOREM 4.4. *Fix $\mu = 0$ and $l = m = 1$. There exists a structurally stable branch of heteroclinic cycles in* (4.3) *for $\lambda > 0$, connecting standing waves* (2) *to standing waves* (3) *through* Fix (9) *and* Fix (10) *if*:

$$p_\lambda^1(0) > 0, \quad p_\lambda^3(0) > 0, \tag{4.18a}$$

$$\varepsilon_{12} < 0, \quad \varepsilon_{13} < 0, \tag{4.18b}$$

$$\operatorname{sgn}(\varepsilon_{32}) = \operatorname{sgn}(\varepsilon_{43}) = -\operatorname{sgn}(\varepsilon_{33}) = -\operatorname{sgn}(\varepsilon_{42}), \tag{4.18c}$$

$$\varepsilon_7 > -2, \quad \varepsilon_8 > -2. \tag{4.18d}$$

Note 4.5. Conditions (4.18a–c) must be assumed in order for the heteroclinic cycle to exist. When $l \neq m$, $\varepsilon_{32} = \varepsilon_{33}$ and hence (4.18c) fails; therefore no such cycle exists.

THEOREM 4.6. *The branch of heteroclinic cycles between standing waves in* (4.3) *found in Theorem* 4.4 *generically consists of asymptotically stable cycles if* (4.19) *and either* (4.20*a*) *or* (4.20*b*) *is valid, where*:

$$\varepsilon_{22} < 0, \quad \varepsilon_{23} < 0, \tag{4.19}$$

$$\min\{-\varepsilon_{12}, -\varepsilon_{42}, \varepsilon_{32} - \varepsilon_{22}\} \cdot \min\{-\varepsilon_{13}, -\varepsilon_{33}, \varepsilon_{43} - \varepsilon_{23}\} > \varepsilon_{32}\varepsilon_{43}, \tag{4.20a}$$

$$\min\{-\varepsilon_{12}, -\varepsilon_{32}, \varepsilon_{42} - \varepsilon_{22}\} \cdot \min\{-\varepsilon_{13}, -\varepsilon_{43}, \varepsilon_{33} - \varepsilon_{23}\} > \varepsilon_{42}\varepsilon_{33}. \tag{4.20b}$$

Remark 4.7. The hypotheses of Theorems 4.4 and 4.6 can be satisfied simultaneously only when $l = m = 1$.

The proofs of Theorems 4.4 and 4.6 are identical in spirit with those of Theorems 4.1 and 4.3. The calculations needed are those corresponding to Table 4.6 and are given in Table 4.7.

4.2.3. *Heteroclinic cycles between rotating and standing waves*

Next, we consider the possible heteroclinic cycles inspired by Figure 4.3. In this case, heteroclinic cycles exist as a primary branch when $l = m = 1$. (This is an important special case as it occurs in Hopf/Hopf mode interactions in the Taylor–Couette system, see **[5]**. It turns out, however, that the specific cases considered in that reference do not lead to heteroclinic cycles of this kind.) Such heteroclinic cycles may exist for other l, m, but we have not pursued their classification here. In particular, these cycles are never asymptotically stable when $l \neq m$.

THEOREM 4.8. *Fix $\mu = 0$. There exists a branch of structurally stable heteroclinic cycles in* (4.3, 4.8) *for $\lambda > 0$, connecting rotating waves* (1) *to standing waves* (2) *to standing waves* (3) *to rotating waves* (4) *and back to rotating waves* (1) *through the*

TABLE 4.8
Other eigenvalues for amplitude equations in fixed-point planes

	Fix (5)	Fix (9) or Fix (10)	Fix (6)	Fix (7) or Fix (8)
(1)	ε_{21}			ε_{31} or ε_{41}
(2)	ε_{22}	ε_{32} or ε_{42}		
(3)		ε_{33} or ε_{43}	ε_{23}	
(4)			ε_{24}	ε_{44} or ε_{34}

subspaces Fix (5), Fix (9) *or* Fix (10), Fix (6), *and* Fix (7) *or* Fix (8), *respectively, if*:

$$p_\lambda^1(0) > 0, \quad p_\lambda^3(0) > 0 \tag{4.21a}$$

$$\varepsilon_{11} < 0, \quad \varepsilon_{12} < 0, \quad \varepsilon_{13} < 0, \quad \varepsilon_{14} < 0 \tag{4.21b}$$

$$\operatorname{sgn}(\varepsilon_{23}) = \operatorname{sgn}(\varepsilon_{21}) \tag{4.21c}$$

$$\textit{either} \quad \operatorname{sgn}(\varepsilon_{32}) = -\operatorname{sgn}(\varepsilon_{33}) = \operatorname{sgn}(\varepsilon_{21}), \quad \varepsilon_7 > -2 \tag{4.21d(i)}$$

$$\textit{or} \quad \operatorname{sgn}(\varepsilon_{42}) = -\operatorname{sgn}(\varepsilon_{43}) = \operatorname{sgn}(\varepsilon_{21}), \quad \varepsilon_8 > -2, \tag{4.21d(ii)}$$

$$\textit{either} \quad \operatorname{sgn}(\varepsilon_{44}) = -\operatorname{sgn}(\varepsilon_{31}) = \operatorname{sgn}(\varepsilon_{21}), \quad \varepsilon_5 > -2 \tag{4.21e(i)}$$

$$\textit{or} \quad \operatorname{sgn}(\varepsilon_{34}) = -\operatorname{sgn}(\varepsilon_{41}) = \operatorname{sgn}(\varepsilon_{21}), \quad \varepsilon_6 > -2. \tag{4.21e(ii)}$$

Proof. Conditions (a), (b) ensure that the trivial solution is stable subcritically and that the branches of rotating waves and standing waves bifurcate supercritically. To determine the existence of the heteroclinic cycle, we need to consider the eigenvalues of the Jacobian in the relevant fixed-point planes. This information is listed in Table 4.8.

As usual, we require that the entries in Table 4.8 are alternately positive and negative in order to get the appropriate saddle-sink connections. This immediately yields (4.21c). The entries ε_{22} and ε_{24} automatically have the correct sign since $\varepsilon_{22} = -2\varepsilon_{21}$ and $\varepsilon_{24} = -\varepsilon_{23}/2$.

The situation is complicated by the fact that there are choices of routes between the rotating waves (1) and (4) and between the standing waves (2) and (3). For existence of the cycle, at least one route must satisfy the appropriate eigenvalue conditions. The choices (4.21d) ensure that (2) and (3) are connected through Fix (9) or Fix (10) with a trajectory going in the correct direction. Similarly, (4.21e) guarantees a connection between (1) and (4) through Fix (7) or Fix (8).

Note that conditions $\varepsilon_k > -2$ are conditions that guarantee boundedness in Fix $(k+2)$. By Remark 2.7, boundedness is automatic in Fix (5) and Fix (6) and there are no submaximal solutions in any of the relevant fixed point planes. □

Each of the cycles whose existence is guaranteed by Theorem 4.8 can be asymptotically stable. Conditions sufficient to imply stability are given in the next theorem.

THEOREM 4.9. *Assume* (4.21) *so that heteroclinic cycles in* (4.3) *exist as guaranteed by Theorem* 4.8. *These cycles may be formed in one of four ways depending on whether* (4.21d(i)) *or* (4.21d(ii)) *is valid and whether* (4.21e(i)) *or*

(4.21e(ii)) is valid. In each case we can have asymptotic stability and this stability may be determined as follows.

At each node in the cycle there is a radial eigenvalue r_i (with eigenvector in $\mathrm{Fix}(\Sigma_i)$*), a contracting eigenvalue c_i and an expanding eigenvalue e_i (with eigenvectors in* $\mathrm{Fix}(T_{i-1}) + \mathrm{Fix}(T_i)$*), and a transverse eigenvalue t_i. Asymptotic stability generically holds if: $r_i < 0$, $c_i < 0$, $e_i > 0$ (which identifies c_i and e_i), $t_i < 0$, and $\prod_{i=1}^4 \min\{r_i, c_i, e_i - t_i\} > \prod_{i=1}^4 e_i$.*

We now identify these eigenvalues in each of the four cases. First, set $r_i = \varepsilon_{1i}$.

If (4.21d(i)) *and* (4.21e(i)) *are valid*:

$$t_1 = \varepsilon_{41}, \quad t_2 = \varepsilon_{42}, \quad t_3 = \varepsilon_{43}, \quad \text{and} \quad t_4 = \varepsilon_{34},$$

and either

$$c_1 = \varepsilon_{31}, \quad c_2 = \varepsilon_{22}, \quad c_3 = \varepsilon_{33}, \quad \text{and} \quad c_4 = \varepsilon_{24},$$

$$e_1 = \varepsilon_{21}, \quad e_2 = \varepsilon_{32}, \quad e_3 = \varepsilon_{23}, \quad \text{and} \quad e_4 = \varepsilon_{44},$$

or the c's and e's can be interchanged.

If (4.21d(i)) *and* (4.21e(ii)) *are valid*:

$$t_1 = \varepsilon_{31}, \quad t_2 = \varepsilon_{42}, \quad t_3 = \varepsilon_{43}, \quad \text{and} \quad t_4 = \varepsilon_{44},$$

and either

$$c_1 = \varepsilon_{41}, \quad c_2 = \varepsilon_{22}, \quad c_3 = \varepsilon_{33}, \quad \text{and} \quad c_4 = \varepsilon_{24},$$

$$e_1 = \varepsilon_{21}, \quad e_2 = \varepsilon_{32}, \quad e_3 = \varepsilon_{23}, \quad \text{and} \quad e_4 = \varepsilon_{34},$$

or the c's and e's can be interchanged.

If (4.21d(ii)) *and* (4.21e(i)) *are valid*:

$$t_1 = \varepsilon_{41}, \quad t_2 = \varepsilon_{32}, \quad t_3 = \varepsilon_{33}, \quad \text{and} \quad t_4 = \varepsilon_{34},$$

and either

$$c_1 = \varepsilon_{31}, \quad c_2 = \varepsilon_{22}, \quad c_3 = \varepsilon_{43}, \quad \text{and} \quad c_4 = \varepsilon_{24},$$

$$e_1 = \varepsilon_{21}, \quad e_2 = \varepsilon_{42}, \quad e_3 = \varepsilon_{23}, \quad \text{and} \quad e_4 = \varepsilon_{44},$$

or the c's and e's can be interchanged.

If (4.21d(ii)) *and* (4.21e(ii)) *are valid*:

$$t_1 = \varepsilon_{31}, \quad t_2 = \varepsilon_{32}, \quad t_3 = \varepsilon_{33}, \quad \text{and} \quad t_4 = \varepsilon_{44},$$

and either

$$c_1 = \varepsilon_{41}, \quad c_2 = \varepsilon_{22}, \quad c_3 = \varepsilon_{43}, \quad \text{and} \quad c_4 = \varepsilon_{24},$$

$$e_1 = \varepsilon_{21}, \quad e_2 = \varepsilon_{42}, \quad e_3 = \varepsilon_{23}, \quad \text{and} \quad e_4 = \varepsilon_{34},$$

or the c's and e's can be interchanged.

Proof. The proof of each case proceeds as the proofs of the previous stability results. The results are more tedious to display than they are to prove.

5. Appendix: Asymptotic stability of heteroclinic cycles

In this section, we obtain sufficient conditions for asymptotic stability of heteroclinic cycles. Although this result is not the best possible, it is sufficient to prove Theorem 2.10. A simpler example of a heteroclinic cycle is studied in **[22]**, and necessary and sufficient conditions for stability are given there. We begin here by first proving the theorem in the case that Γ is finite, and then return to the case where $\dim \Gamma > 0$ at the end.

5.1. The case of finite Γ

We use the notation of Section 2. The heteroclinic cycle consists of (group orbits of) equilibria $A_1, \ldots, A_k$ lying in the flow invariant lines $l_1, \ldots, l_k$ (where $l_j = \mathrm{Fix}(\Sigma_j)$ for some maximal isotropy subgroup Σ_j) and (group orbits of) trajectories $x_j(t)$ joining A_j and A_{j+1}. For $j = 1, \ldots, k$ the lines l_j, l_{j+1} are adjacent in the flow invariant plane P_j (recall that $A_{k+1} = \gamma A_1$ and $l_{k+1} = \gamma l_1$ for some $\gamma \in \Gamma$). For each j, the trajectory $x_j(t)$ is assumed to lie in the invariant plane P_j.

For definiteness, suppose that $x_j(t)$ is forward asymptotic to A_{j+1}, for each j. Then our assumptions are that A_j is a sink in the plane P_{j-1} and a saddle in P_j.

Let a_j, b_j, c_j be the (real) eigenvalues of $(df)_{A_j}$ restricted to the three-dimensional subspace $P_{j-1} + P_j$. We have assumed that only one of these eigenvalues is positive. So, without loss of generality, assume that $a_j < b_j < 0 < c_j$. Suppose that the eigenspace corresponding to the possibly multiple eigenvalue c_j is contained inside the stable manifold of A_{j+1}. Let μ_j be the maximum real part of the remaining eigenvalues of $(df)_{A_j}$ that are not forced to be c_j by the group action. In words, Lemma 5.3 below states that the rate of contraction/expansion in a neighbourhood of the saddle point A_j is bounded by

$$v_j = \min\left\{\frac{-b_j}{c_j}, \frac{c_j - \mu_j}{c_j}\right\}, \tag{5.1}$$

as long as $\mu_j < 0$. In particular, if $-b_j > c_j$, then the flow is contracting near A_j (since $v_j > 1$).

The only singular points in a neighbourhood of the heteroclinic cycle are the equilibria A_j themselves. It follows that the flow from a neighbourhood of A_j to a neighbourhood of A_{j+1} is nonsingular. It turns out that the contraction rates at each saddle point can be combined to give:

THEOREM 5.1. *The heteroclinic cycle* $\{A_1, A_2, \ldots, A_k, A_{k+1}\}$ *is generically asymptotically stable, if*

$$\mu_j > 0, \textit{ for all } j, \textit{ and}$$

$$\prod_{i=1}^{k} \min\{-b_j, c_j - \mu_j\} > \prod_{i=1}^{k} c_j.$$

Remark 5.2. The main idea in the proof is to use Sternberg's theorem to estimate the contraction or expansion rates near each equilibrium and then to combine the rates to obtain an overall estimate (this is where the products appear in the statement of Theorem 5.1). The factors in the product come from estimates around each equilibrium, and these are described in Lemma 5.3 below. An overall contraction corresponds to stability of the cycle.

The proof relies on linearising the flow near each equilibrium by a differentiable change of coordinates. By Sternberg's theorem [**24**, **25**], this can be done, provided finitely many nonresonance conditions are obeyed by the eigenvalues at each equilibrium. Hence we are only able to prove the theorem for generic heteroclinic cycles.

Before proving this theorem, we discuss expression (5.1), with subscript j suppressed, and state Lemma 5.3. Let $L=(df)_A$ be the linearisation of f at a saddle point A. We begin by finding cross-sections θ^i and θ^0 for the inflow and outflow of trajectories near the saddle, suitable for deriving (5.1). We know that L commutes with Σ, the isotropy of A. We assume that L has three eigenvalues a, b, c (in the three-dimensional subspace $P_{j-1}+P_j$) satisfying $a<b<0<c$, and let $d_1, \ldots, d_s$ denote the remaining eigenvalues. Since commutativity of L with Σ may force eigenvalues to be multiple, we make the convention that the d_m's include only the remaining eigenvalues not forced by Σ to be equal to c. In this way we may choose a basis such that

$$L=\begin{bmatrix} a & & & \\ & b & & \\ & & cI_p & \\ & & & D \end{bmatrix},$$

where I_p is the $p\times p$ identity matrix and D is an $s\times s$ matrix with eigenvalues $d_1, \ldots, d_s$, $s=n-p-2$.

Let (w, x, y, z) denote the coordinates corresponding to this form of L, where w corresponds to a, x to b, y to the c's, and z to the d's. Let $\|y\|_0=\max\{|y_m|\}$ and define the cross-sections

$$\Omega^i=\{(w, x, y, z): |w|<1,\ x=1,\ 0<y_m<1,\ \|z\|<1\},$$
$$\Omega^0=\{(w, x, y, z): |w|<1,\ 0<x<1,\ \|y\|_0=1,\ \|z\|<1\},$$

where $\|\ \|$ is the norm on $\mathbb{R}^s$ constructed in [**16**, lemma, p. 145].

LEMMA 5.3. *The first hit map $\Phi:\Omega^i\to\Omega^0$ is well-defined and there exists a norm $|||\ |||$ on Ω^0 such that, in $(n-1)$-dimensional spherical coordinates on Ω^i,*

$$|||\Phi(r, \theta_1, \ldots, \theta_{n-2})|||\leqq Hr^v+o(r^v)$$

for some constant H.

Proof of Theorem 5.1. A change of coordinates linearises the flow in a neighbourhood of each equilibrium. Let $x_{j-1}(t)$ be a heteroclinic trajectory which is forward asymptotic to A_j. Then, as $t\to+\infty$, $x_{j-1}(t)$ is tangential to the eigenvector corresponding to b_j, since that is the *weaker* eigenvalue in P_{j-1}. Similarly, as $t\to-\infty$, $\gamma x_j(t)$ is tangential to an eigenvector corresponding to c_j for each $\gamma\in\Gamma$. It follows from Lemma 5.3 that Φ_j is defined in a deleted neighbourhood of the heteroclinic cycle.

Now define the first hit maps

$$\Psi_j: \bigcup_\gamma \gamma\Omega_j^0\to\bigcup_\gamma \gamma\Omega_{j+1}^i.$$

These first hit maps are diffeomorphisms, so

$$|||\Psi_j \,.\, \Phi_j(r, \underline{\theta})||| \leqq H' r^{v_j} + o(r^{v_j}).$$

Let g be the Poincaré map obtained as the composition of the $\psi_j \,.\, \Phi_j$, $j = 1, \ldots, k$ and let $v = \prod_{j=1}^{k} v_j$. Then

$$|||g(r, \underline{\theta})||| \leqq H'' r^{v} + o(r^{v}),$$

and Theorem 5.1 is proved. □

Proof of Lemma 5.3. The linear flow is

$$(w, x, y, z) \mapsto (e^{at}w, e^{bt}x, e^{ct}y, \exp(Dt) \,.\, z).$$

The first hit map Φ sends (w, l, y, z) into (w', x', y', z') where $\|y'\|_0 = 1$. Hence the 'time of flight' t is given by $\|e^{ct}y\|_0 = 1$ or

$$t = \frac{-1}{c} \log \|y\|_0,$$

which is positive since $0 < y_m < 1$, and $c > 0$. Therefore,

$$\Phi(w, y, z) = (\|y\|_0^{-a/c} w, \|y\|_0^{-b/c}, e^{-D(\log \|y\|_0)/c} z).$$

It follows from the proof of [16, Theorem 1, p. 145] that

$$\begin{aligned} |||e^{-D(\log \|y\|_0)/c} z||| &\leqq e^{-\mu(\log \|y\|_0)/c} \|z\| \\ &= \|y\|_0^{-\mu/c} \|z\|, \end{aligned}$$

where $\mu = \max \operatorname{Re}(d_m)$. Therefore, the first hit map $\Phi : \Omega^i \to \Omega^0$ is well-defined, and moreover

$$|||\Phi(w, y, z)|||^2 \leqq \|y\|_0^{-2a/c} |w|^2 + \|y\|_0^{-2b/c} + \|y\|_0^{-2\mu/c} \|z\|^2.$$

Hence, in spherical coordinates, we obtain the bound

$$|||\Phi(r, \theta_1, \ldots, \theta_{n-2})|||^2 \leqq r^{2.\min\{1-a/c, -b/c, 1-\mu/c\}} h(r, \theta),$$

where h is bounded in θ. The result follows since this minimum is $\leqq v$. □

5.2. The case $\dim \Gamma > 0$

We now sketch the proof of Theorem 2.10. The main differences here are that the nodes in the heteroclinic cycle may be periodic, and that the nodes (whether equilibria or periodic) may lie on continuous group orbits. The solution to both these problems is the same; we must replace the linear analysis around an equilibrium in the analysis above with a linear analysis near the whole group orbit of nodes. We note, however, that in the examples of Sections 3 and 4, we are analysing normal form vector fields where the periodic solutions actually lie on group orbits. Such a flow invariant group orbit is called a *relative* equilibrium.

Krupa [19] shows that if S is a group orbit, then in a neighbourhood of S the vector field f can be decomposed as

$$f = f_N + f_T, \tag{5.2}$$

where both f_T and f_N are equivariant, f_T is tangential to group orbits, and f_N is transverse to group orbits. Moreover, the dynamics of f may be understood as the dynamics of f_N coupled with drift along group orbits. In particular, relative equilibria of f correspond to equilibria of f_N and the (orbital) asymptotic stability of these relative equilibria is given by the asymptotic stability of the equilibria.

It follows from (5.2) that the real parts of the eigenvalues of df_N at S are precisely the real parts of the eigenvalues of df at S which correspond to eigenvectors not tangential to S. These are the eigenvalues which dominate the estimates we need to establish stability in Theorem 2.10.

We end by discussing the term 'genericity' as we apply it to Theorems 3.3, 4.3, 4.6, and 4.9 concerning asymptotic stability. When a system of ODE depends on parameters, generically the eigenvalues of the Jacobian matrix at equilibria will be nonconstant and vary continuously. For such systems, except for a measure zero closed set of parameters, these eigenvalues will satisfy the nonresonance conditions of Sternberg's theorem as required in the proof of Theorem 5.1.

Acknowledgment

We are grateful to Dieter Armbruster and Mike Field for several helpful conversations regarding the work reported here, and to Norman Dancer for pointing out an error in an earlier version of Theorem 5.1. This research was supported in part by the ACMP program of DARPA (MG and PC), by NASA-Ames Grant NAG 2-432 (MG), by NSF Grant DMS-8700897 (MG), and by a University of Houston RIG (IM). PC would also like to thank the University of Houston for a visiting position during which time this research was completed.

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(*Issued* 29 *December* 1989)