

## Bifurcations from Synchrony in Homogeneous Networks: Linear Theory\*

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**Abstract.** A *regular* network is a network with one kind of node and one kind of coupling. We show that a codimension one bifurcation from a synchronous equilibrium in a regular network is at linear level isomorphic to a generalized eigenspace of the adjacency matrix of the network, at least when the dimension of the internal dynamics of each node is greater than 1. We also introduce the notion of a product network—a network where the nodes of one network are replaced by copies of another network. We show that generically the center subspace of a bifurcation in product networks is the tensor product of eigenspaces of the adjacency matrices of the two networks.

**Key words.** coupled systems, bifurcation theory, networks

**AMS subject classifications.** 37G10, 37G40, 34C23

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**1. Introduction.** The fundamental result in this paper, Theorem 2.7, states that the center subspace of a codimension one synchrony-breaking bifurcation in a regular network is isomorphic to a generalized eigenspace of the adjacency matrix of that network, at least when the dimension of the network nodes is greater than 1. Moreover, the action of the Jacobian matrix on that center subspace is conjugate to the action of the adjacency matrix on the associated generalized eigenspace. We now discuss the various terms in these statements.

Stewart and coworkers [13, 21, 16] developed a theory for the dynamics of a network of coupled systems based on network architecture. In this theory a network consists of a finite number of nodes or cells  $C = \{c_1, \dots, c_r\}$  and a finite number of arrows. The cells represent systems of differential equations, and the arrows represent coupling between those systems. More precisely, a *cell* consists of a state space, usually taken to be a Euclidean space  $\mathbf{R}^k$ , and a system of differential equations associated with that cell. The system associated with cell  $j$  has the form

$$\dot{x}_j = f_j(x_j, x_{i_1}, \dots, x_{i_m}),$$

where the first argument in  $f_j$  represents the internal dynamics of the cell and the remaining variables represent coupling. Note that  $x_p$  appears in  $f_j$  only if cell  $p$  is coupled to cell  $j$ . The general theory allows for *self-coupling* (some  $i_q$  equals  $j$ ) and *multiple arrows* (some subsets of indices  $i_q$  are equal). The *network* of a coupled cell system is a graph that shows which cells have the same equations and which cells are coupled to which.

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Examples of such networks have been considered previously by a number of authors; we mention, in particular, Pecora and Carroll [19]. These papers discuss the stability of fully synchronous solutions. In this context the tensor structure of the linearization given in (2.2) has been observed. The master stability function, which recovers the eigenvalues of the Jacobian matrix in terms of the eigenvalues of the adjacency matrix, is related to our Lemma 2.3. Pecora and Carroll use this structure to discuss stability, that is, to determine the maximum real part of the eigenvalues or, in a more general context, the Liapunov exponents. In this paper we use the tensor structure to discuss the surprisingly nontrivial Jordan structure of the Jacobian restricted to center subspaces that is sometimes forced by network architecture. This issue is relevant for bifurcations from synchrony. For example, it has been shown in [10, 8] that the length of Jordan blocks determines the amplitude of certain bifurcating solutions obtained by Hopf bifurcation in feed-forward chains.

It is well known in the pattern formation literature that symmetry-breaking bifurcations from a stable fully symmetric equilibrium in a single reaction-diffusion equation are not possible. Such bifurcations can occur only in systems consisting of several equations. For example, Ermentrout and Lewis [9] show the need for three species to obtain a Turing symmetry-breaking Hopf bifurcation. A similar issue occurs in bifurcation from synchrony in coupled cell systems, where the role of the number of equations in reaction-diffusion systems is replaced by the dimensionality  $k$  of the internal cell dynamics. In section 7 we show that when the eigenvalues of the adjacency matrix are real, then all such steady-state synchrony-breaking bifurcations are possible when  $k \geq 2$  (Theorem 7.1) and all such Hopf bifurcations are possible when  $k \geq 3$  (Theorem 7.2). The last theorem (proved by Amit Vutha) generalizes, in the context of coupled cell systems, the observation in [9].

In this paper we consider only networks in which all cells are *identical*; that is, the state spaces are all identical ( $\mathbf{R}^{k_j} = \mathbf{R}^k$  for all  $j$ ), and the systems of differential equations are all equal ( $f_j = f$  for all  $j$ ); that is,

$$(1.1) \quad \dot{x}_j = f(x_j, x_{i_1}, \dots, x_{i_m}).$$

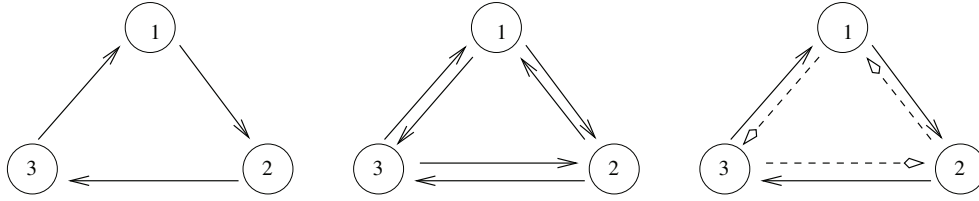
**Definition 1.1.** A homogeneous coupled cell network is a network in which all cells are *identical*.

The diagonal  $\Delta_0 = \{x : x_1 = \dots = x_r\}$  consists of points where the coordinates in each cell are identical and is always flow-invariant in homogeneous cell systems. Solutions in  $\Delta_0$  can be found by solving the single system of differential equations

$$(1.2) \quad \dot{y} = f(y; y, \dots, y).$$

Synchronous equilibria, zeros of  $f$  in (1.2), occur naturally in homogeneous cell systems. In particular, it is reasonable to assume that there is an equilibrium of an admissible vector field in  $\Delta_0$  which we can assume, without loss of generality, is at the origin.

A homogeneous network can be depicted by a graph, which consists of nodes and arrows. This graph has identical node symbols but can have different types of arrows. However, for the network to be homogeneous each node must receive the same number of inputs of each arrow type. Examples of homogeneous three-cell rings are given in Figure 1. Observe that in the third figure there are two types of coupling indicated by solid and dashed arrows.



**Figure 1.** Three-cell homogeneous networks: (left) unidirectional ring; (center) bidirectional ring; (right) directionally coupled ring.

Each network architecture has associated with it a class of admissible vector fields, as described above. The classes of admissible vector fields associated with each of the networks in Figure 1 are given by

$$\begin{array}{lll} \dot{x}_1 = f(x_1, x_3), & \dot{x}_1 = g(x_1, \overline{x_2, x_3}), & \dot{x}_1 = h(x_1, x_2, x_3), \\ \dot{x}_2 = f(x_2, x_1), & \dot{x}_2 = g(x_2, \overline{x_3, x_1}), & \dot{x}_2 = h(x_2, x_3, x_1), \\ \dot{x}_3 = f(x_3, x_2), & \dot{x}_3 = g(x_3, \overline{x_1, x_2}), & \dot{x}_3 = h(x_3, x_1, x_2). \end{array}$$

The overbar in  $g$  indicates that the two couplings to a given cell are identical and, because of this, the coupling arguments can be interchanged; that is,  $g(x, y, z) = g(x, z, y)$ . The theory in [21, 16] explains how the assignment of admissible vector fields to network architecture can be done in a functorial way (the admissible vector fields are just those that commute with the symmetry groupoid of the graph).

**Definition 1.2.** A homogeneous network is regular if all couplings are of the same type. The valency of a regular network is the number of arrows that input to each cell.

Note that the first two networks in Figure 1 are regular, whereas the third is not.

This paper considers the structure of synchrony-breaking bifurcations from a fully synchronous equilibrium in a homogeneous cell system. More specifically, suppose that

$$\dot{x} = F(x)$$

is an admissible system for a homogeneous network with  $F(0) = 0$ . By a *synchrony-breaking* bifurcation we mean that at the origin there is a critical eigenvector in the center subspace  $E^c$  of the Jacobian  $J = (dF)_0$  that is not in  $\Delta_0$ . One goal of this paper is to describe the Jordan normal form that  $J|_{E^c}$  takes at a generic (*codimension one*) synchrony-breaking bifurcation from a synchronous equilibrium. We will show that for regular networks (section 2) synchrony-breaking bifurcations can be identified with the Jordan structure of the adjacency matrix associated with the network architecture.

**Definition 1.3.** The adjacency matrix of a regular network is the matrix  $A = [a_{ij}]$ , where  $a_{ij}$  is the number of arrows from cell  $j$  to cell  $i$ .

Note that the each row sum of the adjacency matrix equals the network valency.

We prove (Theorem 2.7) that when the dimension of the internal dynamics of each cell  $k$  is at least 2, then the center subspace at bifurcation is isomorphic to a generalized eigenspace of the adjacency matrix. This theorem generalizes observations by Leite and Golubitsky [18], who studied bifurcations in three-cell systems. There is a striking analogy between eigenspaces

of the adjacency matrix in synchrony-breaking bifurcations in regular cell networks and irreducible representations in symmetry-breaking bifurcations for equivariant dynamical systems (see Remark 2.9).

Leite and Golubitsky [18] and Elmhirst and Golubitsky [8] analyzed the codimension one steady-state and Hopf synchrony-breaking bifurcations in valency one and two regular three-cell networks. It was shown in [18] that there are 34 different such networks and that all manner of center subspaces are possible in codimension one bifurcations. For simplicity assume that the dimension  $k$  of the internal dynamics phase space is 1. Then [18] shows that there are networks where the center subspace  $E^c$  at a synchrony-breaking bifurcation is one-dimensional, or is two-dimensional with two independent eigenvectors, or is two-dimensional with only one independent eigenvector (the *nilpotent* case). It is even possible for a codimension one bifurcation to have two simultaneously critical eigenvectors, with one in the subspace  $\Delta_0$  and one not in that subspace (see network 12 in [18]). In particular, the Jordan structure of the Jacobian at a synchronous equilibrium in a regular system can be quite complicated in codimension one bifurcations.

After discussing the relationship between synchrony-breaking bifurcations in regular networks and symmetry-breaking bifurcations in equivariant bifurcation theory, we focus on networks that are networks of networks. The product of two regular networks is defined in section 3, and such networks are homogeneous. Product networks have two types of coupling (one from each regular network) and two adjacency matrices. Theorem 5.5 states that generically the center subspaces of codimension one synchrony-breaking bifurcations for product networks are isomorphic to tensor products of eigenspaces of adjacency matrices, one from each of the regular networks.

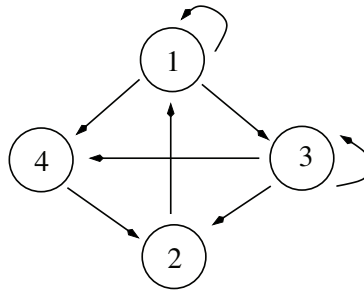
The remainder of the paper is devoted to discussions of partial results for the nonlinear bifurcations whose linear parts have been described in the beginning sections. In particular, sections 6 and 8 discuss examples of (nonlinear) synchrony-breaking bifurcations for regular networks and for product networks. A discussion of a general strategy for analyzing synchrony-breaking bifurcations is given in section 9.

**2. Center subspaces in regular networks.** In this section we discuss the generic structure of critical eigenspaces of regular networks. We fix a regular network with  $r$  cells and, as before, assume that an admissible vector field  $F$  has a synchronous equilibrium at the origin; that is,  $F(0) = 0$ . Suppose that the internal dynamics of each cell has dimension  $k = 1$ . Then

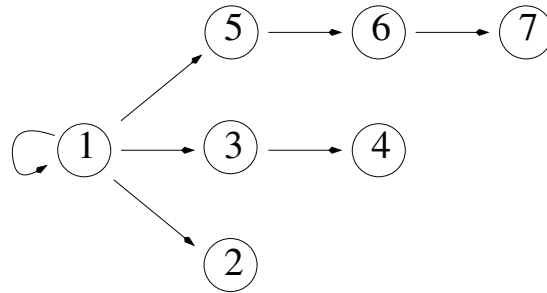
$$(dF)_0 = \alpha I + \beta A,$$

where  $A$  is the adjacency matrix. Every eigenvalue of  $J = (dF)_0$  has the form  $\alpha + \beta\mu$ , where  $\mu$  is an eigenvalue of  $A$ . It also follows that generalized eigenspaces of  $J$  are generalized eigenspaces of  $A$ . In this section, we will prove that generically center subspaces at synchrony-breaking bifurcations are isomorphic to the real parts of generalized eigenspaces of  $A$ , but this conclusion is valid in general only when  $k > 1$ .

*Remark 2.1.* Suppose  $k = 1$ . Then the real parts of the eigenvalues of  $J$  are just  $\alpha + \operatorname{Re}(\mu_j)\beta$ . So, if two of the  $A$  eigenvalues have the same real part, say  $\operatorname{Re}(\mu_1) = \operatorname{Re}(\mu_2)$ , then the real parts of  $\alpha + \mu_1\beta$  and  $\alpha + \mu_2\beta$  will be equal for all  $\alpha$  and  $\beta$ . As an example consider



**Figure 2.** Four-cell network whose adjacency matrix has eigenvalues  $2, 0, \pm i$ .



**Figure 3.** Branching feed-forward network that can exhibit the degenerate Jordan normal form  $J$  in (2.1) in a codimension one bifurcation from a synchronous equilibrium.

the four-cell network shown in Figure 2 whose adjacency matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of  $A$  are  $2, 0, \pm i$ . At a codimension one steady-state synchrony-breaking bifurcation, the center subspace is three-dimensional—the sum of the eigenspaces corresponding to the eigenvalues  $0$  and  $\pm i$ .

*Remark 2.2.* Every Jordan structure  $J$  associated to a  $0$  eigenvalue can be the Jordan structure of a real eigenvalue of an adjacency matrix of a regular network. For example, suppose that

$$(2.1) \quad J = \left[ \begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Then an example of a network whose adjacency matrix has the the Jordan structure  $J$  (corresponding to the zero eigenvalue) is given in Figure 3. We see that in the general case we can always choose that network to be a branching network of feed-forward subnets.

Next assume that the internal dynamics of each cell is  $k$ -dimensional, where  $k > 1$ . At a fully synchronous state, the Jacobian is determined by two  $k \times k$  matrices: the linearized internal dynamics  $\alpha$  and the linearized coupling  $\beta$ . These matrices are found by differentiating  $F$ , as defined in (1.1), with respect to the first two variables. Using the discussion of linearizations given in [18, section 3], we have

$$(2.2) \quad (dF)_0 = \alpha \otimes \mathbf{I}_r + \beta \otimes A.$$

To see this, first consider the linearization when the coupling matrix  $\beta$  is zero. Then the linearization has the form  $\alpha \otimes \mathbf{I}_r$  as in [18]. Assume next that the linearized internal dynamics  $\alpha$  is zero to obtain the second term. Again this proceeds as in [18].

Let  $\mu_1, \dots, \mu_s$  be the distinct eigenvalues of  $A$ . Moreover, let

$$M_{\mu_i} = \alpha + \mu_i \beta.$$

**Lemma 2.3.** *The  $kr$  eigenvalues of the Jacobian  $J$  are the union of the eigenvalues of the  $k \times k$  matrices  $M_{\mu_j}$  for  $1 \leq j \leq s$ . Specifically, suppose  $v \in \mathbf{C}^r$  is an eigenvector of  $A$ . Then*

$$(2.3) \quad J(u \otimes v) = (M_\mu u) \otimes v.$$

Therefore, if  $u \in \mathbf{C}^k$  is an eigenvector of  $M_\mu$ , then  $u \otimes v$  is an eigenvector of  $J$ .

*Proof.* Suppose that  $\mu \in \mathbf{C}$  is an eigenvalue of  $A$  with eigenvector  $v \in \mathbf{C}^r$ . Let

$$Y_v = \{u \otimes v : u \in \mathbf{C}^k\} \subset \mathbf{C}^k \otimes \mathbf{C}^r.$$

We claim that the subspace  $Y_v$  is  $J$ -invariant and that  $J|_{Y_v} = M_\mu$ . It then follows that the  $k$  eigenvalues of  $J|_{Y_v}$  are eigenvalues of  $J$ . To verify the claim, calculate

$$\begin{aligned} J(u \otimes v) &= (\alpha u) \otimes v + (\beta u) \otimes Av \\ &= (\alpha u) \otimes v + \mu(\beta u) \otimes v \\ &= ([\alpha + \mu\beta]u) \otimes v \\ &= (M_\mu u) \otimes v, \end{aligned}$$

thus verifying (2.3). Specifically,  $J|_{Y_v}$  is the matrix  $M_\mu$ . The fact that  $u \otimes v$  is an eigenvector of  $J$  whenever  $u$  is an eigenvector of  $M_\mu$  follows directly from (2.3). ■

Our primary goal is to classify the center subspaces that occur at codimension one bifurcations in regular networks. Let

$$(2.4) \quad \rho_1, \dots, \rho_p \quad \text{and} \quad \eta_1, \dots, \eta_q$$

be distinct eigenvalues of  $A$ . The  $\rho_j$  are the distinct real eigenvalues, and the  $\eta_i$  are the distinct complex eigenvalues of  $A$  whose imaginary parts are positive.

**Definition 2.4.** *A pair of real  $k \times k$  matrices  $\alpha, \beta$  is a codimension one pair if the real parts of the eigenvalues of the matrices  $M_{\rho_1}, \dots, M_{\rho_p}, M_{\eta_1}, \dots, M_{\eta_q}$  are distinct (except for complex conjugate pairs of eigenvalues of the  $M_{\rho_j}$ ). The set of all codimension one pairs is denoted by  $\mathcal{M}_2(k)$ .*

It is clear that  $\mathcal{M}_2(k)$  is an open set of  $L(k)^2$ , where  $L(k)$  is the space of real  $k \times k$  matrices. In Proposition 2.5 we prove that when  $k > 1$  the set  $\mathcal{M}_2(k)$  is also a dense subset. Therefore, the generic codimension one bifurcation will occur with a Jacobian  $J$  associated to a codimension one pair, and  $J$  will have a center subspace that corresponds to one critical eigenvalue.

**Proposition 2.5.** *For  $k > 1$ ,  $\mathcal{M}_2(k) \subset L(k)^2$  is open and dense. For  $k = 1$ ,  $\mathcal{M}_2(k)$  is open and dense when the eigenvalues of  $A$  in (2.4) have distinct real parts.*

The second main result in this section states that over the complex numbers generically the generalized eigenspace of the Jacobian matrix  $J$  is isomorphic to a generalized eigenspace of the adjacency matrix  $A$ . We begin with a straightforward lemma.

**Lemma 2.6.** *Let  $\mu \in \mathbf{C}$  be an eigenvalue of  $A$ , and let  $\mathcal{G}_A(\mu) \subset \mathbf{C}^r$  denote the associated complex generalized eigenspace. Then  $\mathbf{C}^k \otimes \mathcal{G}_A(\mu)$  is invariant under  $J$ .*

*Proof.* Let  $A_\mu = A - \mu \mathbf{I}$  and write

$$J = \alpha \otimes \mathbf{I}_r + \beta \otimes A = M_\mu \otimes \mathbf{I}_r + \beta \otimes A_\mu.$$

Clearly both summands leave the space  $\mathbf{C}^k \otimes \mathcal{G}_A(\mu)$  invariant, and hence so does  $J$ . ■

Let  $\sigma \in \mathbf{C}$  be an eigenvalue of  $J$ , and let  $\mathcal{G}_J(\sigma)$  denote the generalized eigenspace of  $J_\sigma = J - \sigma \mathbf{I}$  restricted to the invariant subspace  $\mathbf{C}^k \otimes \mathcal{G}_A(\mu)$ .

Proposition 2.5 allows us to assume that all eigenvalues of  $M_\mu$  are simple and have distinct real parts. It follows that for each  $\mu$  there is a basis of eigenvectors  $u_1, \dots, u_k$  of  $M_\mu$  in  $\mathbf{C}^k$ . In order to state the next theorem we must impose further genericity assumptions on the coupling matrix  $\beta$  as follows. For each eigenvalue  $\sigma$  of  $M_\mu$  with associated eigenvector  $u$  write

$$(2.5) \quad \beta u = \sum_j \zeta_j u_j.$$

Assume that all of the

$$(2.6) \quad \zeta_j \neq 0 \quad \text{for } j = 1, \dots, k.$$

Observe that the set of  $\beta$  that satisfy (2.5), (2.6) for a fixed  $\sigma$  is open and dense. Since there are a finite number of  $\sigma$ , the set of  $\beta$  that satisfy (2.5), (2.6) for all  $\sigma$  is also open and dense.

**Theorem 2.7.** *Assume (2.5), (2.6). Let  $\mu \in \mathbf{C}$  be an eigenvalue of  $A$ , and let  $\sigma \in \mathbf{C}$  be a simple eigenvalue of  $M_\mu$ . Then there exists an isomorphism*

$$\eta : \mathcal{G}_A(\mu) \rightarrow \mathcal{G}_J(\sigma)$$

such that on  $\mathcal{G}_A(\mu)$

$$(2.7) \quad J_\sigma \circ \eta = \eta \circ A_\mu.$$

**Remark 2.8.** A similar situation occurs if we study synchrony-breaking bifurcations from a fully synchronous limit cycle. Observe that each time dependent matrix in the variational equation has a tensor structure as in (2.2). It follows that the autonomous Floquet system keeps this tensor product structure, which leads to invariant subspaces as in Lemma 2.6.

Since the critical Floquet multiplier corresponds to the trivial eigenvalue of the adjacency matrix, this tensor product structure is inherited by the Poincaré map of the limit cycle. If the linearization of the Poincaré map satisfies the genericity conditions (2.5) and (2.6), then the result of Theorem 2.7 holds in this context as well. Although we do not have a formal proof for this last statement, we believe it is true.

Elmhirst and Golubitsky [8] proved that the amplitude growth of periodic solutions obtained from codimension one synchrony-breaking Hopf bifurcations in feed-forward networks can be unusual. Recently, Comanici [4] showed that a similar effect can be found in codimension one period-doubling bifurcations in maps. These results strongly suggest that the codimension one bifurcations from synchronous limit cycles in feed-forward networks will lead to solutions with interesting amplitude growth. However, here we have not attempted to work out the nonlinear theory that is needed to establish such a result.

*Remark 2.9.* Proposition 2.5 and Theorem 2.7 provide an analogy between synchrony-breaking bifurcations in regular networks and symmetry-breaking bifurcations in equivariant theory; however, to see this analogy clearly, we must discuss center subspaces of the real Jacobian  $J$ , not just eigenspaces of the complexification of  $J$ . In this analogy, center subspaces of  $J$  associated with real eigenvalues of  $A$  correspond to absolutely irreducible representations, and center subspaces of  $J$  associated with complex eigenvalues of  $A$  correspond to non-absolutely irreducible representations.

Proposition 2.5 implies that generically, at a synchrony-breaking bifurcation, the center subspace of  $J$  is determined by the eigenspace corresponding to one eigenvalue  $\mu$  of  $A$  and one simple critical eigenvalue  $\sigma$  of  $M_\mu$ . It then follows, as we explain next, that the center subspaces of  $J$ , and hence the codimension one synchrony-breaking bifurcations, divide into three types, which are described best by considering separately the cases when  $\mu$  is real and when  $\mu$  is not real.

- (a)  $\mu$  is real. In this case the matrix  $M_\mu$  is an arbitrary real  $k \times k$  matrix. As  $J$  varies with a parameter, so does  $M_\mu$ , and a critical eigenvalue  $\sigma$  of  $M_\mu$  is either 0 or purely imaginary.

Suppose that  $\sigma = 0$ . In this case all of the constructions in Theorem 2.7 are real, and the real subspaces  $\mathcal{G}_J(0)$  and  $\mathcal{G}_A(\mu)$  are isomorphic. Hence  $E^c(J) \cong \mathcal{G}_A(\mu)$ .

Suppose that  $\sigma$  is purely imaginary. Theorem 2.7 states that the complex vector space  $\mathcal{G}_J(\sigma)$  and the complexification of  $\mathcal{G}_A(\mu)$  are isomorphic. However, the fact that  $M_\mu$  is a real matrix implies that  $\bar{\sigma}$  is also a critical eigenvalue of  $M_\mu$ . Since the center subspace  $E^c(J)$  is the real part of  $\mathcal{G}_J(\sigma) \oplus \mathcal{G}_J(\bar{\sigma})$ , it follows that  $E^c(J) \cong \mathcal{G}_A(\mu) \oplus \mathcal{G}_A(\mu)$ .

- (b)  $\mu$  is not real. In this case the matrix  $M_\mu$  is an arbitrary complex  $k \times k$  matrix. As  $J$  varies with a parameter, so does  $M_\mu$ , and a critical eigenvalue  $\sigma$  of  $M_\mu$  is generically purely imaginary. Note, however, that generically  $\bar{\sigma}$  is not an eigenvalue of  $M_\mu$ , though it is always an eigenvalue of  $M_{\bar{\mu}}$ . Hence  $E^c(J) \cong \text{Re}(\mathcal{G}_A(\mu) \oplus \mathcal{G}_A(\bar{\mu}))$ .

In  $\Gamma$ -equivariant bifurcation theory, steady-state bifurcations correspond to absolutely irreducible representations, and Hopf bifurcations correspond to  $\Gamma$ -simple representations (see [15]). Recall that  $\Gamma$ -simple representations are either the direct sum of two copies of an absolutely irreducible representation or a single copy of a nonabsolutely irreducible representation. As we have seen, generically there can be synchrony-breaking steady-state and Hopf bifurcations in regular networks that correspond to real eigenvalues of  $A$  and Hopf bifurcations



that correspond to complex eigenvalues of  $A$ .

As an immediate consequence of Remark 2.9 we have the following characterization of center subspaces for codimension one bifurcations in regular networks.

**Corollary 2.10.** *Generically in  $\alpha, \beta$ , the center subspace  $E^c(J)$  at a codimension one steady-state bifurcation in a regular network is isomorphic to  $\mathcal{G}_A(\mu)$  for some real eigenvalue  $\mu$  of  $A$ . At a Hopf bifurcation,  $E^c(J)$  is isomorphic to  $\mathcal{G}_A(\mu) \oplus \mathcal{G}_A(\mu)$  if  $\mu$  is real or to  $\text{Re}(\mathcal{G}_A(\mu) \oplus \mathcal{G}_A(\bar{\mu}))$  if  $\mu$  is not real.*

*Proof of Proposition 2.5.* To prove the proposition we must establish that a certain finite number of sets in the pairs  $\alpha, \beta$  are open and dense. The intersection of these sets will be open and dense and contain the desired pairs of matrices.

To begin, take two eigenvalues  $\nu_a, \nu_b$  of  $A$  from the set (2.4) and consider the two sets of matrix pairs  $\alpha, \beta$ :

$$(2.8) \quad \begin{aligned} D_a &= \{(\alpha, \beta) \in L(k)^2 : M_{\nu_a} \text{ has eigenvalues with distinct real parts}\}, \\ E_{a,b} &= \{(\alpha, \beta) \in D_a \cap D_b : M_{\nu_a}, M_{\nu_b} \text{ have eigenvalues with different real parts}\}. \end{aligned}$$

Note that the sets  $D_a$  and  $E_{a,b}$  are open and that  $\mathcal{M}_2(k)$  consists of matrices in the intersection of the sets  $E_{a,b}$ . Moreover, if the sets  $D_a$  are dense in  $L(k)^2$ , then proving that  $E_{a,b}$  is dense in  $D_a \cap D_b$  suffices to prove that  $E_{a,b}$  is dense in  $L(k)^2$ .

From the point of view of this proposition the distinct eigenvalues of the adjacency matrix  $A$  in (2.4) just form a finite set of real and complex numbers. This set can be shifted by any real number  $c$ , since

$$M_\mu = (\alpha + c\beta) + (\mu - c)\beta.$$

It follows that the sets in (2.8) are open and dense for  $\nu_a, \nu_b$  if and only if they are open and dense for the set  $\nu_a - c, \nu_b - c$ . Similarly, we can scale the set of eigenvalues, since

$$M_\mu = \alpha + (c\mu) \left( \frac{1}{c}\beta \right)$$

for any real constant  $c \neq 0$ .

Thus, when we consider the sets  $D_a$  we may assume, without loss of generality, that  $\nu_a = 0$  or  $\nu_a = i$ . In the first case,  $M_{\nu_a} = \alpha$ . Using real Jordan normal forms, it is easy to show that the small perturbations of  $\alpha$  have eigenvalues all of whose real parts are unequal. In the second case,  $M_{\nu_a} = \alpha + i\beta$ , which is an arbitrary complex matrix. Now using the basis whose existence is guaranteed by complex Jordan normal form, we can see that small perturbations of  $\alpha$  and  $\beta$  will move, arbitrarily, the real parts of the eigenvalues of  $M_{\nu_a}$ . Hence, in all cases,  $D_a$  is dense.

To prove the density of the sets  $E_{a,b}$  we need to consider pairs of eigenvalues  $\nu_a$  and  $\nu_b$  in (2.4). These eigenvalues either have real parts unequal or real parts equal. Therefore, we can shift and scale these eigenvalues so that they are equal to one of the pairs:

$$(2.9) \quad \begin{array}{ll} (\nu_a, \nu_b) & \text{Re}(\nu_a) = 0; \text{Re}(\nu_b) \neq 0, \\ (i, i\omega) & 0 \leq \omega < 1. \end{array}$$

The sets  $E_{a,b}$  can be shown to be dense by explicit constructions in each case. Since  $\alpha$  and  $\beta$  are in  $D_a \cap D_b$ , the matrices  $M_{\nu_a}$  and  $M_{\nu_b}$  have distinct eigenvalues and a basis of eigenvectors.

*Case  $\operatorname{Re}(\nu_a) = 0$ ;  $\operatorname{Re}(\nu_b) \neq 0$ .* In this case we keep  $\alpha$  fixed and perturb  $\beta$  to  $\beta + \varepsilon I_k$ . This perturbation shifts the eigenvalues of  $M_{\nu_a}$  by  $\varepsilon\nu_a$ , which is purely imaginary; hence the perturbation keeps the real parts of the eigenvalues of this matrix fixed. This perturbation also shifts the eigenvalues of  $M_{\nu_b}$  by  $\varepsilon\nu_b$ . Since  $\operatorname{Re}(\nu_b) \neq 0$ , this perturbation shifts the real parts of the eigenvalues by  $\varepsilon \operatorname{Re}(\nu_b)$ . Thus, the sets  $E_{i,\nu_b}$  are dense.

*Case  $\nu_a = i$ ;  $\nu_b = i\omega$ ;  $0 \leq \omega < 1$ .* When  $k = 1$  the real parts of the eigenvalues of  $M_{\nu_a}$  and  $M_{\nu_b}$  are always equal, and separation of the real parts of the eigenvalues is impossible. So we assume  $k \geq 2$ .

The matrix  $M_{\nu_a} = \alpha + i\beta$  is an arbitrary complex matrix. We can assume, as above, that the eigenvalues of  $M_{\nu_a}$  are distinct and, by an arbitrarily small perturbation, that these eigenvalues are all complex. Fix an eigenvalue  $\lambda_a$  of  $M_{\nu_a}$  with eigenvector  $y = y_R + iy_I$ . We can also assume, after an arbitrarily small perturbation, that  $y_R$  and  $y_I$  are linearly independent so that the subspace  $Y = \mathbf{R}\{y_R, y_I\}$  is two-dimensional. Let  $p + iq$  be a perturbation matrix for which

$$(2.10) \quad (p + iq)y = 0.$$

Note that  $(M_{\nu_a} + (p+iq))y = \lambda_a y$ . So the real part of the perturbed eigenvalue of  $M_{\nu_a} + (p+iq)$  is the same as the real part of the eigenvalue  $\lambda_a$  of  $M_{\nu_a}$ .

Next assume that  $M_{\nu_b} = \alpha + i\omega\beta$  (where  $\omega \neq 1$ ) has an eigenvalue  $\lambda_b$  with eigenvector  $z = z_R + iz_I$  and  $\operatorname{Re}(\lambda_a) = \operatorname{Re}(\lambda_b)$ . Assume that  $p, q$  also satisfy

$$(2.11) \quad (p + i\omega q)z = \varepsilon z$$

for some nonzero  $\varepsilon \in \mathbf{R}$ . Equation (2.11) implies that the real part of the eigenvalue of  $M_{\nu_b} + (p + i\omega q)$  associated with the eigenvector  $z$  is  $\lambda_b + \varepsilon$ . Thus the sets  $E_{i,\omega i}$  are dense if the perturbation constraints (2.10) and (2.11) can hold simultaneously.

Assume that  $p$  is invertible, and let  $B = p^{-1}q$ . Then (2.10) can be written as

$$(I + iB)y = 0$$

or as  $By = iy$ . Since  $y_R$  and  $y_I$  are linearly independent, we can choose  $B$  to satisfy  $By_R = -y_I$  and  $By_I = y_R$ . Note that the eigenvalues of  $B$  on  $Y$  are  $\pm i$ . It follows that we can choose  $B$  so that  $I + \omega iB$  is invertible (both on and off  $Y$ ), since  $\omega \neq \pm 1$ . Next observe that (2.11) reduces to

$$p^{-1}z = \frac{1}{\varepsilon}(I + i\omega B)z.$$

Since  $I + i\omega B$  is invertible, we can choose an invertible  $p$  to satisfy this equation. Then we set  $q = pB$ .

Note that the only place where we needed to assume  $k > 1$  was in the second case, the case where two eigenvalues of the adjacency matrix  $A$  have equal real parts. ■

*Proof of Theorem 2.7.* Proposition 2.5 allows us to assume that all eigenvalues of  $M_\mu$  are simple and have distinct real parts. Then for each  $\mu$  there is a basis of eigenvectors  $u_1, \dots, u_k$  of  $M_\mu$  in  $\mathbf{C}^k$ . By assumptions (2.5) and (2.6) we assume that all of the  $\zeta_j$  are nonzero.

For the remainder of the proof we fix the eigenvalue  $\mu$ . Let  $\sigma$  be an eigenvalue of  $M_\mu$  and hence an eigenvalue of  $J$ . Let  $\mathcal{G}_J(\sigma) \subset \mathbf{C}^k \otimes \mathcal{G}_A(\mu)$  denote the generalized eigenspace corresponding to the eigenvalue  $\sigma$  of  $J$ . We prove the theorem by constructing the isomorphism  $\eta$ .

In fact, we need only show that  $\eta$  is an injection. If it is, then  $\dim \mathcal{G}_J(\sigma) \geq \dim \mathcal{G}_A(\mu)$ . Indeed, if some  $\eta$  were not a surjection, then it would follow that

$$\sum_{i=1}^k \dim \mathcal{G}_J(\sigma_i) > k \dim \mathcal{G}_A(\mu),$$

where  $\sigma_1, \dots, \sigma_k$  are the eigenvalues of  $M_\mu$ . However, since

$$\mathcal{G}_J(\sigma_1) \oplus \dots \oplus \mathcal{G}_J(\sigma_k) = \mathbf{C}^k \otimes \mathcal{G}_A(\mu),$$

it follows that

$$\sum_{i=1}^k \dim \mathcal{G}_J(\sigma_i) = k \dim \mathcal{G}_A(\mu).$$

Hence,  $\eta$  must be surjective as well as injective. Observe that as a consequence of the dimension count, we have also proved that all of the eigenvalues of  $J$  are the ones that came from the  $M_\mu$ 's in Lemma 2.3. Since generically the real parts of all of the eigenvalues  $\sigma$  are distinct (Proposition 2.5), it follows that  $E^c = \text{Re}(\mathcal{G}_J(\sigma))$ .

Next we choose a basis in  $\mathcal{G}_A(\mu)$  for which  $A$  is in Jordan normal form. Let

$$\mathcal{G}_A^j(\mu) = \ker(A_\mu^j)$$

for  $j = 1, \dots, s$ , where  $s$  is the smallest integer for which  $\mathcal{G}_A^s(\mu) = \mathcal{G}_A(\mu)$ . Let  $P^1 = \mathcal{G}_A^1(\mu)$  consist of the eigenvectors of  $A$  with eigenvalue  $\mu$ , and choose subspaces  $P^j$  for  $j = 2, \dots, s$  so that

$$\mathcal{G}_A^j(\mu) = P^j \oplus \mathcal{G}_A^{j-1}.$$

Let  $d_j = \dim P^j$  and note that  $d_s \leq \dots \leq d_1$ . Moreover, we can choose bases  $\{v_{j,1}, \dots, v_{j,d_j}\}$  of  $P^j$  so that

$$A_\mu(v_{j,m}) = v_{j-1,m}.$$

We define  $\eta$  on Jordan blocks of  $A$  restricted to  $\mathcal{G}_A(\mu)$ . More precisely, each string of generalized eigenvectors  $v_{1,m}, v_{2,m}, \dots, v_{\ell,m}$  (where  $d_{\ell+1} < m$ ) is a basis for a Jordan block of  $A$ . The construction of  $\eta$  proceeds inductively on each string.

To simplify the notation denote the string with single subscripts by  $V_1, \dots, V_\ell$ . We claim that there exist vectors  $W_1, \dots, W_\ell$  in  $\mathbf{C}^k \otimes \mathcal{G}_A(\mu)$  such that  $W_1 \neq 0$ ,  $J_\sigma(W_1) = 0$ , and

$$J_\sigma(W_j) = W_{j-1}$$

for  $2 \leq j \leq \ell \leq s$ . We then define  $\eta(V_j) = W_j$  and extend by linearity. By construction  $\eta$  satisfies (2.7). Moreover, the set of  $W_j$  so constructed is linearly independent. Note that none of the  $W_j$  equals zero. Next suppose that

$$W = \rho_1 W_1 + \dots + \rho_s W_s = 0,$$

where  $\rho_j \in \mathbf{C}$ . It follows that  $J_\sigma^{s-1}(W) = \rho_s W_1 = 0$ . Hence  $\rho_s = 0$  since  $W_1 \neq 0$ . Since  $\rho_s = 0$ , it follows that  $J_\sigma^{s-2}(W) = \rho_{s-1} W_1 = 0$ . Hence,  $\rho_{s-1} = 0$ . Proceed inductively.

It is also straightforward to show that the union of vectors  $W_j$  that come from all the different strings are also linearly independent. Thus the map  $\eta : \mathcal{G}_A(\mu) \rightarrow \mathcal{G}_J(\sigma)$  is an injection. We now continue with the construction of the  $W_j$ .

Since the eigenvalues of  $M_\mu$  are simple, we can write

$$(2.12) \quad \mathbf{C}^k = \mathbf{C}\{U\} \oplus Q,$$

where  $M_\mu U_1 = \sigma U_1$  is an eigenvector and  $Q$  is the sum of the eigenspaces corresponding to eigenvalues other than  $\sigma$ . Let  $M_{\mu,\sigma} = M_\mu - \sigma \mathbf{I}_k$ . Observe that  $M_{\mu,\sigma} : \mathbf{C}^k \rightarrow Q$  and  $M_{\mu,\sigma}$  restricted to  $Q$  is invertible.

Let  $U$  be an eigenvector of  $M_\mu$  with eigenvalue  $\sigma$ . We claim that we can choose  $U_1^j = \gamma_j U$  where  $\gamma_j \in \mathbf{C}$  is nonzero and  $U_2^j, \dots, U_j^j \in Q$  so that  $W_j$  has the form

$$(2.13) \quad W_j = U_1^j \otimes V_j + \dots + U_j^j \otimes V_1.$$

To begin, define  $W_1 = U_1^1 \otimes V_1$ , where  $U_1^1 = U$ . Since the vector  $V_1$  is an eigenvector of  $A$  with eigenvalue  $\mu$ , Lemma 2.3 states that  $W_1 \neq 0$  is an eigenvector of  $J$  with eigenvalue  $\sigma$ ; that is,  $J_\sigma(W_1) = 0$ .

Next we find  $\gamma_2 \in \mathbf{C}$ ,  $U_2^2 \in Q$ , and  $W_2$ . Compute

$$\begin{aligned} J_\sigma(W_2) &= J_\sigma(U_1^1 \otimes V_2 + U_2^2 \otimes V_1) \\ &= \gamma_2(M_{\mu,\sigma}U) \otimes V_2 + \gamma_2(\beta U) \otimes (A_\mu V_2) + (M_{\mu,\sigma}U_2^2) \otimes V_1 + (\beta U_2^2) \otimes (A_\mu V_1) \\ &= \gamma_2(\beta U) \otimes V_1 + (M_{\mu,\sigma}U_2^2) \otimes V_1 \end{aligned}$$

since  $M_{\mu,\sigma}U = 0$ ,  $A_\mu V_1 = 0$ , and  $A_\mu V_2 = V_1$ . It follows that to solve  $J_\sigma(W_2) = W_1 = U \otimes V_1$  we must solve the equation

$$(2.14) \quad \gamma_2 \beta U + M_{\mu,\sigma}U_2^2 - U = 0$$

for  $U_2^2 \in \mathbf{C}^k$  and  $\gamma_2 \in \mathbf{C}$ . Using the genericity condition (2.5), we can set  $\gamma_2 = \zeta_j^{-1}$ , where  $u_j$  is the eigenvector in (2.5) that corresponds to the eigenvalue  $\sigma$ . It follows that (2.14) can now be written in the form

$$M_{\mu,\sigma}U_2^2 = q,$$

where  $q \in Q$ . Since  $M_{\mu,\sigma}$  is invertible on  $Q$ , this equation can be solved for  $U_2^2 \in Q$ .

Next, we assume that  $W_{j-1}$  has been defined in  $\mathbf{C}^k \otimes \mathcal{G}_A(\mu)$  so that  $J_\sigma(W_{j-1}) = W_{j-2}$ . To complete the induction, we must find  $W_j \in \mathbf{C}^k \otimes \mathcal{G}_A(\mu)$  and  $\gamma_j \in \mathbf{C}$  so that  $J_\sigma(W_j) = W_{j-1}$ . Let  $W_j$  have the form in (2.13). Compute

$$\begin{aligned} J_\sigma(W_j) &= J_\sigma(U_1^j \otimes V_j + \dots + U_j^j \otimes V_1) \\ &= J_\sigma(U_1^j \otimes V_j) + \dots + J_\sigma(U_j^j \otimes V_1) \\ &= \gamma_j(\beta U) \otimes V_{j-1} + \sum_{p=2}^{j-1} ((M_{\mu,\sigma}U_p^j) \otimes V_{j+1-p} + (\beta U_p^j) \otimes V_{j-p}) + (M_{\mu,\sigma}U_j^j) \otimes V_1 \\ &= \sum_{p=1}^{j-1} (\beta U_p^j + M_{\mu,\sigma}U_{p+1}^j) \otimes V_{j-p}. \end{aligned}$$

Recall that  $W_{j-1}$  has the form

$$W_{j-1} = U_1^{j-1} \otimes V_{j-1} + \dots + U_{j-1}^{j-1} \otimes V_1,$$

where  $U_1^{j-1} = \gamma_{j-1}U$  for some  $\gamma_{j-1} \in \mathbf{C}$ . Thus to solve for  $W_j$  such that  $J_\sigma(W_j) = W_{j-1}$ , we need to solve the system of equations

$$\begin{aligned} (\gamma_j\beta - \gamma_{j-1})U + M_{\mu,\sigma}U_2^j &= 0, \\ \beta U_2^j - U_2^{j-1} + M_{\mu,\sigma}U_3^j &= 0, \\ \vdots &= \vdots \\ \beta U_{j-1}^j - U_{j-1}^{j-1} + M_{\mu,\sigma}U_j^j &= 0, \end{aligned}$$

for  $U_2^j, \dots, U_j^j \in Q$  and  $\gamma_j \in \mathbf{C}$ . As in the case  $j = 2$  we can solve the first equation by choosing  $\gamma_j$  so that  $(\gamma_j\beta - \gamma_{j-1})U \in Q$ . Then one can invert  $M_{\mu,\sigma}$  to solve for  $U_2^j \in Q$ . The remaining equations have the form

$$\begin{aligned} M_{\mu,\sigma}U_3^j &= q_3, \\ \vdots &= \vdots \\ M_{\mu,\sigma}U_j^j &= q_j, \end{aligned}$$

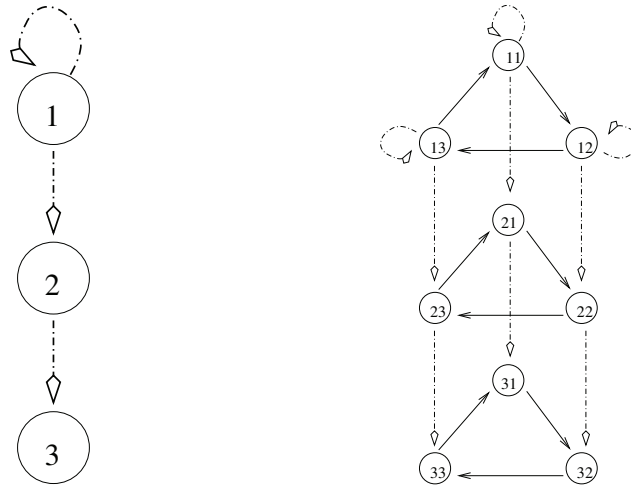
where  $q_3, \dots, q_j \in Q$ , and can also be solved for  $U_3^j, \dots, U_j^j \in Q$  by inverting  $M_{\mu,\sigma}$ .  $\blacksquare$

**3. Product networks.** Suppose that  $N_1$  and  $N_2$  are two regular networks of sizes  $r_1$  and  $r_2$  with cells  $c_1, \dots, c_{r_1}$  and  $d_1, \dots, d_{r_2}$ , respectively. Form the *product* network  $N = N_1 \boxtimes N_2$  of size  $r = r_1 r_2$ , where each node  $c_i$  in network  $N_1$  is replaced by a copy of network  $N_2$ . Let  $p_{ij}$  be the  $j$ th cell in the copy of network  $N_2$  that replaces cell  $c_i$ . So  $p_{ij}$  is a copy of cell  $d_j$ . Examples of product networks from the point of view of symmetry were considered by Dangelmayr, Güttinger, and Wegelin [6] and Dionne, Golubitsky, and Stewart [7]. The stability of synchronous dynamics in products of bidirectional graphs (without self-coupling or multiple arrows) was also considered by Atay and Biyikoğlu [2]. In particular, these authors calculated the eigenvalues of the Laplacian matrix which, in the kinds of graphs they considered, is directly related to the eigenvalues of the adjacency matrix.

We assume that there is an arrow from cell  $p_{ij}$  to cell  $p_{\ell j}$  if and only if there is an arrow from cell  $c_i$  to cell  $c_\ell$  in network  $N_1$ . We also assume that there is an arrow from cell  $p_{ij}$  to cell  $p_{i\ell}$  if and only if there is an arrow from cell  $d_j$  to cell  $d_\ell$  in network  $N_2$ . Finally, we assume that these two types of arrows are different. So there are two arrow types in the homogeneous network  $N$ —those arrows that connect cells within a given copy of  $N_2$  and those arrows that connect cells between copies of  $N_2$ . In this sense, the product networks we consider differ from product networks in the graph theory literature.

As an example, consider the three-cell feed-forward network  $N_1$  pictured in Figure 4(left). Let  $N_2$  be the unidirectional ring pictured in Figure 1(left). Then  $N = N_1 \boxtimes N_2$  is the 9-cell network in Figure 4(right). Note that  $N$  is homogeneous with two types of coupling. The locomotor central pattern generators modeling the gaits of  $n$  legged animals introduced in [14] provide a second example. Those networks are the product of a unidirectional ring of  $n$  cells with a ring of two cells.

It is easy to write the adjacency matrices corresponding to the two types of coupling in terms of tensor products. Let  $A_1, A_2$  be the adjacency matrices for the networks  $N_1, N_2$ . Formally, write  $p_{ij} = c_i \otimes d_j$ . Then the adjacency matrix of couplings in network  $N$  corresponding



**Figure 4.** (Left) three-cell feed-forward network; (right) nine-cell product of three-cell feed-forward network with three-cell unidirectional ring.

to the couplings in network  $N_1$  is  $A_1 \otimes \mathbf{I}_{r_2}$ , and the adjacency matrix corresponding to the couplings in network  $N_2$  is  $\mathbf{I}_{r_1} \otimes A_2$ .

Similarly, if we assume that the internal dynamics of cells in  $N$  is one-dimensional, it is convenient to write the state space of  $N$  as

$$\mathbf{R}^r = \mathbf{R}^{r_1} \otimes \mathbf{R}^{r_2}.$$

Let  $u_1, \dots, u_{r_1}$  be the standard basis for  $\mathbf{R}^{r_1}$  and  $v_1, \dots, v_{r_2}$  be the standard basis for  $\mathbf{R}^{r_2}$ . Then  $u_i \otimes v_j$  is the standard basis for  $\mathbf{R}^r$ .

Next suppose that  $\dot{z} = F(z)$  is an admissible vector field for the product network with  $F(0) = 0$ . Then the Jacobian has the form

$$(3.1) \quad (dF)_0 = \alpha \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} + \beta_1 A_1 \otimes \mathbf{I}_{r_2} + \beta_2 \mathbf{I}_{r_1} \otimes A_2,$$

where  $\alpha$  is the linearized internal dynamics and  $\beta_1, \beta_2$  are the linearized coupling strengths for the couplings in networks  $N_1, N_2$ . It follows that if  $v \in \mathbf{R}^{r_1}$  is an eigenvector of  $A_1$  with eigenvalue  $\mu$  and  $w \in \mathbf{R}^{r_2}$  is an eigenvector of  $A_2$  with eigenvalue  $\nu$ , then  $v \otimes w$  is an eigenvector of  $(dF)_0$  with eigenvalue  $\alpha + \beta_1 \mu + \beta_2 \nu$ .

The assumption that the product network internal dynamics is one-dimensional is convenient because we can then identify the phase space  $P_{ij}$  for cell  $i, j$  in the product network with  $P_i^1 \otimes P_j^2$ , where  $P_i^1$  is the phase space for cell  $i$  in network  $N_1$  and  $P_j^2$  is the phase space for cell  $j$  in network  $N_2$ . This identification, which is possible because  $\mathbf{R} = \mathbf{R} \otimes \mathbf{R}$ , allows us to write the linearization of a vector field at a synchronous equilibrium in the simple form given in (3.1). However, the assumption of one-dimensional internal dynamics is restrictive, because many generic codimension one Hopf bifurcations cannot occur unless the dimension of the internal dynamics is at least 2.

In the theory of coupled cell systems developed in [21, 16] the dimensions of cell phase spaces are fixed but arbitrary. In homogeneous networks, these phase spaces all have the same

dimension. However, in a product network, where the dimension of  $P_{ij}$  is  $k > 1$ , there is no natural way to think of this space as a tensor product. Said more strongly, there is no natural way to think of the admissible vector fields on a product network as tensor products of vector fields on the original networks.

We now assume that  $P_{ij} = \mathbf{R}^k$ , where  $k \geq 1$ . The full phase space of a coupled system corresponding to the product network consists of  $r_1 r_2$  copies of  $\mathbf{R}^k$ . Thus, we can write the total phase as

$$P = \mathbf{R}^k \otimes \mathbf{R}^{r_1} \otimes \mathbf{R}^{r_2},$$

and we can use this tensor product decomposition of phase space to compute the Jacobian of a general admissible vector field at a synchronous equilibrium. To begin, recall from [16] applied to a product network that an admissible vector field has the form

$$\dot{x}_{ij} = f(x_{ij}, x_{I_{ij}}),$$

where  $x_{ij} \in \mathbf{R}^k$  are the coordinates associated to cell  $i, j$ ;  $I_{ij}$  is an index set consisting of cells connected to cell  $i, j$ ; and the function  $f$  is independent of  $i, j$  since the product of homogeneous networks is homogeneous. However, there are two types of coupling in product networks. We can denote the indices of those cells that are coupled to cell  $i, j$  using the couplings in network  $N_1$  by  $I_i, j$ , where  $I_i$  consists of those cells in network  $N_1$  that are coupled to cell  $i$ . Similarly, those that are coupled to cell  $i, j$  using the couplings in network  $N_2$  can be denoted by  $i, J_j$ . Thus, a product network admissible vector field has the form

$$(3.2) \quad \dot{x}_{ij} = f(x_{ij}, \overline{x_{I_i, j}}, \overline{x_{i, J_j}}),$$

where the function  $f$  is invariant under permutation of the coordinates under an overbar.

At a fully synchronous state, the Jacobian is determined by three  $k \times k$  matrices: the linearized internal dynamics  $\alpha$ , the linearized coupling  $\beta_1$  due to the couplings in network  $N_1$ , and the linearized coupling  $\beta_2$  due to the couplings in network  $N_2$ . These matrices are found by differentiating  $f$  in (3.2) in the first, second, and third types of variables, respectively. Using the discussion of linearizations given in [18, section 3], we can now generalize the formula for the Jacobian in (3.1) to the case  $k > 1$ . Specifically,

$$(3.3) \quad (dF)_0 = \alpha \otimes \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} + \beta_1 \otimes A_1 \otimes \mathbf{I}_{r_2} + \beta_2 \otimes \mathbf{I}_{r_1} \otimes A_2.$$

To see this first consider the linearization when the coupling matrices  $\beta_j$  are zero. Then the linearization has the form  $\alpha \otimes \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2}$  as in [18]. Assume next that the linearized internal dynamics  $\alpha$  and one of the coupling matrices are zero to obtain the second and third terms. Again this proceeds as in [18].

It is possible to rewrite (3.3) in a more convenient form for later calculations. Let  $\mu$  and  $\nu$  be complex numbers and let

$$(3.4) \quad M_{\mu, \nu} = \alpha + \mu\beta_1 + \nu\beta_2, \quad A_{1, \mu} = A_1 - \mu\mathbf{I}_{r_1}, \quad A_{2, \nu} = A_2 - \nu\mathbf{I}_{r_2}.$$

Let  $J = (dF)_0$ . Then

$$(3.5) \quad J = M_{\mu, \nu} \otimes \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} + \beta_1 \otimes A_{1, \mu} \otimes \mathbf{I}_{r_2} + \beta_2 \otimes \mathbf{I}_{r_1} \otimes A_{2, \nu}.$$

To verify (3.5) compute

$$\begin{aligned}
J &= \alpha \otimes \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} + \beta_1 \otimes A_1 \otimes \mathbf{I}_{r_2} + \beta_2 \otimes \mathbf{I}_{r_1} \otimes A_2 \\
&= \alpha \otimes \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} + \beta_1 \otimes (A_{1,\mu} + \mu \mathbf{I}_{r_1}) \otimes \mathbf{I}_{r_2} + \beta_2 \otimes \mathbf{I}_{r_1} \otimes (A_{2,\nu} + \nu \mathbf{I}_{r_2}) \\
&= (\alpha + \mu \beta_1 + \nu \beta_2) \otimes \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} + \beta_1 \otimes A_{1,\mu} \otimes \mathbf{I}_{r_2} + \beta_2 \otimes \mathbf{I}_{r_1} \otimes A_{2,\nu} \\
&= M_{\mu,\nu} \otimes \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} + \beta_1 \otimes A_{1,\mu} \otimes \mathbf{I}_{r_2} + \beta_2 \otimes \mathbf{I}_{r_1} \otimes A_{2,\nu}.
\end{aligned}$$

**Proposition 3.1.** *Let  $\mathcal{G}_1(\mu)$  be the complex generalized eigenspace of  $A_1$  corresponding to the eigenvalue  $\mu$ , and let  $\mathcal{G}_2(\nu)$  be the complex generalized eigenspace of  $A_2$  corresponding to the eigenvalue  $\nu$ . Then the following hold:*

(a) *the complex subspace*

$$(3.6) \quad V_{\mu,\nu} = \mathbf{C}^k \otimes \mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu) \subset \mathbf{C}^k \otimes \mathbf{C}^{r_1} \otimes \mathbf{C}^{r_2}$$

*is invariant under  $J$ , where the Jacobian matrix  $J$  is defined in (3.5).*

(b) *Let  $A_1 v = \mu v$  and  $A_2 w = \nu w$ . Then*

$$(3.7) \quad J(u \otimes v \otimes w) = (M_{\mu,\nu} u) \otimes v \otimes w.$$

*Hence every eigenvalue of  $M_{\mu,\nu}$  is an eigenvalue of  $J$ .*

*Proof.* (a) Let  $u \otimes v \otimes w$  be in  $V_{\mu,\nu}$ , and use (3.5) to compute

$$J(u \otimes v \otimes w) = (M_{\mu,\nu} u) \otimes v \otimes w + (\beta_1 u) \otimes (A_{1,\mu} v) \otimes w + (\beta_2 u) \otimes v \otimes (A_{2,\nu} w).$$

Since  $A_{1,\mu} v \in \mathcal{G}_1(\mu)$  and  $A_{2,\nu} w \in \mathcal{G}_2(\nu)$ , invariance of  $V_{\mu,\nu}$  is established.

(b) By assumption,  $A_{1,\mu} v = A_{2,\nu} w = 0$ . Therefore

$$J(u \otimes v \otimes w) = (M_{\mu,\nu} u) \otimes v \otimes w.$$

So every eigenvalue of  $M_{\mu,\nu}$  is an eigenvalue of  $J$ . ■

**Remark 3.2.** Suppose that  $\mu$  and  $\nu$  are both real. Then  $M_{\mu,\nu}$  is a real matrix. Let  $\mathcal{G}_1^R(\mu)$  and  $\mathcal{G}_2^R(\nu)$  denote the real generalized eigenspaces of  $A_1$  and  $A_2$  corresponding to the eigenvalues  $\mu$  and  $\nu$ , respectively. Then the real subspace

$$(3.8) \quad V_{\mu,\nu}^R = \mathbf{R}^k \otimes \mathcal{G}_1^R(\mu) \otimes \mathcal{G}_2^R(\nu) \subset \mathbf{R}^k \otimes \mathbf{R}^{r_1} \otimes \mathbf{R}^{r_2}$$

is invariant under  $J$ .

Let  $\mu_1, \dots, \mu_{s_1}$  be the distinct eigenvalues of the adjacency matrix  $A_1$ , and let  $\nu_1, \dots, \nu_{s_2}$  be the distinct eigenvalues of the adjacency matrix  $A_2$ .

**Definition 3.3.** *A triple of real  $k \times k$  matrices  $\alpha, \beta_1, \beta_2$  is a codimension one triple if, except for complex conjugate eigenvalues, the real parts of the  $s_1 s_2 k$  eigenvalues of the matrices  $M_{\mu_j, \nu_i}$  are all distinct. We denote the set of all codimension one triples by  $\mathcal{M}_3(k)$ .*

It is clear that  $\mathcal{M}_3(k)$  is an open set of  $L(k)^3$ , where  $L(k)$  is the space of  $k \times k$  matrices. In Proposition 3.4 we prove that when  $k > 1$ , the set  $\mathcal{M}_3(k)$  is also a dense subset. Therefore, the generic codimension one bifurcation will occur with a Jacobian  $J$  associated with a codimension one triple, and  $J$  will have a center subspace that corresponds to one critical eigenvalue.



**Proposition 3.4.** For  $k > 1$ ,  $\mathcal{M}_3(k) \subset L(k)^3$  is open and dense. If  $k = 1$ ,  $\mathcal{M}_3(k)$  is open and dense unless either  $A_1$  or  $A_2$  have eigenvalues with equal real parts.

*Proof.* Let  $\mu_1, \mu_2$  be eigenvalues of  $A_1$  and  $\nu_1, \nu_2$  be eigenvalues of  $A_2$ . Assume that  $k = 1$ . Then the corresponding eigenvalues are

$$M_{\mu_1, \nu_1} = \alpha + \mu_1 \beta_1 + \nu_1 \beta_2 \quad \text{and} \quad M_{\mu_2, \nu_2} = \alpha + \mu_2 \beta_1 + \nu_2 \beta_2,$$

and they have equal real part if and only if

$$(3.9) \quad \beta_1(\operatorname{Re}(\mu_1) - \operatorname{Re}(\mu_2)) + \beta_2(\operatorname{Re}(\nu_1) - \operatorname{Re}(\nu_2)) = 0.$$

Identity (3.9) can hold for an open set of  $\beta_1, \beta_2$  if and only if both

$$(3.10) \quad \operatorname{Re}(\mu_1) = \operatorname{Re}(\mu_2) \quad \text{and} \quad \operatorname{Re}(\nu_1) = \operatorname{Re}(\nu_2).$$

Suppose that  $\mu_1$  and  $\mu_2$  have equal real parts and  $\nu_1 = \nu_2$ . Then the real parts of the eigenvalues  $M_{\mu_1, \nu_1}$  and  $M_{\mu_2, \nu_1}$  will be the same independent of the choice  $\alpha, \beta_1, \beta_2$ . A similar conclusion holds if the  $\nu$ 's have equal real parts and the  $\mu$ 's are equal (or, indeed, if both the  $\mu$ 's and  $\nu$ 's have equal real parts).

We can now assume that  $k \geq 2$ . Suppose now that two eigenvalues  $\sigma_1$  from the matrix  $M_{\mu_1, \nu_1}$  and  $\sigma_2$  from the matrix  $M_{\mu_2, \nu_2}$  have equal real part. We can perturb  $\beta_1$  to  $\beta_1 + \varepsilon_1 I_k$  and  $\beta_2$  to  $\beta_2 + \varepsilon_2 I_k$ . The perturbed eigenvalues  $\sigma_j + \varepsilon_1 \mu_j + \varepsilon_2 \nu_j$  can have equal real parts for an open set of  $\varepsilon_j$  if and only if (3.10) is valid. Thus we have reduced the proof to two cases:

- (a)  $\operatorname{Re}(\mu_1) = \operatorname{Re}(\mu_2)$  and  $\nu_1 = \nu_2$ , where  $\mu_1 \neq \mu_2$ , or the reverse in the  $\mu$ 's and  $\nu$ 's.
- (b)  $\operatorname{Re}(\mu_1) = \operatorname{Re}(\mu_2)$  and  $\operatorname{Re}(\nu_1) = \operatorname{Re}(\nu_2)$ , where  $\mu_1 \neq \mu_2$  and  $\nu_1 \neq \nu_2$ .

*Case (a).* As in the proof of Proposition 2.5, we can shift and scale the eigenvalues  $\nu$  so that  $\nu_1 = \nu_2 = 0$  or  $\nu_1 = \nu_2 = i$ . In the first case, the matrices are  $M_{\mu_1, \nu_1} = \alpha + \mu_1 \beta_1$  and  $M_{\mu_2, \nu_2} = \alpha + \mu_2 \beta_1$ . For these matrices we can apply Proposition 2.5 directly to perturb the matrices  $\alpha, \beta_1$  to separate the real parts of the eigenvalues of  $M_{\mu_1, \nu_1}, M_{\mu_2, \nu_2}$ . So we can assume

$$M_{\mu_1, \nu_1} = \alpha + \mu_1 \beta_1 + i \beta_2 \quad \text{and} \quad M_{\mu_2, \nu_2} = \alpha + \mu_2 \beta_1 + i \beta_2.$$

Choose vectors  $v_1, v_2$  so that  $M_{\mu_1, \nu_1} v_1 = \sigma_1 v_1$  and  $M_{\mu_2, \nu_2} v_2 = \sigma_2 v_2$ . Moreover, we can perturb  $\beta_2$  so that the vectors  $\beta_2 v_1$  and  $\beta_2 v_2$  are linearly independent over  $\mathbf{C}$ . Note that  $C = \alpha + i \beta_2$  is an arbitrary complex matrix and that

$$C v_1 = -(\sigma_1 + \beta_2) v_1 \quad \text{and} \quad C v_2 = -(\sigma_2 + \beta_2) v_2.$$

It follows from linear independence that we can perturb  $C$  so that  $C v_2$  is unchanged whereas  $C v_1 = -(\sigma_1 + \varepsilon + \beta_2) v_1$ . It follows that the real parts of the eigenvalues  $\sigma_1, \sigma_2$  are separated.

*Case (b).* Write

$$M_{\mu\nu} = \alpha + \mu \beta_1 + \nu \beta_2$$

and use the scaling argument as before to write

$$M_{\mu, \nu} = (\alpha + c_1 \beta_1 + c_2 \beta_2) + (\mu - c_1) \beta_1 + (\nu - c_2) \beta_2.$$

This way we can achieve that the coefficients of  $\beta_j$  are purely imaginary. By the second scaling,

$$\mu\beta_1 \mapsto (c\mu)\frac{1}{c}\beta_1,$$

and a similar argument for the second coefficient we have

$$\mu_1 = 0 \quad \text{or} \quad \mu_1 = i$$

and similarly

$$\nu_1 = 0 \quad \text{or} \quad \nu_1 = i.$$

Applying the same scalings to  $\mu_2$  and  $\nu_2$  gives us

$$\mu_2 = si \quad \text{and} \quad \nu_2 = ti$$

for some real numbers  $s, t \neq 0, 1$ . We have three cases to distinguish:

1.  $\mu_1 = \nu_1 = 0, \mu_2 = \nu_2 = i$ .
2.  $\mu_1 = 0, \nu_1 = i$  and  $\mu_2 = i, \nu_2 = ti, 0 \neq t \neq 1$ .
3.  $\mu_1 = i, \nu_1 = i$  and  $\mu_2 = si, \nu_2 = ti, 0 \neq s, t \neq 1$ .

In the first case we consider

$$M_{\mu_1, \nu_1} = \alpha \quad \text{and} \quad M_{\mu_2, \nu_2} = \alpha + i(\beta_1 + \beta_2).$$

In this case our perturbation argument is the same as in part (a).

In the second case we look at the matrices

$$\alpha + i\beta_2 \quad \text{and} \quad \alpha + is\beta_1 + it\beta_2,$$

where  $s, t$  are as before. Again the first matrix is an arbitrary complex matrix. We can apply the argument from  $\nu_a = i; \nu_b = i\omega$  given in the second part of the proof of Proposition 2.5.

In the third case, we look at matrices

$$M_{\mu_1, \nu_1} = \alpha + i(\beta_1 + \beta_2) \quad \text{and} \quad M_{\mu_2, \nu_2} = \alpha + is\beta_1 + it\beta_2$$

with  $s, t \neq 0, 1$ . Again we can use the argument given in the second part of the proof of Proposition 2.5 to show that a perturbation exists which splits the real parts of the eigenvalues. ■

**4. Center subspaces for product networks.** As we have seen, the eigenvalues  $\sigma$  of the Jacobian  $J$  of an admissible system of differential equations associated with a product network are the union of the eigenvalues of  $M_{\mu, \nu}$ , where  $\mu, \nu$  vary over the sets of eigenvalues of the adjacency matrices  $A_1$  and  $A_2$ , respectively. Recall that the matrix  $M_{\mu, \nu}$  is defined in (3.7). We assume that the  $k \times k$  matrices  $\alpha, \beta_1, \beta_2$  that define  $J$  form a codimension one triple. Note that

$$\overline{M_{\mu, \nu}} = M_{\bar{\mu}, \bar{\nu}},$$

so that there may be some redundancy in the enumeration of eigenvalues of  $J$ . To eliminate possible redundancies, let  $\mathcal{P}$  be a set of eigenvalue pairs that includes precisely one of the

eigenvalue pairs  $\mu, \nu$  and  $\bar{\mu}, \bar{\nu}$ . It follows from Proposition 3.4 that the eigenvalues  $\sigma$  corresponding to  $M_{\mu, \nu}$  with the pair  $\mu, \nu$  in  $\mathcal{P}$  all have distinct real parts, at least when  $k > 1$ , which we now assume. It is also the case that generically  $\sigma$  is a simple eigenvalue of  $M_{\mu, \nu}$  and generically  $\sigma$  is not real if either  $\mu$  or  $\nu$  is not real.

Given these genericity assumptions, the center subspace of  $J$  is the real part of the generalized eigenspace of  $J$  corresponding to an eigenvalue  $\sigma$  with 0 real part. To make the content of this observation more transparent we distinguish two cases:

- (a)  $\sigma = 0$ ,
- (b)  $\sigma = \omega i$ , where  $\omega \neq 0$ .

In case (a)  $\mu$  and  $\nu$  are real, and the matrix  $M_{\mu, \nu}$  is also real. Proposition 3.4 implies that generically there are no other pairs  $\mu', \nu'$  such that 0 is an eigenvalue of  $M_{\mu', \nu'}$  and that no other eigenvalue of  $J$  has zero real part. Remark 3.2 shows that the real subspace  $V_{\mu, \nu}^R$  defined in (3.8) is  $J$ -invariant. Let  $U^R \subset \mathbf{R}^k$  be the real one-dimensional kernel of  $M_{\mu, \nu}$ ; then the center subspace of  $J$  is the generalized eigenspace corresponding to the eigenvalue 0 and is given by  $U^R \otimes \mathcal{G}_1^R(\mu) \otimes \mathcal{G}_2^R(\nu) \cong \mathcal{G}_1^R(\mu) \otimes \mathcal{G}_2^R(\nu)$ . The last isomorphism is valid because  $U^R$  is real one-dimensional.

Case (b) itself divides into two parts: either  $\mu$  and  $\nu$  are both real or at least one of  $\mu, \nu$  is not real. In the first part  $M_{\mu, \nu}$  is a real matrix and  $-\omega i$  is also an eigenvalue of  $M_{\mu, \nu}$ . So the pair  $\pm \omega i$  are both eigenvalues of  $J$ , and there are no other eigenvalues of  $J$  with 0 real part. Then the real subspace  $V_{\mu, \nu}^R$  is invariant under  $J$ . Let  $U^R$  be the two-dimensional invariant subspace of  $\mathbf{R}^k$  corresponding to the simple eigenvalues  $\pm \omega i$  of  $M_{\mu, \nu}$ ; then the center subspace of  $J$  is the invariant subspace of  $J$  corresponding to these eigenvalues and is given by  $U^R \otimes \mathcal{G}_1^R(\mu) \otimes \mathcal{G}_2^R(\nu) \cong (\mathcal{G}_1^R(\mu) \otimes \mathcal{G}_2^R(\nu))^2$ . The last isomorphism is valid because  $U^R$  is real two-dimensional.

In the second part either  $\mu$  or  $\nu$  or both are not real. Then  $-\omega i$  is an eigenvalue of  $\overline{M_{\mu, \nu}} = M_{\bar{\mu}, \bar{\nu}}$ , and the center subspace of  $J$  corresponds to the generalized eigenspaces of the eigenvalues  $\pm \omega i$  of  $J$ , since no other eigenvalue of  $J$  has 0 real part. More precisely, let  $U \subset \mathbf{C}^k$  be the complex one-dimensional subspace of  $\mathbf{C}^k$  that consists of eigenvectors of  $M_{\mu, \nu}$  corresponding to the eigenvalue  $\omega i$ , and let  $\bar{U}$  be the subspace of eigenvectors of  $M_{\bar{\mu}, \bar{\nu}}$  corresponding to  $-\omega i$ . Then the space

$$(U \otimes \mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu)) \oplus (\bar{U} \otimes \mathcal{G}_1(\bar{\mu}) \otimes \mathcal{G}_2(\bar{\nu}))$$

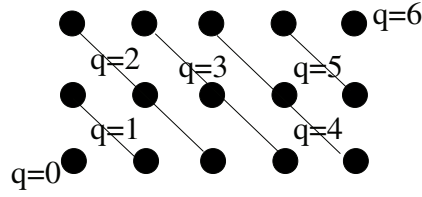
is invariant under  $J$ , and its real part

$$(4.1) \quad \{w + \bar{w} : w \in U \otimes \mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu)\} = \text{Re}(U \otimes \mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu)) \cong \text{Re}(\mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu))$$

is the generalized eigenspace of  $J$  corresponding to the eigenvalues  $\pm \omega i$ . The last isomorphism is valid because  $U$  is complex one-dimensional.

**5. Jordan structure on center subspaces.** Before we prove a result similar to Theorem 2.7 for product networks, we need to develop a deeper understanding of tensor products of Jordan blocks. Specifically, we consider

$$\mathcal{A} : \mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu) \rightarrow \mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu),$$



**Figure 5.** Rectangle of lattice sites in  $\mathbf{R}^2$  with  $\ell = 5$  and  $m = 3$ .

where

$$\mathcal{A} = A_{1,\mu} \otimes \mathbf{I}_{r_2} + \mathbf{I}_{r_1} \otimes A_{2,\nu}.$$

We can write

$$\mathcal{G}_1(\mu) = \bigoplus_{s=1}^{q_1} J_s^1(\mu) \quad \text{and} \quad \mathcal{G}_2(\nu) = \bigoplus_{s=1}^{q_2} J_s^2(\nu),$$

where  $J_s^j$  stands for a Jordan block in  $\mathcal{G}_j$ . Each block has dimension  $d_s(\mu)$  or  $d_s(\nu)$ , and we see easily that the tensor products

$$J_s^1(\mu) \otimes J_t^2(\nu)$$

are invariant under  $\mathcal{A}$ . Moreover,

$$\bigoplus_{s,t} J_s^1(\mu) \otimes J_t^2(\nu) = \mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu).$$

It is important to observe that the tensor products  $J_s^1(\mu) \otimes J_t^2(\nu)$  are not the Jordan blocks corresponding to  $\mathcal{A}$ .

Fix Jordan blocks  $J^1(\mu)$  for  $A_1$  and  $J^2(\nu)$  for  $A_2$ . Let  $v_0$  be an eigenvector of  $A_1$  in  $J^1(\mu)$ , and let  $\ell$  be the smallest positive integer for which  $A_{1,\mu}^\ell$  is identically zero on  $J^1(\mu)$ . Similarly, let  $w_0$  be an eigenvector for  $A_2$  in  $J^2(\nu)$ , and let  $m$  be the smallest positive integer for which  $A_{2,\nu}^m$  is zero on  $J^2(\nu)$ . A basis for the space  $\mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu)$  is given by the tensors  $v_j \otimes w_r$ , where  $0 \leq j \leq \ell - 1$  and  $0 \leq r \leq m - 1$ . For each  $q = 0, \dots, \ell + m - 2$ , let  $Z_q$  be the subspace spanned by the tensors  $v_s \otimes w_t$  with  $s + t = q$ . Then for  $q \geq 1$

$$\mathcal{A} : Z_q \rightarrow Z_{q-1}.$$

We say that a vector in  $Z_q$  has *order*  $q$ .

A good way to think about the subspaces  $Z_q$  is to look at the rectangle of lattice sites in  $\mathbf{R}^2$  (see Figure 5). The pairs  $(j, r)$  for which  $j + r = q$ , which lie on lines with slope  $-1$  in that figure, form a basis for the subspace  $Z_q$ . Without loss of generality assume  $\ell \geq m$ . Then we have a sequence of maps

$$(5.1) \quad Z_{\ell+m-2} \rightarrow Z_{\ell+m-3} \rightarrow \cdots \rightarrow Z_{\ell-1} \rightarrow \cdots \rightarrow Z_{m-1} \rightarrow \cdots \rightarrow Z_0.$$

Writing  $\mathcal{A}$  in coordinates with respect to the basis, we see that  $\mathcal{A}$  has maximal rank as a mapping of  $Z_q \rightarrow Z_{q-1}$  for all  $q$ .

Looking at the geometric picture in Figure 5, we see that

$$1 = \dim Z_{\ell+m-2} < \dim Z_{\ell+m-3} = 2 < \cdots < \dim Z_{\ell-1} = \dim Z_{\ell-2} = \cdots = \dim Z_{m-1}$$

and

$$1 = \dim Z_0 < \cdots < \dim Z_{m-1}.$$

Therefore,  $\mathcal{A} : Z_q \rightarrow Z_{q-1}$  is

- surjective for  $q \leq m$ ,
- an isomorphism for  $m+1 \leq q \leq \ell$ , and
- injective for  $\ell+1 \leq q$ .

So it remains to describe precisely the Jordan blocks. For each  $0 \leq q \leq m-1$  we have to give a vector in the kernel of  $\mathcal{A}$ , and in a similar way we have to give vectors in the complement to the range of  $\mathcal{A}$ , or the *cokernel* of  $\mathcal{A}$ , for  $q > \ell$ .

**Proposition 5.1.** *The kernel of  $\mathcal{A}$  is spanned by the vectors*

$$z_q = \sum_{j=0}^q (-1)^j v_j \otimes w_{q-j},$$

where  $1 \leq q \leq m-1$ , and a basis for the cokernel of  $\mathcal{A}$  is

$$z^q = \sum_{j=0}^q (-1)^j \alpha_j v_{\ell-j-1} \otimes w_{m-q+j-1},$$

with appropriately chosen  $\alpha_j$ . Moreover, we have

$$(5.2) \quad \mathcal{A}^{\ell+m-2(1+q)-1} z^q = z_q.$$

**Remark 5.2.** Observe that the vectors for the complement of the ranges are not uniquely determined. We could set all  $\alpha_j = 1$ ; however, for (5.2) to hold, we need the freedom to choose the  $\alpha_j$ .

**Proof.** Observe that  $A_{1,\mu}, A_{2,\nu}$  are nilpotent, allowing us to use representation theory for  $\mathfrak{sl}(2, \mathbf{R})$ . We decompose the representation, on the tensor product into representations, and we obtain the length of the Jordan blocks from this Clebsch–Gordan decomposition; compare [20, 5]. ■

In order to describe the way that this decomposition carries over to the decomposition into Jordan blocks for  $J_\sigma$  we need some auxiliary results.

**Lemma 5.3.**

$$U \otimes \mathbf{J}^1(\mu) \otimes \mathbf{J}^2(\nu)$$

is invariant under  $J_\sigma$ .

**Proof.** Let us write

$$M_{\mu,\nu}(\sigma) = M_{\mu,\nu} - \sigma \mathbf{I}.$$

Then, for  $u \in U$ ,  $v \in \mathbf{J}^1(\mu)$ ,  $w \in \mathbf{J}^2(\nu)$  we have

$$J_\sigma(u \otimes v \otimes w) = (M_{\mu,\nu}(\sigma)u) \otimes v \otimes w + \beta_1 u \otimes A_{1,\mu} v \otimes w + \beta_2 u \otimes v \otimes A_{2,\nu} w.$$

All three summands are in  $U \otimes J(\mu) \otimes J(\nu)$ , thus proving the invariance.  $\blacksquare$

So we can reduce our attention to tensor products of Jordan blocks. We need one result about being able to solve certain equations up to terms of lower order, which we will state first. Assuming that all eigenvalues of  $M_{\mu,\nu}$  are simple (cf. Proposition 3.4), we can write each  $u \in U$  uniquely as

$$u = \sum_{i=1}^k \xi_i u_i,$$

where each  $u_i$  is an eigenvector of  $M_{\mu,\nu}$ . Fix an eigenvalue  $\sigma$  of  $M_{\mu,\nu}$  and denote the corresponding eigenvector by  $u_0$ .

**Lemma 5.4.** *Given  $p + 1$  vectors  $x_j \in U$  and  $p$  real numbers  $\zeta_1, \dots, \zeta_p$ , where for each vector  $x_j$  the  $u_0$  component is zero. Then for*

$$z = \sum_{j=0}^p (-1)^j x_j \otimes v_j \otimes w_{p-j}$$

there exist vectors

$$y_0, \dots, y_p \in U$$

such that

$$Y = \sum_{j=0}^p y_j \otimes v_j \otimes w_{p-j}$$

solves the equation

$$J_\sigma Y = z$$

up to terms of lower order. That is,

$$z - J_\sigma Y = \sum_{j=0}^{p-1} h_j \otimes v_j \otimes w_{p-1-j},$$

with the additional property that the  $u_0$ -component of each  $h_j$  is  $\zeta_j u_0$ .

*Proof.* We begin with the case  $p = 1$ ; then we have two vectors  $x_1, x_2$ , neither having a component in the direction of  $u_0$ . We look at

$$z = x_0 \otimes v_1 \otimes w_0 - x_1 \otimes v_0 \otimes w_1.$$

Make the ansatz

$$Y = y_0 \otimes v_1 \otimes w_0 - y_1 \otimes v_0 \otimes w_1$$

and compute  $J_\sigma Y - z$ :

$$J_\sigma(y_0 \otimes v_1 \otimes w_0 - y_1 \otimes v_0 \otimes w_1) - x_0 \otimes v_1 \otimes w_0 - x_1 \otimes v_0 \otimes w_1.$$

We find (up to terms of lower order)

$$J_\sigma(y_0 v_1 \otimes w_0 - y_1 v_0 \otimes w_1) - z = (M_{\mu,\nu}(\sigma)y_0 - x_0) \otimes v_1 \otimes w_0 + (M_{\mu,\nu}(\sigma)y_1 - x_1).$$

Solving this equation requires the partial invertibility of  $M_{\mu,\nu}(\sigma)$ . Using the definition of  $Q$  as in (2.12), we find that  $M_{\mu,\nu}$  restricted to  $Q$  is invertible. We write  $M_{\mu,\nu}(\sigma)^{-1}$  for this partial inverse if we obtain solutions

$$y_a = M_{\mu,\nu}(\sigma)^{-1}x_a$$

for  $a = 0, 1$ . Observe for any  $t_a \in \mathbf{R}$

$$M_{\mu,\nu}(\sigma)(y_a + t_a u_0) = x_a.$$

Now consider the application of the remaining part of  $J_\sigma$  to

$$(y_0 + t_0 u_0) \otimes v_1 \otimes w_0 + (y_1 + t_1 u_0) \otimes v_0 \otimes w_1.$$

We obtain for the terms of order  $p - 1$

$$(\beta_1 y_0 + t_0 \beta_1 u_0) \otimes v_0 \otimes w_0 + (\beta_2 y_1 + t_1 \beta_2 u_0) \otimes v_0 \otimes w_0.$$

Now, if we make the nondegeneracy assumption that  $\beta_j u_0$  has a component in the direction of  $u_0$ , we choose  $t_0, t_1$  so that the  $u_0$ -component in this expression is  $\zeta_j u_0$ .

Repeating this computation for general  $q$  gives the induction, and we have proved the claim. ■

Theorem 5.5 needs an assumption on  $\beta_1, \beta_2$  that is similar to the genericity assumptions (2.5) and (2.6). Its precise form will be given in the proof.

**Theorem 5.5.** *Let  $\mu, \nu$  be eigenvalues of  $A_1, A_2$ , and let  $\sigma$  be a simple eigenvalue of  $M_{\mu,\nu}$ . Then generically in  $\alpha, \beta_1, \beta_2$  the corresponding eigenspace of  $J$  is isomorphic to  $\text{Re}(\mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu))$ . The isomorphism sends each eigenvector  $z_q$  of  $\mathcal{A}$  to an eigenvector  $\tilde{z}_q$  of  $J_\sigma$ , where the top part of  $\tilde{z}_q$  is given by  $u_0 \otimes z_q$ .*

As in the case of regular networks we can immediately describe the generic center subspaces for codimension one bifurcations in product networks.

**Corollary 5.6.** *Assume  $k > 1$ . Let  $E^c$  be the center subspace corresponding to a codimension one bifurcation from a fully synchronous equilibrium in a product network. Then, generically in  $\alpha, \beta_1, \beta_2$ ,  $E^c$  is as a vector space isomorphic to  $\text{Re}(\mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu))$  or to  $\text{Re}(\mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu)) \oplus \text{Re}(\mathcal{G}_1(\mu) \otimes \mathcal{G}_2(\nu))$ .*

**Example 5.7.** We consider the case of a product of the three-cell feed-forward network with a unidirectional ring with three cells and a two-dimensional internal dynamics leading to Hopf bifurcation. The eigenvalues of  $A_1$  are  $0, 0, 1$  with a geometrically simple eigenvalue  $0$ . The eigenvalues of  $A_2$  are  $1, \zeta, \zeta^2$  with  $\zeta^3 = 1$ . We consider  $\mu_1 = 0$  and  $\nu_1 = 1$  (any other choice here gives similar results). Then we can find real  $2 \times 2$ -matrices  $\alpha, \beta_1, \beta_2$  such that the corresponding eigenvalue  $\sigma$  is purely imaginary and equal to  $i$ . Then the critical eigenvalues are  $\pm i$  which are algebraically double and geometrically simple.

*Proof.* The proof of the theorem is more involved than that of Theorem 2.7. We concentrate on one tensor block. In a first step, we show that for each element in the kernel of  $\mathcal{A}$  there is an eigenvector of  $J$  corresponding to the eigenvalue  $\sigma$ .

The spaces  $Z_q$  give a grading of the space  $J^1(\mu) \otimes J^2(\nu)$ . We will use this structure and Lemma 5.4 to solve equations recursively on the various levels.

Now let us consider the kernel vectors of  $\mathcal{A}$  within  $Z_q$ , for  $q > 1$ . For the eigenvector we make the ansatz

$$\tilde{z}_q = \sum_{j=0}^q \alpha_j u_0 \otimes v_j \otimes w_{q-j} + \sum_{p < q} \sum_{j=0}^p x_j^p \otimes v_j \otimes w_{p-j}, \quad x_j \in U.$$

We have to show that we can solve the equation  $J_\sigma \tilde{z}_q = 0$ .

In a first step we consider the first term (of order  $q$ ), and we describe how to adjust the other terms  $p < q$  in a second step. Applying  $M_{\mu,\nu}(\sigma)$  to the first term gives 0 since  $u_0$  is an eigenvector of  $M_{\mu,\nu}$  for the eigenvalue  $\sigma$ . Therefore we just have to apply the other two operators. This gives a term of the form

$$\sum_{j=0}^{p-1} (\alpha_j \beta_2 u_0 - \alpha_{j+1} \beta_1 u_0) \otimes v_j \otimes w_{p-1-j}.$$

Observe that we can choose the  $\alpha_j$  to adjust the  $u_0$  contribution in each term to be 0. Set

$$z = \sum_{j=0}^{p-1} (\alpha_j \beta_2 u_0 - \alpha_{j+1} \beta_1 u_0) \otimes v_j \otimes w_{p-1-j}.$$

By the induction step we can choose vectors  $y_j$  such that  $J_\sigma Y - z$  is zero on level  $q - 1$ , and the terms in level  $q - 2$  have no  $u_0$  contribution. We can iterate this procedure to solve the equation  $J_\sigma \tilde{z}_q = 0$ . This proves the result on eigenvectors. ■

**Lemma 5.8.** *Let  $Q(\lambda)$  be a smooth family of nilpotent matrices, where  $\dim \ker Q(\alpha)$  is constant. Then the kernel varies smoothly with  $\lambda$ . If the length of all chains of generalized eigenvectors are different, then the generalized eigenvectors vary smoothly in  $\lambda$ .*

*Proof.* First, we solve the equation  $Q(\lambda)x = 0$  using a Liapunov–Schmidt decomposition. Without loss of generality we consider the problem near  $\lambda = 0$ . For this, let  $K$  be the kernel of  $Q(0)$ , let  $M$  be a complement to  $K$ , let  $R$  be the range of  $Q(0)$ , and let  $N$  be a complement to  $N$ . Let  $E : \mathbf{R}^n \rightarrow R$  and  $(I - E) : \mathbf{R}^n \rightarrow N$  be projections. Write  $u$  for the variable in the kernel,  $w$  for the variable in  $M$ . Then  $Q(\lambda)x = 0$  is equivalent to

$$\begin{aligned} f(u, w, \lambda) &\equiv (I - E)Q(\lambda)(u + w) = 0, \\ g(u, w, \lambda) &\equiv EQ(\lambda)(u + w) = 0. \end{aligned}$$

Then  $g_w : M \rightarrow R$  is an isomorphism which leads to a smooth solution of

$$g(u, w, \lambda) = 0$$

as  $w = w(u, \lambda)$ , where  $w(u, 0) = 0$  and  $w$  is linear in  $u$ . Then we have the equivalent equation

$$(I - E)Q(\lambda)(u + w(u, \lambda)) = 0.$$

From the hypotheses we know that the solution set of this equation has the same dimension as the kernel of  $Q(0)$ . Therefore, the kernel of  $Q(\lambda)$  is given by

$$\ker Q(\lambda) = \{u + w(u, \lambda) : u \in K\}.$$



This is a smooth deformation of  $K$ . Especially we find a smooth family of projection operators  $P(\lambda)$  mapping  $\mathbf{R}^n$  onto  $\ker Q(\lambda)$ .

Secondly we look at the generalized eigenvector  $g$  for  $Q$  which has the longest chain. There is a positive integer  $q$  with

$$Q^q g = 0$$

and

$$Q^i \neq 0$$

for  $1 \leq i < q$ . Now we want to solve an equation of the form

$$Q^q(\lambda)(w) = 0.$$

We solve the equivalent equation

$$P(\lambda) Q^{q-1}(\lambda)x(\lambda) = 0$$

for some  $x(\lambda)$  near  $g$ .  $Q^{q-1}$  is an isomorphism from a complement of the kernel to its range. Write  $w$  for the coordinate in the complement of the kernel. Then the implicit function theorem tells us that we can solve the equation

$$P(\lambda)Q^{q-1}(\lambda)(g + w) = 0$$

uniquely by some  $w(\lambda)$ . Since  $w(0) = 0$ , we have that  $Q^i(\lambda)g + w(\lambda)$  is uniformly near the points  $Q^i g$ . This means that the element  $g + w(\lambda)$  defines a chain of the required length.

By restriction of the problem to a complement of the longest chain, we can iteratively prove the result. ■

**Corollary 5.9.** *Generically, the generalized eigenspaces of  $J_\sigma$  are in 1:1 correspondence with those of  $\mathcal{A}$ .*

*Proof.* We note that if  $\beta_1 = \beta_2$ , the result follows from Theorem 2.7, because there the matrix  $\mathcal{A}$  is the adjacency matrix for the regular network obtained from  $N_1 \boxtimes N_2$  by setting the couplings equal. Next we perturb  $\beta_1, \beta_2$  near the diagonal  $\beta_1 = \beta_2$ . Theorem 5.5 guarantees that the kernel contains at least the given set of vectors. Generic perturbations will not increase the dimension of the kernel, and therefore the dimension stays locally constant. Since the Jordan blocks for  $J_\sigma$  in the case of  $\beta_1 = \beta_2$  have the same structure as those for  $\mathcal{A}$ , which is given by the Clebsch–Gordan theorem, all the length of the chains are different. Therefore the hypotheses of the previous Lemma 5.8 are satisfied, and we obtain that the Jordan structure is the same for  $\|\beta_1 - \beta_2\|$  small. ■

**Remark 5.10.** The Jordan structure within the center subspace  $E^Q(\lambda_1 + \lambda_2)$  can be complicated. Proposition 3.1 gives a bound on the maximal length of Jordan blocks within this space; however, it does not enumerate the number of Jordan blocks. Consider the matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since  $A_1^3 = 0$  and  $A_2^2 = 0$ , Proposition 3.1 states that  $Q^4 = 0$ . In this case the lengths of the Jordan blocks are  $\ell_1 = 3$  and  $\ell_2 = 2$ . Observe that the Jordan block structure of

$Q$  contains two blocks—one of length 4 and one of length 2. In general, when  $A_1$  and  $A_2$  are Jordan blocks of lengths  $\ell_1 \geq \ell_2$ , the tensor product will consist of  $\ell_2$  blocks of length  $\ell_1 - \ell_2 + 1, \ell_1 - \ell_2 + 3, \dots, \ell_1 + \ell_2 - 1$ .

**6. Branching of solutions.** In this section we prove an analogue of the equivariant branching lemma [15] in the context of homogeneous cell systems.

**Definition 6.1.** *Let  $y$  in phase space be given. The isotropy subspace containing  $y$  is the smallest synchrony subspace  $\Delta_y$  that contains  $y$ .*

The subspace  $\Delta = \{(y, \dots, y)\}$  consists of fully synchronous points. Suppose that  $\dot{x} = F(x, \lambda)$  is a coupled cell system and  $x_0 \in \Delta$  is a fully synchronous equilibrium at  $\lambda_0$ ; that is,  $F(x_0, \lambda_0) = 0$ . We call  $x_0$  a *synchrony-breaking* bifurcation point if

$$K = \ker(dF)_{x_0, \lambda_0} \neq 0 \quad \text{and} \quad K \cap \Delta = \{0\}.$$

Without loss of generality, we can assume that  $\lambda_0 = 0$ . At a synchrony-breaking bifurcation  $(dF)_{x_0, 0}|_{\Delta}$  is nonsingular. Therefore, the implicit function theorem guarantees that there is a parameterized family of fully synchronous equilibria  $x(\lambda)$ , where  $x(0) = x_0$ . Again, we can assume, without loss of generality, that  $x(\lambda) = 0$ . It follows that  $x_0 = 0$  and  $F(0, \lambda) = 0$ .

**Definition 6.2.** *An isotropy subspace  $\Delta_y$  is axial if  $\dim(\Delta_y \cap K) = 1$ .*

**Theorem 6.3.** *Let  $\Delta_y$  be an axial polydiagonal, where  $\Delta_y \cap K = \mathbf{R}\{v\}$ . Then generically there exists a unique branch of zeros of  $F$  in  $\Delta_y$ .*

*Proof.* Let  $A$  be the  $r \times r$  adjacency matrix of an network with  $r$  nodes. Let  $\bowtie$  be the equivalence relation corresponding to input equivalence. Considering  $A$  as a linear mapping on  $\mathbf{R}^r$  (or  $\mathbf{C}^r$ ), we have an invariant linear subspace  $S$  corresponding to  $\bowtie$ . We do not distinguish between the real and complex versions of  $S$ . Let  $\mu$  be an eigenvalue of  $A$ ,  $\mathcal{G}_A(\mu)$  be the corresponding generalized eigenspace, and  $E_A(\mu)$  the kernel of  $A - \mu\mathbf{I}$ .

Next we consider the case of a  $k$ -dimensional internal dynamics for the network. The state space is isomorphic to  $\mathbf{R}^k \otimes \mathbf{R}^r$ . Let

$$\overline{S} = \mathbf{R}^k \otimes S,$$

which is invariant under the map  $\mathbf{I} \otimes A$ . We have

$$M_\mu = \alpha + \mu\beta$$

for real  $k \times k$  matrices  $\alpha, \beta$ . Let  $\sigma$  be a simple eigenvalue of  $A$ . Let

$$J = \alpha \otimes \mathbf{I}_r + \beta \otimes A.$$

Then Theorem 2.7 implies that there exists an isomorphism  $\eta$ ,

$$\eta : \mathcal{G}_A(\mu) \rightarrow \mathcal{G}_J(\sigma).$$

We can also consider the spaces

$$\mathcal{G}_A^S(\mu) = S \cap \mathcal{G}_A(\mu)$$

and

$$\mathcal{G}_J^{\overline{S}}(\sigma) = \overline{S} \cap \mathcal{G}_J(\sigma).$$

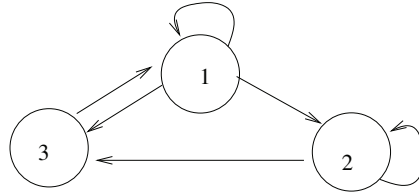


Figure 6. Three-cell network with multiple eigenvalues in the adjacency matrix.

If we apply this same construction, we conclude that in the case that  $\sigma$  is a simple eigenvalue of  $M_\mu$ , then

$$\eta^S : \mathcal{G}_A^S(\mu) \rightarrow \mathcal{G}_J^{\bar{S}}(\sigma)$$

is an isomorphism. Especially we have that

$$\dim E_A^S(\mu) = \dim E_J^{\bar{S}}(\sigma).$$

Let  $\hat{F} = F|_{\Delta_y}$ . Observe that  $\hat{F}(0, \lambda) = 0$  and that  $(d\hat{F})_{0,0}$  has a one-dimensional kernel equal to  $\Delta_y \cap K = \mathbf{R}\{v\}$ . Let  $G : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  be the reduced equation obtained by Liapunov-Schmidt reduction of  $\hat{F}$  to  $\mathbf{R}\{v\}$ . Observe that  $G(0, \lambda) = 0$ . Therefore, by Taylor's theorem,

$$G(t, \lambda) = tH(t, \lambda).$$

Generically,  $H(0, 0)$  is nonzero, and the implicit function theorem guarantees the existence of a branch of solutions to  $H(t, \lambda) = 0$  parametrized by  $t$ . ■

*Example 6.4.* Network 6 in [18], shown in Figure 6, provides an example where Theorem 6.3 can be applied when 0 is a multiple eigenvalue of the adjacency matrix and therefore of  $dF$ . The adjacency matrix is

$$A_6 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

with a double zero eigenvalue and a single nullvector  $v = (1, -1, -1)$ .

Observe that setting cell 2 equal to cell 3 leads to a balanced coloring and that  $\Delta_v \cap K = \mathbf{R}\{v\}$  is one-dimensional. Therefore, Theorem 6.3 implies the existence of a branch of equilibria with  $x_2 = x_3$ . We note that [18, Theorem 4.11] also proves the existence of an asynchronous branch of solutions emanating from this bifurcation.

Suppose that the coupled cell system has a symmetry group  $\Gamma$ , and suppose that  $\Sigma$  is an axial subgroup for the action of  $\Gamma$  on  $K$ . Then  $\text{Fix}_K(\Sigma) = \text{Fix}(\Sigma) \cap K$  is one-dimensional, and Theorem 6.3 reduces to the equivariant branching lemma.

*Example 6.5.* The four-cell network pictured in Figure 7 shows that in principle Theorem 6.3 gives more refined information than does the equivariant branching lemma.

To see this, observe that the adjacency matrix of this network,

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

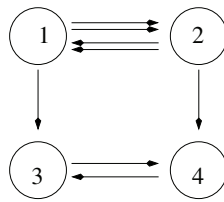


Figure 7. A homogeneous four-cell network with  $\mathbf{Z}_2$  symmetry.

has eigenvectors

$$(1, 1, 1, 1)^t, \quad (1, -1, -1, 1)^t, \quad (0, 0, 1, -1)^t, \quad (0, 0, 1, 1)^t,$$

corresponding to the eigenvalues  $2, -2, -1, 1$ . Bifurcations associated with the corresponding center subspace, which are covered by Theorem 6.3, lead to synchronous solutions. The first eigenvector leads to a fully synchronous solution, and the remaining three eigenvectors lead to solutions in the synchrony subspace corresponding to the balanced colorings in Figure 8. Note that the equivariant branching lemma discusses only the pattern of synchrony corresponding to the eigenvector  $(0, 0, 1, 1)^t$ , which is associated with  $\text{Fix}(\mathbf{Z}_2)$ .

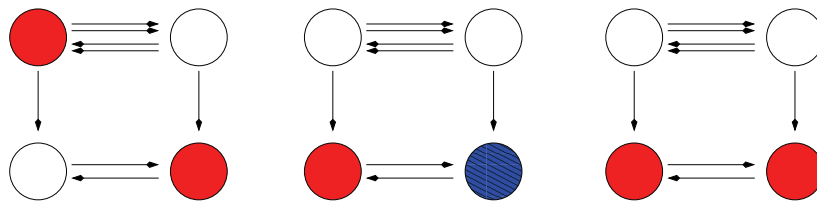


Figure 8. Three nontrivial balanced colorings of the four-cell network with  $\mathbf{Z}_2$  symmetry in Figure 7.

If the internal dynamics is one-dimensional, the center subspace corresponding to the eigenvalues  $-1, 1$  cannot occur as the first bifurcation. However, if two-dimensional internal dynamics is assumed, then a bifurcation corresponding to each eigenvalue of  $A$  can be the first one to undergo bifurcation. Observe that the eigenvalues of  $(dF)_0$  that correspond to the eigenvalues  $2, -2, -1, 1$  of  $A$  are the eigenvalues of the matrices  $\alpha + 2\beta, \alpha - 2\beta, \alpha - \beta,$  and  $\alpha + \beta,$  respectively. So to find a bifurcation associated with the eigenvalue  $1$  of  $A$ , we must find  $\alpha$  and  $\beta$  so that the six eigenvalues of the three  $2 \times 2$  matrices  $\alpha + 2\beta, \alpha - 2\beta, \alpha - \beta$  all have negative real part, and one eigenvalue of  $\alpha + \beta$  is  $0$ , whereas the other one is negative. The matrices

$$\alpha = \begin{bmatrix} -1 & -2 \\ -2.5 & -4.5 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

give such an example.

**Theorem 6.6.** *Suppose that bifurcations with axial polydiagonals in  $E^{A_1}$  and  $E^{A_2}$  lead to solutions with balanced colorings  $\boxtimes_1$  and  $\boxtimes_2$ . Then bifurcations in the product network corresponding to the axial polydiagonal  $E^{A_1} \otimes E^{A_2}$  will lead to solutions with balanced coloring  $\boxtimes = \boxtimes_1 \boxtimes \boxtimes_2$ .*

A general theorem for networks that is analogous to the equivariant Hopf theorem [15] (EHT) is difficult to state. The issue concerns spatio-temporal symmetries. The EHT uses

the fact that Hopf bifurcation itself induces a natural  $\mathbf{S}^1$  action corresponding to phase shift and the interaction of this  $\mathbf{S}^1$  symmetry with the spatial symmetries of the system. In the general network context there is mounting evidence that periodic solutions with (robust) phase shift synchrony can occur only if an associated quotient network has nontrivial symmetry. See Stewart and Parker [22, 23, 24, 25]. However, it is not clear how to state a worthwhile analogue of EHT using this conjecture. Even in the special context of interior symmetries [11], where the analogue of EHT can easily be stated, the proof remained elusive and was given only recently by Antoneli, Dias, and Paiva [1].

Two examples illustrate some of the simpler difficulties: the three-cell unidirectional ring (Figure 1(left)) and the three-cell feed-forward network (Figure 4(left)).

Hopf bifurcation from a synchronous equilibrium in the unidirectional ring can occur with one-dimensional internal dynamics and leads, because of the  $\mathbf{Z}_3$  symmetry, to solutions where any two cells oscillate a third of a period out of phase. It is a curious fact that bifurcation to synchronous oscillation can occur only if the internal dynamics is at least two-dimensional. These facts follow trivially from the center subspaces associated with the adjacency matrix,

$$(6.1) \quad A_{\text{udr}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of  $A_{\text{udr}}$  are  $\mu_1 = 1$ ,  $\mu_2 = (-1 + i\sqrt{3})/2$ , and  $\overline{\mu_2}$  with associated real eigenspaces

$$(6.2) \quad V_1 = \mathbf{R}\{(1, 1, 1)^t\} \quad \text{and} \quad V_2 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}.$$

It was shown in [10, 8] that synchrony-breaking Hopf bifurcation in the feed-forward network leads to two branches of periodic solutions. The amplitude of the solutions on one branch grows at the expected rate of  $\lambda^{\frac{1}{2}}$  (where  $\lambda$  is the bifurcation parameter), and the amplitude of solutions along the other branch grows at the unexpected rate of  $\lambda^{\frac{1}{6}}$ . Neither the existence of multiple solutions nor the rapid growth rate is due to symmetry; rather they are due in part to multiple eigenvalues in the adjacency matrix caused by network architecture. The adjacency matrix of the feed-forward network is

$$(6.3) \quad A_{\text{ff}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of  $A_{\text{ff}}$  are  $\nu_1 = 1$  and  $\nu_2 = 0$  with algebraic multiplicity 2 and geometric multiplicity 1. The associated real (generalized) eigenspaces are

$$(6.4) \quad W_1 = \mathbf{R}\{(1, 1, 1)^t\} \quad \text{and} \quad W_2 = \mathbf{R}\{(0, 1, 0)^t, (0, 0, 1)^t\}.$$

The nontrivial Jordan structure of  $A_{\text{ff}}$  is responsible for the existence of multiple solution branches in synchrony-breaking Hopf bifurcations in this network, but the complete picture is even more complicated. Elmhirst and Golubitsky [8] show that there are other regular networks that have Jordan structures in their adjacency matrices that are identical to that in the feed-forward network, but whose nonlinear bifurcations generate solutions with different multiplicities and different growth rates.

Nevertheless, once these solution structures are understood, they seem to lead naturally to solution structures in the product network, as we discuss in the next section.

**7. Bifurcations from stable synchronous solutions.** Until now we have classified the center subspaces of possible codimension one bifurcations from a fully synchronous equilibrium that can occur in regular cell systems, but we have not discussed whether these bifurcations can lead to stable solutions. More precisely, suppose that  $\mu_1, \dots, \mu_r$  are the distinct eigenvalues of the adjacency matrix. A synchrony-breaking bifurcation associated with the eigenvalue  $\mu_j$  can lead to stable solutions only if there exist  $k \times k$  matrices  $\alpha, \beta$  such that all eigenvalues of the matrices  $M_{\mu_i}$  for  $i \neq j$  have negative real part and all noncritical eigenvalues of  $M_{\mu_j}$  have negative real parts. If this necessary condition is satisfied at a codimension one bifurcation, we call that bifurcation a *first bifurcation*.

We conjecture that if  $k$  is large enough, there can be a codimension one synchrony-breaking first bifurcation associated with any eigenvalue  $\mu_j$  of the adjacency matrix. However, we can prove this conjecture only for those regular networks whose adjacency matrices have real eigenvalues.

**Theorem 7.1.** *Assume that all eigenvalues of the adjacency matrix  $A$  of a regular network are real. Let  $\mu_1 < \dots < \mu_r$  be the distinct eigenvalues of  $A$ . Suppose that  $k = 2$ . Then, for any  $1 \leq j \leq r$ , there exist  $2 \times 2$  matrices  $\alpha$  and  $\beta$  such that the eigenvalues of  $M_{\mu_i}$  have negative real part for all  $i \neq j$  and the eigenvalues of  $M_{\mu_j}$  are 0 and negative.*

*Proof.* Without loss of generality we can translate the numbers  $\mu_i$  by the same constant  $c$ . More precisely,

$$\alpha + \mu\beta = (\alpha + c\beta) + (\mu - c)\beta.$$

So, without loss of generality, we may assume that  $\mu_j = 0$ .

The theorem is easily seen to be valid for  $j = 1$ , even when  $k = 1$ . Just take  $\alpha = 0$  and  $\beta = -1$ . Similarly, when  $j = r$ , set  $\alpha = 0$  and  $\beta = 1$ . So we can assume  $1 < j < r$ . In this case (when  $k = 2$ ) we must find matrices  $\alpha, \beta$  so that  $\text{tr}(M_{\mu_i}) < 0$  for all  $i$  and  $\det(M_{\mu_i}) > 0$  for all  $i \neq j$ . We claim that this can be done by setting

$$\alpha = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} -1 & b \\ 1 & 0 \end{bmatrix},$$

where

$$(7.1) \quad a < \mu_1$$

and

$$(7.2) \quad b < -\frac{a}{\mu_{j-1}}.$$

Observe that

$$\text{tr}(\alpha + \mu\beta) = a - \mu \quad \text{and} \quad \det(\alpha + \mu\beta) = -a\mu - b\mu^2.$$

So assumption (7.1) implies that

$$\text{tr}(M_{\mu_i}) = a - \mu_i < a - \mu_1 < 0$$

for all  $i$ , since  $\mu_1 \leq \mu_i$ . We claim  $\det(M_{\mu_i}) > 0$  if  $i \neq j$ . To verify this point, note that for  $i \neq j$ ,  $\det(M_{\mu_i}) > 0$  if and only if

$$b < (-a)\frac{1}{\mu_i},$$

which is equivalent to

$$-\frac{a}{\mu_i} - b > 0$$

for all  $i \neq j$ . Since  $-a > 0$ , we need to choose  $b$  sufficiently negative, namely, less than  $-a$  times the largest negative value of  $\frac{1}{\mu_i}$ . Because of the ordering of the  $\mu_i$ , that value is  $\frac{1}{\mu_{j-1}}$ , which we have assumed in (7.2). ■

Suppose that the eigenvalues of the adjacency matrix of a regular network are real. When the dimension of the internal dynamics is  $k \geq 2$ , Theorem 7.1 implies that there is a codimension one steady-state first bifurcation associated with any eigenvalue  $\mu$ . This statement is not true for Hopf bifurcations, at least when  $k = 2$ .

Suppose that a regular network has an adjacency matrix  $A$  with a negative eigenvalue  $\mu_1$ , a zero eigenvalue, and a positive eigenvalue  $\mu_r$ . Suppose also that  $k = 2$ . Then a Hopf bifurcation associated with the 0 eigenvalue of  $A$  cannot be a first bifurcation.

To verify this point, note that if the  $2 \times 2$  matrix  $M_0 = \alpha$  has purely imaginary eigenvalues, it follows that  $\text{tr}(\alpha) = 0$  and hence that

$$\text{tr}(M_1) = \mu_1 \text{tr}(\beta) \quad \text{and} \quad \text{tr}(M_r) = \mu_r \text{tr}(\beta).$$

If  $\text{tr}(\beta) \neq 0$ , then the traces of the  $2 \times 2$  matrices  $M_1$  and  $M_r$  have opposite sign, and one of them must have an eigenvalue with positive real part. So the eigenvalue 0 of the adjacency matrix cannot correspond to a first (Hopf) bifurcation. If  $\text{tr}(\beta) = 0$ , then  $\text{tr}(M_1) = 0$  and the bifurcation is either not codimension one (if  $\det(M_1) \geq 0$ ) or not a first bifurcation (if  $\det(M_1) < 0$ ). In short, to have a first Hopf bifurcation associated to a general adjacency matrix eigenvalue  $\mu$ , one must assume that  $k \geq 3$ . Observations such as these have been made previously in the pattern formation literature (see, for example, [9]). On the other hand, if  $k \geq 3$ , then all types of Hopf bifurcation can occur as first bifurcations. The following theorem was proved by Amit Vutha [26].

**Theorem 7.2.** *Assume that all eigenvalues of the adjacency matrix  $A$  of a regular network are real. Let  $\mu_1 < \dots < \mu_r$  be the distinct eigenvalues of  $A$ . Suppose that  $k = 3$ . Then, for any  $1 \leq j \leq r$ , there exist  $3 \times 3$  matrices  $\alpha$  and  $\beta$  such that the eigenvalues of  $M_{\mu_i}$  have negative real part for all  $i \neq j$  and the eigenvalues of  $M_{\mu_j}$  are purely imaginary and negative.*

*Proof.* We begin by recalling the Routh–Hurwitz criterion. Given a polynomial of degree 3,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

All roots of this polynomial have negative real part if and only if

$$a_1 > 0, \quad a_3 > 0, \quad a_1a_2 > a_3.$$

The polynomial has a pair of purely imaginary roots if and only if

$$a_1 > 0, \quad a_3 > 0, \quad a_1a_2 = a_3.$$

We prove that the eigenvalue  $\mu_j$  can have a first Hopf bifurcation when  $k = 3$ . As in the proof of Theorem 7.1, we may translate all of the eigenvalues  $\mu_i$  so that  $\mu_j = 0$ . We wish to find  $3 \times 3$  matrices  $\alpha$  and  $\beta$  such that

- (i)  $\alpha + \mu_i\beta$  has all eigenvalues with negative real parts for  $i \neq j$ ,
- (ii)  $\alpha$  has a pair of purely imaginary eigenvalues and a real negative eigenvalue.

We proceed by first choosing  $a > 0$  satisfying  $a > |\mu_i|$  for all  $i$ . Then let

$$\alpha = \begin{bmatrix} -a & a & -a \\ 0 & 0 & -a \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\alpha + \mu_i\beta = \begin{bmatrix} -a & a + \mu_i & -a \\ 0 & 0 & -a + \mu_i \\ 1 & 0 & 0 \end{bmatrix},$$

whose characteristic polynomial is

$$(7.3) \quad \lambda^3 + a\lambda^2 + a\lambda + (a^2 - \mu_i^2) = 0.$$

For  $i = j$  we have  $\mu_i = 0$ , and (7.3) has roots  $-a$  and  $\pm i\sqrt{a}$ , whereas for  $i \neq j$ , (7.3) has all roots with negative real part by the Routh–Hurwitz criterion. ■

**8. An example of bifurcation in a product network.** In this section we discuss the codimension one synchrony-breaking bifurcations in the nine-cell product network shown in Figure 4(right). For simplicity we assume that the internal dynamics is one-dimensional, even though this restricts some of the bifurcation types that are possible in the general case.

The adjacency matrices for this network are  $A_{\text{udr}} \otimes I_3$  and  $I_3 \otimes A_{\text{ff}}$  defined in (6.1) and (6.3). Suppose that a system of differential equations associated with this network is denoted by  $\dot{X} = F(x)$  and that  $J = (dF)_{X_0}$  is the Jacobian of this system at a synchronous equilibrium, which we can take to be  $X_0 = 0$ . From (3.1) it follows that

$$(8.1) \quad J = \alpha \mathbf{I}_3 \otimes \mathbf{I}_3 + \beta_1 A_{\text{udr}} \otimes \mathbf{I}_3 + \beta_2 \mathbf{I}_3 \otimes A_{\text{ff}},$$

where  $\alpha$  is the linearized internal dynamics and  $\beta_1, \beta_2$  are the linearized coupling strengths for the couplings in the unidirectional ring and the feed-forward networks, respectively. It follows from the discussion in section 3 that the eigenvalues of  $J$  are  $\alpha + \beta_1\mu + \beta_2\nu$ , where  $\mu$  is an eigenvalue of  $A_{\text{udr}}$  and  $\nu$  is an eigenvalue of  $A_{\text{ff}}$ . As noted in section 6, the eigenvalues of  $A_{\text{udr}}$  are  $\mu_1 = 1$ ,  $\mu_2 = (-1 + i\sqrt{3})/2$ , and  $\overline{\mu_2}$ , and the eigenvalues of  $A_{\text{ff}}$  are  $\nu_1 = 1$  and  $\nu_2 = 0$  with algebraic multiplicity 2 and geometric multiplicity 1.

The real parts of the eigenvalues of the  $9 \times 9$  matrix  $J$  are

$$\begin{aligned} \sigma_{11} &= \alpha + \beta_1 + \beta_2, \\ \sigma_{21} &= \alpha - \frac{1}{2}\beta_1 + \beta_2, \\ \sigma_{12} &= \alpha + \beta_1, \\ \sigma_{22} &= \alpha - \frac{1}{2}\beta_1, \end{aligned}$$

with multiplicities 1, 2, 2, and 4. It is possible to choose  $\alpha, \beta_1, \beta_2$  so that any one of the  $\sigma_{ij} = 0$  and the other three are negative. Sample values are given in the following table:

$\alpha$	$\beta_1$	$\beta_2$	$\sigma_{11}$	$\sigma_{21}$	$\sigma_{12}$	$\sigma_{22}$
-4	2	2	0	-2	-1	-3
-3	-2	2	-3	0	-5	-2
-2	2	-2	-2	-5	0	-1
-1	-2	-2	-5	-2	-3	0

(8.2)



It follows that each of the four types of bifurcations can lead, in principle, to branches of stable solutions.

If  $\sigma_{11}$  is the critical eigenvalue, then the bifurcation occurs in the fully synchronous subspace with all coordinates equal. Generically we expect a saddle node bifurcation and no synchrony-breaking.

If  $\sigma_{21}$  is the critical eigenvalue, then the bifurcation occurs in the synchrony subspace where all cell coordinates in each ring are equal. In this case the quotient network is the feed-forward network, and the generic (steady-state) bifurcation was analyzed in [18]. Indeed, [18, Theorem 4.6] proves that two nontrivial branches of solutions bifurcate from the trivial branch, one transcritical and one a pitchfork. Moreover, when the internal dynamics is one-dimensional, generically, the bifurcating solutions are unstable. Note that if the product network has two-dimensional internal dynamics, then Hopf bifurcation associated with this center subspace is possible and periodic solutions with the  $\lambda^{\frac{1}{6}}$  growth rate may occur.

If  $\sigma_{12}$  is the critical eigenvalue, then the bifurcation occurs in the synchrony subspace where all cell coordinates in each feed-forward network are equal. In this case the quotient network is the unidirectional ring, and we have purely imaginary eigenvalues at criticality. Hopf bifurcation to discrete rotating waves occurs.

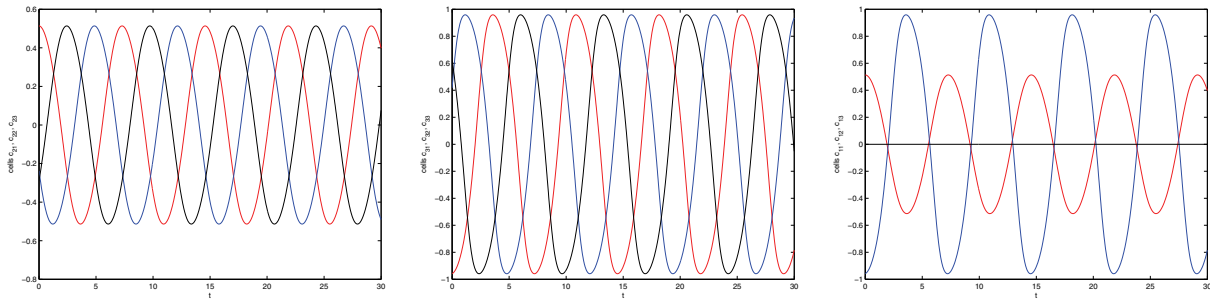
If  $\sigma_{22}$  is the critical eigenvalue, then bifurcation occurs with four purely imaginary eigenvalues, and the center subspace is a four-dimensional real subspace. This is an example of (4.1) where  $U$  is complex one-dimensional,  $\mathcal{G}_1(\mu)$  is complex two-dimensional, and  $\mathcal{G}_2(\nu)$  is complex one-dimensional. The tensor product in (4.1) is complex two-dimensional, and the center subspace is real four-dimensional.

The Jordan form at bifurcation is identical to the Hopf bifurcation in the feed-forward network. Of course, the nonlinear bifurcation need not be identical—but it is. In particular, there are two solution branches. The first is found by setting the coordinates in the first two rings to 0. Then there is a standard Hopf bifurcation to a branch of discrete rotating waves in the third ring (because of the  $\mathbf{Z}_3$  symmetry), whose amplitude grows at rate  $\lambda^{\frac{1}{2}}$ . Second, setting the coordinates in the upper ring to 0 implies that the middle ring equations undergo a standard Hopf bifurcation to a branch of discrete rotating waves (again because of the  $\mathbf{Z}_3$  symmetry), whose amplitude also grows at rate  $\lambda^{\frac{1}{2}}$ . Finally, the equations in the third ring are forced by the periodic solutions in the middle ring, and just as in the three-cell feed-forward network there is a unique branch of periodic solutions in the bottom ring. These solutions also form a discrete rotating wave (because of symmetry and uniqueness) with amplitude growth  $\lambda^{\frac{1}{6}}$ . This may be checked using [8] or by a numerical experiment (we did the latter). Simulations using the cell vector field

$$f(y_1, y_2, y_3) = -0.3y_1 - y_2 - y_3 - y_1^3,$$

that is,  $\beta_1 = \beta_2 = -1$ ,  $\alpha = -0.5$ , and  $\lambda = 0.2$ , are shown in Figure 9.

**9. Conclusions.** The number of different regular networks grows superexponentially in the number of cells  $r$  (even for fixed valency). Therefore, the bifurcation analysis for three-cell systems given in [18, 8] cannot be repeated for  $r$ -cell systems if the bifurcation analysis is truly different for each network. The results of this paper and those in [8] give some hope



**Figure 9.** Simulations from product network. (Left, center) Discrete rotating waves are seen in second and third rings. (Right) Time series from cells in the first feed-forward network; note the increase in amplitude from the second to the third cell in that chain.

that bifurcations in general cell systems can be understood in a way analogous to equivariant bifurcation theory.

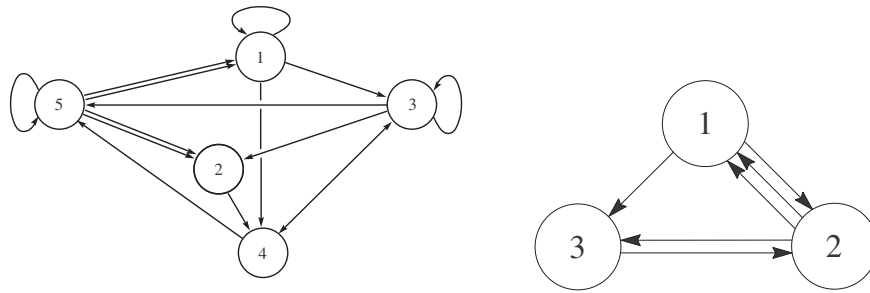
In equivariant bifurcation theory [15, 3, 12] it is known that each irreducible representation of a given group  $\Gamma$  leads to a type of Hopf bifurcation and that each absolutely irreducible representation of  $\Gamma$  leads to a type of steady-state bifurcation theory. Each of these bifurcations is analyzed using a set of techniques that have now become standard (normal form theory, singularity theory, equivariant branching lemma, the EHT, etc.). The important point is that the number of different bifurcations that need to be analyzed is indexed by the number of distinct irreducible representations of  $\Gamma$  and not by the number of representations of  $\Gamma$ .

Here we suggest that the number of bifurcations in regular networks that need to be analyzed is indexed by the number of Jordan structures of adjacency matrices and not by the number of networks. But the situation is not so clear as in the equivariant case. Elmhirst and Golubitsky [8] study *nilpotent Hopf bifurcations*, that is, those Hopf bifurcations where the algebraic multiplicities of the critical eigenvalues are each two but the geometric multiplicities are one. They show that there are (at least) three or four different (nonlinear) bifurcation scenarios in regular networks that can occur with such a fixed center subspace structure. We conjecture that each type of center subspace leads to just a small finite number of different bifurcation scenarios (unlike in equivariant bifurcations, where each irreducible representation leads to one bifurcation scenario), and we explain our reasoning using nilpotent Hopf bifurcations.

The codimension one nilpotent Hopf bifurcation scenarios for regular networks analyzed in [8] can be described by the number of bifurcating branches of periodic solutions and the growth rate of the amplitudes of solutions along these branches. The four possibilities are

- (a) two branches with growth rates  $\lambda^{\frac{1}{2}}$  and  $\lambda^{\frac{1}{6}}$ ,
- (b) two branches each with growth rate  $\lambda^1$ ,
- (c) two or four branches each with growth rate  $\lambda^{\frac{1}{2}}$ , and
- (d) two branches with growth rates  $\lambda^{\frac{1}{2}}$  and  $\lambda^{\frac{1}{4}}$ ,

where  $\lambda$  is the bifurcation parameter. The three-cell feed-forward network in Figure 4(left) is an example of (a). Other examples are given in [8]. The five-cell network in Figure 10(left) is an example where scenario (b) occurs. The three-cell network in Figure 10(right) is an example of scenario (c); other examples are given in [8]. As of now no examples of a network



**Figure 10.** Regular networks that exhibit nilpotent Hopf bifurcations in codimension one.

that yields scenario (d) are known.

Krupa [17] observed that if one analyzes the normal form of the center manifold reduction to the four-dimensional center manifold of a nilpotent Hopf bifurcation, then all scenarios can be seen in the third order truncation. This suggests more strongly that generically, for any network, codimension one nilpotent Hopf bifurcations occur in at most four types in any regular network, but we have no proof of this fact. Moreover, the stability of these solutions is sometimes determined at order greater than 3. Nevertheless, these results together give some credence to the conjecture we have made in this section.

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