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HOPF BIFURCATION IN THE PRESENCE OF SYMMETRY, CENTER MANIFOLD AND LIAPUNOV-SCHMIDT REDUCTION

Pascal Chossat

Martin Golubitsky

ABSTRACT: Assume that the linear part of a vector field X is semisimple and has eigenvalues at $\pm \omega i$. We show that if the quadratic terms of X vanish when restricted to the center subspace, then to third order the Liapunov-Schmidt reduction for finding periodic solutions of X is identical to a center manifold reduction of X followed by putting the reduction in normal form. Several examples of systems with symmetry that satisfy this hypothesis are discussed.

I. INTRODUCTION

Local bifurcation theory, as developed through the Liapunov-Schmidt decomposition method [9], provides a simple and efficient way for computing zeroes of families of nonlinear maps near singularities. However, when applied to evolution equations (ODE's or PDE's) this method does not, in general, give information on the dynamics close to the bifurcated solutions, except in situations where certain exchange of stability principles are valid, such as at simple eigenvalues [9]. The "natural" setting for the study of dynamics close to a bifurcation is a combination of a center manifold reduction along with a normal form analysis [11]. The relationship between the Liapunov-Schmidt decomposition and the stability of bifurcated solutions has been studied, from a general point of view, in [10]. Our purpose is different. We shall show that if a simple condition is realized (no quadratic terms for the Taylor expansion of the equation on the center manifold), the Liapunov- Schmidt method for Hopf bifurcation leads to an equation which is identical to the normal form equation on the center manifold, up to cubic order (assuming semisimple critical eigenvalues). This can be interesting in problems invariant under a symmetry group, because symmetry can force the

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quadratic terms to be identically zero, as we shall see in Section III. We now recall the Liapunov-Schmidt method and the center manifold reduction process.

Let $f(u,\lambda)$ be a C^k map $(k \geq 1)$ from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^n$ such that $f(0,0) = 0$ and $L = D_u f(0,0)$ is not invertible. Let V be the kernel of L and let P,Q be projections in $\mathbb{R}^n$ such that $\operatorname{Im} P = V$ and $\operatorname{Ker} Q = \operatorname{range} L$. We note that, if 0 is a semisimple eigenvalue of L one can choose $P \equiv Q$. Now set $u = x+y$ where $x \in V$ and $y \in \ker P$; then the equation $f(u,\lambda) = 0$ splits as follows

$$Qf(x+y,\lambda) = 0 \tag{1.1}$$

$$(1-Q)f(x+y,\lambda) = 0 \tag{1.2}$$

Notice that we can write equation (1.2) as follows:

$$(1-Q)Ly = (1-Q)N(x+y,\lambda) \tag{1.3}$$

where

$$N(u,\lambda) = f(u,\lambda) - Lu.$$

It is easy to check that the operator $(1-Q)L$ restricted to a complement of V in $\mathbb{R}^m$ is invertible. Hence the implicit function theorem applies to (1.3) and gives $y = y(x,\lambda)$ of class C^k in a neighborhood of $(0,0)$. Replacing y by $y(x,\lambda)$ in (1.1) leads to the "bifurcation equation":

$$g(x,\lambda) = QN(x+y(x,\lambda),\lambda) = 0 \tag{1.4}$$

This method has proved to be powerful. For example, the Hopf bifurcation theorem can be proved by looking for the zeroes of a functional equation, within a class of periodic functions. Other methods are possible, but the Liapunov-Schmidt method has the advantage of providing a simple and systematic rule for computing the Taylor expansion of the solution just by matching powers in (1.3) and (1.4).

On the other hand, let X_0 be a smooth (C^k) vector field in $\mathbb{R}^n$, and let 0 be a singular point for X_0, i.e., $X_0(0) = 0$. It is well-known that if no eigenvalues of $A = DX_0(0)$ have zero real part, then the dynamics of the flow of X_0 in a neighborhood of 0 is determined by the eigenvalues of A. In particular, if all of the eigenvalues have negative real part, then 0 is an attractor or sink of X_0. If, however, some eigenvalues lie on the imaginary axis, then the dynamics of the flow near 0 will be determined by higher order terms in the Taylor expansion of X_0. More precisely, let V_0 be the (generalized) eigenspace of A associated to eigenvalues with real part equal to zero, and let P_0 be the projection onto V_0 such that $AP_0 = P_0A$. Then there exists a C^k map on $\mathbb{R}^n$

$$\phi_0 : V_0 \rightarrow (1-P_0)\mathbb{R}^n,$$

such that $\phi_0(0) = D\phi_0(0) = 0$ and whose graph is a center manifold M_0, i.e. M_0 is a locally invariant manifold for the flow defined by X_0 [11]. If, in addition, the remaining eigenvalues of A are in the left-half plane, then M_0 is also an attractor for the flow. We can therefore restrict the study of the flow to its action on M_0. In the case of a family X_λ of vector fields ($\lambda \in \mathbb{R}^p$), there exists a map $\phi : V_0 \times \mathbb{R}^p \rightarrow (1-P_0)\mathbb{R}^n$ such that $\phi(.,0) = \phi_0$. The Taylor expansion of $\phi(x,\lambda)$ can be computed by a matching rule, slightly more complicated than the rule for $y(x,\lambda)$ in the Liapunov-Schmidt method. Also, the study of the dynamics generally requires the vector field on M_λ to be in "normal form" (the normal form is a k-th order Taylor expansion obtained from X_λ by near identity changes of variables and which is as simple as possible [8]).

II. STATEMENT AND PROOF OF THE MAIN RESULT

Let us consider the equation

$$\frac{du}{dt} = F(u,\lambda), \tag{2.1}$$

where $F : D \times \mathbb{R} \rightarrow E$ of class C^{k+1} ($k \geq 3$) in a neighborhood of $(0,0)$, $F(0,0) = 0$, E is a real Banach space and $D \subset E$ is the domain of the linear operator $A = D_uF(0,0)$. We assume that standard hypotheses are valid for A and F, so that the center manifold theorem applies to (2.1) at $(u,\lambda) = (0,0)$ [11]. We also assume that A has semisimple eigenvalues $\pm i\omega_0$ of finite multiplicity, and that the rest of the spectrum lies in the half-space $\{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta \leq \xi < 0\}$. We denote the invariant subspace associated with eigenvalues $\pm i\omega_0$ by V_0, and let $E = V_0 \oplus W$, where W is such that $A(W \cap D) \subset W$. Finally we let

$$x = P_0 u \quad \text{and} \quad y = (1-P_0)u$$

where P_0 is the projection on V_0 associated to the foregoing decomposition of E. From the center manifold theorem, equation (2.1) is reduced to the equation:

$$\frac{dx}{dt} = G(x,\lambda) \tag{2.2}$$

where

$$G(x,\lambda) = P_0F(x+y(x,\lambda),\lambda)$$

and $y(x,\lambda)$ is the C^k solution of

$$\frac{dy}{dt} = (1-P_0)F(x+y,\lambda).$$

We now write

$$G(x,\lambda) = Lx + R(x,\lambda)$$

where L is the linear part of G. The remainder term R has the form

$$R(x,\lambda) = \sum_{\substack{p,q \\ p+q\geq 2}} \lambda^p R_{pq}(x)$$

where $R_{p,q}(x)$ is a polynomial homogeneous of degree q in x.

Theorem 2.1: If $R_{02} \equiv 0$, then the normal form of (2.2) and the Taylor expansion of the Liapunov-Schmidt bifurcation equation are identical at order $0(|\lambda||x| + |x|^3)$.

Remark 2.2: The simplest way to check that $R_{02} = 0$ is as follows. Let $\rho:E \to V_0$ be a projection and let $G:V_0\times\mathbb{R} \to V_0$ be ρ composed with the restiction of F to $V_0\times\mathbb{R}$. If the quadratic terms of G are all zero, then R_{02} also vanishes. That is, if the restiction of (2.1) to its center subspace has no quadratic terms, then the Liapunov-Schmidt reduction gives the same equations as reduction to the center manifold coupled with changes of coordinates to normal form.

We split the proof of Theorem 2.1 into two lemmas. In the following we consider the action of the circle group S^1 defined on V_0 by

$$(\phi,x) \to e^{\phi L/\omega_0}x \quad \text{for all} \quad \phi\varepsilon S^1,\ x\varepsilon V_0.$$

For every vector field X in V, we set

$$\rho(X)[x] = \omega_0/2\pi \int_0^{2\pi/w_0} e^{-\phi L/\omega_0} X(e^{\phi L/\omega_0}x)d\phi.$$

The operator ρ is a projection onto the space of S^1-equivariant vector fields.

Lemma 2.3: The normal form equation for (2.2) with $R_{02} = 0$ takes the form

$$\frac{dx}{dt} = Lx + \lambda\rho(R_{11})(x) + \rho(R_{03})(x) + 0(|\lambda|^2|x| + |\lambda||x|^3 + |x|^4).$$

Proof: According to the Normal Form process [8], we look for a near identity change of variables with Taylor expansion

$$x \to \tilde{x} = x + \lambda h_1(x) + h_3(x,x,x) + \ldots .$$

where h_p is symmetric and p-linear. Then, the new vector field has the following Taylor expansion up to order $|\lambda||x|$ and $|x|^3$:

$$\tilde{R}_{11}(x) = R_{11}(\tilde{x}) - ad_L(h_1)(\tilde{x})$$

$$\tilde{R}_{03}(x) = R_{03}(\tilde{x}) - ad_L(h_3)(\tilde{x})$$

where

$$ad_L(X)(\tilde{x}) = DX(\tilde{x})L\tilde{x} - LX(\tilde{x}).$$

Denoting by $\mathbb{H}_k$ ($\mathbb{G}_k$) the space of vector fields in V_0 whose components are homogeneous polynomials of degree k (S^1-equivariant such vector fields), we know [5] that $\mathbb{H}_k = \mathrm{Im}\, ad_L \oplus \mathbb{G}_k$, the corresponding projection being precisely ρ. Hence $\tilde{R}_{11}=\rho(R_{11})$ and $\tilde{R}_{03} = \rho(R_{03})$. □

We now recall the Liapunov-Schmidt decomposition for equation (2.2). By the change of scale $s = \omega_0 t$, equation (2.2) is rewritten as

$$Jx = R(x,\lambda) - (\omega-\omega_0)\frac{dx}{ds} \tag{2.3}$$

where ω is the unknown frequency and $J \equiv \omega_0\, d/ds - L$. We look for a function $x(s)$ in $C^1(S^1,V_0)$ where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let $\{\zeta_j\}$ be an orthonormal basis of the eigenspace associated to the eigenvalue $\omega_0 i$. Then, in $C^0(S^1,V_0)$

$$\ker J = \{\sum_j (x_j e^{is}\zeta_j + \bar{x}_j e^{-is}\bar{\zeta}_j) : x_j \in \mathbb{C}\}.$$

For

$$x(s) = \sum_j (x_j(s)\zeta_j + \bar{x}_j(s)\bar{\zeta}_j)$$

we define

$$(Px)(s) = \sum_j (\alpha_j e^{is}\zeta_j + \bar{\alpha}_j e^{-is}\bar{\zeta}_j) \tag{2.4}$$

where

$$\alpha_j = \frac{1}{2\pi}\int_0^{2\pi} x_j(s)e^{-is}ds.$$

This projection on $\ker J$ satisfies the properties required for decomposing (2.3) according to the Liapunov-Schmidt method: set

$$(Px)(s) = \hat{x}(s) \quad \text{and} \quad y = (I-P)x;$$

Thus, $x = \hat{x} + y$ and we obtain

$$\begin{aligned} &(a) \quad y = (I-P)\tilde{J}^{-1}\left[R(\hat{x}+y,\lambda) - (\omega-\omega_0)\frac{dy}{ds}\right] \\ &(b) \quad 0 = PR(\hat{x}+y,\lambda) - (\omega-\omega_0)\frac{d\hat{x}}{ds}. \end{aligned} \tag{2.5}$$

where $\tilde{J}^{-1}$ is the inverse of J defined on $(I-P)C^0(S^1,V_0)$. Equation (2.5a) is solved for $y = y(\hat{x},\omega,\lambda)$ and then equation (2.5b) gives the bifurcation equation on ker J.

Lemma 2.4: Let $f:V_0 \to V_0$ and $\hat{x} \in \ker J$. Then

$$P[f(\hat{x}(s))] = \rho(f)(\hat{x}(s)).$$

Proof: Note that we can write

$$\hat{x}(s) = e^{sL/\omega_0}\,\hat{x}_0$$

for some $\hat{x}_0 \in V_0$. We now set $f = \sum_j f_j \zeta_j$. Then

$$P[f(\hat{x}(s))] = \sum_j \frac{1}{2\pi}\left(\int_0^{2\pi} f_j(e^{tL/\omega_0}\,\hat{x}_0)e^{-i(t-s)}dt\right)\zeta_j$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_j f_j(e^{(\tilde{t}+s)L/\omega_0}\,\hat{x}_0)e^{-i\tilde{t}}\,\zeta_j\right)d\tilde{t}$$

where $\tilde{t} = t-s$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-\tilde{t}L/\omega_0}\, f(e^{\tilde{t}L/\omega_0}\,\hat{x}(s))d\tilde{t} = \rho(f)[\hat{x}(s)].$$ □

We can now write the expansion of the bifurcation equation to third order in $\hat{x}$ and λ. Since $R_{02} \equiv 0$ and $\frac{d\hat{x}}{ds} = \omega_0^{-1}L\,\hat{x}$, it is easy to check that

$$\omega\frac{d\hat{x}}{ds} = L\hat{x} + \lambda\rho(R_{11})(\hat{x}) + \rho(R_{03})(\hat{x}) + \dots .$$

We therefore obtain the same equation as in Lemma 2.1, which proves Theorem 2.1 upon substituting x for $\hat{x}$. □

Remarks 2.5: (1) We would obtain the same result by applying the Liapunov-Schmidt reduction directly to (2.1) instead of to (2.2). In this case, the projection P on ker J has to be replaced by the projection $P'=PP_0$ of $C^0(S^1,E)$ onto ker J. Then everything else works as above.

(2) We obtain the same result if L has an additional zero eigenvalue or two pairs of imaginary eigenvalues. Such a situation occurs in multiparameter bifurcation of the Couette-Taylor system considered in Example 3.4 below. In the first case, for example, V_0 is the eigenspace associated to the eigenvalues 0 and $\pm\,\omega i$,

$$x(s) = \sum_j (x_j(s)\zeta_j + \bar{x}_j(s)\bar{\zeta}_j) + \sum_k y_k\xi_k$$

where $\{\xi_k\}$ is an orthonormal basis of ker L, and

$$(Px)(s) = \sum_j (\alpha_j e^{is}\zeta_j + \bar{\alpha}_j e^{-is}\bar{\zeta}_j) + \sum_k \beta_k \xi_k$$

with

$$\beta_k = \frac{1}{2\pi}\int_0^{2\pi} y_k(s)ds.$$

The proof of Theorem 2.1 given above then applies to this case.

III. APPLICATIONS OF THEOREM 1

Assume there exits a group Γ acting on E such that (2.1) is invariant under this action, that is, we assume

$$F(\gamma u,\lambda) = \gamma F(u,\lambda), \quad \forall\ \gamma \in \Gamma \text{ and } (u,\lambda) \in D\times\mathbb{R}^p$$

(F is Γ-equivariant). It is well known that the invariance properties carry over to the bifurcation equation [14], which we now write as:

$$g(x,\lambda) = 0, \quad x \in V_0,\ \lambda \in \mathbb{R}\ .$$

We assume $V_0 = V\oplus V$, where V is Γ-invariant and Γ acts diagonally on $V\oplus V$ (see [14] for a discussion of this point).

<u>Lemma 3.1</u>: Let $P_2(\Gamma,V)$ ($P_2(\Gamma,V_0)$) be the space of Γ-equivariant quadratic maps of $V \to V$ ($V_0 \to V_0$). Then:

$$\dim P_2(\Gamma,V_0) = 6 \dim P_2(\Gamma,V).$$

<u>Proof</u>: Let $f:V\oplus V \to V\oplus V$ be Γ-equivariant. We write $f = (f_1,f_2)$ where $f_j:V\oplus V \to V$ is Γ-equivariant. We need only show that

$$\dim P_2(\Gamma,V\oplus V,V) = 3 \dim P_2(\Gamma,V)$$

where $P_2(\Gamma,V\oplus V,V)$ is the space of quadratic Γ-equivariant maps $V\oplus V \to V$. Now, define $\phi:\{P_2(\Gamma,V)\}^3 \to P_2(\Gamma,V\oplus V,V)$ by

$$\phi(a,b,c)(v+w) = \tilde{a}(v,v) + \tilde{b}(v,w) + \tilde{c}(w,w),\ \forall\ (v,w) \in V\oplus V,$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are the symmetric bilinear forms associated with the quadratic mappings a,b,c respectively. We claim that ϕ is a linear isomorphism, as we now prove.

(i) ϕ is 1:1. Evaluating separately at $w=0$ and at $v=0$ shows that

$$\phi(a,b,c) = 0 \Rightarrow a=b=c=0.$$

(ii) ϕ is onto. Let A be in $P_2(\Gamma,V\oplus V,V)$, then

$$A(v+w) = \tilde{A}(v,v) + 2\tilde{A}(v,w) + \tilde{A}(w,w)$$

where $\tilde{A}$ is the symmetric bilinear form associated with A. Hence, $A =$

$\phi(A,2A,A)$. Clearly ϕ itself is Γ-equivariant; hence we have proved the claim.

Corollary 3.2: $P_2(\Gamma, V \oplus V)$ is nonzero $\Leftrightarrow$ $P_2(\Gamma, V)$ is nonzero.

This corollary gives a simple way to characterize the vanishing of the quadratic part due to symmetry. We now turn to some examples.

Example 3.3: Hopf bifurcation with $O(2)$ or D_n symmetry $(n \geq 4)$. Assume that equation (2.1) is invariant under a representation of the group $\Gamma = O(2)$ or D_n such that its restriction to V is irreducible and 2-dimensional. Then there exists a basis of V such that the $O(2)$ action in V is generated by the matrices

$$R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where either $\phi \in SO(2)$ or $\phi = 2\pi/n \in D_n$. This corresponds to the case of critical eigenvalues $\pm i\omega_0$ which are semisimple and double; the assumption of irreducibility of V is, in some sense, generic (see [13]). This problem was studied in [5] and [6], where the stability of bifurcated solutions was analysed by means of normal form. The stability conditions involve only terms up to cubic order in equation (2.2). There is no quadratic term in (2.2) if $\Gamma = O(2)$ or $\Gamma = D_n$ with $n \geq 4$ (this can be easily checked). Therefore, using Corollary 3.2 and Theorem 2.1, we conclude that the stability conditions in [5] and [6] are directly obtained from the Liapunov-Schmidt bifurcation equation. The case $\Gamma = D_3$ should provide an example where the Liapunov-Schmidt method does not give the correct stability assignments for bifurcating branches of periodic solutions.

Example 3.4: The Couette-Taylor problem. This is the problem of the onset of vortex flow in a fluid between two coaxial cylinders which rotate at independent constant speeds. When the cylinders are counterrotating, the following critical situations can occur:

(1) one pair of eigenvalues $\pm i\omega_0$,

(2) one 0 eigenvalue and one pair $\pm i\omega_0$ eigenvalues,

(3) two pairs of eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, $\omega_1 \neq \omega_2$.

Case 1 was studied in [4], case 2 (in a theoretical framework) in [7], and case 3 in [2]. The symmetries are the following: $SO(2)$ (rotation around the axis of cylinders) and O(2) (translations and reflections along the axis of cylinders). The O(2) symmetry comes from the assumption of infinite cylinders and periodicity of the flow in the axial direction (translations are then equivalent to rotations). The eigenvalues are double as a consequence of the reflectional symmetry and hence the dimension of equation (2.2) can be 4,6 or 8. In cases 1 and 3, the analysis in [4] and [2] was done directly on equation (2.2). We remark that equation (2.2) is already in normal form in this problem (for analysis of periodic bifurcations), because the rotational symmetry acts identically with the S^1 action required for the normal form. Here again, one can easily check that, because of symmetry, quadratic terms vanish in (2.2), hence stability conditions are obtained directly from the bifurcation equation (see Remark 2.5 (2) in case 2; an equivalent remark could be given for case 3).

Example 3.5: Onset of convection for a fluid in a rotating self-gravitating spherical shell [1]. When the shell does not rotate, bifurcation occurs at a zero eigenvalue with multiplicity $2\ell+1$, where ℓ is the "degree" of the associated irreducible representation of the group of rotations SO(3) (under which the problem is invariant). If the system is now slowly rotating, the 0 eigenvalue splits in one real eigenvalue and ℓ pairs of complex eigenvalues, leading to a complex diagram of Hopf bifurcations and interactions. In this problem, quadratic terms can be zero due to the SO(3) symmetry if ℓ is odd (see [14]) or, if ℓ is even. because of some physical condition in the problem (see [1]). In such a case, the stability analysis reduces to the bifurcation equations in the $(2\ell+1)$-dimensional invariant subspace of the nonrotating limit case. Numerical applications are given in [3].

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Pascal Chossat
Laboratoire de Mathematiques
Universite de Nice
Parc Valrose
F-06034 Nice Cedex
France

Martin Golubitsky
Department of Mathematics
University of Houston
University Park
Houston, TX 77004
USA