

The Abelian Hopf $H \bmod K$ Theorem*

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Abstract. We study the symmetries of periodic solutions obtained from Hopf bifurcation in systems with finite abelian symmetries. The $H \bmod K$ theorem gives necessary and sufficient conditions for the existence of periodic solutions with spatial symmetries K and spatio-temporal symmetries H in systems with finite symmetry group Γ . Our main result, the abelian Hopf $H \bmod K$ theorem, gives necessary and sufficient conditions for when these $H \bmod K$ periodic solutions can occur by Hopf bifurcation when Γ is a finite abelian group. We give examples of our results in the case when the symmetry group $\Gamma = \mathbf{Z}_l \times \mathbf{Z}_k$ acts on $\mathbf{R}^l \times \mathbf{R}^k$ by permutation of coordinates. In this case, we classify the $H \bmod K$ periodic solutions that are obtainable by a generic Hopf bifurcation and show that there exist families of $H \bmod K$ periodic solutions that cannot be obtained by Hopf bifurcation.

Key words. Hopf bifurcation, spatio-temporal symmetry, periodic solutions

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1. Introduction. In equivariant systems there are two ways of obtaining periodic solutions: the $H \bmod K$ theorem [1, 3] and the equivariant Hopf bifurcation theorem [4, 3]. When the symmetry group Γ is finite, the $H \bmod K$ theorem gives necessary and sufficient conditions for the existence of a Γ -equivariant system of ODEs and a periodic solution $x(t)$ to that system with specified spatio-temporal symmetries. The group of spatio-temporal symmetries of $x(t)$ is defined to be the subgroup $H \subseteq \Gamma$ that leaves the solution trajectory of $x(t)$ invariant. The subgroup $K \subseteq H$ is defined to be the symmetries that fix the solution trajectory pointwise. In addition, the $H \bmod K$ theorem classifies the spatio-temporal symmetries that are possible. In contrast, the equivariant Hopf theorem [3] guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for each \mathbf{C} -axial subgroup of $\Gamma \times \mathbf{S}^1$.

In [3] Golubitsky and Stewart showed that there are examples of periodic solutions with spatio-temporal symmetries guaranteed by the $H \bmod K$ theorem that cannot be obtained by generic Hopf bifurcation. In this article, we ask the more general question: Which periodic solution types, whose existence is guaranteed by the $H \bmod K$ theorem, are obtainable by Hopf bifurcation? Our main result, the abelian Hopf $H \bmod K$ theorem (see Theorem 5.1 and [2]), answers the question when the symmetry group Γ is finite abelian.

There are two important observations concerning Hopf bifurcation in systems with abelian symmetries: first, the group of spatio-temporal symmetries H of bifurcating periodic solutions

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is the isotropy subgroup of the point x_0 where Hopf bifurcation occurs; second, the center subspace is an H -simple representation with kernel K . Recall from [4] that a representation is H -simple if either it is the sum of two isomorphic absolutely irreducible representations or it is itself irreducible but not absolutely irreducible. Observations (3.1) and (3.2), together with the four conditions of the $H \bmod K$ theorem (Theorem 5.1), comprise the necessary and sufficient conditions of the abelian Hopf $H \bmod K$ theorem.

We outline the four steps in constructing $H \bmod K$ periodic solutions from Hopf bifurcation. The first step is the construction of an equivariant vector field that has a stable equilibrium with the desired isotropy. When Γ is finite, we show that every isotropy subgroup of Γ is the symmetry group for a stable equilibrium. Second, we show that the Jacobian of a Γ -equivariant vector field, with an equilibrium at x_0 , can be any linear map that commutes with the isotropy subgroup of x_0 .

Third, choose an appropriate Jacobian and apply the equivariant Hopf theorem to construct a locally H -equivariant vector field that yields a branch of $H \bmod K$ periodic solutions emanating from generic Hopf bifurcation at x_0 . Note that since $H = \Sigma_{x_0}$ at the point of generic Hopf bifurcation, x_0 , we apply the equivariant Hopf theorem extended to a point $x_0 \in \mathbf{R}^n$ that is not necessarily of full isotropy; that is, for each \mathbf{C} -axial subgroup of $H \times \mathbf{S}^1$, there exists a unique branch of periodic solutions with \mathbf{C} -axial spatio-temporal symmetries, emanating from x_0 .

Fourth, we show that an H -equivariant map, g , on an H -invariant subspace with an equilibrium at x_0 , can be extended to a Γ -equivariant mapping on all of \mathbf{R}^n . Moreover, the center subspace of the extended vector field at x_0 equals the center subspace of g at x_0 . In this case, we show that genericity of Σ_{x_0} -simple subspaces is given by Γ -equivariant mappings. The resulting Γ -equivariant vector field is defined on all of \mathbf{R}^n and yields families of periodic solutions from Hopf bifurcation with specified \mathbf{C} -axial spatio-temporal symmetries.

We end this paper with the example of $\Gamma = \mathbf{Z}_l \times \mathbf{Z}_k$ acting on $\mathbf{R}^l \times \mathbf{R}^k$. In this case, we classify the $H \bmod K$ periodic solutions that can be obtained by Hopf bifurcation and show that there are other types that cannot be obtained by generic Hopf bifurcation. This result generalizes the examples in [3].

2. The $H \bmod K$ theorem. The $H \bmod K$ theorem gives necessary and sufficient conditions for the existence of a periodic solution to some Γ -equivariant system of ODEs with specified spatio-temporal symmetries $K \subset H \subset \Gamma$. Recall that the *isotropy subgroup* Σ_x of a point $x \in \mathbf{R}^n$ consists of group elements that fix x , that is, $\Sigma_x = \{\sigma \in \Gamma : \sigma x = x\}$. Let $N(H)$ denote the *normalizer* of H in Γ , that is, $N(H) = \{\gamma \in \Gamma : \gamma H = H\gamma\}$. Finally, $\text{Fix}(K) = \{x \in \mathbf{R}^n : kx = x \ \forall k \in K\}$.

Theorem 2.1 ($H \bmod K$ [3, p. 65]). *Let Γ be a finite group acting on \mathbf{R}^n . There is a periodic solution to some Γ -equivariant system of ODEs on \mathbf{R}^n with spatial symmetries K and spatio-temporal symmetries H if and only if the following hold:*

- (a) H/K is cyclic.
- (b) K is an isotropy subgroup.
- (c) $\dim \text{Fix}(K) \geq 2$. If $\dim \text{Fix}(K) = 2$, then either $H = K$ or $H = N(K)$.
- (d) H fixes a connected component of $\text{Fix}(K) \setminus L_K$, where L_k is defined in (2.1) below.

Moreover, if (a)–(d) hold, the system can be chosen so that the periodic solution is stable.

Definition 2.2. Let $K \subset \Gamma$ be an isotropy subgroup. The variety L_k is defined by

$$(2.1) \quad L_k = \bigcup_{\gamma \notin K} \text{Fix}(\gamma) \cap \text{Fix}(K).$$

Definition 2.3. We call a pair of subgroups (H, K) admissible if the pair satisfies hypotheses (a)–(d) of Theorem 2.1, that is, if there exist periodic solutions to some Γ -equivariant system with (H, K) symmetry.

We can say more about admissible (H, K) pairs; they are robust to small perturbations in the following sense.

Proposition 2.4 (see [3, p. 68]). Let $x(t)$ be a hyperbolic periodic solution to a system of ODEs $\dot{x} = f(x)$ with spatio-temporal symmetries (H, K) . Let \tilde{f} be a small Γ -equivariant perturbation of f with a hyperbolic periodic solution $\tilde{x}(t)$ that is a small perturbation of $x(t)$. Then the spatio-temporal symmetries of \tilde{x} are also (H, K) .

The proof can be found in [3].

3. Hopf bifurcation with abelian symmetries. In this section we discuss the equivariant Hopf bifurcation theorem in the case when the symmetry group Γ is finite abelian. In doing so we extend the statement in [4, p. 264] in three ways. First, we allow the Hopf bifurcation to occur at any point x_0 with isotropy subgroup Σ_{x_0} ; second, we do not assume that the bifurcation has been reduced to the center manifold; and third, we identify the symmetry groups H and K . The first two extensions are well known (but not usually stated explicitly) and are included here for completeness. The third extension is needed in order to state our main result, Theorem 5.1. Let V be the center subspace of the bifurcation at x_0 , and note that Σ_{x_0} acts on V .

Theorem 3.1 (abelian Hopf theorem). In systems with abelian symmetry, generically, Hopf bifurcation at a point x_0 occurs with simple eigenvalues, and there exists a unique branch of small-amplitude periodic solutions emanating from x_0 . Moreover, the spatio-temporal symmetries of the bifurcating periodic solutions are

$$(3.1) \quad H = \Sigma_{x_0}$$

and

$$(3.2) \quad K = \ker_V(H),$$

and H acts H -simply on V .

The following two lemmas prove that center manifolds can be transformed into center subspaces by an equivariant change of coordinates. Let $x_0 \in \mathbf{R}^n$. Suppose that V is an Σ_{x_0} -invariant subspace of \mathbf{R}^n . Let $\hat{V} = x_0 + V$, and observe that \hat{V} is also Σ_{x_0} -invariant.

Lemma 3.2. Let g be an Σ_{x_0} -equivariant map on \hat{V} such that $g(x_0) = 0$. Then g extends to a Γ -equivariant mapping f on \mathbf{R}^n so that the center subspace of $(df)_{x_0}$ equals the center subspace of $(dg)_{x_0}$.

Proof. Let $W = V^\perp$ and $\hat{W} = x_0 + W$. Let \hat{U} be an H -invariant neighborhood of x_0 in \hat{V} , and let Z be a cylinder over \hat{U} ; that is,

$$Z = \{z = (v, w) \in \hat{U} \times \hat{W} : w \text{ is small}\}.$$

Choose \hat{U} and Z small enough so that $\gamma Z \cap Z = \emptyset$ when $\gamma \notin \Sigma_{x_0}$ and $\gamma Z = Z$ when $\gamma \in \Sigma_{x_0}$. Extend g to an Σ_{x_0} -equivariant function f on Z defined by

$$f(v, w) = (g(v), -w).$$

Next, extend f to

$$\Gamma Z \equiv \bigcup_{\gamma \in \Gamma} \gamma(Z)$$

by $f(\gamma z) = \gamma f(z)$ on ΓZ . The H -equivariance of g guarantees that f is well defined and Γ -equivariant on ΓZ . Now extend f to \mathbf{R}^n in any way and average over Γ to get a Γ -equivariant function h on all of \mathbf{R}^n ; that is,

$$h(x) = \sum_{\Gamma} \gamma^{-1} f(\gamma x).$$

It is straightforward to check that h is f restricted to \hat{V} . ■

Next we show that center manifolds can be linearized in an equivariant fashion.

Lemma 3.3. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be Γ -equivariant, and let $f(x_0) = 0$. Let V be the center subspace of $(df)_{x_0}$. Then there exists a Γ -equivariant diffeomorphism $\psi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $\psi(x_0) = x_0$ and the center manifold of the transformed vector field*

$$\psi_* f(x) \equiv (d\psi)_{\psi(x)}^{-1} f(\psi(x))$$

is \hat{V} .

Proof. Let M be a center manifold of f at x_0 . According to the center manifold theorem with symmetry [6] we may assume that M is H -invariant. In the coordinates used in Lemma 3.2 we can write M as the graph of an H -equivariant function $\varphi : \hat{V} \rightarrow \hat{W}$; that is, $(v, \varphi(v))$ parametrizes the center manifold on a neighborhood of x_0 . Define the H -invariant vector field p on a cylinder Z by $p(v, w) = (0, \varphi(v))$. Note that the time 1 map of p restricted to \hat{V} maps \hat{V} onto a neighborhood of x_0 in M .

Extend p to a Γ -invariant vector field on \mathbf{R}^n , and let ψ^{-1} be the Γ -equivariant time 1 diffeomorphism of p on \mathbf{R}^n . Then ψ is the desired diffeomorphism. ■

Proof of Theorem 3.1. We begin by proving that generically equivariant Hopf bifurcation leads to a unique branch of small-amplitude periodic solutions emanating from x_0 when Γ is finite abelian. It follows from Lemma 3.2 that we may assume without loss of generality that the bifurcation point $x_0 = 0$ and hence that $\Gamma = \Sigma_{x_0}$. It follows from Lemma 3.3 that, by reducing to the center manifold, we may assume that $\mathbf{R}^n = V$. In this case generically the center subspace V at a point of Hopf bifurcation is Γ -simple [4, p. 264]. Since the irreducible representations of abelian groups are one-dimensional and absolutely irreducible or two-dimensional and nonabsolutely irreducible [4, p. 63], it follows that V is two-dimensional and, hence, that the eigenvalues of the Jacobian are simple. The standard Hopf theorems now apply to obtain the unique branch of periodic solutions.

Next we discuss the forms of H and K . Let λ be the bifurcation parameter, and let $x(t, \lambda)$ be the unique smooth branch of small-amplitude periodic solutions that emanate from a generic Hopf bifurcation at the point x_0 . Note that for each t ,

$$x_0 = \lim_{\lambda \rightarrow 0} x(t, \lambda).$$

Let H be the spatio-temporal symmetry subgroup of $x(\cdot, \lambda)$, and let $\Phi : H \rightarrow \mathbf{S}^1$ be the homomorphism that associates a symmetry $h \in H$ with a phase shift $\Phi(h) \in \mathbf{S}^1$. We first prove that $H \subset \Sigma_{x_0}$. Note that

$$\begin{aligned} hx_0 &= \lim_{\lambda \rightarrow 0} hx(0, \lambda) \quad \text{by continuity of } h \\ &= \lim_{\lambda \rightarrow 0} x(\Phi(h), \lambda) \quad \text{by definition of spatio-temporal symmetries} \\ &= x_0. \end{aligned}$$

Therefore, $h \in \Sigma_{x_0}$.

Next we show that $\Sigma_{x_0} \subset H$. Let $\gamma \in \Sigma_{x_0}$. Since $\gamma \in \Sigma_{x_0} \subseteq \Gamma$ is a symmetry, it follows that $\gamma x(t, \lambda)$ is also a periodic solution. Since we have proved that there is a unique solution trajectory for each λ , it follows that

$$\gamma\{x(t, \lambda)\} = \{x(t, \lambda)\}$$

and hence $\gamma \in H$.

Using Lemma 3.3, we may assume that the center manifold at x_0 is $\hat{V} = V + x_0$, which may be identified with V . Hence, we may assume that V is H -invariant. Furthermore, we have shown that V is H -simple since Γ is abelian.

Finally, we verify (3.2) by showing that the kernel of the action of H on V is K . Since \hat{V} is the center manifold of f at x_0 , \hat{V} is f invariant and f is H -equivariant. It follows that bifurcating solutions remain in \hat{V} . Let $x(t) \in \hat{V}$ be a periodic solution emanating from x_0 with spatio-temporal symmetries H . Since $x(t)$ is fixed by K , $\dim(V) = 2$ implies that V is fixed by K . Therefore, $K \subset \ker_V H$. Conversely, if $\gamma \in \ker H$ on V , then for all t , $\gamma x(t) = x(t)$ on \hat{V} . Therefore, $\gamma \in K$ and $K = \ker_V H$. ■

4. Constructing symmetric systems near Hopf points. When Γ is finite abelian, a key step in constructing H mod K periodic solutions from Hopf bifurcation at x_0 is the construction of a locally Σ_{x_0} -equivariant vector field.

We first construct, for finite symmetry groups, a Γ -equivariant vector field that has a stable equilibrium, $x_0 \in \mathbf{R}^n$, with the desired isotropy.

Theorem 4.1. *Let Γ be a finite group acting on \mathbf{R}^n , and let $x_0 \in \mathbf{R}^n$. Then there exists a Γ -equivariant system of ODEs on \mathbf{R}^n with a stable equilibrium x_0 .*

We use the following interpolation lemma from [5] in the proof.

Lemma 4.2 (see [5, pp. 306–307]). *For any finite set of distinct points y_1, \dots, y_l , vectors v_1, \dots, v_l in \mathbf{R}^n , and matrices $A_1, \dots, A_l \in \text{GL}(n)$, there exists a polynomial map $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $g(y_j) = v_j$ and $(dg)_{y_j} = A_j$.*

Proof of Theorem 4.1. Let smooth $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a polynomial map as in Lemma 4.2 such that for all $\gamma \in \Gamma$, $g(\gamma x_0) = 0$ and $(dg)_{\gamma x_0} = -I$. Define f to be the average of g over the group Γ ; that is,

$$(4.1) \quad f(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^{-1} g(\gamma x).$$

See [3, p. 67]. Then f is Γ -equivariant, $f(x_0) = 0$, and

$$(df)_{x_0} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^{-1} (dg)_{\gamma x_0} \gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (-I) = -I.$$

Therefore, $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is Γ -equivariant and has x_0 as a stable equilibrium. ■

Theorem 4.1 shows that any point $x_0 \in \mathbf{R}^n$ can be a stable equilibrium for some Γ -equivariant vector field $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$. We know that $(df)_{x_0}$ must commute with the isotropy subgroup Σ_{x_0} of x_0 [3, p. 37]. The following is an analogous result that states that the Jacobian at x_0 can be any linear map that commutes with the isotropy subgroup, which we now prove.

Theorem 4.3. *Let $x_0 \in \mathbf{R}^n$, and let $A : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a linear map that commutes with the isotropy subgroup Σ_{x_0} . Then there exists a polynomial Γ -equivariant vector field $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $f(x_0) = 0$ and $(df)_{x_0} = A$.*

Proof. By Lemma 4.2 there is a polynomial $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $g(\gamma x_0) = 0$ and

$$(dg)_{\gamma x_0} = \gamma A \gamma^{-1}$$

for all $\gamma \in \Gamma$. This assignment is possible because A commutes with Σ_{x_0} . Obtain f from g by averaging over Γ as in (4.1). It follows that f is Γ -equivariant, x_0 is an equilibrium of f , and

$$(df)_{x_0} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^{-1} (dg)_{\gamma x_0} \gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^{-1} (\gamma A \gamma^{-1}) \gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} A = A,$$

thus proving the lemma. ■

When constructing Hopf bifurcation at points $x_0 \in \mathbf{R}^n$ that are not necessarily of full isotropy, we note that genericity of Σ_{x_0} -simple subspaces at points of Hopf bifurcation is given by Γ -equivariant mappings. We prove this point in the following lemma.

Lemma 4.4. *Let Γ act on \mathbf{R}^n , and fix $x_0 \in \mathbf{R}^n$. Let V be a Σ_{x_0} -invariant neighborhood of x_0 such that $\gamma \bar{V} \cap \bar{V} = \emptyset$ for any $\gamma \in \Gamma \setminus \Sigma_{x_0}$. Let $g : \bar{V} \times \mathbf{R} \rightarrow \mathbf{R}^n$ be a smooth Σ_{x_0} -equivariant vector field. Then there exists an extension of g to a smooth Γ -equivariant vector field $f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$.*

Proof. Define g on γV by $g(x, \lambda) = \gamma g(\gamma^{-1} x, \lambda)$, and f by averaging g over Γ as in (4.1). Note that the extended g is well defined by Σ_{x_0} -equivariance. It is straightforward that f is a smooth extension of g and that f is Γ -equivariant. ■

5. The abelian Hopf $H \bmod K$ theorem. The abelian Hopf $H \bmod K$ theorem gives necessary and sufficient conditions for when an $H \bmod K$ periodic solution $x(t)$ can occur by Hopf bifurcation from a point x_0 .

Theorem 5.1 (abelian Hopf $H \bmod K$ theorem). *Let Γ be a finite abelian group acting on \mathbf{R}^n . There is an $H \bmod K$ periodic solution that arises by a generic Hopf bifurcation if and only if the following six conditions hold: Theorem 2.1(a)–(d), H is an isotropy subgroup, and there exists an H -simple subspace V such that $K = \ker_V(H)$.*

Definition 5.2. *We call a pair of subgroups (H, K) of an abelian group Hopf-admissible if the pair satisfies the six hypotheses of the abelian Hopf $H \bmod K$ theorem, Theorem 5.1.*

Proof. The proof of necessity follows directly from the $H \bmod K$ theorem (Theorem 2.1) and the abelian Hopf theorem (Theorem 3.1). Next, we show sufficiency. Let $x_0 \in \mathbf{R}^n$

such that $H = \Sigma_{x_0}$. Let W be a Σ_{x_0} -simple representation. Since Γ is abelian, W is two-dimensional. Define the linear maps $A(\lambda) : W \rightarrow W$ by

$$A(\lambda) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}.$$

Then $A(\lambda)$ commutes with $H = \Sigma_{x_0}$ on W .

By Theorem 4.3 extended to a bifurcation problem as in Lemma 4.4, we can let $f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ be a Γ -equivariant polynomial such that for all $\gamma \in \Gamma$, $f(\gamma x_0, \lambda) = 0$ and $(df)_{\gamma x_0, \lambda}|_{\gamma W} = \gamma A(\lambda) \gamma^{-1}$. Let $g = f|_{W+x_0}$. By construction of f , g is H -equivariant on $W + x_0$ and $g(x_0) = 0$. Lemma 3.2 implies that W is the center subspace of $(dg)_{x_0, 0}$.

Consider $(dg)_{x_0, \lambda}|_W$. Its eigenvalues are $\sigma(\lambda) \pm i\rho(\lambda)$ such that $\sigma(0) = 0$, $\rho(0) = 1$, and $\sigma'(0) \neq 0$. The equivariant Hopf theorem extended to a point $x_0 \in \mathbf{R}^n$ [2, p. 40] implies the existence of small-amplitude periodic solutions emanating from x_0 with spatio-temporal symmetries $H = \Sigma_{x_0}$ and space symmetries K . ■

Suppose that $x(t)$ is a periodic solution with spatio-temporal symmetries H ; then $\gamma x(t)$ is also a periodic solution with spatio-temporal symmetries H , since Γ is abelian. Since $\{\gamma x(t)\} = \{x(t)\}$ when $\gamma \in H$, it follows that there are $|\Gamma/H|$ distinct (conjugate) periodic trajectories.

6. Examples. Some examples of pairs of subgroups that are admissible but not Hopf-admissible were exhibited in [3]. In this section we use the abelian Hopf H mod K theorem (Theorem 5.1) to exhibit classes of admissible symmetry pairs that are not Hopf-admissible. Specifically, let \mathbf{Z}_l act on \mathbf{R}^l by cyclic permutation of coordinates, and $\Gamma = \mathbf{Z}_l \times \mathbf{Z}_k$ act on $\mathbf{R}^l \times \mathbf{R}^k$ by the diagonal action, where $l, k > 1$. We exhibit admissible but not Hopf-admissible pairs for this action of Γ by classifying in Theorem 6.1 all Hopf admissible pairs $K \subset H \subset \Gamma$ and showing that there are admissible pairs that are not on this list.

Theorem 6.1. *The (H, K) Hopf-admissible pairs in Γ are $(\mathbf{Z}_m \times \mathbf{Z}_n, \mathbf{Z}_m \times \mathbf{Z}_q)$, where q divides n except when $q = \frac{k}{2}$ and $n = k$, and $(\mathbf{Z}_m \times \mathbf{Z}_n, \mathbf{Z}_p \times \mathbf{Z}_n)$, where p divides m except when $p = \frac{l}{2}$ and $m = l$.*

It is not difficult to find $K \subset H$ pairs that are admissible but are not Hopf-admissible. For example, let $K = 1$ and $H = \mathbf{Z}_l \times \mathbf{Z}_k$, where $k, l \geq 3$ are coprime. Observe that $H/K \cong \mathbf{Z}_{lk}$ is cyclic, K is an isotropy subgroup, and $\dim \text{Fix}(K) = l + k \geq 7 > 2$. Our discussion at the end of the proof of Theorem 6.1 shows that the last condition of the H mod K theorem is also satisfied. It is much more complicated to classify all possible admissible pairs.

We begin by discussing some necessary conditions for the pair (H, K) to be Hopf-admissible. Theorem 2.1(b) and (3.1) imply that both H and K must be isotropy subgroups. Moreover, all isotropy subgroups of this Γ -action have the form $\mathbf{Z}_m \times \mathbf{Z}_n$. Thus we may assume that $H = \mathbf{Z}_m \times \mathbf{Z}_n$ and $K = \mathbf{Z}_p \times \mathbf{Z}_q$. Note that $K \subseteq H$ if p divides m and q divides n .

In this example, (3.2) states that there is a $\mathbf{Z}_m \times \mathbf{Z}_n$ -simple representation $U \subset \mathbf{R}^l \times \mathbf{R}^k$ such that $\mathbf{Z}_p \times \mathbf{Z}_q = \ker_U(\mathbf{Z}_m \times \mathbf{Z}_n)$. If $H = K$, then we can choose the trivial representation since $\dim \text{Fix}(H) \geq 2$. If $H \supset K$, then we note that the possible irreducible representations are subspaces of $(\mathbf{R}^l, 0)$ or $(0, \mathbf{R}^k)$. To see this, observe that H decomposes $\mathbf{R}^l \times \mathbf{R}^k$ into isotypic components as follows:

$$\mathbf{R}^l \times \mathbf{R}^k = \text{Fix}(H) \oplus (V_1, 0) \oplus \cdots \oplus (V_s, 0) \oplus (0, W_1) \oplus \cdots \oplus (0, W_t),$$

where each V_i is an isotypic component of a nontrivial representation of \mathbf{Z}_m acting on \mathbf{R}^l and each W_j is an isotypic component of a nontrivial representation of \mathbf{Z}_n acting on \mathbf{R}^k . This decomposition follows directly from the fact that each subspace is H -invariant and no representation V_i is H -isomorphic to a W_j . Note also that the kernel of H acting on $(V_i, 0)$ contains $\{0\} \times \mathbf{Z}_n$, and the kernel of H acting on $(0, W_j)$ contains $\mathbf{Z}_m \times \{0\}$. It follows from this decomposition that, for $\mathbf{Z}_p \times \mathbf{Z}_q$ to be the kernel of an H -simple representation, either $p = m$ or $q = n$. Thus, Theorem 2.1(a), which states that H/K must be cyclic, is always valid, since $H/K \cong Z_a$, where either $a = \frac{m}{p}$ or $a = \frac{n}{q}$.

Moreover, to answer the question of whether there is an H -simple representation with kernel exactly K , we need only consider the simpler question for the action of \mathbf{Z}_k on \mathbf{R}^k . The result is as follows.

Proposition 6.2. *Let \mathbf{Z}_k act on \mathbf{R}^k by cyclic permutation of coordinates. Let $\mathbf{Z}_q \subseteq \mathbf{Z}_n \subseteq \mathbf{Z}_k$. Then there is a \mathbf{Z}_n -simple representation with kernel \mathbf{Z}_q with the single exception when $n = k$ is even and $q = \frac{k}{2}$.*

Proof. Consider the action of \mathbf{Z}_N on \mathbf{R}^N by cyclic permutation. It is well known (see [4]) that when $N > 2$ this representation contains the standard irreducible representation of \mathbf{Z}_N on \mathbf{R}^2 and that this representation is nonabsolutely irreducible. Hence, there is a \mathbf{Z}_N -simple representation with kernel 1. When $N = 2$, the (unique) nontrivial irreducible representation of \mathbf{Z}_2 is one-dimensional and absolutely irreducible. Since this representation occurs only once in \mathbf{R}^2 , there is no \mathbf{Z}_2 -simple representation with kernel 1.

Next consider the action of \mathbf{Z}_N on $(\mathbf{R}^S)^N$, which cycles the \mathbf{R}^S factors. It follows from the previous discussion that the natural representation of \mathbf{Z}_N occurs S times. Hence, when $S > 1$ there is a \mathbf{Z}_N -simple representation with kernel 1 for all $N \geq 2$.

Finally, we can divide by the kernel \mathbf{Z}_q to look for a \mathbf{Z}_N -simple representation, where $N = \frac{n}{q}$, on the space $\text{Fix}(\mathbf{Z}_q) = \mathbf{R}^{\frac{k}{q}} = (\mathbf{R}^S)^N$, where $S = \frac{k}{n}$, with kernel 1. Our previous discussion shows that we always have such a \mathbf{Z}_N -simple representation unless $n = k$ (that is, $S = 1$) and $q = \frac{k}{2}$ (which requires that k be even). ■

Let $\rho_k \in \mathbf{Z}_k$ act on \mathbf{R}^k by

$$(6.1) \quad \rho_k(x_1, x_2, \dots, x_k) = (x_2, \dots, x_k, x_1).$$

Suppose that s is a positive integer that divides k . Then

$$(6.2) \quad \text{Fix}(\rho_k^s) = \{(x_1, \dots, x_s, x_1, \dots, x_s, \dots, x_1, \dots, x_s)\}$$

and

$$(6.3) \quad \dim \text{Fix}(\rho_k^s) = s.$$

Note that q divides k (since $\mathbf{Z}_q \subset \mathbf{Z}_k$) and that a generator for \mathbf{Z}_q is ρ_k^d , where $d = \frac{k}{q}$. Therefore, $\dim \text{Fix}(\mathbf{Z}_q) = d \geq 1$.

Lemma 6.3. *Assume that s divides k . Let C be the codimension of $\text{Fix}(\rho_k^a) \cap \text{Fix}(\rho_k^s)$ in $\text{Fix}(\rho_k^s)$. Then*

$$C = \begin{cases} 0 & \text{if } a \text{ is a multiple of } s, \\ 1 & \text{if } a \text{ is odd and } s = 2, \\ \geq 2 & \text{otherwise.} \end{cases}$$

Proof. The fixed-point subspace of ρ_k^s is s -dimensional and has the form in (6.2). We can assume that $0 < a < s$ by writing $a = a' + bs$ and observing that ρ_k^{bs} acts as the identity on $\text{Fix}(\rho_k^s)$. If $a = 0$ (or $a = s$), the codimension is 0. If $a = 1$ and $s = 2$, then the desired codimension is 1. Otherwise the number of equalities forced on the coordinates of points in (6.2) is greater than 1, and the codimension is at least 2. ■

Proof of Theorem 6.1. By construction the pair H and K defined in the statement of this theorem satisfies Theorem 2.1(a)–(b), (3.1), and (3.2). We must show that H and K also satisfy Theorem 2.1(c)–(d).

Since $\text{Fix}(K) = \text{Fix}(\mathbf{Z}_p) \oplus \text{Fix}(\mathbf{Z}_q)$, it follows that $\dim \text{Fix}(K) = c + d \geq 2$, where $c = \frac{l}{p}$ and $d = \frac{k}{q}$. Thus, if $\dim \text{Fix}(K) = 2$, we see that $c = d = 1$ and $K = \Gamma$. So either $H = K = \Gamma$ or $\dim \text{Fix}(K) > 2$. In either case Theorem 2.1(c) is satisfied.

Finally, we consider the restrictions placed on the pair (H, K) by Theorem 2.1(d). Since K fixes $\text{Fix}(K)$ pointwise, every element in K fixes every connected component of the complement of L_K . Therefore $H = K$ always satisfies Theorem 2.1(d).

There is a possible nontrivial restriction given by Theorem 2.1(d) only if there is $\sigma \notin K$ such that $\text{Fix}(\sigma) \cap \text{Fix}(K)$ has codimension 1 in $\text{Fix}(K)$. We can write any $\sigma \in \Gamma$ as $\sigma = (\rho_l^\alpha, \rho_k^\beta)$. Note that

$$\text{Fix}(\sigma) \cap \text{Fix}(K) = [\text{Fix}(\rho_l^\alpha) \cap \text{Fix}(\rho_l^c), \text{Fix}(\rho_k^\beta) \cap \text{Fix}(\rho_k^d)].$$

It follows that $\text{Fix}(\sigma) \cap \text{Fix}(K)$ has codimension 1 in $\text{Fix}(K)$ if $\text{Fix}(\rho_l^\alpha) \cap \text{Fix}(\rho_l^c)$ has codimension 0 in $\text{Fix}(\rho_l^c)$ (that is, $\text{Fix}(\rho_l^\alpha) \supset \text{Fix}(\rho_l^c)$) and $\text{Fix}(\rho_k^\beta) \cap \text{Fix}(\rho_k^d)$ has codimension 1 in $\text{Fix}(\rho_k^d)$ or vice versa. The only way this can happen is if

$$\alpha \text{ is a multiple of } c \quad \text{and} \quad d = 2 \text{ and } \beta \text{ is odd}$$

or

$$\beta \text{ is a multiple of } d \quad \text{and} \quad c = 2 \text{ and } \alpha \text{ is odd.}$$

So there can be a new restriction given by Theorem 2.1(d) only if $d = 2$ or $c = 2$ (that is, $k = 2q$ or $l = 2p$).

Suppose $d = 2$. Then the pairs that might have difficulty are $K = \mathbf{Z}_m \times \mathbf{Z}_{\frac{k}{2}} = H$, $K = \mathbf{Z}_m \times \mathbf{Z}_{\frac{k}{2}} \subset H = \mathbf{Z}_m \times \mathbf{Z}_k$, and $K = \mathbf{Z}_p \times \mathbf{Z}_{\frac{k}{2}} \subset H = \mathbf{Z}_m \times \mathbf{Z}_{\frac{k}{2}}$. The first pair satisfies $H = K$, which is always admissible. The second pair has already been excluded. A calculation shows that the third pair satisfies Theorem 2.1(d) except in the case $p = \frac{l}{2}$ and $m = l$, which was previously excluded. A similar argument works when $c = 2$. ■

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